

Equivariant Cohomological Chern Characters

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Abstract

We construct for an equivariant cohomology theory for proper equivariant CW -complexes an equivariant Chern character, provided that certain conditions about the coefficients are satisfied. These conditions are fulfilled if the coefficients of the equivariant cohomology theory possess a Mackey structure. Such a structure is present in many interesting examples.

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0. Introduction

The purpose of this paper is to construct an equivariant Chern character for a proper equivariant cohomology theory $\mathcal{H}_?^*$ with values in R -modules for a commutative associative ring R with unit which satisfies $\mathbb{Q} \subseteq R$. It is a natural transformation of equivariant cohomology theories

$$\text{ch}_?^*: \mathcal{H}_?^* \rightarrow \mathcal{BH}_?^*$$

for a given equivariant cohomology theory $\mathcal{H}_?^*$. Here $\mathcal{BH}_?^*$ is the associated equivariant cohomology theory which is defined by the Bredon cohomology with coefficients coming from the coefficients of $\mathcal{H}_?^*$. The notion of an equivariant cohomology theory and examples for it are presented in Section 1 and the associated Bredon cohomology is explained in Section 3. The point is that $\mathcal{BH}_?^*$ is

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much simpler and easier to compute than $\mathcal{H}_?^*$. If $\mathcal{H}_?^*$ satisfies the disjoint union axiom, then

$$\text{ch}_G^n(X, A): \mathcal{H}_G^n(X, A) \xrightarrow{\cong} \mathcal{B}\mathcal{H}_G^n(X, A)$$

is bijective for every discrete group G , proper G -CW-pair (X, A) and $n \in \mathbb{Z}$.

The Chern character $\text{ch}_?^*$ is only defined if the coefficients of $\mathcal{H}_?^*$ satisfy a certain injectivity condition (see Theorem 4.6). This condition is fulfilled if the coefficients of $\mathcal{H}_?^*$ come with a Mackey structure (see Theorem 5.5) what is the case in many interesting examples.

The equivariant cohomological Chern character is a generalization to the equivariant setting of the classical non-equivariant Chern character for a (non-equivariant) cohomology theory \mathcal{H}^* (see Example 4.1)

$$\text{ch}^n(X, A): \mathcal{H}^n(X, A) \xrightarrow{\cong} \prod_{p+q=n} H^p(X, A, \mathcal{H}^q(*)).$$

The equivariant cohomological Chern character has already been constructed in the special case, where $\mathcal{H}_?^*$ is equivariant topological K -theory $K_?^*$, in [12]. Its homological version has already been treated in [8] and plays an important role in the computation of the source of the assembly maps appearing in the Farrell-Jones Conjecture for $K_n(RG)$ and $L_n^{\langle -\infty \rangle}(RG)$ and the Baum-Connes Conjecture for $K_n(C_r^*(G))$ (see also [9]).

The detailed formulation of the main result of this paper is presented in Theorem 5.5.

The equivariant Chern character will play a key role in the proof of the following result which will be presented in [11].

Theorem 0.1 (Rational computation of the topological K -theory of BG). *Let G be a discrete group. Suppose that there is a finite G -CW-model for the classifying space $\underline{E}G$ for proper G -actions. Then there is a \mathbb{Q} -isomorphism*

$$\begin{aligned} \overline{\text{ch}}_G^n: K^n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \xrightarrow{\cong} \prod_{i \in \mathbb{Z}} H^{2i+n}(BG; \mathbb{Q}) \times \prod_{p \text{ prime}} \prod_{(g) \in \text{con}_p(G)} H^{2i+n}(BC_G(g); \widehat{\mathbb{Q}}_p). \end{aligned}$$

Here $\text{con}_p(G)$ is the set of conjugacy classes (g) of elements $g \in G$ of order p^m for some integer $m \geq 1$ and $C_G(g)$ is the centralizer of the cyclic subgroup generated by g in G .

The assumption in Theorem 0.1 that there is a finite G -CW-model for the classifying space $\underline{E}G$ for proper G -actions is satisfied for instance, if G is word-hyperbolic in the sense of Gromov, if G is a cocompact subgroup of a Lie group with finitely many path components, if G is a finitely generated one-relator group, if G is an arithmetic group, a mapping class group of a compact surface or the group of outer automorphisms of a finitely generated free group. For more information about $\underline{E}G$ we refer for instance to [1] and [10].

A group G is always understood to be discrete and a ring R is always understood to be associative with unit throughout this paper.

The paper is organized as follows:

1. Equivariant Cohomology Theories
 2. Modules over a Category
 3. The Associated Bredon Cohomology Theory
 4. The Construction of the Equivariant Cohomological Chern Character
 5. Mackey Functors
 6. Multiplicative Structures
- References

1. Equivariant Cohomology Theories

In this section we describe the axioms of a (proper) equivariant cohomology theory. They are dual to the ones of a (proper) equivariant homology theory as described in [8, Section 1].

Fix a group G and a commutative ring R . A G -CW-pair (X, A) is a pair of G -CW-complexes. It is *proper* if all isotropy groups of X are finite. It is *relative finite* if X is obtained from A by attaching finitely many equivariant cells, or, equivalently, if $G \backslash (X/A)$ is compact. Basic information about G -CW-pairs can be found for instance in [7, Section 1 and 2]. A G -cohomology theory \mathcal{H}_G^* with values in R -modules is a collection of covariant functors \mathcal{H}_G^n from the category of G -CW-pairs to the category of R -modules indexed by $n \in \mathbb{Z}$ together with natural transformations $\delta_G^n(X, A): \mathcal{H}_G^n(X, A) \rightarrow \mathcal{H}_G^{n+1}(A) := \mathcal{H}_G^{n+1}(A, \emptyset)$ for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

- G -homotopy invariance

If f_0 and f_1 are G -homotopic maps $(X, A) \rightarrow (Y, B)$ of G -CW-pairs, then $\mathcal{H}_G^n(f_0) = \mathcal{H}_G^n(f_1)$ for $n \in \mathbb{Z}$;

- Long exact sequence of a pair

Given a pair (X, A) of G -CW-complexes, there is a long exact sequence

$$\dots \xrightarrow{\delta_G^{n-1}} \mathcal{H}_G^n(X, A) \xrightarrow{\mathcal{H}_G^n(j)} \mathcal{H}_G^n(X) \xrightarrow{\mathcal{H}_G^n(i)} \mathcal{H}_G^n(A) \xrightarrow{\delta_G^n} \dots,$$

where $i: A \rightarrow X$ and $j: X \rightarrow (X, A)$ are the inclusions;

- Excision

Let (X, A) be a G -CW-pair and let $f: A \rightarrow B$ be a cellular G -map of G -CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a G -CW-pair. Then the canonical map $(F, f): (X, A) \rightarrow (X \cup_f B, B)$ induces an isomorphism

$$\mathcal{H}_G^n(F, f): \mathcal{H}_G^n(X, A) \xrightarrow{\cong} \mathcal{H}_G^n(X \cup_f B, B).$$

Sometimes also the following axiom is required.

- Disjoint union axiom

Let $\{X_i \mid i \in I\}$ be a family of G -CW-complexes. Denote by $j_i: X_i \rightarrow \coprod_{i \in I} X_i$ the canonical inclusion. Then the map

$$\prod_{i \in I} \mathcal{H}_G^n(j_i): \mathcal{H}_G^n\left(\prod_{i \in I} X_i\right) \xrightarrow{\cong} \prod_{i \in I} \mathcal{H}_G^n(X_i)$$

is bijective.

If \mathcal{H}_G^* is defined or considered only for proper G -CW-pairs (X, A) , we call it a *proper G -cohomology theory \mathcal{H}_G^* with values in R -modules*.

The role of the disjoint union axiom is explained by the following result. Its proof for non-equivariant cohomology theories (see for instance [16, 7.66 and 7.67]) carries over directly to G -cohomology theories.

Lemma 1.1. *Let \mathcal{H}_G^* and \mathcal{K}_G^* be (proper) G -cohomology theories. Then*

- (a) *Suppose that \mathcal{H}_G^* satisfies the disjoint union axiom. Then there exists for every (proper) G -CW-pair (X, A) with an exhaustion by subcomplexes $A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq \bigcup_{n \geq -1} X_n = X$ a natural short exact sequence*

$$0 \rightarrow \lim_{n \rightarrow \infty}^1 \mathcal{H}_G^{p-1}(X_n \cup A, A) \rightarrow \mathcal{H}^p(X, A) \rightarrow \lim_{n \rightarrow \infty} \mathcal{H}_G^p(X_n \cup A, A) \rightarrow 0;$$

- (b) *Let $T^*: \mathcal{H}_G^* \rightarrow \mathcal{K}_G^*$ be a transformation of (proper) G -cohomology theories, i.e. a collection of natural transformations $T^n: \mathcal{H}_G^n \rightarrow \mathcal{K}_G^n$ of contravariant functors from the category of (proper) G -CW-pairs to the category of R -modules indexed by $n \in \mathbb{Z}$ which is compatible with the boundary operator associated to (proper) G -CW-pairs. Suppose that $T^n(G/H)$ is bijective for every (proper) homogeneous space G/H and $n \in \mathbb{Z}$.*

Then $T^n(X, A): \mathcal{H}_G^n(X, A) \rightarrow \mathcal{K}_G^n(X, A)$ is bijective for all $n \in \mathbb{Z}$ provided that (X, A) is relative finite or that both \mathcal{H}^ and \mathcal{K}^* satisfy the disjoint union axiom.*

Remark 1.2 (The disjoint union axiom is not compatible with $-\otimes_{\mathbb{Z}} \mathbb{Q}$).

Let \mathcal{H}_G^* be a G -cohomology theory with values in \mathbb{Z} -modules. Then $\mathcal{H}_G^* \otimes_{\mathbb{Z}} \mathbb{Q}$ is a G -cohomology theory with values in \mathbb{Q} -modules since \mathbb{Q} is flat as \mathbb{Z} -module. However, even if \mathcal{H}^* satisfies the disjoint union axiom, $\mathcal{H}_G^* \otimes_{\mathbb{Z}} \mathbb{Q}$ does not satisfy the disjoint union axiom since $-\otimes_{\mathbb{Z}} \mathbb{Q}$ is not compatible with products over arbitrary index sets.

Example 1.3 (Rationalizing topological K -theory). Consider for instance the (non-equivariant) cohomology theory with values in \mathbb{Z} -modules satisfying the disjoint union axiom given by topological K -theory K^* . Let $K^*(-; \mathbb{Q})$ be the cohomology theory associated to the rationalization of the K -theory spectrum. This is a (non-equivariant) cohomology theory with values in \mathbb{Q} -modules satisfying the disjoint union axiom. There is a natural transformation

$$T^*(X): K^*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K^*(X; \mathbb{Q})$$

of (non-equivariant) cohomology theory with values in \mathbb{Q} -modules. The \mathbb{Q} -map $T^n(\{\text{pt.}\})$ is bijective for all $n \in \mathbb{Z}$. Hence $T^n(X)$ is bijective for all finite CW -complexes by Lemma 1.1 (b). Notice that $K^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ does not satisfy the disjoint union axiom in contrast to $K^*(X; \mathbb{Q})$. Hence we cannot expect $T^n(X)$ to be bijective for all CW -complexes. Consider the case $X = BG$ for a finite group G . Since $H^p(BG; \mathbb{Q}) \cong H^p(\{\text{pt.}\}; \mathbb{Q})$ for all $p \in \mathbb{Z}$, one obtains $K^p(BG; \mathbb{Q}) \cong K^p(\{\text{pt.}\}; \mathbb{Q})$ for all $p \in \mathbb{Z}$. By the Atiyah-Segal Completion Theorem $K^p(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K^p(\{\text{pt.}\}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is only true if and only if the finite group G is trivial.

Let $\alpha: H \rightarrow G$ be a group homomorphism. Given an H -space X , define the *induction of X with α* to be the G -space $\text{ind}_{\alpha} X$ which is the quotient of $G \times X$ by the right H -action $(g, x) \cdot h := (g\alpha(h), h^{-1}x)$ for $h \in H$ and $(g, x) \in G \times X$. If $\alpha: H \rightarrow G$ is an inclusion, we also write ind_H^G instead of ind_{α} .

