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The type of the classifying space for a family of subgroups

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Abstract

The classifying space $E(\Gamma,\mathcal{F})$ for a family $\mathcal{F}$ of subgroups of a group Γ is defined up to Γ-homotopy as a Γ-CW-complex $E(\Gamma,\mathcal{F})$ such that $E(\Gamma,\mathcal{F})^H$ is contractible if H belongs to $\mathcal{F}$ and is empty otherwise. We investigate the question whether there is a finite-dimensional Γ-CW-model, a finite Γ-CW-model or a Γ-CW-model of finite type for $E(\Gamma,\mathcal{F})$ focusing on the case where $\mathcal{F}$ is the family of finite subgroups.

MSC: 55R35

0. Introduction

Let Γ be a group and $\mathcal{F}$ be a *family of subgroups*, i.e. a set of subgroups of Γ which is closed under conjugation and taking subgroups.

Definition 0.1. A *classifying space* $E(\Gamma,\mathcal{F})$ for $\mathcal{F}$ is a Γ-CW-complex such that $E(\Gamma,\mathcal{F})^H$ is contractible for $H\in\mathcal{F}$ and empty otherwise.

If $\mathcal{F}$ is the family $\mathcal{FIN}$ of finite subgroups, we abbreviate

$$\underline{E}\Gamma = E(\Gamma,\mathcal{FIN}).$$

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The existence of $E(\Gamma,\mathcal{F})$ and proofs that for any Γ-CW-complex X whose isotropy groups belong to $\mathcal{F}$ there is precisely one Γ-map up to Γ-homotopy from X to $E(\Gamma,\mathcal{F})$ and thus that two such classifying spaces are Γ-homotopy equivalent, are presented in [9] and [10, I.6]. A functorial "bar-type" construction is given in [8, Section 7]. These classifying spaces occur in the Isomorphism-Conjectures in algebraic K- and L-theory [12, 1.6 on p. 257] for $\mathcal{F}$ the family of virtually cyclic groups and in the Baum-Connes-Conjecture [2, Conjecture 3.15 on p. 254] for $\mathcal{F}$ the family $\mathcal{FIN}$ of finite subgroups. The space $\underline{E}\Gamma$ plays also a role in the extension of the Atiyah–Segal Completion Theorem from finite to infinite groups [17]. Sometimes $\underline{E}\Gamma$ is also called the classifying space for proper actions. Notice that $E(\Gamma,\mathcal{F})$ for $\mathcal{F}$ the family consisting of one element, namely the trivial subgroup, is the same as the total space $E\Gamma$ of the universal principal Γ-bundle $E\Gamma \to B\Gamma$. If Γ is torsionfree, then $\underline{E}\Gamma$ is the same as $E\Gamma$. If Γ is finite, then the one-point-space is a model for $\underline{E}\Gamma$.

In this paper we are dealing with the question whether one can find a d-dimensional Γ-CW-model, a finite Γ-CW-model or a Γ-CW-model of finite type for $\underline{E}\Gamma$. Recall that a Γ-CW-complex X is *finite* if and only if it consists of finitely many Γ-equivariant cells, or, equivalently, $\Gamma\backslash X$ is compact. It is called *of finite type* if each skeleton is finite. More information about Γ-CW-complexes can be found in [10, II.1 and II.2] and [15, Sections 1 and 2]. A survey of groups for which nice geometric Γ-CW-models for $\underline{E}\Gamma$ exist can be found in [2, Section 2]. These include for instance (i) word-hyperbolic groups Γ for which the Rips complex yields a finite Γ-CW-model for $\underline{E}\Gamma$, (ii) discrete subgroups $\Gamma \subset G$ of a Lie group G with finitely many path components for which G/K with the left Γ-action for a maximal compact subgroup $K \subset G$ is a Γ-CW-model for $\underline{E}\Gamma$ and (iii) groups Γ acting cellularly (without inversion) on trees with finite isotropy groups.

In Section 1 we give necessary conditions for the existence of a Γ-CW-model of $\underline{E}\Gamma$ with prescribed dimensions of the H-fixed point sets $\underline{E}\Gamma^H$ in terms of the Borel cohomology of the posets of non-trivial finite subgroups of the Weyl groups WH of the finite subgroups $H \subset \Gamma$ (see Theorem 1.6). The *length* $l(H) \in \{0,1,\ldots\}$ of a finite group H is the supremum over all p for which there is a nested sequence $H_0 \subset H_1 \subset \cdots \subset H_p$ of subgroups H_i of H with $H_i \neq H_{i+1}$. We do not know whether the necessary conditions above are also sufficient in general. However, if there is an upper bound l on the length $l(H)$ of finite subgroups H of Γ, then the necessary conditions above are also sufficient (see Theorem 1.6).

We also give a necessary algebraic condition $B(d)$ for a non-negative integer d for the existence of a d-dimensional Γ-CW-model for $\underline{E}\Gamma$, namely that for each finite subgroup $H \subset \Gamma$ a $\mathbb{Z}WH$-module M whose restriction to $\mathbb{Z}K$ is projective for any finite subgroup $K \subset WH$ has a d-dimensional projective $\mathbb{Z}WH$-resolution (Notation 1.4. and Lemma 1.5). If there is an upper bound l on the length $l(H)$ of finite subgroups H of Γ and $B(d)$ is satisfied we will prove that there is a $(\max\{3,d\}+l(d+1))$-dimensional Γ-CW-model for $\underline{E}\Gamma$ (Theorem 1.10). Such a result has already been proven for certain classes of groups by Kropholler and Mislin [13, Theorem B], for an in l exponential dimension estimate.

In Section 2 we show that $\underline{E}\Gamma$ has a dm-dimensional Γ-CW-model if there is a subgroup $\Delta \subset \Gamma$ of finite index d with m-dimensional Δ-CW-model for $\underline{E}\Delta$ (Theorem 2.4).

In Section 3 we show for an exact sequence $1 \to \Delta \to \Gamma \to \pi \to 1$ such that there is an upper bound on the orders of finite subgroups of π that $\underline{E}\Gamma$ has a finite-dimensional Γ-CW-model if $\underline{E}\Delta$ and $\underline{E}\pi$ respectively have a finite-dimensional Δ-CW-model and a finite-dimensional π-CW-model respectively (Theorem 3.1). We discuss to which extend a statement like this holds if we ask for finite models or models of finite type (Theorems 3.2 and 3.3).

In Section 4 we prove that $\underline{E}\Gamma$ has a Γ-CW-model of finite type if and only if there is a CW-model of finite type for BWH for all finite subgroups $H \subset \Gamma$ and Γ contains only finitely many conjugacy classes of finite subgroups (Theorem 4.2).

In Section 5 we deal with the question whether there is a finitely dominated or finite Γ-CW-complex model for $\underline{E}\Gamma$ (Theorem 5.1 and Remark 5.2).

In Section 6 we consider the special case of groups of finite virtual cohomological dimension. Provided that Γ contains a torsionfree group of finite index, Γ satisfies $B(d)$ if and only if Γ has virtual cohomological dimension $\le d$ (Theorem 6.3). If l is an upper bound on the length $l(H)$ of finite subgroups H of Γ and Γ has virtual cohomological dimension $\le d$, then we will prove that there is a $(\max\{3,d\}+l)$-dimensional Γ-CW-model for $\underline{E}\Gamma$ (Theorem 6.4).

Finally we discuss some open problems in Section 7.

We will always work in the category of compactly generated spaces (see [20] and [22, I.4]). A Γ-space or a $\mathbb{Z}\Gamma$-module respectively is always to be understood as a left Γ-space or left $\mathbb{Z}\Gamma$-module, respectively. The letter Γ stands always for a discrete group.

1. Finite-dimensional classifying spaces

In this section we deal with the question whether there are finite-dimensional Γ-CW-models for $\underline{E}\Gamma$.

Define the Γ-poset

$$\mathscr{P}(\Gamma) := \{K \mid K \subset \Gamma \text{ finite}, K \neq 1\}. \tag{1.1}$$

An element $\gamma \in \Gamma$ sends K to $\gamma K \gamma^{-1}$ and the poset-structure comes from inclusion of subgroups. Denote by $|\mathscr{P}(\Gamma)|$ the geometric realization of the category given by the poset $\mathscr{P}(\Gamma)$. This is a Γ-CW-complex but in general not proper, i.e. it can have points with infinite isotropy groups.

Let NH be the normalizer and let $WH := NH/H$ be the *Weyl group* of $H \subset \Gamma$. Notice for a Γ-space X that X^H inherits a WH-action. Denote by $\sigma_\Gamma(X)$ the *singular set* of the

Γ-space X, i.e. the set of points with non-trivial isotropy groups. Notice that $\sigma_{\Gamma'}(X)$ may differ from $\sigma_\Gamma(X)$ for a subgroup $\Gamma' \subset \Gamma$. Denote by CX the *cone* over X. Notice that $C\emptyset$ is the one-point-space. The meaning of $|\mathscr{P}(\Gamma)|$ lies in the following result which follows from [7, Lemma 2.4].

Lemma 1.2. *There is a Γ-equivariant map*

$$f : (\underline{E}\Gamma, \sigma_\Gamma(\underline{E}\Gamma)) \to (C|\mathscr{P}(\Gamma)|, |\mathscr{P}(\Gamma)|),$$

which is a (*non-equivariant*) *homotopy equivalence.*

If H and K are subgroups of Γ and H is finite, then Γ/K^H is a finite union of WH-orbits of the shape WH/L for finite $L \subset WH$. Now one easily checks

Lemma 1.3. *The WH-space $\underline{E}\Gamma^H$ is a WH-CW-model for $\underline{E}WH$. In particular, if $\underline{E}\Gamma$ has a Γ-CW-model which is finite, of finite type or d-dimensional respectively, then there is a WH-model for $\underline{E}WH$ which is finite, of finite type or d-dimensional respectively.*

Notation 1.4. Let $d \geq 0$ be an integer. A group Γ satisfies the condition $b(d)$ or $b(<\infty)$, respectively if any $\mathbb{Z}\Gamma$-module M with the property that M restricted to $\mathbb{Z}K$ is projective for all finite subgroups $K \subset \Gamma$ has a projective $\mathbb{Z}\Gamma$-resolution of dimension d or of finite dimension, respectively. A group Γ satisfies the condition $B(d)$ if WH satisfies the condition $b(d)$ for any finite subgroup $H \subset \Gamma$.

The *length* $l(H) \in \{0, 1, \ldots\}$ of a finite group H is the supremum over all p for which there is a nested sequence $H_0 \subset H_1 \subset \cdots \subset H_p$ of subgroups H_i of H with $H_i \neq H_{i+1}$.

