L^2-invariants

Wolfgang Lück*
Fachbereich Mathematik und Informatik
Westfälische Wilhelms-Universität
Münster
Einsteinstr. 62
48149 Münster
Germany
lueck@math.uni-muenster.de
http://www.math.uni-muenster.de/u/lueck

July 2003

1. Basic Introduction to L^2-Betti Numbers

Principle 1.1 Given an invariant for finite CW-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.

Examples:

Classical notion	generalized version
Homology with $\mathbb{Z}$-coefficients	Homology with coefficients in representations
Euler characteristic $\in \mathbb{Z}$	Walls finiteness obstruction in $K_0(\mathbb{Z}\pi)$
Lefschetz numbers $\in \mathbb{Z}$	Generalized Lefschetz invariants in $\mathbb{Z}\pi_\phi$
Signature $\in \mathbb{Z}$	Surgery invariants in $L_*(\mathbb{Z}G)$
—	torsion invariants

Goal 1.2 *Apply this principle to (classical) Betti numbers*

$$b_p(X) = \dim_{\mathbb{C}}(H_p(X;\mathbb{C})).$$

Here are two naive attempts which fail:

- $\dim_{\mathbb{C}}(H_p(\widetilde{X};\mathbb{C}))$
- $\dim_{\mathbb{C}\pi}(H_p(\widetilde{X};\mathbb{C}))$,
  where $\dim_{\mathbb{C}\pi}(M)$ for a $\mathbb{C}[\pi]$-module could be choosen as $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}G} M)$.

We will use the following successful approach essentially due to Atiyah:

Throughout these lectures let G be a discrete group.

Denote by $l^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$.

Definition 1.3 *Define the* **group von Neumann algebra**

$$\mathcal{N}(G) \; := \; \mathcal{B}(l^2(G))^G$$

to be the algebra of bounded G-equivariant operators $l^2(G) \to l^2(G)$.

The **von Neumann trace** *is defined by*

$$\mathrm{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{l^2(G)}.$$

Example 1.4 If G is finite, then $\mathbb{C}G = l^2(G) = \mathcal{N}(G)$. The trace $\mathrm{tr}_{\mathcal{N}(G)}$ assigns to $\sum_{g \in G} \lambda_g \cdot g$ the coefficient λ_e.

Example 1.5 Let G be $\mathbb{Z}^n$.

Let $L^2(T^n)$ be the Hilbert space of L^2-integrable functions $T^n \to \mathbb{C}$.

Let $L^\infty(T^n)$ be the Banach space of essentially bounded functions $f \colon T^n \to \mathbb{C} \coprod \{\infty\}$. An element $(k_1, \ldots, k_n)$ in $\mathbb{Z}^n$ acts isometrically on $L^2(T^n)$ by pointwise multiplication with the function $T^n \to \mathbb{C}$ which maps $(z_1, z_2, \ldots, z_n)$ to $z_1^{k_1} \cdot \ldots \cdot z_n^{k_n}$.

Fourier transform yields an isometric $\mathbb{Z}^n$-equivariant isomorphism

$$l^2(\mathbb{Z}^n) \xrightarrow{\cong} L^2(T^n).$$

We obtain an isomorphism

$$L^\infty(T^n) \xrightarrow{\cong} \mathcal{N}(\mathbb{Z}^n) = \mathcal{B}(L^2(T^n))^{\mathbb{Z}^n}$$

by sending $f \in L^\infty(T^n)$ to the $\mathbb{Z}^n$-operator

$$M_f \colon L^2(T^n) \to L^2(T^n), \quad g \mapsto g \cdot f.$$

Under this identification the trace becomes

$$\mathrm{tr}_{\mathcal{N}(\mathbb{Z}^n)} \colon L^\infty(T^n) \to \mathbb{C}, \qquad f \mapsto \int_{T^n} f d\mu.$$

Definition 1.6 *A* **finitely generated Hilbert $\mathcal{N}(G)$-module** V *is a Hilbert space* V *together with a linear isometric* G*-action such that there exists an isometric linear* G*-embedding of* V *into* $l^2(G)^n$ *for some* $n \geq 0$.