A (proper) equivariant cohomology theory $\mathcal{H}_?^*$ with values in R -modules consists of a collection of (proper) G -cohomology theory \mathcal{H}_G^* with values in R -modules for each group G together with the following so called *induction structure*: given a group homomorphism $\alpha: H \rightarrow G$ and a (proper) H - CW -pair (X, A) such that $\ker(\alpha)$ acts freely on X , there are for each $n \in \mathbb{Z}$ natural isomorphisms

$$\text{ind}_{\alpha}: \mathcal{H}_G^n(\text{ind}_{\alpha}(X, A)) \xrightarrow{\cong} \mathcal{H}_H^n(X, A) \quad (1.4)$$

satisfying

- (a) Compatibility with the boundary homomorphisms

$$\delta_H^n \circ \text{ind}_{\alpha} = \text{ind}_{\alpha} \circ \delta_G^n;$$

- (b) Functoriality

Let $\beta: G \rightarrow K$ be another group homomorphism such that $\ker(\beta \circ \alpha)$ acts freely on X . Then we have for $n \in \mathbb{Z}$

$$\text{ind}_{\beta \circ \alpha} = \text{ind}_{\alpha} \circ \text{ind}_{\beta} \circ \mathcal{H}_K^n(f_1): \mathcal{H}_H^n(\text{ind}_{\beta \circ \alpha}(X, A)) \rightarrow \mathcal{H}_K^n(X, A),$$

where $f_1: \text{ind}_{\beta} \text{ind}_{\alpha}(X, A) \xrightarrow{\cong} \text{ind}_{\beta \circ \alpha}(X, A)$, $(k, g, x) \mapsto (k\beta(g), x)$ is the natural K -homeomorphism;

- (c) Compatibility with conjugation

For $n \in \mathbb{Z}$, $g \in G$ and a (proper) G -CW-pair (X, A) the homomorphism $\text{ind}_{c(g): G \rightarrow G}: \mathcal{H}_G^n(\text{ind}_{c(g): G \rightarrow G}(X, A)) \rightarrow \mathcal{H}_G^n(X, A)$ agrees with $\mathcal{H}_G^n(f_2)$, where f_2 is the G -homeomorphism $f_2: (X, A) \rightarrow \text{ind}_{c(g): G \rightarrow G}(X, A)$, $x \mapsto (1, g^{-1}x)$ and $c(g)(g') = gg'g^{-1}$.

This induction structure links the various G -cohomology theories for different groups G . It will play a key role in the construction of the equivariant Chern character even if we want to carry it out only for a fixed group G . In all of the relevant examples the induction homomorphism ind_α of (1.4) exists for every group homomorphism $\alpha: H \rightarrow G$, the condition that $\ker(\alpha)$ acts freely on X is only needed to ensure that ind_α is bijective. If α is an inclusion, we sometimes write ind_H^G instead of ind_α .

We say that $\mathcal{H}_?^*$ satisfies the *disjoint union axiom* if for every group G the G -cohomology theory \mathcal{H}_G^* satisfies the disjoint union axiom.

We will later need the following lemma whose elementary proof is analogous to the one in [8, Lemma 1.2].

Lemma 1.5. *Consider finite subgroups $H, K \subseteq G$ and an element $g \in G$ with $gHg^{-1} \subseteq K$. Let $R_{g^{-1}}: G/H \rightarrow G/K$ be the G -map sending $g'H$ to $g'g^{-1}K$ and $c(g): H \rightarrow K$ be the homomorphism sending h to ghg^{-1} . Let $\text{pr}: (\text{ind}_{c(g): H \rightarrow K}\{\text{pt.}\}) \rightarrow \{\text{pt.}\}$ be the projection. Then the following diagram commutes*

$$\begin{array}{ccc} \mathcal{H}_G^n(G/K) & \xrightarrow{\mathcal{H}_G^n(R_{g^{-1}})} & \mathcal{H}_G^n(G/H) \\ \text{ind}_K^G \downarrow \cong & & \text{ind}_H^G \downarrow \cong \\ \mathcal{H}_K^n(\{\text{pt.}\}) & \xrightarrow{\text{ind}_{c(g)} \circ \mathcal{H}_K^n(\text{pr})} & \mathcal{H}_H^n(\{\text{pt.}\}) \end{array}$$

Example 1.6 (Borel cohomology). Let \mathcal{K}^* be a cohomology theory for (non-equivariant) CW-pairs with values in R -modules. Examples are singular cohomology and topological K -theory. Then we obtain two equivariant cohomology theories with values in R -modules by the following constructions

$$\begin{aligned} \mathcal{H}_G^n(X, A) &= \mathcal{K}^n(G \setminus X, G \setminus A); \\ \mathcal{H}_G^n(X, A) &= \mathcal{K}^n(EG \times_G (X, A)). \end{aligned}$$

The second one is called the *equivariant Borel cohomology associated to \mathcal{K}^** . In both cases \mathcal{H}_G^* inherits the structure of a G -cohomology theory from the cohomology structure on \mathcal{K}^* .

The induction homomorphism associated to a group homomorphism $\alpha: H \rightarrow G$ is defined as follows. Let $a: H \setminus X \xrightarrow{\cong} G \setminus (G \times_\alpha X)$ be the homeomorphism sending Hx to $G(1, x)$. Define $b: EH \times_H X \rightarrow EG \times_G G \times_\alpha X$ by sending (e, x) to $(E\alpha(e), 1, x)$ for $e \in EH$, $x \in X$ and $E\alpha: EH \rightarrow EG$ the α -equivariant map induced by α . The desired induction map ind_α is given by $\mathcal{K}^*(a)$ and $\mathcal{K}^*(b)$. If the kernel $\ker(\alpha)$ acts freely on X , the map b is a homotopy equivalence and hence in both cases ind_α is bijective.

If \mathcal{K}^* satisfies the disjoint union axiom, the same is true for the two equivariant cohomology theories constructed above.

Example 1.7 (Equivariant K -theory). In [12] G -equivariant topological (complex) K -theory $K_G^*(X, A)$ is constructed for any proper G - CW -pair (X, A) and shown that K_G^* defines a proper G -cohomology theory satisfying the disjoint union axiom. Given a group homomorphism $\alpha: H \rightarrow G$, it induces an injective group homomorphism $\bar{\alpha}: H/\ker(\alpha) \rightarrow G$. Let

$$\text{Infl}_{H/\ker(\alpha)}^H: K_{H/\ker(\alpha)}^*(\ker(\alpha)\backslash X) \rightarrow K_H^*(X)$$

be the inflation homomorphism of [12, Proposition 3.3] and

$$\text{ind}_{\bar{\alpha}}: K_{H/\ker(\alpha)}^*(\ker(\alpha)\backslash X) \xrightarrow{\cong} K_G^*(\text{ind}_{\bar{\alpha}}(\ker(\alpha)\backslash X))$$

be the induction isomorphism of [12, Proposition 3.2 (b)]. Define the induction homomorphism

$$\text{ind}_{\alpha}: K_G^*(\text{ind}_{\alpha} X) \rightarrow K_H^*(X)$$

by $\text{Infl}_{H/\ker(\alpha)}^H \circ (\text{ind}_{\bar{\alpha}})^{-1}$, where we identify $\text{ind}_{\alpha} X = \text{ind}_{\bar{\alpha}}(\ker(\alpha)\backslash X)$. On the level of complex finite-dimensional vector bundles the induction homomorphism ind_{α} corresponds to considering for a G -vector bundle E over $G \times_{\alpha} X$ the H -vector bundle obtained from E by the pullback construction associated to the α -equivariant map $X \rightarrow G \times_{\alpha} X$, $x \mapsto (1, x)$.

Thus we obtain a proper equivariant cohomology theory $K_{\mathbb{Z}}^*$ with values in \mathbb{Z} -modules which satisfies the disjoint union axiom. There is also a real version $KO_{\mathbb{Z}}^*$.

Example 1.8 (Equivariant cohomology theories and spectra). Denote by GROUPOIDS the category of small groupoids. Let Ω -SPECTRA be the category of Ω -spectra, where a morphism $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$ is a sequence of maps $f_n: E_n \rightarrow F_n$ compatible with the structure maps and we work in the category of compactly generated spaces (see for instance [3, Section 1]). A contravariant GROUPOIDS- Ω -spectrum is a contravariant functor $\mathbf{E}: \text{GROUPOIDS} \rightarrow \Omega\text{-SPECTRA}$.

Next we explain how we can associate to it an equivariant cohomology theory $H_{\mathbb{Z}}^*(-; \mathbf{E})$ satisfying the disjoint union axiom, provided that \mathbf{E} respects equivalences, i.e. it sends an equivalence of groupoids to a weak equivalence of spectra. This construction is dual to the construction of an equivariant homology theory associated to a covariant GROUPOIDS-spectrum as explained in [13, Section 6.2], [14, Theorem 2.10 on page 21].

Fix a group G . We have to specify a G -cohomology theory $\mathcal{H}_G^*(-; \mathbf{E})$. Let $\text{Or}(G)$ be the *orbit category* whose set of objects consists of homogeneous G -spaces G/H and whose morphisms are G -maps. For a G -set S we denote by $\mathcal{G}^G(S)$ its associated *transport groupoid*. Its objects are the elements of S . The set of morphisms from s_0 to s_1 consists of those elements $g \in G$ which satisfy $gs_0 = s_1$. Composition in $\mathcal{G}^G(S)$ comes from the multiplication in G . Thus we obtain for a group G a covariant functor

$$\mathcal{G}^G: \text{Or}(G) \rightarrow \text{GROUPOIDS}, \quad G/H \mapsto \mathcal{G}^G(G/H), \quad (1.9)$$

and a contravariant $\text{Or}(G)$ - Ω -spectrum $\mathbf{E} \circ \mathcal{G}^G$. Given a G - CW -pair (X, A) , we obtain a contravariant pair of $\text{Or}(G)$ - CW -complexes $(X^?, A^?)$ by sending G/H to $(\text{map}_G(G/H, X), \text{map}_G(G/H, A)) = (X^H, A^H)$. The contravariant $\text{Or}(G)$ -spectrum $\mathbf{E} \circ \mathcal{G}^G$ defines a cohomology theory on the category of contravariant $\text{Or}(G)$ - CW -complexes as explained in [3, Section 4]. Its value at $(X^?, A^?)$ is defined to be $H_G^*(X, A; \mathbf{E})$. Explicitly, $H_G^n(X, A; \mathbf{E})$ is the $(-n)$ -th homotopy group of the spectrum $\text{map}_{\text{Or}(G)} \left(X_+^? \cup_{A_+^?} \text{cone}(A_+^?), \mathbf{E} \circ \mathcal{G}^G \right)$. We need Ω -spectra in order to ensure that the disjoint union axiom holds.

We briefly explain for a group homomorphism $\alpha: H \rightarrow G$ the definition of the induction homomorphism $\text{ind}_\alpha: \mathcal{H}_G^n(\text{ind}_\alpha X; \mathbf{E}) \rightarrow \mathcal{H}_H^n(X; \mathbf{E})$ in the special case $A = \emptyset$. The functor induced by α on the orbit categories is denoted in the same way

$$\alpha: \text{Or}(H) \rightarrow \text{Or}(G), \quad H/L \mapsto \text{ind}_\alpha(H/L) = G/\alpha(L).$$

There is an obvious natural transformation of functors $\text{Or}(H) \rightarrow \text{GROUPOIDS}$

$$T: \mathcal{G}^H \rightarrow \mathcal{G}^G \circ \alpha.$$

Its evaluation at H/L is the functor of groupoids $\mathcal{G}^H(H/L) \rightarrow \mathcal{G}^G(G/\alpha(L))$ which sends an object hL to the object $\alpha(h)\alpha(L)$ and a morphism given by $h \in H$ to the morphism $\alpha(h) \in G$. Notice that $T(H/L)$ is an equivalence if $\ker(\alpha)$ acts freely on H/L . The desired isomorphism

$$\text{ind}_\alpha: H_G^n(\text{ind}_\alpha X; \mathbf{E}) \rightarrow H_H^n(X; \mathbf{E})$$

is induced by the following map of spectra

$$\begin{aligned} & \text{map}_{\text{Or}(G)} \left(\text{map}_G(-, \text{ind}_\alpha X_+), \mathbf{E} \circ \mathcal{G}^G \right) \\ & \xrightarrow{\cong} \text{map}_{\text{Or}(G)} \left(\alpha_* (\text{map}_H(-, X_+)), \mathbf{E} \circ \mathcal{G}^G \right) \\ & \xrightarrow{\cong} \text{map}_{\text{Or}(H)} \left(\text{map}_H(-, X_+), \mathbf{E} \circ \mathcal{G}^G \circ \alpha \right) \\ & \xrightarrow{\text{map}_{\text{Or}(H)}(\text{id}, \mathbf{E}(T))} \text{map}_{\text{Or}(H)} \left(\text{map}_H(-, X_+), \mathbf{E} \circ \mathcal{G}^H \right). \end{aligned}$$

Here $\alpha_* \text{map}_H(-, X_+)$ is the pointed $\text{Or}(G)$ -space which is obtained from the pointed $\text{Or}(H)$ -space $\text{map}_H(-, X_+)$ by induction, i.e. by taking the balanced product over $\text{Or}(H)$ with the $\text{Or}(H)$ - $\text{Or}(G)$ bimodule $\text{mor}_{\text{Or}(G)}(??, \alpha(?))$ [3, Definition 1.8]. The second map is given by the adjunction homeomorphism of induction α_* and restriction α^* (see [3, Lemma 1.9]). The first map comes from the homeomorphism of $\text{Or}(G)$ -spaces

$$\alpha_* \text{map}_H(-, X_+) \rightarrow \text{map}_G(-, \text{ind}_\alpha X_+)$$

which is the adjoint of the obvious map of $\text{Or}(H)$ -spaces $\text{map}_H(-, X_+) \rightarrow \alpha^* \text{map}_G(-, \text{ind}_\alpha X_+)$ whose evaluation at H/L is given by ind_α .