Lemma 1.5. *Suppose that there is a d-dimensional Γ-CW-complex X with finite isotropy groups such that $H_p(X;\mathbb{Z}) = H_p(*,\mathbb{Z})$ for all $p \geq 0$ holds. This assumption is for instance satisfied if there is a d-dimensional Γ-CW-model for $\underline{E}\Gamma$. Then Γ satisfies condition $B(d)$.*

Proof. Let $H \subset \Gamma$ be finite. Then X/H satisfies $H_p(X/H;\mathbb{Z}) = H_p(*,\mathbb{Z})$ for all $p \geq 0$ [4, III.5.4 on p. 131]. Let C_* be the cellular $\mathbb{Z}WH$-chain complex of X/H. This is a d-dimensional resolution of the trivial $\mathbb{Z}WH$-module $\mathbb{Z}$ and each chain module is a sum of $\mathbb{Z}WH$-modules of the shape $\mathbb{Z}[WH/K]$ for some finite subgroup $K \subset WH$. Let N be a $\mathbb{Z}WH$-module such that N is projective over $\mathbb{Z}K$ for any finite subgroup $K \subset WH$. Then $C_* \otimes_{\mathbb{Z}} N$ with the diagonal WH-operation is a d-dimensional projective $\mathbb{Z}WH$-resolution of N. $\square$

Theorem 1.6. *Let Γ be a group. Suppose that we have for any finite subgroup $H \subset \Gamma$ an integer $d(H) \geq 3$ such that $d(H) \geq d(K)$ for $H \subset K$ and $d(H) = d(K)$ if H and K are conjugated in Γ. Consider the following statements*:

(1) *There is a Γ-CW-model $\underline{E}\Gamma$ such that for any finite subgroup $H \subset \Gamma$*

$$\dim(\underline{E}\Gamma^H) = d(H).$$

(2) *We have for any finite subgroup $H \subset \Gamma$ and for any $\mathbb{Z}WH$-module M*

$$H^{d(H)+1}_{\mathbb{Z}WH}(EWH \times (C|\mathscr{P}(WH)|, |\mathscr{P}(WH)|); M) = 0.$$

(3) *We have for any finite subgroup $H \subset \Gamma$ that its Weyl group WH satisfies b $(<\infty)$ and that there is a subgroup $\Delta(H) \subset WH$ of finite index such that for any $\mathbb{Z}\Delta(H)$-module M*

$$H^{d(H)+1}_{\mathbb{Z}\Delta(H)}(E\Delta(H) \times (C|\mathscr{P}(WH)|, |\mathscr{P}(WH)|); M) = 0.$$

Then (1) *implies both* (2) *and* (3). *If there is an upper bound on the length $l(H)$ of the finite subgroups H of Γ, then these statements* (1), (2) *and* (3) *are equivalent.*

In the case that Γ has finite virtual cohomological dimension a similar result is proven in [7, Theorem III].

Example 1.7. Suppose that Γ is torsionfree. Then Theorem 1.6 reduces to the well-known result [6, Theorem VIII.3.1 on page 190, Theorem VIII.7.1 on page 205] that the following assertions are equivalent for an integer $d \geq 3$:

1. There is a d-dimensional CW-model for $B\Gamma$;
2. Γ has cohomological dimension $\leq d$;
3. Γ has virtual cohomological dimension $\leq d$.

Example 1.8. In this example we use the notation of Theorem 1.6. If WH is torsion-free, then

$$H^{d(H)+1}_{\mathbb{Z}WH}(EWH \times (C|\mathscr{P}(WH)|, |\mathscr{P}(WH)|); M) = H^{d(H)+1}_{\mathbb{Z}WH}(EWH; M);$$

$$H^{d(H)+1}_{\mathbb{Z}\Delta(H)}(E\Delta(H) \times (C|\mathscr{P}(WH)|, |\mathscr{P}(WH)|); M) = H^{d(H)+1}_{\mathbb{Z}\Delta(H)}(E\Delta(H); M),$$

and the condition that $H^{d(H)+1}_{\mathbb{Z}WH}(EWH \times (C|\mathscr{P}(WH)|, |\mathscr{P}(WH)|); M) = 0$ or $H^{d(H)+1}_{\mathbb{Z}\Delta(H)}(E\Delta(H) \times (C|\mathscr{P}(WH)|, |\mathscr{P}(WH)|); M) = 0$ respectively holds for all $\mathbb{Z}WH$-modules M or $\mathbb{Z}\Delta(H)$-modules M respectively is equivalent to the existence of a $d(H)$-dimensional model for BWH or $B\Delta(H)$, respectively.

If WH contains a non-trivial normal finite subgroup L, then $|\mathscr{P}(WH)|$ is contractible and

$$H^{d(H)+1}_{\mathbb{Z}WH}(EWH \times (C|\mathscr{P}(WH)|, |\mathscr{P}(WH)|); M) = 0;$$

$$H^{d(H)+1}_{\mathbb{Z}\Delta(H)}(E\Delta(H) \times (C|\mathscr{P}(WH)|, |\mathscr{P}(WH)|); M) = 0.$$

Namely, define maps of posets $C,F:\mathscr{P}(WH)\to\mathscr{P}(WH)$ by $C(K)=L$ and $F(K)=\langle K,L\rangle$ where the subgroup $\langle K,L\rangle$ generated by K and L is finite since L is normal in WH. Since $K\subset F(K)$ and $L\subset F(K)$ holds for all finite subgroups K, there are natural transformations between the functors id and F and the functors C and F of the category given by the poset $\mathscr{P}(WH)$ and hence the maps induced by these functors on the geometric realizations are homotopic. Hence the identity is homotopic to the constant map which is the map induced by C.

The proof of Theorem 1.6 needs some preparation. The next lemma deals with the question whether a Γ-CW-complex can be made contractible by attaching free cells of bounded dimension and is the key ingredient in the induction step in the proof of Theorem 1.6.

Lemma 1.9. *Let X be a Γ-CW-complex and $d\geq 3$ be an integer. Consider the following statements*:

(1) *There is a Γ-CW-complex Y which is obtained from X by attaching free cells $\Gamma\times D^n$ of dimension $n\leq d$ and is contractible.*

(2) *There is a Γ-CW-complex Y which is obtained from $E\Gamma\times X$ by attaching free cells $\Gamma\times D^n$ of dimension $n\leq d$ and is contractible.*

(3) *$H_n(X;\mathbb{Z})=0$ for $n\geq d$ and for any $(d-2)$-connected Γ-CW-complex Z which is obtained from X by attaching free cells $\Gamma\times D^n$ of dimension $n\leq d-1$ the $\mathbb{Z}\Gamma$-module $H_{d-1}(Z;\mathbb{Z})$ is $\mathbb{Z}\Gamma$-projective.*

(4) *$H_n(X;\mathbb{Z})=0$ for $n\geq d$ and there is a $(d-2)$-connected Γ-CW-complex Z which is obtained from X by attaching free cells $\Gamma\times D^n$ of dimension $n\leq d-1$ such that the $\mathbb{Z}\Gamma$-module $H_{d-1}(Z;\mathbb{Z})$ is $\mathbb{Z}\Gamma$-projective.*

(5) *The $\mathbb{Z}\Gamma$-chain complex $C_*(E\Gamma\times(CX,X))$ is $\mathbb{Z}\Gamma$-chain homotopy equivalent to a d-dimensional projective $\mathbb{Z}\Gamma$-chain complex.*

(6) *$H_n(X;\mathbb{Z})=0$ for $n\geq d+1$ and $H^{d+1}_{\mathbb{Z}\Gamma}(E\Gamma\times(CX,X);M)=0$ for any $\mathbb{Z}\Gamma$-module M.*

(7) *$H_n(X;\mathbb{Z})=0$ for $n\geq d+1$, the group Γ satisfies b $(<\infty)$ and there is a subgroup $\Gamma_0\subset\Gamma$ of finite index such that $H^{d+1}_{\mathbb{Z}\Gamma_0}(E\Gamma_0\times(CX,X);M)=0$ holds for any $\mathbb{Z}\Gamma_0$-module M.*

(8) *There is a non-negative integer e such that $H_n(X;\mathbb{Z})=0$ for $n\geq d-e$ and the group Γ satisfies $b(e)$.*

Then we have

$$(1)\Leftrightarrow(2)\Leftrightarrow(3)\Leftrightarrow(4)\Leftrightarrow(5)\Leftrightarrow(6).$$

If X is finite-dimensional and all isotropy groups of X are finite then $(1)\Rightarrow(7)$.

If X^H is contractible for all non-trivial finite subgroups $H\subset\Gamma$, then $(7)\Rightarrow(1)$ *and* $(8)\Rightarrow(1)$.

Proof. $(1)\Rightarrow(2)$ Let $Z\supset X$ be the given extension. Then one constructs a pair of Γ-CW-complexes $(Z',E\Gamma\times X)$ such that Z' is obtained from $E\Gamma\times X$ by attaching free cells $\Gamma \times D^n$ of dimension $n \leq d$ and for the Γ-space Z'' defined by the Γ-pushout

$$\begin{array}{ccc} E\Gamma \times X & \longrightarrow & Z' \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \overline{\mathrm{pr}}} \\ X & \longrightarrow & Z'' \end{array}$$

there is a Γ-homotopy equivalence of pairs relative X from (Z'',X) to (Z,X). Then Z' is contractible and the desired Γ-CW-complex.

In fact there is a bijective correspondence between the cells $\Gamma \times D^n$ of $Z' - (E\Gamma \times X)$ and of $Z - X$ for all n. The construction is based on the observation that for any Γ-map $f : A \to B$ of Γ-spaces which is a (non-equivariant) homotopy equivalence any Γ-map $\Gamma \times S^n \to B$ lifts up to Γ-homotopy to a Γ-map $\Gamma \times S^n \to A$ and changing the Γ-homotopy type of the attaching maps does not change the Γ-homotopy type of a Γ-CW-complex. Details can be found in [15, pp. 288–289].

(2) $\Rightarrow$ (5) There is a Γ-homotopy equivalence $(f, \mathrm{id}) : (Y, E\Gamma \times X) \to E\Gamma \times (CX, X)$ since both $E\Gamma \times CX$ and Y are contractible.