A **map of finitely generated Hilbert $\mathcal{N}(G)$-modules** $f\colon V \to W$ *is a bounded* G*-equivariant operator.*

Definition 1.7 *Let* V *be a finitely generated Hilbert* $\mathcal{N}(G)$*-module. Choose a* G*-equivariant projection* $p\colon l^2(G)^n \to l^2(G)^n$ *with* $\operatorname{im}(p) \cong_{\mathcal{N}(G)} V$. *Define the* **von Neumann dimension** *of* V *by*

$$\begin{aligned}\dim_{\mathcal{N}(G)}(V) &:= \operatorname{tr}_{\mathcal{N}(G)}(p) \\ &:= \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(p_{i,i}) \ \in [0,\infty).\end{aligned}$$

Example 1.8 For finite G a finitely generated Hilbert $\mathcal{N}(G)$-module V is the same as a unitary finite dimensional G-representation and

$$\dim_{\mathcal{N}(G)}(V) \;=\; \frac{1}{|G|} \cdot \dim_{\mathbb{C}}(V).$$

Example 1.9 Let G be $\mathbb{Z}^n$. Let $X \subset T^n$ be any measurable set with characteristic function $\chi_X \in L^\infty(T^n)$. Let $M_{\chi_X} \colon L^2(T^n) \to L^2(T^n)$ be the $\mathbb{Z}^n$-equivariant unitary projection given by multiplication with χ_X. Its image V is a Hilbert $\mathcal{N}(\mathbb{Z}^n)$-module with

$$\dim_{\mathcal{N}(\mathbb{Z}^n)}(V) = \mathrm{vol}(X).$$

In particular each $r \in [0, \infty)$ occurs as $r = \dim_{\mathcal{N}(\mathbb{Z}^n)}(V)$.

Definition 1.10 *A sequence of Hilbert* $\mathcal{N}(G)$-*modules* $U \xrightarrow{i} V \xrightarrow{p} W$ *is* **weakly exact** *at* V *if the kernel* $\ker(p)$ *of* p *and the closure* $\mathrm{clos}(\mathrm{im}(i))$ *of the image* $\mathrm{im}(i)$ *of* i *agree.*

A map of Hilbert $\mathcal{N}(G)$-*modules* $f \colon V \to W$ *is a* **weak isomorphism** *if it is injective and has dense image.*

Example 1.11 The morphism of $\mathcal{N}(\mathbb{Z})$-Hilbert modules

$$M_{z-1} \colon l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), \quad u \mapsto (z-1) \cdot u$$

is a weak isomorphism, but not an isomorphism.

Theorem 1.12 *1.* **Faithfulness**
We have for a Hilbert $\mathcal{N}(G)$-module V

$$V = 0 \iff \dim_{\mathcal{N}(G)}(V) = 0;$$

2. **Additivity**
If $0 \to U \to V \to W \to 0$ is a weakly exact sequence of Hilbert $\mathcal{N}(G)$-modules, then

$$\begin{aligned}\dim_{\mathcal{N}(G)}(U) + \dim_{\mathcal{N}(G)}(W) & \\ = \ & \dim_{\mathcal{N}(G)}(V);\end{aligned}$$

3. **Cofinality**
Let $\{V_i \mid i \in I\}$ be a directed system of Hilbert $\mathcal{N}(G)$- submodules of V, directed by $\subset$. Then

$$\begin{aligned}\dim_{\mathcal{N}(G)}(\operatorname{clos}(\cup_{i\in I} V_i)) & \\ = \ & \sup\{\dim_{\mathcal{N}(G)}(V_i) \mid i \in I\};\end{aligned}$$

Definition 1.13 *A* G*-*CW**-complex** X *is a* G*-space with a* G*-invariant filtration*

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \cup_{n \geq 0} X_n = X$$

such that X *carries the colimit topology and* X_n *is obtained from* X_{n-1} *by attaching equivariant* n*-dimensional cells, i.e. there exists a* G*-pushout*

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow[\coprod_{i \in I_n} Q_i]{} & X_n \end{array}$$

We call X **finite** *if it is built by finitely many equivariant cells. We call* X **of finite type** *if each skeleton* X_n *is finite.*

Definition 1.14 *Let X be a free G-CW-complex of finite type. Denote by $C_*(X)$ its* **cellular $\mathbb{Z}G$-chain complex**. *Define its* **cellular L^2-chain complex** $C_*^{(2)}(X)$ *to be the Hilbert $\mathcal{N}(G)$-chain complex*

$$C_*^{(2)}(X) \ := \ l^2(G) \otimes_{\mathbb{Z}G} C_*(X).$$

Define its **p-th L^2-homology** *to be the finitely generated Hilbert $\mathcal{N}(G)$-module*

$$H_p^{(2)}(X;\mathcal{N}(G)) := \ker(C_p^{(2)})/\overline{\mathrm{im}(c_{p+1}^{(2)})}.$$