2. Modules over a Category

In this section we give a brief summary about modules over a small category \mathcal{C} as far as needed for this paper. They will appear in the definition of the equivariant Chern character.

Let \mathcal{C} be a small category and let R be a commutative ring. A *contravariant RC -module* is a contravariant functor from \mathcal{C} to the category R -MOD of R -modules. Morphisms of contravariant RC -modules are natural transformations. Given a group G , let \widehat{G} be the category with one object whose set of morphisms is given by G . Then a contravariant $R\widehat{G}$ -module is the same as a right RG -module. Therefore we can identify the abelian category $\text{MOD-}R\widehat{G}$ with the abelian category of right RG -modules $\text{MOD-}RG$ in the sequel. Many of the constructions, which we will introduce for RC -modules below, reduce in the special case $\mathcal{C} = \widehat{G}$ to their classical versions for RG -modules. The reader should have this example in mind. There is also a covariant version. In the sequel RC -module means contravariant RC -module unless stated explicitly differently.

The category $\text{MOD-}RC$ of RC -modules inherits the structure of an abelian category from R -MOD in the obvious way, namely objectwise. For instance a sequence $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ of contravariant RC -modules is called *exact* if its evaluation at each object in \mathcal{C} is an exact sequence in R -MOD. The notion of an injective and of a projective RC -module is now clear. For a set S denote by RS the free R -module with S as basis. An RC -module is *free* if it is isomorphic to RC -module of the shape $\bigoplus_{i \in I} R \text{mor}_{\mathcal{C}}(?, c_i)$ for some index set I and objects $c_i \in \mathcal{C}$. Notice that by the Yoneda-Lemma there is for every RC -module N and every object c a bijection of sets

$$\text{hom}_{RC}(R \text{mor}_{\mathcal{C}}(?, c), N) \xrightarrow{\cong} N(c), \quad \phi \mapsto \phi(\text{id}_x).$$

This implies that every free RC -module is projective and a RC -module is projective if and only if it is a direct summand in a free RC -module. The category of RC -modules has enough projectives and injectives (see [7, Lemma 17.1] and [17, Example 2.3.13]).

Given a contravariant RC -module M and a covariant RC -module N , their *tensor product over RC* is defined to be the following R -module $M \otimes_{RC} N$. It is given by

$$M \otimes_{RC} N = \bigoplus_{c \in \text{Ob}(\mathcal{C})} M(c) \otimes_R N(c) / \sim,$$

where \sim is the typical tensor relation $mf \otimes n = m \otimes fn$, i.e. for every morphism $f: c \rightarrow d$ in \mathcal{C} , $m \in M(d)$ and $n \in N(c)$ we introduce the relation $M(f)(m) \otimes n - m \otimes N(f)(n) = 0$. The main property of this construction is that it is adjoint to the hom_R -functor in the sense that for any R -module L there are natural isomorphisms of R -modules

$$\text{hom}_R(M \otimes_{RC} N, L) \xrightarrow{\cong} \text{hom}_{RC}(M, \text{hom}_R(N, L)); \quad (2.1)$$

$$\text{hom}_R(M \otimes_{RC} N, L) \xrightarrow{\cong} \text{hom}_{RC}(N, \text{hom}_R(M, L)). \quad (2.2)$$

Consider a functor $F: \mathcal{C} \rightarrow \mathcal{D}$. Given a RD -module M , define its *restriction with F* to be $F^*M := M \circ F$. Given a contravariant RC -module M , its *induction with F* is the contravariant RD -module F_*M given by

$$(F_*M)(??) := M(?) \otimes_{RC} R \operatorname{mor}_{\mathcal{D}}(??, F(?)), \quad (2.3)$$

and *coinduction with F* is the contravariant RD -module $F_!M$ given by

$$(F_!M)(??) := \operatorname{hom}_{RC}(R \operatorname{mor}_{\mathcal{D}}(F(?), ??), M(?)). \quad (2.4)$$

Restriction with F can be written as $F^*N(?) = \operatorname{hom}_{RD}(R \operatorname{mor}_{\mathcal{D}}(??, F(?)), N(??))$, the natural isomorphism sends $n \in N(F(?))$ to the map

$$R \operatorname{mor}_{\mathcal{D}}(??, F(?)) \rightarrow N(??), \quad \phi: ?? \rightarrow F(?) \mapsto N(\phi)(n).$$

Restriction with F can also be written as $F^*N(?) = R \operatorname{mor}_{\mathcal{D}}(F(?), ??) \otimes_{RD} N(??)$, the natural isomorphism sends $\phi \otimes_{RD} n$ to $N(\phi)(n)$. We conclude from (2.2) that (F_*, F^*) and $(F^*, F_!)$ form adjoint pairs, i.e. for a RC -module M and a RD -module N there are natural isomorphisms of R -modules

$$\operatorname{hom}_{RD}(F_*M, N) \xrightarrow{\cong} \operatorname{hom}_{RC}(M, F^*N); \quad (2.5)$$

$$\operatorname{hom}_{RD}(F^*N, M) \xrightarrow{\cong} \operatorname{hom}_{RC}(N, F_!M). \quad (2.6)$$

Consider an object c in \mathcal{C} . Let $\operatorname{aut}(c)$ be the group of automorphism of c . We can think of $\widehat{\operatorname{aut}(c)}$ as a subcategory of \mathcal{C} in the obvious way. Denote by

$$i(c): \widehat{\operatorname{aut}(c)} \rightarrow \mathcal{C}$$

the inclusion of categories and abbreviate the group ring $R[\operatorname{aut}(c)]$ by $R[c]$ in the sequel. Thus we obtain functors

$$i(c)^*: \operatorname{MOD-RC} \rightarrow \operatorname{MOD-R}[c]; \quad (2.7)$$

$$i(c)_*: \operatorname{MOD-R}[c] \rightarrow \operatorname{MOD-RC}; \quad (2.8)$$

$$i(c)_!: \operatorname{MOD-R}[c] \rightarrow \operatorname{MOD-RC}. \quad (2.9)$$

The *projective splitting functor*

$$S_c: \operatorname{MOD-RC} \rightarrow \operatorname{MOD-R}[c] \quad (2.10)$$

sends M to the cokernel of the map

$$\bigoplus_{\substack{f: c \rightarrow d \\ f \text{ not an isomorphism}}} M(f): \bigoplus_{\substack{f: c \rightarrow d \\ f \text{ not an isomorphism}}} M(d) \rightarrow M(c).$$

The *injective splitting functor*

$$T_c: \operatorname{MOD-RC} \rightarrow \operatorname{MOD-R}[c] \quad (2.11)$$

sends M to the kernel of the map

$$\prod_{\substack{f: d \rightarrow c \\ f \text{ not an isomorphism}}} M(f): M(c) \rightarrow \prod_{\substack{f: d \rightarrow c \\ f \text{ not an isomorphism}}} M(d).$$

From now on suppose that \mathcal{C} is an EI-category, i.e. a small category such that endomorphisms are isomorphisms. Then we can define the *inclusion functor*

$$I_c: \text{MOD-}R[c] \rightarrow \text{MOD-}RC \quad (2.12)$$

by $I_c(M)(?) = M \otimes_{R[c]} R \text{mor}(?, c)$ if $c \cong ?$ in \mathcal{C} and by $I_c(M)(?) = 0$ otherwise. Let B be the RC - $R[c]$ -bimodule, covariant over \mathcal{C} and a right module over $R[c]$, given by

$$B(c, ?) = \begin{cases} R \text{mor}_{\mathcal{C}}(c, ?) & \text{if } c \cong ?; \\ 0 & \text{if } c \not\cong ?. \end{cases}$$

Let C be the $R[c]$ - RC -bimodule, contravariant over \mathcal{C} and a left module over $R[c]$, given by

$$C(?, c) = \begin{cases} R \text{mor}_{\mathcal{C}}(?, c) & \text{if } c \cong ?; \\ 0 & \text{if } c \not\cong ?. \end{cases}$$

One easily checks that there are natural isomorphisms

$$\begin{aligned} S_c M &\cong M \otimes_{RC} B; \\ I_c N &\cong \text{hom}_{R[c]}(B, N); \\ T_c M &\cong \text{hom}_{RC}(C, M); \\ I_c N &\cong N \otimes_{R[c]} C. \end{aligned}$$

Lemma 2.13. *Let \mathcal{C} be an EI-category and c, d objects in \mathcal{C} .*

- (a) *We obtain adjoint pairs $(i(c)_*, i(c)^*)$, $(i(c)^*, i(c)!)$, (S_c, I_c) and (I_c, T_c) ;*
- (b) *There are natural equivalences of functors $S_c \circ i(c)_* \xrightarrow{\cong} \text{id}$ and $T_c \circ i(c)! \xrightarrow{\cong} \text{id}$ of functors $\text{MOD-}R[c] \rightarrow \text{MOD-}R[c]$. If $c \not\cong d$, then $S_c \circ i(d)_* = T_c \circ i(d)! = 0$;*
- (c) *The functors S_c and $i(c)_*$ send projective modules to projective modules. The functors I_c and $i(c)!$ send injective modules to injective modules.*

Proof. (a) follows from (2.5), (2.6) and (2.1).

(b) This follows in the case $T_c \circ i(d)!$ from the following chain of canonical isomorphisms

$$\begin{aligned} T_c \circ i(d)!(M) &= \text{hom}_{RC}(C(?, c), \text{hom}_{R[d]}(R \text{mor}_{\mathcal{C}}(d, ?), M)) \\ &\xrightarrow{\cong} \text{hom}_{R[d]}(C(?, c) \otimes_{RC} R \text{mor}_{\mathcal{C}}(d, ?), M) \xrightarrow{\cong} \text{hom}_{R[c]}(C(c, d), M), \end{aligned}$$

and analogously for $S_d \circ i(c)_*$.

(c) The functors S_c and $i(c)_*$ are left adjoint to an exact functor and hence respect projective. The functors T_c and $i(c)!$ are right adjoint to an exact functor and hence respect injective. \square

The *length* $l(c) \in \mathbb{N} \cup \{\infty\}$ of an object c is the supremum over all natural numbers l for which there exists a sequence of morphisms $c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} \dots \xrightarrow{f_l} c_l$ such that no f_i is an isomorphism and $c_l = c$. The *colength* $col(c) \in \mathbb{N} \cup \{\infty\}$ of an object c is the supremum over all natural numbers l for which there exists a sequence of morphisms $c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} \dots \xrightarrow{f_l} c_l$ such that no f_i is an isomorphism and $c_0 = c$. If each object c has length $l(c) < \infty$, we say that \mathcal{C} has *finite length*. If each object c has colength $col(c) < \infty$, we say that \mathcal{C} has *finite colength*.