(5) $\Rightarrow$ (3) Define a Γ-CW-complex Y by the Γ-pushout

$$\begin{array}{ccc} E\Gamma \times X & \longrightarrow & E\Gamma \times CX \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \overline{\mathrm{pr}}} \\ X & \longrightarrow & Y \end{array}$$

for $\mathrm{pr} : E\Gamma \times X \to X$ the projection. Since the inclusion of $E\Gamma \times X$ into $E\Gamma \times CX$ is a cofibration and pr is a (non-equivariant) homotopy equivalence, $\overline{\mathrm{pr}}$ is a (non-equivariant) homotopy equivalence. Hence Y is contractible. Obviously $C_*(Y,X)$ and $C_*(E\Gamma \times (CX,X))$ are $\mathbb{Z}\Gamma$-chain isomorphic. Choose a Γ-map $f : Z \to Y$ which induces the identity on X and a $\mathbb{Z}\Gamma$-chain homotopy equivalence $g_* : C_*(Y,X) \to P_*$ for a d-dimensional projective $\mathbb{Z}\Gamma$-chain complex P_*. Let C_* be the mapping cone of the $\mathbb{Z}\Gamma$-chain map $g_* \circ C_*(f, \mathrm{id}) : C_*(Z,X) \to P_*$. Since the relative dimension of (Z,X) is $d-1$ and the dimension of P_* is d, C_* is a d-dimensional projective $\mathbb{Z}\Gamma$-chain complex. Since Z is $(d-2)$-connected and Y is contractible, $H_j(C_*)$ is trivial for $j \leq d-1$ and is $H_{d-1}(Z;\mathbb{Z})$ for $j = d$. Hence we obtain an exact sequence of $\mathbb{Z}\Gamma$-modules

$$0 \to H_{d-1}(Z;\mathbb{Z}) \to C_d \to C_{d-1} \to \cdots \to C_0 \to 0.$$

This implies that $H_{d-1}(Z;\mathbb{Z})$ is a projective $\mathbb{Z}\Gamma$-module. We conclude $H_j(X;\mathbb{Z}) = 0$ for $j \geq d$ from the long homology sequence associated to $E\Gamma \times (CX,X)$ and the fact that $H_j(E\Gamma \times (CX,X);\mathbb{Z}) = 0$ for $j \geq d+1$.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1) By the Eilenberg swindle there exists a free $\mathbb{Z}\Gamma$-module F such that $F \oplus H_{d-1}(Z)$ is free. Hence we can achieve by attaching free $(d-1)$-dimensional cells $\Gamma \times D^{d-1}$ with trivial attaching maps to Z that $H_{d-1}(Z)$ is free. The Hurewicz homomorphism $\pi_{d-1}(Z) \to H_{d-1}(Z;\mathbb{Z})$ is bijective since Z is $(d-2)$-connected and $d \geq 3$. Now choose a $\mathbb{Z}\Gamma$-basis for $\pi_{d-1}(Z)$ and attach for each basis element which is given by a map $g: S^{d-1} \to Z$ a free Γ-cell $\Gamma \times D^d$ to Z by the attaching map $\Gamma \times S^{d-1} \to Z$ $(\gamma, s) \mapsto \gamma \cdot g(s)$. The resulting space Y is obtained from X by attaching free Γ-cells $\Gamma \times D^n$ of dimension $n \leq d$. It is $(d-2)$-connected and in particular simply-connected by construction. By inspecting the long homology sequence of (Y,Z) one concludes that the homology $H_j(Y;\mathbb{Z})$ is trivial for $1 \leq j \leq d-1$ and the map induced by the inclusion $H_d(Z;\mathbb{Z}) \to H_d(Y;\mathbb{Z})$ is surjective. Using the long exact sequences of the pair (Z,X) one concludes that $H_d(Y;\mathbb{Z})$ is trivial. Since $H_j(Y;\mathbb{Z}) = H_j(X;\mathbb{Z}) = 0$ holds for $j \geq d+1$, the space Y must be contractible.

(5) $\Leftrightarrow$ (6) If we would substitute in (6) the condition $H_n(X;\mathbb{Z}) = 0$ for $n \geq d+1$ by the sharper condition $H_n(X;\mathbb{Z}) = 0$ for $n \geq d$, (5) $\Leftrightarrow$ (6) would follow from [15, Proposition 11.10 on p. 221] or [21]. Hence it remains to show that $H_d(X;\mathbb{Z}) = 0$ holds provided that $H^{d+1}_{\mathbb{Z}\Gamma}(E\Gamma \times (CX,X); M) = 0$ for any $\mathbb{Z}\Gamma$-module M. Let A be any abelian group. Equip $i_\sharp A := \hom_{\mathbb{Z}}(\mathbb{Z}\Gamma, A)$ with the left $\mathbb{Z}\Gamma$-module structure given by $(\gamma \cdot f)(x) = f(x \cdot \gamma)$ for $\gamma \in \Gamma$, $f \in i_\sharp A$ and $x \in \mathbb{Z}\Gamma$. This is the coinduction of A with respect to the inclusion of the trivial group in Γ. Since coinduction is the right adjoint of restriction, we get

$$\begin{aligned} H^d(X;A) &\cong H^{d+1}_{\mathbb{Z}}(E\Gamma \times (CX,X); A) \\ &\cong H^{d+1}_{\mathbb{Z}\Gamma}(E\Gamma \times (CX,X); i_\sharp A) \\ &\cong 0. \end{aligned}$$

From the universal coefficient theorem we conclude $\hom_{\mathbb{Z}}(H_d(X;\mathbb{Z}), A) = 0$ for any abelian group A. This implies $H_d(X;\mathbb{Z}) = 0$.

Now we show (1) $\Rightarrow$ (7) provided that X is finite-dimensional and the isotropy groups of X are finite. If $i: \Gamma_0 \to \Gamma$ is the inclusion and $i_\sharp$ denotes coinduction with i, we have

$$H^{d+1}_{\mathbb{Z}\Gamma_0}(E\Gamma_0 \times (CX,X); M) = H^{d+1}_{\mathbb{Z}\Gamma}(E\Gamma \times (CX,X); i_\sharp M)$$

for any $\mathbb{Z}\Gamma_0$-module M. Since we already know (1) $\Rightarrow$ (6), it suffices to show that Γ satisfies $b(d)$ if (1) holds. Then $C_*(Y)$ is a finite-dimensional $\mathbb{Z}\Gamma$-resolution of the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ and each $C_n(Y)$ is a direct sum of $\mathbb{Z}\Gamma$-modules of the shape $\mathbb{Z}\Gamma/L$ for appropriate finite subgroups $L \subset \Gamma$. If N is a $\mathbb{Z}\Gamma$-module which is projective over $\mathbb{Z}H$ for all finite subgroups $H \subset \Gamma$, then $C_*(X) \otimes_{\mathbb{Z}} N$ with the diagonal Γ-action is a finite-dimensional projective $\mathbb{Z}\Gamma$-resolution of N.

From now on suppose that X^K is contractible for any non-trivial finite subgroup $K \subset \Gamma$. Notice that then for any non-trivial finite subgroup $H \subset \Gamma$ the singular set $\sigma_H(X)$ of the H-space X is contractible because of [7, Lemma 2.5 on p. 22] because

$\sigma_H(X)$ is the union of the sets X^K for $K \subset H$ with $K \neq 1$ and $X^{K_1} \cap X^{K_2} = X^{\langle K_1, K_2 \rangle}$, where $\langle K_1, K_2 \rangle$ is the subgroup of H generated by K_1 and K_2.

(7) $\Rightarrow$ (3) If the Γ-space X satisfies (7), the Γ_0-space X satisfies (6) and hence (3). Therefore $H_n(X;\mathbb{Z}) = 0$ for $n \geq d$ and $H_{d-1}(Z;\mathbb{Z})$ is projective over $\mathbb{Z}\Gamma_0$. Hence it remains to show that $H_{d-1}(Z;\mathbb{Z})$ is projective over $\mathbb{Z}\Gamma$. Because of [7, Lemma 4.1 (a) on p. 26] it suffices to show that the cohomological dimension of the $\mathbb{Z}\Gamma$-module $H_{d-1}(Z;\mathbb{Z})$ is finite. As Γ satisfies $b(<\infty)$ by assumption it suffices to prove for any finite subgroup $H \subset \Gamma$ that the $\mathbb{Z}H$-module $H_{d-1}(Z;\mathbb{Z})$ is projective. Since the singular set $\sigma_H(X)$ of the H-space X is contractible and agrees with the singular set $\sigma_H(Z)$ of the H-space Z we conclude $H_j(Z;\mathbb{Z}) = H_j(Z, \sigma_H(Z);\mathbb{Z})$ for $j \geq 1$. Since Z is $(d-2)$-connected and the relative dimension of $(Z, \sigma_H(Z))$ is less or equal to $d-1$, we obtain an exact sequence of $\mathbb{Z}H$-modules

$$0 \to H_{d-1}(Z;\mathbb{Z}) \to C_{d-1}(Z, \sigma_H(Z)) \to \cdots \to C_0(Z, \sigma_H(Z)) \to 0.$$

Since each $C_j(Z, \sigma_H(Z))$ is $\mathbb{Z}H$-free, the $\mathbb{Z}H$-module $H_{d-1}(Z;\mathbb{Z})$ is projective.

(8) $\Rightarrow$ (4) Assume that the Γ-CW-complex Z' is obtained from X by attaching free cells $\Gamma \times D^n$ of dimension $n \leq d-e-1$ and is $(d-e-2)$-connected. By the argument above $H_{d-e-1}(Z';\mathbb{Z})$ is $\mathbb{Z}H$-projective for any finite subgroup $H \subset \Gamma$. Since Γ satisfies $b(e)$ by assumption, the cohomological dimension of the $\mathbb{Z}\Gamma$-module $H_{d-e-1}(Z';\mathbb{Z})$ is less or equal to e. Now attach free cells $\Gamma \times D^n$ of dimension $d-e \leq n \leq d-1$ to Z' such that the resulting Γ-CW-complex Z is $(d-2)$-connected. We conclude from the long exact homology sequences associated to (Z,Z'), (Z,X) and (Z',X) that $H_j(Z,Z';\mathbb{Z}) = 0$ for $d-e+1 \leq j \leq d-2$, $H_{d-1}(Z;\mathbb{Z}) = H_{d-1}(Z,Z';\mathbb{Z})$ and $H_{d-e-1}(Z';\mathbb{Z}) = H_{d-e}(Z,Z';\mathbb{Z})$ hold. Hence we obtain an exact sequence

$$0 \to H_{d-1}(Z;\mathbb{Z}) \to C_{d-1}(Z,Z') \to \cdots \to C_{d-e}(Z,Z') \to H_{d-e-1}(Z';\mathbb{Z}) \to 0.$$

This implies that $H_{d-1}(Z;\mathbb{Z})$ is a projective $\mathbb{Z}\Gamma$-module. This finishes the proof of Lemma 1.9. $\square$

Proof of Theorem 1.6. Now we are ready to give the proof of Theorem 1.6. First we show that (1) implies both (2) and (3). This follows directly from Lemmas 1.2, 1.3 and the implication (1) $\Rightarrow$ (6) and (1) $\Rightarrow$ (7) of Lemma 1.9 applied to the singular set $X = \sigma_{WH}(\underline{E}WH)$ of the WH-space $\underline{E}WH$. It remains to show that (1) holds provided that there is an upper bound l on the length $l(H)$ of finite subgroups $H \subset \Gamma$ and (2) or (3) is true.

We construct inductively a nested sequence of Γ-CW-complexes

$$\emptyset = X[l+1] \subset X[l] \subset \cdots \subset X[0]$$

with the following properties. The Γ-CW-complex $X[n]$ is obtained from $X[n+1]$ by attaching cells of the type $\Gamma/H \times D^k$ for finite subgroups $H \subset \Gamma$ with $l(H) = n$ and $k \leq d(H)$ and $X[n]^H$ is contractible for any finite subgroup $H \subset \Gamma$ with $l(H) \geq n$. Then $X[0]$ is the desired model for $\underline{E}\Gamma$. The induction begin $n = l+1$ is obvious, the induction step from $n+1$ to n done as follows.