Define its **p-th L^2-Betti number**

$$b_p^{(2)}(X;\mathcal{N}(G)) \ = \ \dim_{\mathcal{N}(G)}\left(H_p^{(2)}(X;\mathcal{N}(G))\right).$$

Remark 1.15 *Notice that $C_p(X) = \oplus_{I_p}\mathbb{Z}G$. Hence $C_p^{(2)}(X) = \oplus_{I_p} l^2(G)$. In particular*

$$\dim_{\mathcal{N}(G)}(C_p^{(2)}(X)) \ = \ |I_p| = |\{p\text{-cells in } G\backslash X\}|.$$

Each differential $c_p^{(2)}$ is a morphism of finitely generated Hilbert $\mathcal{N}(G)$-modules since each I_p is finite by assumption.

Theorem 1.16 (Cellular L^2-Betti numbers)

1. **Homotopy invariance**

   Let $f\colon X \to Y$ be a G-map of free G-CW-complexes of finite type. If f is a weak homotopy equivalence (after forgetting the G-action), then

   $$b_p^{(2)}(X) \;=\; b_p^{(2)}(Y);$$

2. **Euler-Poincaré formula, Atiyah**

   Let X be free finite G-CW-complex. Let $\chi(G\backslash X)$ be the Euler characteristic of the finite CW-complex $G\backslash X$. Then

   $$\chi(G\backslash X) \;=\; \sum_{p\geq 0}(-1)^p \cdot b_p^{(2)}(X);$$

3. **Poincaré duality**

   Let M be a cocompact free proper G-manifold of dimension n which is orientable. Then

   $$b_p^{(2)}(M) \;=\; b_{n-p}^{(2)}(M, \partial M);$$

4. **Künneth formula, Zucker**

   Let X be a free G-CW-complex of finite type and Y be a free H-CW-complex of finite type. Then we get for all $n \geq 0$

$$b_n^{(2)}(X \times Y) \;=\; \sum_{p+q=n} b_p^{(2)}(X) \cdot b_q^{(2)}(Y);$$

5. **Wedges**

   Let X_1, X_2, ..., X_r be connected CW-complexes of finite type and $X = \bigvee_{i=1}^r X_i$ be their wedge. Then

$$\begin{aligned} & b_1^{(2)}(\widetilde{X}) - b_0^{(2)}(\widetilde{X}) \\ =\; & r - 1 + \sum_{j=1}^r \left(b_1^{(2)}(\widetilde{X_j}) - b_0^{(2)}(\widetilde{X_j}) \right); \end{aligned}$$

   and for $2 \leq p$

$$b_p^{(2)}(\widetilde{X}) \;=\; \sum_{j=1}^r b_p^{(2)}(\widetilde{X_j});$$

6. **Morse inequalities, Novikov-Shubin**

Let X be a free G-CW-complex of finite type. Then we get for $n \geq 0$

$$\begin{aligned} &\sum_{p=0}^{n} (-1)^{n-p} \cdot b_p^{(2)}(X) \\ \leq \quad &\sum_{p=0}^{n} (-1)^{n-p} \cdot |\{p\text{-cells of } G\backslash X)\}|; \end{aligned}$$

7. **Zero-th L^2-Betti number**

Let X be a connected free G-CW-complex of finite type. Then

$$b_0^{(2)}(X) \;=\; \frac{1}{|G|};$$

8. **Restriction**

Let X be a free G-CW-complex of finite type and let $H \subset G$ be a subgroup of finite index $[G:H]$. Then

$$\begin{aligned} &[G:H] \cdot b_p^{(2)}(X;\mathcal{N}(G)) \\ &\qquad\qquad = \; b_p^{(2)}(\mathrm{res}_G^H X;\mathcal{N}(H)); \end{aligned}$$

9. **Induction**

Let $H \subseteq G$ and let X be a free H-CW-complex of finite type. Then

$$b_p^{(2)}(G \times_H X; \mathcal{N}(G)) \; = \; b_p^{(2)}(X; \mathcal{N}(H)).$$

Example 1.17 If G is finite and X is a free G-CW-complex of finite type, then $b_p^{(2)}(X)$ is the classical p-th Betti number of X multiplied with $\frac{1}{|G|}$.

Example 1.18 *Consider the $\mathbb{Z}$-CW-complex $\widetilde{S^1}$. We get for $C_*^{(2)}(\widetilde{S^1})$*

$$\ldots \to 0 \to l^2(\mathbb{Z}) \xrightarrow{M_{z-1}} l^2(\mathbb{Z}) \to 0 \to \ldots$$

and hence $H_p^{(2)}(\widetilde{S^1}) = 0$ and $b_p^{(2)}(\widetilde{S^1}) = 0$ for all p.