Theorem 2.14. (Structure theorem for projective and injective RC-modules). *Let \mathcal{C} be an EI-category. Then*

- (a) *Suppose that \mathcal{C} has finite colength. Let M be a contravariant RC-module such that the $\text{Raut}(c)$ -module $S_c M$ is projective for all objects c in \mathcal{C} . Let $\sigma_c: S_c M \rightarrow M(c)$ be an $\text{Raut}(c)$ -section of the canonical projection $M(c) \rightarrow S_c M$. Consider the map of RC-modules*

$$\mu(M): \bigoplus_{(c) \in \text{Is}(\mathcal{C})} i(c)_* S_c M \xrightarrow{\bigoplus_{(c) \in \text{Is}(\mathcal{C})} i(c)_* \sigma_c} \bigoplus_{(c) \in \text{Is}(\mathcal{C})} i(c)_* M(c) \xrightarrow{\bigoplus_{(c) \in \text{Is}(\mathcal{C})} \alpha(c)} M,$$

where $\alpha(c): i(c)_* M(c) = i(c)_* i(c)^* M \rightarrow M$ is the adjoint of the identity $i(c)^* M \rightarrow i(c)^* M$ under the adjunction (2.5). The map $\mu(M)$ is always surjective. It is bijective if and only if M is a projective RC-module;

- (b) *Suppose that \mathcal{C} has finite length. Let M be a contravariant RC-module such that the $\text{Raut}(c)$ -module $T_c M$ is injective for all objects c in \mathcal{C} . Let $\rho_c: M(c) \rightarrow T_c M$ be an $\text{Raut}(c)$ -retraction of the canonical injection $T_c M \rightarrow M(c)$. Consider the map of RC-modules*

$$\nu(M): M \xrightarrow{\prod_{(c) \in \text{Is}(\mathcal{C})} \beta(c)} \prod_{(c) \in \text{Is}(\mathcal{C})} i(c)_! M(c) \xrightarrow{\prod_{(c) \in \text{Is}(\mathcal{C})} i(c)_! \rho_c} \prod_{(c) \in \text{Is}(\mathcal{C})} i(c)_* T_c M$$

where $\beta(c): M \rightarrow i(c)_! i(c)^* M = i(c)_! M(c)$ is the adjoint of the identity $i(c)^* M \rightarrow i(c)^* M$ under the adjunction (2.6). The map $\nu(M)$ is always injective. It is bijective if and only if M is an injective RC-module.

Proof. (a) A contravariant RC-module is the same as covariant RC^{op} -module, where \mathcal{C}^{op} is the opposite category of \mathcal{C} , just invert the direction of every morphisms. The corresponding covariant version of assertion (a) is proved in [8, Theorem 2.11].

(b) is the dual statement of assertion (a). We first show that $\nu(M)$ is always

injective. We show by induction over the length $l(x)$ of an object $x \in \mathcal{C}$ that $\nu(M)(x)$ is injective. Let u be an element in the kernel of $\nu(M)(x)$. Consider a morphism $f: y \rightarrow x$ which is not an isomorphism. Then $l(y) < l(x)$ and by induction hypothesis $\nu(M)(y)$ is injective. Since the composite $\nu(M)(y) \circ M(f)$ factorizes through $\nu(M)(x)$, we have $u \in \ker(M(f))$. This implies $u \in T_x M$. Consider the composite

$$T_x M \xrightarrow{i} M(x) \xrightarrow{\nu(M)(x)} \prod_{(c) \in \text{Is}(\mathcal{C})} i(c)! T_c M(x) \xrightarrow{\text{pr}_x} i(x)! T_x M(x) \xrightarrow{j} T_x M,$$

where i is the inclusion, pr_x is the projection onto the factor belonging to the isomorphism class of x and j is the isomorphism $\text{hom}_{R[x]}(R \text{mor}_{\mathcal{C}}(x, x), T_x M) \xrightarrow{\cong} T_x M$ sending ϕ to $\phi(\text{id}_x)$. Since this composite is the identity on $T_x M$ and u lies in the kernel of $\nu(M)(x)$, we conclude $u = 0$.

In particular we see that an injective RC -module M is trivial if and only if $i(d)! T_d M(x)$ is trivial for all objects $d \in \mathcal{C}$.

If $\nu(M)$ is bijective and each $T_c M$ is an injective $R[c]$ -module, then M is an injective RC -module, since $i(c)!$ sends injective $R[c]$ -modules to injective RC -modules by Lemma 2.13 (c) and the product of injective modules is again injective.

Now suppose that M is injective. Let N be the cokernel of $\nu(M)$. We have the exact sequence

$$0 \rightarrow M \xrightarrow{\nu(M)} \prod_{(c) \in \text{Is}(\mathcal{C})} i(c)_* T_c M \xrightarrow{\text{pr}} N \rightarrow 0. \quad (2.15)$$

Since M is injective, this is a split exact sequence of injective RC -modules. Fix an object d . The functors $i(d)!$ and T_d send split exact sequences to split exact sequences. Therefore we obtain a split exact sequence if we apply $i(d)! T_d$ to (2.15). Using Lemma 2.13 (b) the resulting exact sequence is isomorphic to the exact sequence

$$0 \rightarrow i(d)! T_d M \xrightarrow{\text{id}} i(d)! T_d M \rightarrow i(d)! T_d N \rightarrow 0.$$

Hence $i(d)! T_d N$ vanishes for all objects d . This implies that N is trivial and because of (2.15) that $\nu(M)$ is bijective. \square

For more details about modules over a category we refer to [7, Section 9A].

3. The Associated Bredon Cohomology Theory

Given a proper equivariant cohomology theory with values in R -modules, we can associate to it another proper equivariant cohomology theory with values in R -modules satisfying the disjoint union axiom called Bredon cohomology, which

is much simpler. The equivariant Chern character will identify this simpler proper equivariant cohomology theory with the given one.

The *orbit category* $\text{Or}(G)$ has as objects homogeneous spaces G/H and as morphisms G -maps. Let $\text{Sub}(G)$ be the category whose objects are subgroups H of G . For two subgroups H and K of G denote by $\text{conhom}_G(H, K)$ the set of group homomorphisms $f: H \rightarrow K$, for which there exists an element $g \in G$ with $gHg^{-1} \subseteq K$ such that f is given by conjugation with g , i.e. $f = c(g): H \rightarrow K$, $h \mapsto ghg^{-1}$. Notice that f is injective and $c(g) = c(g')$ holds for two elements $g, g' \in G$ with $gHg^{-1} \subseteq K$ and $g'H(g')^{-1} \subseteq K$ if and only if $g^{-1}g'$ lies in the centralizer $C_G H = \{g \in G \mid gh = hg \text{ for all } h \in H\}$ of H in G . The group of inner automorphisms of K acts on $\text{conhom}_G(H, K)$ from the left by composition. Define the set of morphisms

$$\text{mor}_{\text{Sub}(G)}(H, K) := \text{Inn}(K) \backslash \text{conhom}_G(H, K).$$

There is a natural projection $\text{pr}: \text{Or}(G) \rightarrow \text{Sub}(G)$ which sends a homogeneous space G/H to H . Given a G -map $f: G/H \rightarrow G/K$, we can choose an element $g \in G$ with $gHg^{-1} \subseteq K$ and $f(g'H) = g'g^{-1}K$. Then $\text{pr}(f)$ is represented by $c(g): H \rightarrow K$. Notice that $\text{mor}_{\text{Sub}(G)}(H, K)$ can be identified with the quotient $\text{mor}_{\text{Or}(G)}(G/H, G/K)/C_G H$, where $g \in C_G H$ acts on $\text{mor}_{\text{Or}(G)}(G/H, G/K)$ by composition with $R_{g^{-1}}: G/H \rightarrow G/H$, $g'H \mapsto g'g^{-1}H$.

Denote by $\text{Or}(G, \mathcal{F}) \subseteq \text{Or}(G)$ and $\text{Sub}(G, \mathcal{F}) \subseteq \text{Sub}(G)$ the full subcategories, whose objects G/H and H are given by finite subgroups $H \subseteq G$. Both $\text{Or}(G, \mathcal{F})$ and $\text{Sub}(G, \mathcal{F})$ are EI-categories of finite length.

Given a proper G -cohomology theory \mathcal{H}_G^* with values in R -modules we obtain for $n \in \mathbb{Z}$ a contravariant $R\text{Or}(G, \mathcal{F})$ -module

$$\mathcal{H}_G^n(G/?): \text{Or}(G, \mathcal{F}) \rightarrow R\text{-MOD}, \quad G/H \mapsto \mathcal{H}_G^n(G/H). \quad (3.1)$$

Let (X, A) be a pair of proper G -CW-complexes. Then there is a canonical identification $X^H = \text{map}(G/H, X)^G$. Thus we obtain contravariant functors

$$\begin{aligned} \text{Or}(G, \mathcal{F}) &\rightarrow \text{CW-PAIRS}, & G/H &\mapsto (X^H, A^H); \\ \text{Sub}(G, \mathcal{F}) &\rightarrow \text{CW-PAIRS}, & G/H &\mapsto C_G H \backslash (X^H, A^H), \end{aligned}$$

where CW-PAIRS is the category of pairs of CW -complexes. If we compose them with the covariant functor $\text{CW-PAIRS} \rightarrow \mathbb{Z}\text{-CHCOM}$ sending (Z, B) to its cellular \mathbb{Z} -chain complex, then we obtain the contravariant $\mathbb{Z}\text{Or}(G, \mathcal{F})$ -chain complex $C_*^{\text{Or}(G, \mathcal{F})}(X, A)$ and the contravariant $\mathbb{Z}\text{Sub}(G, \mathcal{F})$ -chain complex $C_*^{\text{Sub}(G, \mathcal{F})}(X, A)$. Both chain complexes are free in the sense that each chain module is a free $\mathbb{Z}\text{Or}(G, \mathcal{F})$ -module resp. $\mathbb{Z}\text{Sub}(G, \mathcal{F})$ -module. Namely, if X_n is obtained from $X_{n-1} \cup A_n$ by attaching the equivariant cells $G/H_i \times D^n$ for $i \in I_n$, then

$$C_n^{\text{Or}(G, \mathcal{F})}(X, A) \cong \bigoplus_{i \in I_n} \mathbb{Z} \text{mor}_{\text{Or}(G, \mathcal{F})}(G/?, G/H_i); \quad (3.2)$$

$$C_n^{\text{Sub}(G, \mathcal{F})}(X, A) \cong \bigoplus_{i \in I_n} \mathbb{Z} \text{mor}_{\text{Sub}(G, \mathcal{F})}(?, H_i). \quad (3.3)$$

Given a contravariant $R\text{Or}(G, \mathcal{F})$ -module M , the *equivariant Bredon cohomology* (see [2]) of a pair of proper G -CW-complexes (X, A) with coefficients in M is defined by

$$H_{\text{Or}(G, \mathcal{F})}^n(X, A; M) := H^n \left(\text{hom}_{\mathbb{Z}\text{Or}(G, \mathcal{F})} (C_*^{\text{Or}(G, \mathcal{F})}(X, A), M) \right). \quad (3.4)$$

This is indeed a proper G -cohomology theory satisfying the disjoint union axiom. Hence we can assign to a proper G -homology theory \mathcal{H}_G^* another proper G -cohomology theory which we call the *associated Bredon cohomology*

$$\mathcal{B}\mathcal{H}_G^n(X, A) := \prod_{p+q=n} H_{\text{Or}(G, \mathcal{F})}^p(X, A; \mathcal{H}_G^q(G/?)). \quad (3.5)$$

There is an obvious $\mathbb{Z}\text{Sub}(G; \mathcal{F})$ -chain map

$$\text{pr}_* C_*^{\text{Or}(G, \mathcal{F})}(X, A) \xrightarrow{\cong} C_*^{\text{Sub}(G, \mathcal{F})}(X, A)$$

which is bijective because of (3.2), (3.3) and the canonical identification

$$\text{pr}_* \mathbb{Z}\text{mor}_{\text{Or}(G, \mathcal{F})}(G/? , G/H_i) = \mathbb{Z}\text{mor}_{\text{Sub}(G, \mathcal{F})}(?, H_i).$$

Given a covariant $\mathbb{Z}\text{Sub}(G, \mathcal{F})$ -module M , we get from the adjunction $(\text{pr}_*, \text{pr}^*)$ (see Lemma 2.13 (a)) natural isomorphisms

$$\begin{aligned} H_{R\text{Or}(G, \mathcal{F})}^n(X, A; \text{res}_{\text{pr}} M) \\ \xrightarrow{\cong} H^n \left(\text{hom}_{\mathbb{Z}\text{Sub}(G, \mathcal{F})} \left(C_*^{\text{Sub}(G, \mathcal{F})}(X, A), M \right) \right). \end{aligned} \quad (3.6)$$

This will allow us to work with modules over the category $\text{Sub}(G; \mathcal{F})$ which is smaller than the orbit category and has nicer properties from the homological algebra point of view. The main advantage of $\text{Sub}(G; \mathcal{F})$ is that the automorphism groups of every object is finite.