Let $H \subset \Gamma$ be a finite subgroup with $l(H) = n$. Since $d(K) \leq d(H)$ holds for all finite subgroups $K \subset \Gamma$ with $H \subset K$, the dimension of $X[n+1]^H$ is less or equal to $d(H)$. In particular $H_n(X[n+1]^H; \mathbb{Z}) = 0$ for $n \geq d(H) + 1$. Fix any $\mathbb{Z}WH$-module M. Let $f: X[n+1] \to \underline{E}\Gamma$ be a Γ-map. It induces a homotopy equivalence f^K for all finite subgroups $K \subset \Gamma$ with $l(K) \geq n+1$. Hence the WH-map $\sigma_{WH}(f^H): \sigma_{WH}(X[n+1]^H) \to \sigma_{WH}(\underline{E}\Gamma^H)$ induced by f is a (non-equivariant) homotopy equivalence by [7, Lemma 2.5 on p. 22] since $\sigma_{WH}(X[n+1]^H)$ is the union of the spaces $X[n+1]^K$ for all finite subgroup $K \subset \Gamma$ with $H \subset K, H \neq K \cap NH$ and similiar for $\sigma_{WH}(\underline{E}\Gamma^H)$ and any such group K satisfies $l(K) \geq n+1$. We get from Lemmas 1.2 and 1.3 an isomorphism

$$\begin{aligned}
&H^{d(H)+1}_{\mathbb{Z}WH}(EWH \times (C\sigma_{WH}(X[n+1]^H), \sigma_{WH}(X[n+1]^H)); M) \\
&\xrightarrow{\cong} H^{d(H)+1}_{\mathbb{Z}WH}(EWH \times (C\sigma_{WH}(\underline{E}\Gamma^H), \sigma_{WH}(\underline{E}\Gamma^H)); M) \\
&\xrightarrow{\cong} H^{d(H)+1}_{\mathbb{Z}WH}(EWH \times (C\sigma_{WH}(\underline{E}WH), \sigma_{WH}(\underline{E}WH)); M) \\
&\xrightarrow{\cong} H^{d(H)+1}_{\mathbb{Z}WH}(EWH \times (C|\mathscr{P}(WH)|, |\mathscr{P}(WH)|); M) \\
&= 0.
\end{aligned}$$

Since $X[n+1]^H$ is obtained from $\sigma_{WH}(X[n+1]^H)$ by attaching free WH-cells $WH \times D^k$ for $k \leq d(H)$, we get for any $\mathbb{Z}WH$-module M an isomorphism

$$\begin{aligned}
&H^{d(H)+1}_{\mathbb{Z}WH}(EWH \times (CX[n+1]^H, X[n+1]^H); M) \\
&\xrightarrow{\cong} H^{d(H)+1}_{\mathbb{Z}WH}(EWH \times (C\sigma_{WH}(X[n+1]^H), \sigma_{WH}(X[n+1]^H)); M) = 0.
\end{aligned}$$

Similarly we get $H^{d(H)+1}_{\mathbb{Z}\Delta(H)}(E\Delta(H) \times (CX[n+1]^H, X[n+1]^H); M) = 0$ for any $\mathbb{Z}\Delta(H)$-module M. Now the implications (6) $\Rightarrow$ (1) and (7) $\Rightarrow$ (1) of Lemma 1.9 yield the existence of a contractible WH-CW-complex $Y(H)$ which is obtained from $X[n+1]^H$ by attaching free cells $WH \times D^m$ of dimension $m \leq d(H)$. Now define $X[n]$ as the Γ-push out

$$\begin{array}{ccc}
\coprod_{(H)} \Gamma \times_{NH} X[n+1]^H & \xrightarrow{i} & \coprod_{(H)} \Gamma \times_{NH} Y(H) \\
{\scriptstyle p}\downarrow & & \downarrow \\
X[n+1] & \longrightarrow & X[n]
\end{array}$$

where (H) runs through the conjugacy classes of finite subgroups $H \subset \Gamma$ with $l(H) = n$, i is the obvious inclusion and p sends $(\gamma, x) \in \Gamma \times_{NH} X[n+1]^H$ to $\gamma x \in X[n+1]$. Since $X[n]^H = Y(H)$ and $X[n]^H = X[n+1]^H$ respectively for any finite subgroup $H \subset \Gamma$ with $l(H) = n$ and $l(H) > n$ respectively, one easily verifies that $X[n]$ has the desired properties. This finishes the proof of Theorem 1.6. $\square$

The next result has already been proven by Kropholler and Mislin [13, Theorem B] for certain classes of groups and in l exponential dimension estimate (see also Remark 1.12). We will give an in l-linear one.

Theorem 1.10. *Let Γ be a group and let $l \geq 0$ and $d \geq 0$ be integers such that the length $l(H)$ of any finite subgroup $H \subset \Gamma$ is bounded by l and Γ satisfies $B(d)$. Then there is a Γ-CW-model for $\underline{E}\Gamma$ such that for any finite subgroup $H \subset \Gamma$*

$$\dim(\underline{E}\Gamma^H) = \max\{3, d\} + (l - l(H))(d+1)$$

holds. In particular $\underline{E}\Gamma$ has dimension $\max\{3, d\} + l(d+1)$.

Proof. The proof is a variation of the proof of Theorem 1.6. now using in the induction step the implication (8) $\Rightarrow$ (1) proven in Lemma 1.9 instead of the implications (6) $\Rightarrow$ (1) and (7) $\Rightarrow$ (1). $\square$

Remark 1.11. There are groups for which there is a 1-dimensional Γ-CW-model for $\underline{E}\Gamma$ but no upper bound l on the length $l(H)$ of finite subgroups. Namely, any group whose rational cohomological dimension is less or equal to 1 admits a 1-dimensional model for $\underline{E}\Gamma$ [11]. In particular any countable locally finite group has a 1-dimensional model for $\underline{E}\Gamma$.

Remark 1.12. In [13] the class $\mathscr{HFIN}$ of hierarchically decomposable groups and the $\mathbb{Z}\Gamma$-module $B(\Gamma, \mathbb{Z})$ of bounded functions from Γ to $\mathbb{Z}$ is investigated and related to the question of the existence of a finite-dimensional Γ-CW-model for $\underline{E}\Gamma$. By definition $\mathscr{HFIN}$ is the smallest class of group which contains all finite groups and which contains Γ if there is a finite-dimensional contractible Γ-CW-complex whose isotropy groups belong to the class. Define the following classes of groups where hdim means the homological dimension:

$$\mathscr{A} := \{\Gamma \mid \mathrm{hdim}_{\mathbb{Z}\Gamma}(B(\Gamma, \mathbb{Z})) < \infty\};$$

$$\mathscr{B} := \{\Gamma \mid \Gamma \text{ satisfies } b(d)\};$$

$$\mathscr{C} := \{\Gamma \mid \Gamma \text{ satisfies } B(d)\};$$

$$\mathscr{D} := \{\Gamma \mid \text{there is a finite-dimensional } \Gamma\text{-}CW\text{-complex } X \text{ with finite isotropy groups and } H_p(X;\mathbb{Z}) = H_p(*,\mathbb{Z}) \text{ for } p \geq 0\};$$

$$\mathscr{E} := \{\Gamma \mid \text{there is a finite-dimensional } \Gamma\text{-}CW\text{-model for } \underline{E}\Gamma\};$$

$$\mathscr{L} := \{\Gamma \mid \text{there is an upper bound } l \text{ on the length } l(H) \text{ of finite subgroups } H \subset \Gamma\}.$$

Then

$$\mathscr{A} \cap \mathscr{HFIN} = \mathscr{B} \cap \mathscr{HFIN} = \mathscr{C} \cap \mathscr{HFIN}$$

follows directly from [13, Proposition 6.1, Lemma 7.3] and the fact that $B(\Gamma,\mathbb{Z})$ is projective over $\mathbb{Z}H$ for any finite subgroup $H\subset\Gamma$ [14]. We conclude

$$\mathscr{A}\cap\mathscr{HFIN}\cap\mathscr{L}=\mathscr{B}\cap\mathscr{HFIN}\cap\mathscr{L}=\mathscr{C}\cap\mathscr{L}=\mathscr{D}\cap\mathscr{L}=\mathscr{E}\cap\mathscr{L}$$

from Lemma 1.5 and Theorem 1.10.

2. Coinduction

In this section we study coinduction and what it does to a classifying space for a family.

Definition 2.1. Let $i:\Delta\subset\Gamma$ be an inclusion of groups. Define the *coinduction with i* of a Δ-space X to be the Γ-space

$$i_\sharp X:=\operatorname{map}_\Delta(\Gamma,X),$$

where $\operatorname{map}_\Delta(\Gamma,X)$ is the space of Δ-equivariant maps of Δ-spaces. The Γ-action is given by $(\gamma f)(\gamma_0):=f(\gamma_0\gamma)$ for $f:\Gamma\to X$ and $\gamma,\gamma_0\in\Gamma$. Given a family $\mathscr{F}$ of subgroups of Δ, define the family $i_\sharp\mathscr{F}$ of subgroups of Γ by the set of subgroups $H\subset\Gamma$ for which $\gamma^{-1}H\gamma\cap\Delta$ belongs to $\mathscr{F}$ for all $\gamma\in\Gamma$.

Notice that $i_\sharp$ is the right adjoint of the restriction functor which sends a Γ-space Y to the Δ-space i^*Y, namely, there is a natural homeomorphism

$$\operatorname{map}_\Delta(i^*Y,X)\xrightarrow{\cong}\operatorname{map}_\Gamma(Y,i_\sharp X). \tag{2.2}$$

The next result reduces for torsionfree Γ to Serre's Theorem that there is a finite-dimensional model for $B\Gamma$ if Γ contains a subgroup Δ of finite index with finite-dimensional $B\Delta$ (see [6, Theorem VIII.3.1. on p. 190, Corollary VIII.7.2 on p. 205] and [18]).

Theorem 2.3. *Let $i:\Delta\subset\Gamma$ be an inclusion of groups and let $\mathscr{F}$ be a family of subgroups of Δ. Then*

(1) *If X is a Δ-CW-complex of dimension d and the index $[\Gamma:\Delta]$ is finite, then $i_\sharp X$ is Γ-homotopy equivalent to a $d\cdot[\Gamma:\Delta]$-dimensional Γ-CW-complex*;

(2) *$i_\sharp E(\Delta,\mathscr{F})^H$ is empty for $H\notin i_\sharp\mathscr{F}$ and is contractible for $H\in i_\sharp\mathscr{F}$.*

Proof. (1) Let l be the index $[\Gamma:\Delta]$. Fix elements $\gamma_1,\gamma_2,\dots,\gamma_l$ in Γ such that $\Delta\backslash\Gamma=\{\Delta\gamma_k\mid k=1,2,\dots,l\}$. Evaluation at γ_k defines a homeomorphism

$$\operatorname{ev}:i_\sharp X\xrightarrow{\cong}\prod_{k=1}^{l}X.$$

Let G be the semi-direct product of $\Delta^l=\prod_{k=1}^l\Delta$ and the symmetric group S_l in l-letters with respect to the action of S_l on Δ^l given by $\sigma\cdot(\delta_k)=(\delta_{\sigma^{-1}(k)})$. The group

G acts in the obvious way on the target $\prod_{k=1}^{l} X$ of ev. Let $s:\Gamma\to S_k$ be the group homomorphism satisfying $\Delta\gamma_{s(\gamma)(k)}\gamma=\Delta\gamma_k\in\Delta\backslash\Gamma$ for all $\gamma\in\Gamma$ and $k\in\{1,\ldots,l\}$. Then we obtain an embedding of group

$$i:\Gamma\to G \quad \gamma\mapsto(\gamma_k\gamma\gamma_{s(\gamma)^{-1}(k)}^{-1})_k\cdot s(\gamma)$$

such that ev is a Γ-homeomorphism with the given Γ-action on the source and the restricted Γ-action on the target. Hence it suffices to show that $\prod_{k=1}^{l} X$ is G-homotopy equivalent to a dl-dimensional G-CW-complex Y. Recall that a G-CW-structure is the same as a CW-structure such that the G-action permutes the cells and for any $g\in G$ and cell e with $g\cdot e=e$ multiplication with g induces the identity on e. Notice that $\prod_{k=1}^{l} X$ has an obvious Δ^l-CW-structure. However, the G-action does respect the skeleta and permutes the cells but it happens that multiplication with an element $g\in G$ maps a cell to itself but not by the identity so that this does not give the structure of a G-CW-complex on $\prod_{k=1}^{l} X$. This problem can be solved by using simplicial complexes with equivariant simplicial actions using the fact that the second barycentric subdivision is an equivariant CW-complex but then one has to deal with product of simplicial complexes. We give another approach.