Example 1.19 Let F_g be the orientable closed surface. For $F_0 = S^2$ we get

$$b_p^{(2)}(\widetilde{F_0}) = b_p(S^2) \;=\; \begin{cases} 1 & \text{if } p = 0, 2; \\ 0 & \text{otherwise.} \end{cases}$$

If $g \geq 0$, then $|\pi_1(F_g)| = \infty$ and hence $b_0^{(2)}(\widetilde{F_g}) = 0$. By Poincaré duality $b_2^{(2)}(\widetilde{F_g}) = 0$. As $\dim(F_g) = 2$, we get $b_p^{(2)}(\widetilde{F_g}) = 0$ for $p \geq 3$. The Euler-Poincaré formula shows

$$\begin{aligned} b_1^{(2)}(\widetilde{F_g}) &= -\chi(F_g) \;=\; 2g-2; \\ b_p^{(2)}(\widetilde{F_0}) &= 0 \;\text{ for } p \neq 1. \end{aligned}$$

Example 1.20 Let $X \to Y$ be a covering with d-sheets of connected CW-complexes of finite type. Theorem 1.16 (8) implies

$$b_p^{(2)}(\widetilde{Y}) = d \cdot b_p^{(2)}(\widetilde{X}).$$

In particular we get for a connected CW-complex X of finite type, for which there is a selfcovering $X \to X$ with d-sheets for some integer $d \geq 2$, that for $p \geq 0$

$$b_p^{(2)}(\widetilde{X}) = 0.$$

This implies for each finite CW-complex Y of finite type

$$b_p^{(2)}(\widetilde{S^1 \times Y}) \;=\; 0.$$

Theorem 1.21 (Vanishing of L^2-Betti numbers of mapping tori, L.)
Let $f\colon X \to X$ be a cellular selfhomotopy equivalence of a connected CW-complex X of finite type. Then we get for all $p \geq 0$

$$b_p^{(2)}(\widetilde{T_f}) \;=\; 0.$$

Proof: As $T_{f^d} \to T_f$ has d sheets, we get

$$b_p^{(2)}(\widetilde{T_f}) \;=\; \frac{b_p^{(2)}(\widetilde{T_{f^d}})}{d}.$$

If $\beta_p(X)$ is the number of p-cells, then there is up to homotopy equivalence a CW-structure on T_{f^d} with $\beta(T_{f^d}) = \beta_p(X) + \beta_{p-1}(X)$. We have

$$\begin{aligned} b_p^{(2)}(\widetilde{T_{f^d}}) \;&=\; \dim_{\mathcal{N}(G)}\left(H_p^{(2)}(C_p^{(2)}(\widetilde{T_{f^d}})\right) \\ &\leq\; \dim_{\mathcal{N}(G)}\left(C_p^{(2)}(\widetilde{T_{f^d}})\right) \;=\; \beta_p(T_{f^d}). \end{aligned}$$

This implies for all $d \geq 1$

$$b_p^{(2)}(\widetilde{T_f}) \;\leq\; \frac{\beta_p(X) + \beta_{p-1}(X)}{d}.$$

Taking the limit for $d \to \infty$ yields the claim.
$\square$

2. Further Results about L^2-Betti Numbers

Theorem 2.1 (Long weakly exact L^2-homology sequence, Cheeger-Gromov)
Let $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \to 0$ *be an exact sequence of chain complexes of finitely generated Hilbert* $\mathcal{N}(G)$*-modules. Then there is a long weakly exact homology sequence*

$$\ldots \xrightarrow{H_{n+1}^{(2)}(p_*)} H_{n+1}^{(2)}(E_*) \xrightarrow{\partial_{n+1}} H_n^{(2)}(C_*)$$
$$\xrightarrow{H_n^{(2)}(i_*)} H_n^{(2)}(D_*) \xrightarrow{H_n^{(2)}(p_*)} H_n^{(2)}(E_*) \xrightarrow{\partial_n} \ldots .$$

Theorem 2.2 (L^2-Betti numbers and S^1-actions, L.)
Let X *be a connected* S^1*-*CW*-complex of finite type, for instance a connected compact manifold with* S^1*-action. Suppose that for one (and hence all)* $x \in X$ *the map* $S^1 \to X$, $z \mapsto zx$ *is* π_1*-injective. (In particular the* S^1*-action has no fixed points.) Then we get for all* $p \geq 0$

$$b_p^{(2)}(\widetilde{X}) = 0.$$

Theorem 2.3 (L^2-Betti numbers and aspherical S^1-manifolds, L.)
Let M be an aspherical closed manifold with non-trivial S^1-action. Then