Suppose, we are given a proper equivariant cohomology theory $\mathcal{H}_?^*$ with values in R -modules. We get from (3.1) for each group G and $n \in \mathbb{Z}$ a covariant $R\text{Sub}(G, \mathcal{F})$ -module

$$\mathcal{H}_G^n(G/?): \text{Sub}(G, \mathcal{F}) \rightarrow R\text{-MOD}, \quad H \mapsto \mathcal{H}_G^n(G/H). \quad (3.7)$$

We have to show that for $g \in C_G H$ the G -map $R_{g^{-1}}: G/H \rightarrow G/H$, $g'H \rightarrow g'g^{-1}H$ induces the identity on $\mathcal{H}_G^n(G/H)$. This follows from Lemma 1.5. We will denote the covariant $R\text{Or}(G, \mathcal{F})$ -module obtained by restriction with $\text{pr}: \text{Or}(G, \mathcal{F}) \rightarrow \text{Sub}(G, \mathcal{F})$ from the $R\text{Sub}(G, \mathcal{F})$ -module $\mathcal{H}_G^n(G/?)$ of (3.7) again by $\mathcal{H}_G^n(G/?)$ as introduced already in (3.1).

It remains to show that the collection of G -cohomology theories $\mathcal{B}\mathcal{H}_G^*(X, A)$ defined in (3.4) inherits the structure of a proper equivariant cohomology theory, i.e. we have to specify the induction structure. We leave it to the reader to carry out the obvious dualization of the construction for homology in [8, Section 3] and to check the disjoint union axiom.

4. The Construction of the Equivariant Cohomological Chern Character

We begin with explaining the cohomological version of the homological Chern character due to Dold [4].

Example 4.1 (The non-equivariant Chern character). Consider a (non-equivariant) cohomology theory \mathcal{H}^* with values in R -modules. Suppose that $\mathbb{Q} \subseteq R$. For a space X let X_+ be the pointed space obtained from X by adding a disjoint base point $*$. Since the stable homotopy groups $\pi_p^s(S^0)$ are finite for $p \geq 1$ by results of Serre [15], the condition $\mathbb{Q} \subseteq R$ imply that the Hurewicz homomorphism induces isomorphisms

$$\text{hur}_R: \pi_p^s(X_+) \otimes_{\mathbb{Z}} R \xrightarrow{\text{hur} \otimes_{\mathbb{Z}} \text{id}_R} H_p(X) \otimes_{\mathbb{Z}} R \xrightarrow{\cong} H_p(X; R)$$

and that the canonical map

$$\alpha: H^p(X; \mathcal{H}^q(\{\text{pt.}\})) \xrightarrow{\cong} \text{hom}_{\mathbb{Q}}(H_p(X; \mathbb{Q}), \mathcal{H}^q(\{\text{pt.}\})) \xrightarrow{\cong} \text{hom}_R(H_p(X; R), \mathcal{H}^q(\{\text{pt.}\}))$$

is bijective. Define a map

$$D^{p,q}: \mathcal{H}^{p+q}(X) \rightarrow \text{hom}_R(\pi_p^s(X_+) \otimes_{\mathbb{Z}} R, \mathcal{H}^q(\{\text{pt.}\})) \quad (4.2)$$

as follows. Denote in the sequel by σ^k the k -fold suspension isomorphism. Given $a \in \mathcal{H}^{p+q}(X)$ and an element in $\pi_p^s(X_+, *)$ represented by a map $f: S^{p+k} \rightarrow S^k \wedge X_+$, we define $D^{p,q}(a)([f]) \in \mathcal{H}^q(\{\text{pt.}\})$ as the image of a under the composite

$$\begin{aligned} \mathcal{H}^{p+q}(X) &\xrightarrow{\cong} \tilde{\mathcal{H}}^{p+q}(X_+) \xrightarrow{\sigma^k} \tilde{\mathcal{H}}^{p+q+k}(S^k \wedge X_+) \xrightarrow{\tilde{\mathcal{H}}^{p+q+k}(f)} \tilde{\mathcal{H}}^{p+q+k}(S^{p+k}) \\ &\xrightarrow{(\sigma^{p+k})^{-1}} \tilde{\mathcal{H}}^q(S^0) \xrightarrow{\cong} \mathcal{H}^q(\{\text{pt.}\}). \end{aligned}$$

Then the (non-equivariant) Chern character for a CW -complex X is given by the following composite

$$\begin{aligned} \text{ch}^n(X): \mathcal{H}^n(X) &\xrightarrow{\prod_{p+q=n} D^{p,q}} \prod_{p+q=n} \text{hom}_R(\pi_p^s(X_+, *) \otimes_{\mathbb{Z}} R, \mathcal{H}^q(*)) \\ &\xrightarrow{\prod_{p+q=n} \text{hom}_R(\text{hur}_R^{-1}, \text{id})} \prod_{p+q=n} \text{hom}_R(H_p(X; R), \mathcal{H}^q(*)) \\ &\xrightarrow{\prod_{p+q=n} \alpha^{-1}} \prod_{p+q=n} H^p(X, \mathcal{H}^q(*)). \end{aligned}$$

There is an obvious version for a pair of CW -complexes

$$\text{ch}^n(X, A): \mathcal{H}^n(X, A) \xrightarrow{\cong} \prod_{p+q=n} H^p(X, A, \mathcal{H}^q(*)).$$

We get a natural transformation ch^* of cohomology theories with values in R -modules. One easily checks that it is an isomorphism in the case $X = \{\text{pt.}\}$. Hence $\text{ch}^n(X, A)$ is bijective for all relative finite CW -pairs (X, A) and $n \in \mathbb{Z}$ by Lemma 1.1 (b). If \mathcal{H}^* satisfies the disjoint union axiom, then $\text{ch}^n(X, A)$ is bijective for all CW -pairs (X, A) and $n \in \mathbb{Z}$ by Lemma 1.1 (b).

Let R be a commutative ring with $\mathbb{Q} \subseteq R$. Consider an equivariant cohomology theory $\mathcal{H}_?^*$ with values in R -modules. Let G be a group and let (X, A) be a proper G - CW -pair. We want to construct an R -homomorphism for a finite subgroup $H \subseteq G$

$$\begin{aligned} \underline{\text{ch}}_G^{p,q}(X, A)(H) : \mathcal{H}_G^{p+q}(X, A) \\ \rightarrow \text{hom}_R(H_p(C_G H \setminus (X^H, A^H); R), \mathcal{H}_G^q(G/H)). \end{aligned} \quad (4.3)$$

We define it only in the case $A = \emptyset$, the general case is completely analogous.

$$\begin{aligned} & \mathcal{H}_G^{p+q}(X) \\ & \mathcal{H}_G^{p+q}(v_H) \downarrow \\ & \mathcal{H}_G^{p+q}(\text{ind}_{m_H} X^H) \\ & \mathcal{H}_G^{p+q}(\text{ind}_{m_H} \text{pr}_2) \downarrow \\ & \mathcal{H}_G^{p+q}(\text{ind}_{m_H} EG \times X^H) \\ & \text{ind}_{m_H} \downarrow \cong \\ & \mathcal{H}_{C_G H \times H}^{p+q}(EG \times X^H) \\ & (\text{ind}_{\text{pr}: C_G H \times H \rightarrow H})^{-1} \downarrow \cong \\ & \mathcal{H}_H^{p+q}(EG \times_{C_G H} X^H) \\ & D_H^{p,q}(EG \times_{C_G H} X^H) \downarrow \\ & \text{hom}_R(\pi_p^s((EG \times_{C_G H} X^H)_+) \otimes_{\mathbb{Z}} R, \mathcal{H}_H^q(\{\text{pt.}\})) \\ & \text{hom}_R(\text{hur}_R(EG \times_{C_G H} X^H), \text{id})^{-1} \downarrow \\ & \text{hom}_R(H_p(EG \times_{C_G H} X^H; R), \mathcal{H}_H^q(\{\text{pt.}\})) \\ & \text{hom}_R(H_p(\text{pr}_1; R), \text{id})^{-1} \downarrow \\ & \text{hom}_R(H_p(C_G H \setminus X^H; R), \mathcal{H}_H^q(\{\text{pt.}\})) \\ & \text{hom}_R(\text{id}; (\text{ind}_H^G)^{-1}) \downarrow \\ & \text{hom}_R(H_p(C_G H \setminus X^H; R), \mathcal{H}_G^q(G/H)) \end{aligned}$$

Here are some explanations, more details can be found in [8, Section 4].

We have a left free $C_G H$ -action on $EG \times X^H$ by $g(e, x) = (eg^{-1}, gx)$ for $g \in C_G H$, $e \in EG$ and $x \in X^H$. The map $\text{pr}_1: EG \times_{C_G H} X^H \rightarrow C_G H \setminus X^H$ is the

canonical projection. Since the projection $BL \rightarrow \{\text{pt.}\}$ induces isomorphisms $H_p(BL; R) \xrightarrow{\cong} H_p(\{\text{pt.}\}; R)$ for all $p \in \mathbb{Z}$ and finite groups L because of $\mathbb{Q} \subseteq R$, we obtain for every $p \in \mathbb{Z}$ an isomorphism

$$H_p(\text{pr}_1; R): H_p(EG \times_{C_G H} X^H; R) \xrightarrow{\cong} H_p(C_G H \backslash X^H; R).$$

The group homomorphism $\text{pr}: C_G H \times H \rightarrow H$ is the obvious projection and the group homomorphism $m_H: C_G H \times H \rightarrow G$ sends (g, h) to gh . The $C_G H \times H$ -action on $EG \times X^H$ comes from the obvious $C_G H$ -action and the trivial H -action. In particular we equip $EG \times_{C_G H} X^H$ with the trivial H -action. The kernels of the two group homomorphisms pr and m_H act freely on $EG \times X^H$. We denote by $\text{pr}_2: EG \times X^H \rightarrow X^H$ the canonical projection. The G -map $v_H: \text{ind}_{m_H} X^H = G \times_{m_H} X^H \rightarrow X$ sends (g, x) to gx .

Since H is a finite group, a CW -complex Z equipped with the trivial H -action is a proper H - CW -complex. Hence we can think of \mathcal{H}_H^* as an (non-equivariant) homology theory if we apply it to a CW -pair Z with respect to the trivial H -action. Define the map

$$D_H^{p,q}(Z): \mathcal{H}_H^{p+q}(Z) \rightarrow \text{hom}_R(\pi_p^s(Z_+) \otimes_{\mathbb{Z}} R, \mathcal{H}_H^q(\{\text{pt.}\}))$$

for a CW -complex Z by the map $D^{p,q}$ of (4.2).

A calculation similar to the one in [8, Lemma 4.3] shows that the system of maps $\underline{\text{ch}}_G^{p,q}(X, A)(H)$ (4.3) fit together to an in X natural R -homomorphism

$$\begin{aligned} \underline{\text{ch}}_G^{p,q}(X, A): \mathcal{H}_G^{p+q}(X, A) \\ \rightarrow \text{hom}_{\text{Sub}(G; \mathcal{F})}(H_p(C_G ? \backslash X^?; R), \mathcal{H}_G^q(G/?)). \end{aligned} \quad (4.4)$$

For any contravariant $R\text{Sub}(G; \mathcal{F})$ -module M and $p \in \mathbb{Z}$ there is an in (X, A) natural R -homomorphism

$$\begin{aligned} \alpha_G^p(X, A; M): H_{R\text{Sub}(G; \mathcal{F})}^p(X, A; M) &\rightarrow \text{hom}_{\mathbb{Q}\text{Sub}(G; \mathcal{F})}(H_p(C_G ? \backslash X^?; \mathbb{Q}), M) \\ &\xrightarrow{\cong} \text{hom}_{R\text{Sub}(G; \mathcal{F})}(H_p(C_G ? \backslash X^?; R), M) \end{aligned} \quad (4.5)$$

which is bijective if M is injective as $\mathbb{Q}\text{Sub}(G; \mathcal{F})$ -module.