We next construct a nested sequence of G-CW-complexes $\emptyset=Y_{-1}\subset Y_0\subset\cdots\subset Y_{ld}$ together with G-homotopy equivalence $f_n:(\prod_{k=1}^{l} X)_n\to Y_n$ such that Y_n is n-dimensional and f_n is an extension of f_{n-1} for all $n\geq 0$. Then f_{ld} is the desired Γ-homotopy equivalence. The begin $n=-1$ of the induction is trivial, the induction step from $n-1$ to n done as follows:

For $p\geq 0$ choose a Δ-push out

$$\begin{array}{ccc}
\coprod_{\alpha\in I_p}\Delta/L_\alpha\times S^{p-1} & \longrightarrow & X_{p-1}\\
\big\downarrow & & \big\downarrow\\
\coprod_{\alpha\in I_p}\Delta/L_\alpha\times D^{p} & \longrightarrow & X_{p}
\end{array}$$

Define an index set

$$J_n:=\left\{(\underline{i},\underline{\alpha})\mid \underline{i}=(i_1,i_2,\ldots,i_l),i_k=0,\ldots,n,\sum_{k=1}^{l} i_k=n,\underline{\alpha}=(\alpha_1,\ldots,\alpha_l)\in I_{i_1}\times\cdots\times I_{i_l}\right\}.$$

Consider the space

$$A=\coprod_{(\underline{i},\underline{\alpha})\in J_n}\left(\prod_{k=1}^{l}\Delta/L_{\alpha_k}\times D^{i_k}\right).$$

There is an obvious S_l-action on J_n given by permutating the coordinates which induces an S_l-action on A. Together with the obvious Δ^l-action this yields a G-action on A.

Next we show that $(A, \partial A)$ carries the structure of a pair of G-CW-complexes with dimension n where ∂A is the topological boundary of A. Fix $(\underline{i}, \underline{\alpha}) \in J_n$. Let $S_l' \subset S_l$ be the isotropy group of $(\underline{i}, \underline{\alpha})$ under the S_l-action on J_n. Then S_l' acts on $\prod_{k=1}^l D^{i_k}$ by permuting the coordinates. Equip the pair $(\prod_{k=1}^l D^{i_k}, \partial \prod_{k=1}^l D^{i_k})$ with a S_l'-CW-complex structure. This induces a G-CW-structure on the pair

$$\coprod_{(\underline{i}', \underline{\alpha}') \in J_n, S_l(\underline{i}', \underline{\alpha}') = S_l(\underline{i}, \underline{\alpha}) \in S_l \backslash J_n} \prod_{k=1}^{l} \Delta/L_{\alpha_k} \times D^{i_k}, \partial \left(\coprod_{(\underline{i}', \underline{\alpha}') \in J_n, S_l(\underline{i}', \underline{\alpha}') = S_l(\underline{i}, \underline{\alpha}) \in S_l \backslash J_n} \prod_{k=1}^{l} \Delta/L_{\alpha_k} \times D^{i_k} \right).$$

Since $(A, \partial A)$ is the disjoint union of such pairs, it carries a G-CW-structure. We have the G-push out

$$\begin{array}{ccc} \partial A & \xrightarrow{q} & (\prod_{k=1}^l X)_{n-1} \\ \downarrow & & \downarrow \\ A & \longrightarrow & (\prod_{k=1}^l X)_n \end{array}$$

By the equivariant cellular approximation theorem [10, II.2] we can choose a G-homotopy h from $f_{n-1} \circ q$ to a cellular G-map $c : \partial A \to Y_{n-1}$. Define Y_n by the G-push out

$$\begin{array}{ccc} \partial A & \xrightarrow{c} & Y_{n-1} \\ \downarrow & & \downarrow \\ A & \longrightarrow & Y_n \end{array}$$

Since c is cellular, $(A, \partial A)$ is a pair of n-dimensional G-CW-complexes and Y_{n-1} an $(n-1)$-dimensional G-CW-complex, Y_n is an n-dimensional G-CW-complex. Using [15, Lemma 2.13 on page 38] and the G-homotopy h one constructs the desired G-homotopy equivalence $f_n : (\prod_{k=1}^l X)_n \to Y_n$. This finishes the proof of Theorem 2.3.1.

(2) Let Z be any (non-equivariant) space and let $H \subset \Gamma$ be a subgroup. We have the isomorphism of Δ-sets

$$\coprod_{k=1}^{l} \Delta/(\gamma_k H \gamma_k^{-1} \cap \Delta) \xrightarrow{\cong} i^* \Gamma/H \quad \delta(\gamma_k H \gamma_k^{-1} \cap \Delta) \mapsto \delta \gamma_k H.$$

We get from (2.2) the following composition of homeomorphisms

$$\begin{aligned}
\mathrm{map}(Z,(i_\sharp E(\Delta,\mathscr{F}))^H) &\cong \mathrm{map}_\Gamma(\Gamma/H\times Z, i_\sharp E(\Delta,\mathscr{F}))\\
&\cong \mathrm{map}_\Delta(i^*\Gamma/H\times Z, E(\Delta,\mathscr{F}))\\
&\cong \mathrm{map}_\Delta\left(\coprod_{k=1}^{l}\Delta/(\gamma_k H\gamma_k^{-1}\cap\Delta)\times Z, E(\Delta,\mathscr{F})\right)\\
&\cong \prod_{k=1}^{l}\mathrm{map}_\Delta(\Delta/(\gamma_k H\gamma_k^{-1}\cap\Delta)\times Z, E(\Delta,\mathscr{F}))\\
&\cong \prod_{k=1}^{l}\mathrm{map}(Z, E(\Delta,\mathscr{F})^{\gamma_k H\gamma_k^{-1}\cap\Delta}).
\end{aligned}$$

Notice that H belongs to $i_\sharp\mathscr{F}$ if and only if $\gamma_k H\gamma_k^{-1}\cap\Delta$ belongs to $\mathscr{F}$ for $k=1,\ldots,l$. Hence for any space Z the space $\mathrm{map}(Z,(i_\sharp E(\Delta,\mathscr{F}))^H)$ is empty if H does not belong to $i_\sharp\mathscr{F}$ and is path-connected otherwise. This finishes the proof of Theorem 2.3. □

Theorem 2.3 implies

Theorem 2.4. *Let $i:\Delta\subset\Gamma$ be an inclusion of a subgroup of finite index $[\Gamma:\Delta]$ and $\mathscr{F}$ be a family of subgroups of Δ. Suppose that $E(\Delta,\mathscr{F})$ has a Δ-CW-model of dimension d. Then $E(\Gamma,i_\sharp\mathscr{F})$ has a Γ-CW-model of dimension $d\cdot[\Gamma:\Delta]$.*

If we take in Theorem 2.4 the family $\mathscr{F}$ as the family of finite subgroups of Δ or virtually cyclic subgroup of Δ respectively, then $i_\sharp\mathscr{F}$ is the family of finite subgroups of Γ or virtually cyclic subgroup of Γ, respectively.

3. Short exact sequences

Let $1\to\Delta\to\Gamma\to\pi\to 1$ be an exact sequence of groups. In this section we want to investigate whether finiteness conditions about the type of a classifying space for $\mathscr{FIN}$ for Δ and π carry over to the one of Γ.

Theorem 3.1. *Let $1\to\Delta\xrightarrow{i}\Gamma\xrightarrow{p}\pi\to 1$ be an exact sequence of groups. Suppose that there exists a positive integer d which is an upper bound on the orders of finite subgroups of π. Suppose that $\underline{E}\Delta$ has a k-dimensional Δ-CW-model and $\underline{E}\pi$ has a m-dimensional π-CW-model. Then $\underline{E}\Gamma$ has a $(dk+m)$-dimensional Γ-CW-model.*

Proof. Let $\underline{E}\pi$ be a m-dimensional π-CW-model for the classifying space of π for the family of finite subgroups. Let $p^*\underline{E}\pi$ be the Γ-space obtained from $\underline{E}\pi$ by restriction with p. We will construct for each $-1\le n\le m$ a $(dk+n)$-dimensional Γ-CW-

complex X_n and a Γ-map $f_n : X_n \to p^*\underline{E}\pi_n$ to the n-skeleton of $p^*\underline{E}\pi$ such that X_{n-1} is a Γ-CW-subcomplex of X_n, f_n restricted to X_{n-1} is f_{n-1}, for each finite subgroup $H \subset \Gamma$ the map $f_n^H : X_n^H \to p^*\underline{E}\pi_n^H$ is a homotopy equivalence and all isotropy groups of X_n are finite. Then X_m will be the desired $(dk+m)$-dimensional Γ-CW-model for $\underline{E}\Gamma$. The begin $n = -1$ is given by the identity on the empty set, the induction step from $n-1$ to $n \geq 0$ is done as follows.