Theorem 4.6 (The equivariant Chern character). *Let R be a commutative ring R with $\mathbb{Q} \subseteq R$. Let $\mathcal{H}_?^*$ be a proper equivariant cohomology theory with values in R -modules. Suppose that the $R\text{Sub}(G; \mathcal{F})$ -module $\mathcal{H}_G^q(G/?)$ of (3.7), which sends G/H to $\mathcal{H}_G^q(G/H)$, is injective as $\mathbb{Q}\text{Sub}(G; \mathcal{F})$ -module for every group G and every $q \in \mathbb{Z}$.*

Then we obtain a transformation of proper equivariant cohomology theories with values in R -modules

$$\text{ch}_?^*: \mathcal{H}_?^* \xrightarrow{\cong} \mathcal{B}\mathcal{H}_?^*,$$

if we define for a group G and a proper G - CW -pair (X, A)

$$\text{ch}_G^n(X, A): \mathcal{H}_G^n(X, A) \rightarrow \mathcal{B}\mathcal{H}_G^n(X, A) := \prod_{p+q=n} H_{R\text{Sub}(G; \mathcal{F})}^p(X, A; \mathcal{H}_G^q(G/?))$$

by the composite

$$\begin{aligned} \mathcal{H}_G^n(X, A) &\xrightarrow{\prod_{p+q=n} \underline{\text{ch}}_G^{p,q}(X, A)} \prod_{p+q=n} \text{hom}_{R\text{Sub}(G; \mathcal{F})} (H_p(C_G? \setminus X^?; R), \mathcal{H}_G^q(G/?)) \\ &\xrightarrow{\prod_{p+q=n} \alpha_G^p(X, A; \mathcal{H}_G^q(G/?))^{-1}} \prod_{p+q=n} H_{R\text{Sub}(G; \mathcal{F})}^p(X, A; \mathcal{H}_G^q(G/?)) \end{aligned}$$

of the maps defined in (4.4) and (4.5).

The R -map $\text{ch}_G^n(X, A)$ is bijective for all proper relative finite G -CW-pairs (X, A) and $n \in \mathbb{Z}$. If \mathcal{H}_*^* satisfies the disjoint union axiom, then the R -map $\text{ch}_G^n(X, A)$ is bijective for all proper G -CW-pairs (X, A) and $n \in \mathbb{Z}$.

Proof. First one checks that ch_G^* defines a natural transformation of proper G -cohomology theories. One checks for each finite subgroup $H \subseteq G$ and $n \in \mathbb{Z}$ that $\text{ch}_G^n(G/H)$ is the identity if we identify for any $R\text{Sub}(G; \mathcal{F})$ -module M

$$\begin{aligned} H_{R\text{Sub}(G; \mathcal{F})}^p(G/H; M) &= H^p \left(\text{hom}_{R\text{Sub}(G; \mathcal{F})} (C_*^{R\text{Sub}(G; \mathcal{F})}(G/H), M) \right) \\ &= \begin{cases} \text{hom}_{R\text{Sub}(G; \mathcal{F})} (R \text{mor}_{\text{Sub}(G; \mathcal{F})} (?, G/H), M) = M(G/H) & \text{if } p = 0; \\ 0 & \text{if } p \neq 0. \end{cases} \end{aligned}$$

Finally apply Lemma 1.1 (b). \square

Remark 4.7 (The Atiyah-Hirzebruch spectral sequence for equivariant cohomology). There exists a Atiyah-Hirzebruch spectral sequence for equivariant cohomology (see [3, Theorem 4.7 (2)]). It converges to $\mathcal{H}_G^{p+q}(X, A)$ and has as E_2 -term the Bredon cohomology groups $H_{R\text{Sub}(G; \mathcal{F})}^p(X, A; \mathcal{H}_G^q(G/?))$. The conclusion of Theorem 4.6 is that the spectral sequences collapses.

Example 4.8 (Equivariant Chern character for $\mathcal{K}^*(G \setminus (X, A))$). Let \mathcal{K}^* be a (non-equivariant) cohomology theory with values in R -modules for a commutative ring with $\mathbb{Q} \subseteq R$. In Example 1.6 we have assigned to it an equivariant cohomology theory by

$$\mathcal{H}_G^n(X, A) = \mathcal{K}^n(G \setminus (X, A)).$$

We claim that the assumptions appearing in Theorem 4.6 are satisfied. We have to show that the constant functor

$$\underline{\mathcal{K}^q(\{\text{pt.}\})}: \text{Sub}(G; \mathcal{F}) \rightarrow \mathbb{Q}\text{-MOD}, \quad H \mapsto \mathcal{K}^q(\{\text{pt.}\})$$

is injective. Let $i: \text{Sub}(\{1\}) \rightarrow \text{Sub}(G; \mathcal{F})$ the obvious inclusion of categories. Since the object $\{1\}$ is an initial object in $\text{Sub}(G; \mathcal{F})$, the $\mathbb{Q}\text{Sub}(G; \mathcal{F})$ -modules $i_!(\mathcal{K}^q(\{\text{pt.}\}))$ and $\mathcal{K}^q(\{\text{pt.}\})$ are isomorphic. Since $i_!$ sends an injective \mathbb{Q} -module to an injective $R\text{Sub}(G; \mathcal{F})$ -module by Lemma 2.13 (c) and $\mathcal{H}^q(\{\text{pt.}\})$ is injective

as \mathbb{Q} -module, $\overline{\mathcal{K}^q(\{\text{pt.}\})}$ is injective as $\mathbb{Q}\text{Sub}(G; \mathcal{F})$ -module. From Theorem 4.6 we get a transformation of equivariant cohomology theories

$$\begin{aligned} \text{ch}_G^n(X, A): \mathcal{K}^n(G \backslash (X, A)) &\xrightarrow{\cong} \prod_{p+q=n} H_{R\text{Sub}(G; \mathcal{F})}^p(X, A; \overline{\mathcal{H}^q(\{\text{pt.}\})}) \\ &= \prod_{p+q=n} H^p(G \backslash (X, A); \mathcal{H}^q(\{\text{pt.}\})). \end{aligned}$$

One easily checks that this is precisely the Chern character of Example 4.1 applied to \mathcal{K}^* and the CW -pair $G \backslash (X, A)$.

5. Mackey Functors

In Theorem 4.6 the assumption appears that the contravariant $R\text{Sub}(G; \mathcal{F})$ -module $\mathcal{H}_G^q(G/?)$ is injective for each $q \in \mathbb{Z}$. We want to give a criterion which ensures that this assumption is satisfied and which turns out to apply to all cases of interest.

Let R be a commutative ring. Let FGINJ be the category of finite groups with injective group homomorphisms as morphisms. Let $M: \text{FGINJ} \rightarrow R\text{-MOD}$ be a bifunctor, i.e. a pair (M_*, M^*) consisting of a covariant functor M_* and a contravariant functor M^* from FGINJ to $R\text{-MOD}$ which agree on objects. We will often denote for an injective group homomorphism $f: H \rightarrow G$ the map $M_*(f): M(H) \rightarrow M(G)$ by ind_f and the map $M^*(f): M(G) \rightarrow M(H)$ by res_f and write $\text{ind}_H^G = \text{ind}_f$ and $\text{res}_G^H = \text{res}_f$ if f is an inclusion of groups. We call such a bifunctor M a *Mackey functor* with values in R -modules if

- (a) For an inner automorphism $c(g): G \rightarrow G$ we have $M_*(c(g)) = \text{id}: M(G) \rightarrow M(G)$;
- (b) For an isomorphism of groups $f: G \xrightarrow{\cong} H$ the composites $\text{res}_f \circ \text{ind}_f$ and $\text{ind}_f \circ \text{res}_f$ are the identity;
- (c) Double coset formula

We have for two subgroups $H, K \subseteq G$

$$\text{res}_G^K \circ \text{ind}_H^G = \sum_{KgH \in K \backslash G/H} \text{ind}_{c(g): H \cap g^{-1}Kg \rightarrow K} \circ \text{res}_H^{H \cap g^{-1}Kg},$$

where $c(g)$ is conjugation with g , i.e. $c(g)(h) = ghg^{-1}$.

Let G be a group. In the sequel we denote for a subgroup $H \subseteq G$ by $N_G H$ the normalizer and by $C_G H$ the centralizer of H in G and by $W_G H$ the quotient $N_G H/H \cdot C_G H$. Notice that $W_G H$ is finite if H is finite. Let R be a commutative ring. Let M be a Mackey functor with values in R -modules. It induces a contravariant $R\text{Sub}(G, \mathcal{F})$ -module denoted in the same way

$$M: \text{Sub}(G, \mathcal{F}) \rightarrow R\text{-MOD}, \quad (f: H \rightarrow K) \mapsto (M^*(f): M(H) \rightarrow M(K)).$$

We want to use Theorem 2.14 (b) to show that M is injective and analyse its structure. The $R[W_G H]$ -module $T_H M$ introduced in (2.11) is the same as the kernel of

$$\prod_{K \subsetneq H} M(i_K): M(H) \rightarrow \prod_{K \subsetneq H} M(K),$$

where for each subgroup $K \subsetneq H$ different from H we denote by i_K the inclusion. Suppose that $R[W_G H]$ -module $T_H M$ is injective for every finite subgroup $H \subseteq G$. For every finite subgroup $H \subseteq G$ choose a retraction $\rho_H: M(H) \rightarrow T_H M$ of the inclusion $T_H M \rightarrow M(H)$. Denote by $I = \text{Is}(\text{Sub}(G, \mathcal{F}))$ the set of isomorphism classes of objects in $\text{Sub}(G, \mathcal{F})$ which is the same as the set of conjugacy classes (H) of finite subgroups H of G . Let

$$\nu = \nu(M): M \rightarrow \prod_{(K) \in I} i(K)_! \circ T_K(M) \quad (5.1)$$

be the homomorphism of $R\text{Sub}(G, \mathcal{F})$ -modules uniquely determined by the property that for any $(K) \in I$ its composition with the projection onto the factor indexed by (K) is the adjoint of $\rho_K: M(K) \rightarrow T_K M$ for the adjoint pair $(i(K)^*, i(K)_!)$.

Theorem 5.2 (Injectivity and Mackey functors). *Let G be a group and let R be a commutative ring such that the order of every finite subgroup of G is invertible in R . Suppose that the $R[W_G H]$ -module $T_H M$ is injective for each finite subgroup $H \subseteq G$. Then M is injective as $R\text{Sub}(G, \mathcal{F})$ -module and the map ν of (5.1) is bijective.*

Proof. The map ν of (5.1) is the map $\nu(M)$ appearing in Theorem 2.14 (b). Because of Theorem 2.14 (b) it suffices to show for each finite subgroup $H \subseteq G$ that $\nu(M)(H)$ is surjective.

Fix for any $(K) \in I$ a representative K . Then choose for any $W_G K \cdot f \in W_G K \setminus \text{mor}(K, H)$ an element $f \in \text{conhom}(K, H)$ which represents a morphism $f: K \rightarrow H$ in $\text{Sub}(G, \mathcal{F})$ which belongs to $W_G K \cdot f \in W_G K \setminus \text{mor}(K, H)$. Notice that $W_G K$ is the automorphism group of the object K in $\text{Sub}(G, \mathcal{F})$ and $W_G K$, $\text{mor}(K, H)$ and $W_G K \setminus \text{mor}(K, H)$ are finite. With these choices we get for every object H in $\text{Sub}(G, \mathcal{F})$ an identification

$$i(K)_! T_K M(H) = \text{hom}_{RW_G K}(R \text{mor}(K, H), T_K M) = \prod_{\substack{W_G K \cdot f \in \\ W_G K \setminus \text{mor}(K, H)}} T_K M^{W_G K_f}$$

where $W_G K_f \subseteq W_G K$ is the isotropy group of f under the $W_G K$ -action on $\text{mor}(K, H)$. Under this identification $\nu(H)$ becomes the map

$$\nu(H): M(H) \rightarrow \prod_{(K) \in I} \prod_{\substack{W_G K \cdot f \in \\ W_G K \setminus \text{mor}(K, H)}} T_K M^{W_G K_f}$$

for which the component of $\nu(H)(m)$, which belongs $(K) \in I$ and $W_G K \cdot f \in W_G K \setminus \text{mor}(K, H)$, is $\rho_K \circ \text{res}_f(m)$ for $m \in M(H)$. Notice that the image of

res_f always is contained in $M(K)^{W_G K_f}$. Next we define a map

$$\mu(H): \bigoplus_{\substack{(K) \in I, \\ (K) \leq (H)}} \bigoplus_{\substack{W_G K \cdot f \in \\ W_G K \setminus \text{mor}(K, H)}} T_K M^{W_G K_f} \rightarrow M(H)$$

by requiring that its restriction to the summand, which belongs to $(K) \in I$ and $W_G K \cdot f \in W_G K \setminus \text{mor}(K, H)$, is the composite of the inclusion $T_K M^{W_G K_f} \rightarrow M(K)$ with $\text{ind}_f: M(K) \rightarrow M(H)$. We want to show that the composite