Choose a Γ-pushout

$$\begin{tikzcd}
\coprod_{i\in I} \Gamma/L_i \times S^{n-1} \arrow[r, "\coprod_{i\in I} q_i"] \arrow[d] & p^*\underline{E}\pi_{n-1} \arrow[d] \\
\coprod_{i\in I} \Gamma/L_i \times D^n \arrow[r] & p^*\underline{E}\pi_n
\end{tikzcd}$$

There is a canonical projection $\mathrm{pr}_i : \Gamma \times_{L_i} \underline{E}L_i \to \Gamma/L_i$. Fix a map of sets $s : \Gamma/L_i \to \Gamma$ whose composition with the projection $\Gamma \to \Gamma/L_i$ is the identity on Γ/L_i. Given an L_i-space Y, there is a natural homeomorphism

$$\coprod_{w\in\Gamma/L_i,\, s(w)^{-1}Hs(w) \subset L_i} Y^{s(w)^{-1}Hs(w)} \xrightarrow{\cong} (\Gamma \times_{L_i} Y)^H$$

which sends $y \in Y^{s(w)^{-1}Hs(w)}$ to $(s(w), y)$. Hence pr_i^H is a homotopy equivalence for all finite subgroups $H \subset \Gamma$ and the isotropy groups of $\Gamma \times_{L_i} \underline{E}L_i$ are finite. Since f_{n-1}^H is a homotopy equivalence for all finite subgroups $H \subset \Gamma$, there is a cellular Γ-map

$$r_i : (\Gamma \times_{L_i} \underline{E}L_i) \times S^{n-1} \to X_{n-1}$$

together with a Γ-homotopy

$$h : (\Gamma \times_{L_i} \underline{E}L_i) \times S^{n-1} \times [0,1] \to p^*\underline{E}\pi_{n-1}$$

from $f_{n-1} \circ r_i$ to $q_i \circ (\mathrm{pr}_i \times \mathrm{id})$ [15, Proposition 2.3 on p. 35]. Define the following three Γ-spaces by the Γ-push outs

$$\begin{tikzcd}
\coprod_{i\in I} (\Gamma \times_{L_i} \underline{E}L_i) \times S^{n-1} \arrow[r, "\coprod_{i\in I} q_i \circ (\mathrm{pr}_i \times \mathrm{id})"] \arrow[d] & p^*\underline{E}\pi_{n-1} \arrow[d] \\
\coprod_{i\in I} (\Gamma \times_{L_i} \underline{E}L_i) \times D^n \arrow[r] & X_n'
\end{tikzcd}$$

and

$$\begin{array}{ccc}
\coprod_{i\in I}(\Gamma\times_{L_i}\underline{E}L_i)\times S^{n-1} & \xrightarrow{\coprod_{i\in I} f_{n-1}\circ r_i} & p^*\underline{E}\pi_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{i\in I}(\Gamma\times_{L_i}\underline{E}L_i)\times D^n & \longrightarrow & X_n''
\end{array}$$

and

$$\begin{array}{ccc}
\coprod_{i\in I}(\Gamma\times_{L_i}\underline{E}L_i)\times S^{n-1} & \xrightarrow{\coprod_{i\in I} r_i} & X_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{i\in I}(\Gamma\times_{L_i}\underline{E}L_i)\times D^n & \longrightarrow & X_n
\end{array}$$

We obtain a Γ-map $f_n':X_n'\to p^*\underline{E}\pi_n$ by the Γ-pushout property and the Γ-maps $\coprod_{i\in I}\mathrm{pr}_i\times\mathrm{id}_{D^n}$, $\coprod_{i\in I}\mathrm{pr}_i\times\mathrm{id}_{S^{n-1}}$ and $\mathrm{id}_{p^*\underline{E}\pi_{n-1}}$. Let $f_n'':X_n'':X_n''\to X_n'$ be the Γ-homotopy equivalence relative $p^*\underline{E}\pi_{n-1}$ induced by the Γ-homotopy $\coprod_{i\in I}h_i$. Let $f_n''':X_n\to X_n''$ be the Γ-map induced by the Γ-maps $\coprod_{i\in I}\mathrm{id}_{(\Gamma\times_{L_i}\underline{E}L_i)\times D^n}$, $\coprod_{i\in I}\mathrm{id}_{(\Gamma\times_{L_i}\underline{E}L_i)}\times S^{n-1}$ and f_{n-1} and the Γ-pushout property. We define the Γ-map $f_n:X_n\longrightarrow p^*\underline{E}\pi_n$ by the composition $f_n'\circ f_n''\circ f_n'''$. Since $(f_n')^H$ and $(f_n''')^H$ are homotopy equivalences for finite $H\subset\Gamma$ as they are push outs of homotopy equivalences [15, Lemma 2.13 on p. 38] and f_n'' is a Γ-homotopy equivalence, f_n^H is a homotopy equivalence for all finite $H\subset\Gamma$.

We conclude that the resulting Γ-CW-complex X_n has only finite isotropy groups and f^H is a homotopy equivalence for each finite $H\subset\Gamma$. Notice that each cell $\Gamma/L_i\times D^n$ of $p^*\underline{E}\pi$ satisfies $L_i=p^{-1}(H_i)$ for some finite subgroup $H_i\subset\pi$ and hence L_i contains Δ as subgroup of finite index $|H_i|$. Theorem 2.4 implies that $\underline{E}L_i$ can be choosen as a $m|H_i|$-dimensional L_i-CW-complex. Hence the dimension of X_n is at most $dk+n$. $\square$

Theorem 3.2. *Let $1\to\Delta\xrightarrow{i}\Gamma\xrightarrow{p}\pi\to 1$ be an exact sequence of groups. Suppose for any finite subgroup $\pi'\subset\pi$ and any extension $1\to\Delta\to\Delta'\to\pi'\to 1$ that $\underline{E}\Delta'$ has a finite Δ'-CW-model or a Δ'-CW-model of finite type respectively and suppose that $\underline{E}\pi$ has a finite π-CW-model or a π-CW-model of finite type respectively. Then $\underline{E}\Gamma$ has a finite Γ-CW-model or a Γ-CW-model of finite type, respectively.*

Proof. The proof is exactly the same as the one of Theorem 3.1 except for the very last step. Namely, we have to know that $\underline{E}L_i$ has a finite L_i-CW-model or a L_i-CW-model of finite type respectively and this follows from the assumptions. $\square$

Notice that the assumption about $\underline{E}\Delta'$ in Theorem 3.2 would follow from the assumption that $\underline{E}\Delta$ has a finite Δ-CW-model or a Δ-CW-model of finite type respectively if Problem 7.2 has an affirmative answer. If π is torsionfree, this assumption about $\underline{E}\Delta'$ reduces to the assumption that there is a finite Δ-CW-model or a Δ-CW-model of finite type for $\underline{E}\Delta$. If Δ is word-hyperbolic, then its Rips complex yields a finite Δ-CW-model for $\underline{E}\Delta$. Moreover a group is word-hyperbolic if it contains a word-hyperbolic subgroup of finite index, because the property word-hyperbolic is a quasi-isometry invariant. It is not hard to check that any virtually poly-cyclic group Δ has a finite Γ-CW-model for $\underline{E}\Gamma$. Hence we conclude from Theorem 3.2.

Theorem 3.3. *Let $1 \to \Delta \xrightarrow{i} \Gamma \xrightarrow{p} \pi \to 1$ be an exact sequence of groups. Suppose that Δ is word-hyperbolic or virtually poly-cyclic. Suppose that $\underline{E}\pi$ has a finite π-CW-model or a π-CW-model of finite type respectively. Then $\underline{E}\Gamma$ has a finite Γ-CW-model or a Γ-CW-model of finite type respectively.*

4. Classifying spaces of finite type

In this section we deal with the question whether there are Γ-CW-models of finite type for $\underline{E}\Gamma$.

Lemma 4.1. *If there is a Γ-CW-complex Y of finite type which has only finite isotropy groups and is (non-equivariantly) contractible, then $E\Gamma$ has a Γ-CW-model of finite type.*

Proof. The idea is to replace a cell $\Gamma/H_i \times D^n$ in Y by the Γ-space $\Gamma \times_{H_i} EH_i \times D^n$. The construction is an obvious modification of the construction in the proof of Theorem 3.1 using the fact that EH_i can be choosen as a H_i-CW-complex of finite type. Namely, we construct for $n \geq -1$ a Γ-CW-complex X_n and a Γ-map $f_n : X_n \longrightarrow Y_n$ to the n-skeleton of Y such that X_n is obtained from X_{n-1} by a Γ-push out of the shape

$$\begin{array}{ccc}
\coprod_{i\in I}(\Gamma \times_{H_i} EH_i) \times S^{n-1} & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{i\in I}(\Gamma \times_{H_i} EH_i) \times D^{n} & \longrightarrow & X_{n}
\end{array}$$

and f_n extends f_{n-1} and is a non-equivariant homotopy equivalence. Then the colimit $X = \operatorname{colim}_{n\to\infty} X_n$ is the desired Γ-CW-model for $E\Gamma$. $\square$

Theorem 4.2. *The following statements are equivalent for the group Γ.*

(1) *There is a Γ-CW-model for $\underline{E}\Gamma$ of finite type*;

(2) *There are only finitely many conjugacy classes of finite subgroups of Γ and for any finite subgroup $H \subset \Gamma$ there is a CW-model for BWH of finite type*;

(3) *There are only finitely many conjugacy classes of finite subgroups of Γ and for any finite subgroup $H \subset \Gamma$ the Weyl group WH is finitely presented and is of type FP_∞, i.e. there is a projective $\mathbb{Z}WH$-resolution of finite type of the trivial $\mathbb{Z}WH$-module $\mathbb{Z}$.*

Proof. (1)$\Rightarrow$(2) Let $\underline{E}\Gamma$ be a Γ-CW-model of finite type. Let $\Gamma/H_1,\dots,\Gamma/H_n$ be the finitely many equivariant 0-cells. Then any other equivariant Γ-cell $\Gamma/K \times D^n$ in $\underline{E}\Gamma$ must have the property that K is subconjugated to one of the H_i's because the existence of a Γ-map from Γ/K to Γ/L is equivalent to K being subconjugated to L. Since each H_i is finite and has only finitely many distinct subgroups, we conclude that there are only finitely many conjugacy classes of finite subgroups in Γ. Now apply Lemmas 1.3 and 4.1.

(2)$\Rightarrow$(1) Let $(H_1),\dots,(H_r)$ be the conjugacy classes of finite subgroups with a numeration such that (H_i) is subconjugated to (H_j) only if $i \geq j$. We construct inductively Γ-CW-complexes $X_0, X_1,\dots,X_r$ such that X_0 is empty, X_n is obtained from X_{n-1} by attaching cells $\Gamma/H_n \times D^m$ for $m \geq 0$, X_n is of finite type and $X_n^{H_n}$ is contractible. Notice that then X_r is a Γ-CW-model of finite type for $\underline{E}\Gamma$ because $X_n^{H_i} = X_{n-1}^{H_i}$ for $i < n$.

In the induction step from $n-1$ to n it suffices to construct an extension of WH_n-CW-complexes $X_{n-1}^{H_n} \xrightarrow{i} Z$ such that Z is contractible, obtained from $X_{n-1}^{H_n}$ by attaching free WH_n-cells and of finite type. Then one defines X_n as the Γ-pushout

$$\begin{tikzcd}
\Gamma \times_{NH_n} X_{n-1}^{H_n} \arrow[r, "\mathrm{id}_\Gamma \times_{NH_n} i"] \arrow[d, "j"] & \Gamma \times_{NH_n} Z \arrow[d] \\
X_{n-1} \arrow[r] & X_n
\end{tikzcd}$$

where j maps (γ, x) to γx. Notice that $X_n^{H_n} = Z^{H_n}$. Since each isotropy group of the WH_n-space $X_{n-1}^{H_n}$ is finite, we can construct a free WH_n-CW-complex Y together with a WH_n-map $h: Y \to X_{n-1}^{H_n}$ which is a (non-equivariant) homotopy equivalence by substituting each cell $WH_n/L \times D^m$ by $WH_n \times_L EL \times D^m$. This construction is an easy modification of the construction in the proof of Theorem 3.1. Since L is finite and hence EL can be choosen as a free L-CW-complex of finite type, Y is of finite type. Let $f: Y \to EWH_n$ be the classifying WH_n-map. We can choose EWH_n of finite type by assumption. Let $i: Y \to \mathrm{cyl}(f)$ be the inclusion into the mapping cylinder of f. Notice that $\mathrm{cyl}(f)$ is a WH_n-CW-model for EWH_n of finite type. Define Z by the

WH_n-pushout

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "h"'] & \mathrm{cyl}(f) \arrow[d, "g"] \\ X_{n-1}^{H_n} \arrow[r] & Z \end{tikzcd}$$

where i is the inclusion. As h is a (non-equivariant) homotopy equivalence, g is a (non-equivariant) homotopy equivalence. Hence Z is contractible and of finite type.

(2) $\Leftrightarrow$ (3) is a variation of the proof of [6, Theorem VIII.7.1 on p. 205] or follows from [15, Proposition 14.9 on p. 182]. □

5. Finitely dominated and finite classifying spaces

In this section we deal with the question whether there are finitely dominated or finite Γ-CW-models for $\underline{E}\Gamma$. Recall that a Γ-CW-complex X is called *finitely dominated* if there are a finite Γ-CW-complex Y and Γ-maps $r: Y \to X$ and $i: X \to Y$ such that $r \circ i$ is Γ-homotopic to the identity.

Theorem 5.1. *The following statements are equivalent for the group Γ.*

(1) *There is a finitely dominated Γ-CW-model for $\underline{E}\Gamma$;*

(2) *There are only finitely many conjugacy classes of finite subgroups of Γ and for any finite subgroup $H \subset \Gamma$ the Weyl group WH is finitely presented, is of type FP_∞ and satisfies condition $b(d)$ for some $d \geq 0$.*

Proof. A finitely dominated Γ-CW-complex is Γ-homotopy equivalent to a Γ-CW-complex of finite orbit type [15, Proposition 2.12 on p. 38]. A Γ-CW-complex X of finite orbit type is finitely dominated if and only if it is Γ-homotopy equivalent to both a Γ-CW-complex Y of finite type and to a finite-dimensional Γ-CW-complex Z. The argument in the proof of [15, Proposition 14.9a on p. 282] applies to the general case. Hence the claim follows from Lemma 1.5, Theorems 1.10 and 4.2. □

If one knows that $\underline{E}\Gamma$ is finitely dominated, then there is the *equivariant finiteness obstruction*

$$\widetilde{o}^{\Gamma}(\underline{E}\Gamma) \in \widetilde{K}_0(\mathbb{Z}\,\mathrm{Or}(\Gamma, \mathcal{FIN})) \cong \bigoplus_{\{(H)\,|\,H \in \mathcal{FIN}\}} \widetilde{K}_0(\mathbb{Z}WH)$$

whose vanishing is a necessary and sufficient condition for $\underline{E}\Gamma$ being Γ-homotopy equivalent to a finite Γ-CW-complex [15, Theorem 14.6 on p. 278, Theorem 10.34 on p. 196].

Remark 5.2. If Δ is a word-hyperbolic group, its Rips complex yields a finite Γ-CW-model for $\underline{E}\Gamma$. If Γ is a discrete cocompact subgroup of a Lie group G with finitely many components, then G/K with the left Γ-action for $K \subset G$ a maximal compact subgroup is a finite Γ-CW-model for $\underline{E}\Gamma$ [1]. If Γ contains $\mathbb{Z}^n$ as subgroup of finite index, there exists an epimorphism of Γ to a crystallographic group with finite kernel (as pointed out to us by Frank Connolly) and hence an n-dimensional Γ-CW-model for $\underline{E}\Gamma$ with $\mathbb{R}^n$ as underlying space. Any virtually poly-cyclic group Γ has a finite Γ-CW-model for $\underline{E}\Gamma$. If Γ is $\Gamma_1 *_{\Gamma_0} \Gamma_2$ for finite groups Γ_i, then there is a finite 1-dimensional Γ-CW-model for $\underline{E}\Gamma$ [19, Theorem 7 in I.4.1 on p. 32].

6. Groups with finite virtual cohomological dimension

In this section we investigate the condition $b(d)$ and $B(d)$ of Notation 1.4 and explain how our results specialize in the case of a group of finite virtual cohomological dimension.

Lemma 6.1. *If Γ satisfies $b(d)$ or $B(d)$, respectively, then any subgroup Δ of Γ satisfies $b(d)$ or $B(d)$, respectively.*

Proof. We begin with $b(d)$. Let M be a $\mathbb{Z}\Delta$-module which is projective over $\mathbb{Z}H$ for all finite subgroups $H \subset \Delta$. For a subgroup $K \subset \Gamma$ the double coset formula gives an isomorphism of $\mathbb{Z}K$-modules

$$\operatorname{res}_\Gamma^K \operatorname{ind}_\Delta^\Gamma M \cong \bigoplus_{K\gamma\Delta \in K\backslash\Gamma/\Delta} \operatorname{ind}_{\gamma\Delta\gamma^{-1}\cap K}^K \operatorname{res}_{c(\gamma)} M, \tag{6.2}$$

where $c(\gamma): \gamma\Delta\gamma^{-1} \cap K \to \Delta$ maps δ to $\gamma^{-1}\delta\gamma$. If K is finite, each subgroup $\Delta \cap \gamma^{-1}K\gamma$ of Δ is finite and hence the $\mathbb{Z}\gamma\Delta\gamma^{-1} \cap K$-module $\operatorname{res}_{c(\gamma)} M$ is projective. We conclude from (6.2) that the $\mathbb{Z}K$-module $\operatorname{res}_\Gamma^K \operatorname{ind}_\Delta^\Gamma M$ is projective for all finite subgroups $K \subset \Gamma$. Since Γ satisfies $b(d)$ by assumption the $\mathbb{Z}\Gamma$-module $\operatorname{ind}_\Delta^\Gamma M$ has a projective resolution of dimension d. If one applies (6.2) to $K = \Delta$, then one concludes that the $\mathbb{Z}\Delta$-module M is a direct summand in $\operatorname{res}_\Gamma^\Delta \operatorname{ind}_\Delta^\Gamma M$. Hence M has a projective $\mathbb{Z}\Delta$-resolution of dimension d. This finishes the proof for $b(d)$. The claim for $B(d)$ follows since the Weyl group $W_\Delta H$ of H in Δ is a subgroup of the Weyl group $W_\Gamma H$ of H in Γ for any subgroup $H \subset \Delta$. □

Recall that a group Γ has *virtual cohomological dimension* $\leq d$ if and only if it contains a torsionfree subgroup Δ of finite index such that the trivial $\mathbb{Z}\Delta$-module $\mathbb{Z}$ has a projective $\mathbb{Z}\Delta$-resolution of dimension d.

Theorem 6.3. *Let Γ be a group which contains a torsionfree subgroup Δ of finite index. Then the following assertions are equivalent:*

(1) *Γ satisfies $B(d)$;*
(2) *Γ satisfies $b(d)$;*
(3) *Γ has virtual cohomological dimension $\leq d$.*

Proof. $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (3)$ The subgroup Δ satisfies $b(d)$ by Lemma 6.1. Since Δ is torsionfree this shows that the virtual cohomological dimension of Γ is less or equal to d.

$(3) \Rightarrow (1)$ Suppose that the virtual cohomological dimension of Γ is $\leq d$. Next we show that then Γ satisfies $b(d)$. Let M be a $\mathbb{Z}\Gamma$-module such that $\operatorname{res}_\Gamma^H M$ is projective for all finite subgroups $H \subset \Gamma$. Then M has a d-dimensional projective $\mathbb{Z}\Gamma$-resolution by [5, Theorem VI.8.12 on p. 152 and Proposition X.5.2 and Theorem X.5.3 on p. 287]. Hence Γ satisfies $b(d)$. Next we show that Γ satisfies $B(d)$. Let $H \subset \Gamma$ be a finite subgroup. Then $\Delta \cap N_\Gamma H$ is a subgroup of Γ and hence has virtual cohomological dimension d. Since $\Delta \cap H$ is trivial, $\Delta \cap N_\Gamma H$ is a subgroup in $W_\Gamma H$ of finite index. Hence $W_\Gamma H$ satisfies $b(d)$ by the assertion for $b(d)$ we have just proven above. Therefore Γ satisfies $B(d)$. This finishes the proof of Theorem 6.3. $\square$

We rediscover from Theorem 5.1 and Theorem 6.3 the result of [7, Theorem I on p. 18] that a group Γ with finite virtual cohomological dimension has a finitely dominated Γ-CW-model for $\underline{E}\Gamma$ if and only if Γ has only finitely many conjugacy classes of finite subgroups and for each finite subgroup $H \subset \Gamma$ its Weyl group WH is finitely presented and of type FP_∞.

Next we improve Theorem 1.10 in the case of groups with finite virtual cohomological dimension. Notice that for such a group there is an upper bound on the length $l(H)$ of finite subgroups $H \subset \Gamma$.

Theorem 6.4. *Let Γ be a group with virtual cohomological dimension $\leq d$. Let $l \geq 0$ be an integer such that the length $l(H)$ of any finite subgroup $H \subset \Gamma$ is bounded by l. Then there is a Γ-CW-model for $\underline{E}\Gamma$ such that for any finite subgroup $H \subset \Gamma$*

$$\dim(\underline{E}\Gamma^H) = \max\{3, d\} + l - l(H)$$

holds. In particular $\underline{E}\Gamma$ has dimension $\max\{3, d\} + l$.

Proof. We want to use the implication $(3) \Rightarrow (1)$ of Theorem 1.6 where the subgroup $\Delta(H) \subset WH$ is given by the image of $\Delta \cap NH$ under the projection $NH \to WH$. Because of Theorem 6.3 the group Γ satisfies $B(d)$. Hence it remains to show

$$H_{\mathbb{Z}\Delta(H)}^{\max\{3,d\}+l-l(H)+1}(E\Delta(H) \times (C|\mathscr{P}(WH)|, |\mathscr{P}(WH)|); M) = 0$$

for any $\mathbb{Z}\Delta(H)$-module M. Notice that $\Delta(H)$ is isomorphic to $\Delta \cap NH$ since Δ is torsionfree and H is finite and hence there is a d-dimensional model for $E\Delta(H)$. Therefore it suffices to show that $|\mathscr{P}(WH)|$ can be choosen to be $(l - l(H) - 1)$-dimensional because then $H_{\mathbb{Z}\Delta(H)}^{\max\{3,d\}+l-l(H)}(E\Delta(H) \times |\mathscr{P}(WH)|; M) = 0$. This follows from

$$\max\{l(K) \mid K \subset WH, K \text{ finite}\} \leq l - l(H);$$
$$\max\{l(K) \mid K \subset WH, K \text{ finite}\} = \dim(|\mathscr{P}(WH)|) + 1. \qquad \square$$

7. Problems

In this section we discuss some open problems. We emphasize that we state most of the problems not because we have a strategy to prove them, but because we lack counterexamples.