$$\begin{aligned} \nu(H) \circ \mu(H): & \bigoplus_{\substack{(K) \in I, \\ (K) \leq (H)}} \bigoplus_{\substack{W_G K \cdot f \in \\ W_G K \setminus \text{mor}(K, H)}} T_K M^{W_G K_f} \\ \rightarrow \prod_{(K) \in I} \prod_{\substack{W_G K \cdot f \in \\ W_G K \setminus \text{mor}(K, H)}} T_K M^{W_G K_f} &= \bigoplus_{\substack{(K) \in I, \\ (K) \leq (H)}} \bigoplus_{\substack{W_G K \cdot f \in \\ W_G K \setminus \text{mor}(K, H)}} T_K M^{W_G K_f} \end{aligned}$$

is bijective. If K is subconjugated to H , we write $(K) \leq (H)$. Fix $(K), (L) \in I$ with $(K) \leq (H)$ and $(L) \leq (H)$ and $W_G K \cdot f \in \text{mor}(K, H)$ and $W_G L \cdot g \in \text{mor}(L, H)$. Then the homomorphism $T_K M^{W_G K_f} \rightarrow T_L M^{W_G L_g}$ given by $\nu(H) \circ \mu(H)$ and the summands corresponding to (K, f) and (L, g) is induced by the composite

$$\begin{aligned} \alpha_{(K,f),(L,g)}: T_K M^{W_G K_f} &\xrightarrow{i} M(K) \xrightarrow{\text{ind}_f: K \rightarrow \text{im}(f)} M(\text{im}(f)) \xrightarrow{\text{ind}_{\text{im}(f)}^H} M(H) \\ &\xrightarrow{\text{res}_H^{\text{im}(g)}} M(\text{im}(g)) \xrightarrow{\text{res}_g: L \rightarrow \text{im}(g)} M(L) \xrightarrow{\rho_L} T_L M, \end{aligned} \quad (5.3)$$

where i is the inclusion. The double coset formula implies

$$\begin{aligned} &\text{res}_H^{\text{im}(g)} \circ \text{ind}_{\text{im}(f)}^H \\ &= \sum_{\substack{\text{im}(g)h \text{ im}(f) \in \\ \text{im}(g) \setminus H / \text{im}(f)}} \text{ind}_{c(h)}: \text{im}(f) \cap h^{-1} \text{im}(g)h \rightarrow \text{im}(g) \circ \text{res}_{\text{im}(f)}^{\text{im}(f) \cap h^{-1} \text{im}(g)h}. \end{aligned} \quad (5.4)$$

The composite

$$\begin{aligned} T_K M^{W_G K_f} &\xrightarrow{i} M(K) \xrightarrow{\text{ind}_f: K \rightarrow \text{im}(f)} M(\text{im}(f)) \\ &\xrightarrow{\text{res}_{\text{im}(f)}^{\text{im}(f) \cap h^{-1} \text{im}(g)h}} M(\text{im}(f) \cap h^{-1} \text{im}(g)h) \end{aligned}$$

is trivial by the definition of $T_K M$ if $\text{im}(f) \cap h^{-1} \text{im}(g)h \neq \text{im}(f)$ holds. Hence $\alpha_{(K,f),(L,g)} \neq 0$ is only possible if $\text{im}(f) \cap h^{-1} \text{im}(g)h = \text{im}(f)$ for some $h \in H$ and hence $(K) \leq (L)$ hold.

Suppose that $(K) = (L)$. Then $K = L$ by our choice of representatives. Suppose that $\alpha_{(K,f),(K,g)} \neq 0$. We have already seen that this implies $\text{im}(f) \cap h^{-1} \text{im}(g)h = \text{im}(f)$ for some $h \in H$. Since $|\text{im}(f)| = |K| = |\text{im}(g)|$ we conclude

$h^{-1} \text{im}(g)h = \text{im}(f)$ and therefore $W_G K \cdot f = W_G K \cdot g$ in $W_G K \setminus \text{mor}(K, H)$. This implies already $f = g$ as group homomorphism $K \rightarrow H$ by our choice of representatives. The double coset formula (5.4) implies that $\alpha_{(K,f),(K,f)}$ is $|H \cap N_G \text{im}(f)| \cdot \text{id}_{T_K M^{W_G K_f}}$ since for all $h \in N_G \text{im}(f) \cap H$ the composite

$$T_K M^{W_G K_f} \xrightarrow{i} M(K) \xrightarrow{\text{ind}_f: K \rightarrow \text{im}(f)} M(\text{im}(f)) \xrightarrow{\text{ind}_{c(h): \text{im}(f) \rightarrow \text{im}(f)}} M(\text{im}(f))$$

agrees with $T_K M^{W_G K_f} \xrightarrow{i} M(K) \xrightarrow{\text{ind}_f: K \rightarrow \text{im}(f)} M(\text{im}(f))$. Since the order of $|H \cap N_G \text{im}(f)|$ is invertible in R by assumption, $\alpha_{(K,f),(K,f)}$ is bijective.

We conclude that $\nu(H) \circ \mu(H)$ can be written as a matrix of maps which has upper triangular form and isomorphisms on the diagonal. Therefore $\nu(H) \circ \mu(H)$ is surjective. This shows that $\nu(H)$ is surjective. This finishes the proof of Theorem 5.2. \square

Theorem 5.5 (The equivariant Chern character and Mackey structures). *Let R be a commutative ring with $\mathbb{Q} \subseteq R$. Let $\mathcal{H}_?^*$ be a proper equivariant cohomology theory. Define a contravariant functor*

$$\mathcal{H}_?^q(\{\text{pt.}\}): \text{FGINJ} \rightarrow R\text{-MOD}$$

by sending a homomorphism $\alpha: H \rightarrow K$ to the composite

$$\mathcal{H}_K^q(\{\text{pt.}\}) \xrightarrow{\mathcal{H}^q(\text{pr})} \mathcal{H}_K^q(K/H) \xrightarrow{\text{ind}_\alpha} \mathcal{H}_H^q(\{\text{pt.}\})$$

where $\text{pr}: H/K = \text{ind}_\alpha(\{\text{pt.}\}) \rightarrow \{\text{pt.}\}$ is the projection and ind_α comes from the induction structure of $\mathcal{H}_?^*$. Suppose that it extends to a Mackey functor for every $q \in \mathbb{Z}$. Then

- (a) For every group G the $R\text{Sub}(G; \mathcal{F})$ -module $\mathcal{H}_G^q(G/?)$ of (3.7) is injective as $R\text{Sub}(G; \mathcal{F})$ -module, provided that R is semisimple;
- (b) We obtain a natural transformation of proper equivariant cohomology theories with values in R -modules

$$\text{ch}_?^*(X, A): \mathcal{H}_?^* \rightarrow \mathcal{B}\mathcal{H}_?^*.$$

In particular we get for every proper G -CW-pair (X, A) and every $n \in \mathbb{Z}$ a natural R -homomorphism

$$\begin{aligned} \text{ch}_G^n(X, A): \mathcal{H}_G^n(X, A) \\ \rightarrow \mathcal{B}\mathcal{H}_G^n(X, A) := \prod_{p+q=n} H_{R\text{Sub}(G; \mathcal{F})}^p(X, A; \mathcal{H}_G^q(G/?)). \end{aligned}$$

It is bijective for all proper relative finite G -CW-pairs (X, A) and $n \in \mathbb{Z}$. If $\mathcal{H}_?^*$ satisfies the disjoint union axiom, it is bijective for all proper G -CW-pairs (X, A) and $n \in \mathbb{Z}$;

(c) Define for finite subgroup $H \subseteq G$ the $R[W_G H]$ -module $T_H(\mathcal{H}_H^q(\{\text{pt.}\}))$ by

$$\ker \left(\prod_{L \subsetneq H} \text{ind}_L^K \circ \mathcal{H}^q(\text{pr}: H/L \rightarrow \{\text{pt.}\}): \mathcal{H}_H^q(\{\text{pt.}\}) \rightarrow \prod_{L \subsetneq H} \mathcal{H}_L^q(\{\text{pt.}\}) \right).$$

Then the Bredon cohomology $\mathcal{B}\mathcal{H}_G^n(X, A)$ of a proper G -CW-pair (X, A) is naturally R -isomorphic to

$$\prod_{p+q=n} \prod_{(H), H \subseteq G \text{ finite}} H_{W_G H}^p(C_G H \setminus (X^H, A^H); T_H(\mathcal{H}_H^q(\{\text{pt.}\})))$$

provided that R is semisimple.

Proof. (a) This follows from Theorem 5.2 since for every finite subgroup $H \subseteq G$ the group $W_G H$ is finite and hence the ring $R[W_G H]$ is semisimple and every $R[W_G H]$ -module is injective.

(b) This follows from assertion (a) applied in the case $R = \mathbb{Q}$ together with Theorem 4.6.

(c) Since R is semisimple and contains \mathbb{Q} and $W_G H$ is finite, the ring $R[W_G H]$ is semisimple and every $R[W_G H]$ -module is injective for every finite subgroup H . Because the map ν of (5.1) is an isomorphism by Theorem 5.2, it remains to show for a $C_G H$ -module N

$$H_{R\text{Sub}(G; \mathcal{F})}^p(X, A; i(H)_! N) \cong H_{W_G H}^p(C_G H \setminus (X^H, A^H); N)$$

This follows from the adjunction $(i(H)^*, i(H)_!)$ of Lemma 2.13 (a). \square

Example 5.6 (Mackey structures for Borel cohomology). Let \mathcal{K}^* be a cohomology theory for (non-equivariant) CW-pairs with values in R -modules for a commutative ring R such that $\mathbb{Q} \subseteq R$ and R is semisimple. In Example 1.6 we have assigned to it an equivariant cohomology theory called equivariant Borel cohomology by

$$\mathcal{H}_G^n(X, A) = \mathcal{K}^n(EG \times_G (X, A)).$$

We claim that the assumptions appearing in Theorem 5.5 are satisfied. Namely, the contravariant functor

$$\text{FGINJ} \rightarrow R\text{-MOD}, \quad H \mapsto \mathcal{K}^n(BH)$$

extends to a Mackey functor, the necessary covariant functor comes from the Becker-Gottlieb transfer (see for instance [5] and [6, Corollary 6.4 on page 206]). Hence we get from Theorem 5.5 for every group G and every proper G -CW-pair (X, A) natural R -maps

$$\begin{aligned} \text{ch}_G^n(X, A): \mathcal{K}^n(EG \times_G (X, A)) &\xrightarrow{\cong} \prod_{p+q=n} H_{R\text{Sub}(G; \mathcal{F})}^p(X, A; \mathcal{K}^q(B?)) \\ &\cong \prod_{p+q=n} \prod_{(H), H \subseteq G \text{ finite}} \text{hom}_{RW_G H}(H_p(C_G H \setminus X^H; R), T_H(\mathcal{K}^q(BH))), \end{aligned}$$

if we define

$$T_H(\mathcal{K}^q(BH)) := \ker \left(\prod_{K \subsetneq H} \mathcal{K}^q(BK \rightarrow BH) : \mathcal{K}^q(BH) \rightarrow \prod_{K \subsetneq H} \mathcal{K}^q(BK) \right).$$

If (X, A) is relative finite or if \mathcal{K}^* satisfies the disjoint union axiom, then these maps $\text{ch}_G^n(X, A)$ are bijective.

Remark 5.7. We mention that this does *not* prove Theorem 0.1 since we cannot apply it to $\mathcal{K}^* := K^* \otimes_{\mathbb{Z}} \mathbb{Q}$. The problem is that $K^* \otimes_{\mathbb{Z}} \mathbb{Q}$ defines all axioms of a cohomology theory but not the disjoint union axiom. But this is needed if we want to deal with classifying spaces BG of groups which are not finite CW -complexes, for instance of groups containing torsion (see also Remark 1.2 and Example 1.3).

A proof of Theorem 0.1 will be given in [11].

Example 5.8 (Equivariant K -theory and Mackey structures). In Example 1.7 we have introduced the equivariant cohomology theory $K_?^*$ given by topological K -theory. Recall that it takes values in R -modules for $R = \mathbb{Z}$. Notice that for a finite group H we get an identification of $K_H^0(\{\text{pt.}\})$ with the complex representation ring $R(H)$ and the associated contravariant functor

$$K_?^q: \text{FGINJ} \rightarrow R\text{-MOD}, \quad H \mapsto K_H^q(\{\text{pt.}\}) = R(?)$$

sends an injective group homomorphism $\alpha: H \rightarrow G$ of finite groups to the homomorphism of abelian groups $R(G) \rightarrow R(H)$ given by restriction with α . Induction with α induces a covariant functor $H \mapsto R(H)$ and it turns out that this defines a Mackey structure on $K_?^q$.