Problem 7.1. *Which of the results in this paper carry over to a Lie group or more general to a (locally compact) topological group G and the classifying space $E(G,\mathscr{K})$ for the family $\mathscr{K}$ of compact subgroups?*

Problem 7.2. *Let $\Delta \subset \Gamma$ be a subgroup of finite index. Suppose that $\underline{E}\Delta$ has a Δ-CW-model of finite type or a finite Δ-CW-model respectively. Does then $\underline{E}\Gamma$ have a Γ-CW-model of finite type or a finite Γ-CW-model, respectively?*

Remark 7.3. If Problem 7.2 has a positive answer, then the following is true for an exact sequence of groups $1 \to \Delta \to \Gamma \to \pi \to 1$. If $\underline{E}\Delta$ has a finite Δ-CW-model or a Δ-CW-model of finite type respectively and $\underline{E}\pi$ has a finite π-CW-model or a π-CW-model of finite type respectively, then $\underline{E}\Gamma$ has a finite Γ-CW-model or a Γ-CW-model of finite type, respectively (cf. Theorem 3.2).

There are some cases where the answer to Problem 7.2 is positive, for example if Δ is word-hyperbolic or virtually poly-cyclic (see Theorem 3.3). Suppose for instance that Γ is torsionfree. There is a normal subgroup $\Delta' \subset \Gamma$ of finite index such that $\Delta' \subset \Delta$ holds. If $B\Delta$ is of finite type or is finite respectively, then $B\Delta'$ is of finite type or is finite, respectively. If π is the finite group Γ/Δ', then we have a fibration $B\Delta \to B\Gamma \to B\pi$. Since $B\Delta'$ and $B\pi$ are of finite type, $B\Gamma$ is of finite type [16, Lemma 7.2]. We have shown that $B\Gamma$ is of finite type if Γ is torsionfree and $B\Delta$ is of finite type. If $B\Delta$ is finite and Γ is torsionfree, then $B\Gamma$ is finitely dominated since it is of finite type and by Theorem 2.4 finite-dimensional [15, Proposition 14.9 on p. 282]. In order to check that $B\Gamma$ is finite, one has to compute its finiteness obstruction $\widetilde{o}(B\Gamma)$ which takes values in $\widetilde{K}_0(\mathbb{Z}\Gamma)$. Notice that there is the conjecture that $\widetilde{K}_0(\mathbb{Z}\Gamma)$ vanishes for torsionfree groups Γ. These considerations show that for torsionfree Γ the answer to Problem 7.2 is positive if one asks for "finite type" and very likely to be positive if one asks for "finite".

There are some reasons to believe that the answer to Problem 7.2 is not always positive or that its proof is difficult. Consider the special case where $\phi : \Delta \to \Delta$ is an automorphism of a group Δ with $\underline{E}\Delta$ of finite type such that $\phi^n = \mathrm{id}$, the group Γ is the semi-direct product of Δ and $\mathbb{Z}/n$ with respect to ϕ, and $H \subset \Gamma$ is just $\mathbb{Z}/n$. Let CH be the centralizer of H in Γ. Since H is finite and hence has only finitely many automorphisms, CH has finite index in NH. Then $CH \cap \Delta$ is the fixed point set $\mathrm{Fix}(\phi)$ of ϕ and has finite index in NH. Hence because of Theorem 4.2 and [16, Lemma 7.2] a positive answer to Problem 7.2 would imply for any periodic automorphism of a group Δ with $\underline{E}\Delta$ of finite type that $B\,\mathrm{Fix}(\phi)$ has finite type and in particular that $\mathrm{Fix}(\phi)$ is finitely presented.

A positive answer to Problem 7.2 would imply also a positive answer to the following problem by Theorem 4.2.

Problem 7.4. *If the group Γ contains a subgroup of finite index Δ which has a Δ-CW-model of finite type for $\underline{E}\Delta$, does then Γ contain only finitely many conjugacy classes of finite subgroups?*

To our knowledge Problem 7.4 is open even for torsionfree Δ (see also [6, Lemma IX.13.2 on p. 267]).

In view of Theorem 4.2 a positive answer to the next problem implies a positive answer to Problem 7.2 in the case of finite type provided that Γ has only finitely many conjugacy classes of finite subgroups because $B\Gamma$ is of finite type if for some subgroup Δ of finite index $B\Delta$ is of finite type [16, Lemma 7.2].

Problem 7.5. *Let Γ be a group such that $B\Gamma$ is of finite type. Is then BWH of finite type for any finite subgroup $H \subset \Gamma$?*

An algebraic analogoue of Problem 7.5 would be the question whether for a group Γ of type FP_∞ and any finite subgroup $H \subset \Gamma$ the Weyl group WH is of type FP_∞.

Remark 7.6. If Problem 7.5 has a positive answer, then Theorem 4.2 would imply that there is a Γ-CW-model of finite type for $\underline{E}\Gamma$ if and only if Γ is finitely presented and of type FP_∞ and there are only finitely many conjugacy classes of finite subgroups in Γ.

Problem 7.7. *Let $1 \to \Delta \xrightarrow{i} \Gamma \xrightarrow{p} \pi \to 1$ be an exact sequence of groups. Suppose that there is a Δ-CW-model of finite type for $\underline{E}\Delta$ and a Γ-CW-model of finite type for $\underline{E}\Gamma$. Is then there a π-CW-model of finite type for $\underline{E}\pi$?*

The answer to Problem 7.7 is yes if Δ, Γ and π are torsionfree [16, Lemma 7.2]. Since for instance the kernel of the obvious epimorphism from the free group of rank 2 to its abelianization is not even finitely generated, it does not make sense to ask a similiar question where one has information about the classifying spaces for Γ and π and wants to conclude something for the one of Δ.

Problem 7.8. *Suppose that there is a finite-dimensional Γ-CW-complex or finite Γ-CW-complex or Γ-CW-complex of finite type respectively whose isotropy groups are all finite and which is contractible. Is then there a finite-dimensional Γ-CW-model or finite Γ-CW-model or Γ-CW-model of finite type respectively for $\underline{E}\Gamma$?*

The answer to Problem 7.8 is positive in the case finite-dimensional if Γ contains a normal torsionfree subgroup of finite index by Theorem 2.4. A positive answer to Problem 7.8 in the case finite-dimensional is given for groups of type FP_∞ in [14, Theorem A]. More generally we get a positive answer to Problem 7.8 in the case finite-dimensional from Lemma 1.5 and Theorem 1.10 provided that there is an upper bound l on the length $l(H)$ of the finite subgroups $H \subset \Gamma$. Problem 7.8 in the case of finite type is equivalent to Problem 7.5 by Lemma 4.1 and Theorem 4.2 provided that there are only finitely many conjugacy classes of finite subgroups of Γ.

Recall that the condition $B(d)$ is necessary for the existence of a d-dimensional Γ-CW-model for $\underline{E}\Gamma$ (see Lemma 1.5). This is not true for the condition appearing in Theorem 1.6 and Theorem 1.10 that there is an upper bound l on the length $l(H)$ of the finite subgroups $H \subset \Gamma$ (see Example 1.11). Therefore the question arises:

Problem 7.9. *For which groups Γ the following is true*:

(1) *For $d \geq 3$ the condition $B(d)$ (or the condition $b(d)$ alone) is equivalent to the existence of a d-dimensional Γ-CW-model for $\underline{E}\Gamma$*;

(2) *The statements* (1), (2) *and* (3) *appearing in Theorem 1.6 are always equivalent (without the assumption that there is an upper bound on the length $l(H)$ of finite subgroups $H \subset \Gamma$).*

Because of Theorem 6.4 Problem 7.9 (1) reduces in the case that Γ contains a torsionfree subgroup of finite index to the problem stated by Brown [5, p. 32].

Problem 7.10. *For which groups of virtual cohomological dimension $\leq d$ there exists a d-dimensional Γ-CW-model for $\underline{E}\Gamma$.*

More generally one may ask

Problem 7.11. *Suppose that Γ contains a subgroup Δ of finite index such that there is a d-dimensional Δ-CW-model for $\underline{E}\Delta$. Is then there a d-dimensional Γ-CW-model for $\underline{E}\Gamma$?*

Remark 7.12. A positive answer to Problem 7.11 would imply for an extension of groups $1 \to \Delta \to \Gamma \to \pi \to 1$ that $\underline{E}\Gamma$ has a $d+e$-dimensional Γ-CW-model if $\underline{E}\Delta$ has a d-dimensional Δ-CW-model and $\underline{E}\pi$ has a e-dimensional π-CW-model. In this context the question is interesting whether Γ satisfies $B(d+e)$ if Δ satisfies $B(d)$ and π satisfies $B(e)$.

Problem 7.13. *Can one give nice algebraic conditions which ensure the vanishing of the finiteness obstruction $\tilde{o}^\Gamma(\underline{E}\Gamma)$ provided that $\underline{E}\Gamma$ is finitely dominated? Is there at all an example where $\underline{E}\Gamma$ is finitely dominated but does not admit a finite Γ-CW-model?*

Here is a suggestion for a solution of Problem 7.13. A (*finite*) permutation $\mathbb{Z}\Gamma$-module is a $\mathbb{Z}\Gamma$-module M which is isomorphic to a (finite) sum of $\mathbb{Z}\Gamma$-modules of the shape $\mathbb{Z}\Gamma/H$ for some finite subgroup $H \subset \Gamma$.

Problem 7.14. *Suppose that Γ satisfies the following condition: There are only finitely many conjugacy classes of finite subgroups in Γ and for each finite subgroup $H \subset \Gamma$ the trivial $\mathbb{Z}WH$-module $\mathbb{Z}$ has a finite-dimensional $\mathbb{Z}WH$-resolution by finite $\mathbb{Z}WH$-permutation modules. Is then there a finite Γ-CW-model for $\underline{E}\Gamma$?*

Notice that the condition in Problem 7.14 is necessary for the existence of a finite Γ-CW-model for $\underline{E}\Gamma$ by Lemma 1.3 and Theorem 5.1 since such a resolution of $\mathbb{Z}$ is given by the $\mathbb{Z}WH$-chain complex of $\underline{E}WH$. Moreover, if the condition is satisfied, $\underline{E}\Gamma$ is at least finitely dominated by Theorem 5.1. Problem 7.14 has a positive answer if for each finite subgroup $H \subset \Gamma$ and each exact $\mathbb{Z}WH$-sequence $0 \to Q \to P_1 \to P_2 \to \cdots \to P_n \to 0$ such that Q is finitely generated projective and P_i is a finite permutation $\mathbb{Z}WH$-module for $1 \leq i \leq n$ the $\mathbb{Z}WH$-module Q is stably free, i.e. becomes free after taking a direct sum with a finitely generated free $\mathbb{Z}WH$-module. The most optimistic version based on Remark 7.6 and Problem 7.9 would be

Problem 7.15. *For which groups are the following statements true for an integer $d \geq 3$?*

(1) *There is a d-dimensional Γ-CW-model for $\underline{E}\Gamma$ if and only Γ satisfies $b(d)$.*

(2) *There is a Γ-CW-model of finite type for $\underline{E}\Gamma$ if and only if Γ is finitely presented, has only finitely many conjugacy classes of finite subgroups and is of type FP_∞.*

(3) *There is a d-dimensional finite Γ-CW-model for $\underline{E}\Gamma$ if and only if Γ is finitely presented, has only finitely many conjugacy classes of finite subgroups, is of type FP_∞ and satisfies $b(d)$.*

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