For rationalized equivariant topological K -theory $K_?^* \otimes_{\mathbb{Z}} \mathbb{Q}$ the equivariant Chern character of Theorem 5.5 can be identified with the one constructed in [12] for proper relative finite G - CW -pairs (X, A) .

6. Multiplicative Structures

Next we want to introduce a multiplicative structure on a proper equivariant cohomology theory $\mathcal{H}_?^*$ and show that it induces one on the associated Bredon cohomology $\mathcal{BH}_?^*$ such that the equivariant Chern character is compatible with it.

We begin with the non-equivariant case. Let \mathcal{H}^* be a (non-equivariant) cohomology theory with values in R -modules. A *multiplicative structure* assigns to a CW -complex X with CW -subcomplexes $A, B \subseteq X$ natural R -homomorphisms

$$\cup: \mathcal{H}^n(X, A) \otimes_R \mathcal{H}^{n'}(X, B) \rightarrow \mathcal{H}^{n+n'}(X, A \cup B). \quad (6.1)$$

This product is required to be compatible with the boundary homomorphism of the long exact sequence of a pair, to be graded commutative, to be associative and to have a unit $1 \in \mathcal{H}^0(\{\text{pt.}\})$.

Given a multiplicative structure on \mathcal{H}^* , we obtain for every $p, q \in \mathbb{Z}$ a pairing

$$\cup: \mathcal{H}^q(\{\text{pt.}\}) \otimes_R \mathcal{H}^{q'}(\{\text{pt.}\}) \rightarrow \mathcal{H}^{q+q'}(\{\text{pt.}\}).$$

It yields on singular (or equivalently cellular) cohomology a product

$$H^p(X, A; \mathcal{H}^q(\{\text{pt.}\})) \otimes_R H^{p'}(X, B; \mathcal{H}^{q'}(\{\text{pt.}\})) \rightarrow H^{p+p'}(X, A \cup B; \mathcal{H}^{q+q'}(\{\text{pt.}\})).$$

The collection of these pairings induce a multiplicative structure on the cohomology theory given by $\prod_{p+q=n} H^p(X, A; \mathcal{H}^q(\{\text{pt.}\}))$. The straightforward proof of the next lemma is left to the reader.

Lemma 6.2. *Let R be a commutative ring with $\mathbb{Q} \subseteq R$. Let \mathcal{H}^* be a (non-equivariant) cohomology theory satisfying the disjoint union axiom which comes with a multiplicative structure.*

Then the (non-equivariant) Chern character of Example 4.1

$$\text{ch}^n(X, A): \mathcal{H}^n(X, A) \xrightarrow{\cong} \prod_{p+q=n} H^p(X, A, \mathcal{H}^q(*))$$

is compatible with the given multiplicative structure on \mathcal{H}^ and the induced multiplicative structure on the target.*

Next we deal with the equivariant version. We only deal with the proper case, the definitions below make also sense without this condition.

Let \mathcal{H}_G^* be a proper G -cohomology theory. A *multiplicative structure* assigns to a proper G -CW-complex X with G -CW-subcomplexes $A, B \subseteq X$ natural R -homomorphisms

$$\cup: \mathcal{H}_G^n(X, A) \otimes_R \mathcal{H}_G^{n'}(X, B) \rightarrow \mathcal{H}_G^{n+n'}(X, A \cup B). \quad (6.3)$$

This product is required to be compatible with the boundary homomorphism of the long exact sequence of a G -CW-pair, to be graded commutative, to be associative and to have a unit $1 \in \mathcal{H}_G^0(X)$ for every proper G -CW-complex X .

Let $\mathcal{H}_?^*$ be a proper equivariant cohomology theory. A *multiplicative structure* on it assigns a multiplicative structure to the associated proper G -cohomology theory \mathcal{H}_G^* for every group G such that for each group homomorphism $\alpha: H \rightarrow G$ the maps given by the induction structure of (1.4)

$$\text{ind}_\alpha: \mathcal{H}_G^n(\text{ind}_\alpha(X, A)) \xrightarrow{\cong} \mathcal{H}_H^n(X, A)$$

are in the obvious way compatible with the multiplicative structures on \mathcal{H}_G^* and \mathcal{H}_H^* .

Next we explain how a given multiplicative structure on $\mathcal{H}_?^*$ induces one on $\mathcal{BH}_?^*$. We have to specify for every group G a multiplicative structure on

the G -cohomology theory \mathcal{BH}_G^* . Consider a G -CW-complex X with G -CW-subcomplexes $A, B \subseteq X$. For two contravariant $R\text{Or}(G; \mathcal{F})$ -chain complexes C_* and D_* define the contravariant $R\text{Or}(G; \mathcal{F})$ -chain complexes $C_* \otimes_R D_*$ by sending G/H to the tensor product of R -chain complexes $C_*(G/H) \otimes_R D_*(G/H)$. Let

$$a_* : C_*^{R\text{Or}(G; \mathcal{F})}(X, A) \otimes_R C_*^{R\text{Or}(G; \mathcal{F})}(X, B) \xrightarrow{\cong} C_*^{R\text{Or}(G; \mathcal{F})}((X, A) \times (X, B))$$

be the isomorphism of $R\text{Or}(G; \mathcal{F})$ -chain complexes which is given for an object G/H by the natural isomorphism of cellular R -chain complexes

$$C_*(X^H, A^H) \otimes_R C_*(X^H, B^H) \xrightarrow{\cong} C_*((X^H, A^H) \times (X^H, B^H)).$$

The multiplicative structure on \mathcal{H}_G^* yields a map of contravariant $R\text{Or}(G; \mathcal{F})$ -modules

$$c : \mathcal{H}_G^q(G/?) \otimes_R \mathcal{H}_G^q(G/?) \rightarrow \mathcal{H}_G^{q+q}(G/?).$$

Let

$$\Delta : (X; A \cup B) \rightarrow (X, A) \times (X, B), \quad x \mapsto (x, x)$$

be the diagonal embedding. Define a R -cochain map by the composite

$$\begin{aligned} b^* : & \text{hom}_{R\text{Or}(G; \mathcal{F})} \left(C_*^{R\text{Or}(G; \mathcal{F})}(X, A), \mathcal{H}_G^q(G/?) \right) \\ & \otimes_R \text{hom}_{R\text{Or}(G; \mathcal{F})} \left(C_*^{R\text{Or}(G; \mathcal{F})}(X, B), \mathcal{H}_G^{q'}(G/?) \right) \xrightarrow{\otimes_R} \\ & \text{hom}_{R\text{Or}(G; \mathcal{F})} \left(C_*^{R\text{Or}(G; \mathcal{F})}(X, A) \otimes_R C_*^{R\text{Or}(G; \mathcal{F})}(X, B), \mathcal{H}_G^q(G/?) \otimes_R \mathcal{H}_G^{q'}(G/?) \right) \\ & \xrightarrow{\text{hom}_{R\text{Or}(G; \mathcal{F})}((a_*)^{-1}, c)} \text{hom}_{R\text{Or}(G; \mathcal{F})} \left(C_*^{R\text{Or}(G; \mathcal{F})}((X, A) \times (X, B)), \mathcal{H}_G^{q+q'}(G/?) \right) \\ & \xrightarrow{\text{hom}_{R\text{Or}(G; \mathcal{F})}(C_*^{R\text{Or}(G; \mathcal{F})}(\Delta), \text{id})} \text{hom}_{R\text{Or}(G; \mathcal{F})} \left(C_*^{R\text{Or}(G; \mathcal{F})}(X, A \cup B), \mathcal{H}_G^{q+q'}(G/?) \right). \end{aligned}$$

There is a canonical R -map

$$H^*(C^* \otimes_R D^*) \rightarrow H^*(C^* \otimes_R D^*)$$

for two R -cochain complexes C^* and D^* . This map together with the map induced by b^* on cohomology yields an R -homomorphism

$$\begin{aligned} H_{R\text{Or}(G; \mathcal{F})}^p(X, A; \mathcal{H}_G^q(G/?)) \otimes_R H_{R\text{Or}(G; \mathcal{F})}^{p'}(X, B; \mathcal{H}_G^{q'}(G/?)) \\ \rightarrow H_{R\text{Or}(G; \mathcal{F})}^{p+p'}(X, A \cup B; \mathcal{H}_G^{q+q'}(G/?)). \end{aligned}$$

The collection of these R -homomorphisms yields the desired multiplicative structure on \mathcal{BH}_G^* . We leave it to the reader to check that the axioms of a multiplicative structure on \mathcal{BH}_G^* are satisfied and that all these are compatible with the induction structure so that we obtain a multiplicative structure on the equivariant cohomology theory \mathcal{BH}_G^* .

We also omit the lengthy but straightforward proof of the following result which is based on Theorem 4.6, Lemma 6.2 and the compatibility of the multiplicative structure with the induction structure.

Theorem 6.4 (The equivariant Chern character and multiplicative structures). *Let R be a commutative ring such that $\mathbb{Q} \subseteq R$. Suppose that $\mathcal{H}_?^*$ is a proper cohomology theory with values in R -modules which comes with a multiplicative structure. Suppose that the $R\text{Sub}(G; \mathcal{F})$ -module $\mathcal{H}_G^q(G/?)$ of (3.7), which sends G/H to $\mathcal{H}_G^q(G/H)$, is injective for each $q \in \mathbb{Z}$.*

Then the natural transformation of equivariant cohomology theories appearing in Theorem 4.6

$$\text{ch}_?^*: \mathcal{H}_?^* \rightarrow \mathcal{BH}_?^*$$

is compatible with the given multiplicative structure on $\mathcal{H}_?^$ and the induced multiplicative structure on $\mathcal{BH}_?^*$.*

Remark 6.5 (External products and restriction structures). One can also define an *external product* for a proper equivariant cohomology theory $\mathcal{H}_?^*$ with values in R -modules. It assigns to every two groups G and H , a proper G - CW -pair (X, A) , a proper H - CW -pair (Y, B) and $p, q \in \mathbb{Z}$ an R -homomorphism

$$\times: \mathcal{H}_G^p(X, A) \otimes_R \mathcal{H}_H^q(Y, B) \rightarrow \mathcal{H}_{G \times H}^{p+q}((X, A) \times (Y, B)).$$

One requires graded commutativity, associativity, the existence of a unit $1 \in \mathcal{H}_{\{1\}}^0(\{\text{pt.}\})$ and compatibility with the induction structure and the boundary homomorphism associated to a pair. One can show that $\mathcal{BH}_?^*$ inherits an external product and prove the analog of Theorem 6.4 for external products.

One can also introduce the notion of a *restriction structure* on $\mathcal{H}_?^*$. It yields for every injective group homomorphism $\alpha: H \rightarrow G$, every proper G - CW -pair (X, A) and $p \in \mathbb{Z}$ an R -homomorphism

$$\text{res}_\alpha: \mathcal{H}_G^p(X, A) \rightarrow \mathcal{H}_H^p(\text{res}_\alpha(X, A)).$$

Again certain axioms are required such as compatibility with the boundary homomorphism associated to pair, compatibility with induction for group isomorphisms $\alpha: H \xrightarrow{\cong} G$, compatibility with conjugation, the double coset formula and compatibility for projections onto quotients under free actions. One can show that $\mathcal{BH}_?^*$ inherits a restriction structure and prove the analog of Theorem 6.4 for restriction structures.

An external product together with a restriction structure yields a multiplicative structure as follows. Consider G - CW -pairs (X, A) and (X, B) . Let $d: G \rightarrow G \times G$ and $D: (X, A \cup B) \rightarrow (X, A) \times (X, B)$ be the diagonal maps. Define

$$\cup: \mathcal{H}_G^m(X, A) \otimes_R \mathcal{H}_G^n(X, A) \rightarrow \mathcal{H}_G^{m+n}(X, A \cup B) \quad (6.6)$$

to be the composite

$$\begin{aligned} \mathcal{H}_G^m(X, A) \otimes_R \mathcal{H}_G^n(X, A) &\xrightarrow{\times} \mathcal{H}_{G \times G}^{m+n}((X, A) \times (X, B)) \\ &\xrightarrow{\text{res}_d} \mathcal{H}_G^{m+n}((X, A) \times (X, B)) \xrightarrow{\mathcal{H}_G^{m+n}(D)} \mathcal{H}_G^{m+n}(X, A \cup B). \end{aligned}$$

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