1. *The action has no fixed points;*

2. *The map $S^1 \to X,\ z \mapsto zx$ is π_1-injective for $x \in X$;*

3. *$b_p^{(2)}(\widetilde{M}) = 0$ for $p \geq 0$ and $\chi(M) = 0$.*

Theorem 2.4 (L^2-Hodge-de Rham Theorem, Dodziuk)
Let M be a cocompact free proper G-manifold with G-invariant Riemannian metric and $\partial M = \emptyset$. Let K be an equivariant smooth triangulation of M. Put

$$\mathcal{H}^p_{(2)}(M) = \{\omega \in \Omega^p(M) \mid \Delta_p(\omega) = 0, ||\omega||_{L^2} < \infty\}.$$

Then integration defines an isomorphism of finitely generated Hilbert $\mathcal{N}(G)$-modules

$$\mathcal{H}^p_{(2)}(M) \xrightarrow{\cong} H^p_{(2)}(K).$$

Corollary 2.5

$$b_p^{(2)}(M) \;=\; \lim_{t\to\infty}\int_{\mathcal{F}} \mathrm{tr}_{\mathbb{C}}(e^{-t\Delta_p}(x,x))\; dvol.$$

where $\mathcal{F}$ is a fundamental domain for the G-action and $e^{-t\Delta_p}(x,y)$ is the heat kernel on $\widetilde{M}$.

Theorem 2.6 (Dodziuk)

Let M be a hyperbolic closed Riemannian manifold of dimension n. Then:

$$b_p^{(2)}(\widetilde{M}) \quad \begin{cases} = 0 & \text{, if } 2p \neq n; \\ > 0 & \text{, if } 2p = n. \end{cases}$$

Proof: A direct computation shows that $\mathcal{H}^p_{(2)}(\mathcal{H}^n)$ is not zero if and only if $2p = n$. Notice that M is hyperbolic if and only if $\widetilde{M}$ is isometrically diffeomorphic to the standard hyperbolic space $\mathcal{H}^n$. $\square$

Corollary 2.7 *Let M be a hyperbolic closed manifold of dimension n. Then*

1. *If $n = 2m$ is even, then*

$$(-1)^m \cdot \chi(M) > 0;$$

2. *M carries no non-trivial S^1-action.*

Proof: (1) We get from the Euler-Poincaré formula and Theorem 2.6

$$(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M}) > 0.$$

(2) We give the proof only for $n = 2m$ even. Then $b_m^{(2)}(\widetilde{M}) > 0$. Since $\widetilde{M} = \mathcal{H}^n$ is contractible, M is aspherical. Now apply Theorem 2.3. $\square$

Theorem 2.8 (L^2-Betti numbers of 3-manifolds, Lott-L.)
Let M be the connected sum $M_1 \sharp \ldots \sharp M_r$ of (compact connected orientable) prime 3-manifolds M_j which are non-exceptional. Assume that $\pi_1(M)$ is infinite. Then

$$\begin{aligned}
b_0^{(2)}(\widetilde{M}) &= 0; \\
b_1^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} - \chi(M) \\
&\qquad + \Big|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\Big|; \\
b_2^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} \\
&\qquad + \Big|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\Big|; \\
b_3^{(2)}(\widetilde{M}) &= 0.
\end{aligned}$$

Lemma 2.9 *Let X be a free $\mathbb{Z}^n$-CW-complex of finite type. Then*

$$b_p^{(2)}(X) = \dim_{\mathbb{C}[\mathbb{Z}^n]^{(0)}} \left(\mathbb{C}[\mathbb{Z}^n]^{(0)} \otimes_{\mathbb{Z}[\mathbb{Z}^n]} H_p(X)\right),$$

where $\mathbb{C}[\mathbb{Z}^n]^{(0)}$ is the quotient field of $\mathbb{C}[\mathbb{Z}^n]$.

Example 2.10 In general there are no relations between the Betti numbers $b_p(X)$ and the L^2-Betti numbers $b_p^{(2)}(\widetilde{X})$ for a connected CW-complex X of finite type.

Given an integer $l \geq 1$ and a sequence r_1, r_2, ..., r_l of non-negative rational numbers, we can construct a group G such that BG is of finite type and

$$\begin{aligned} b_p^{(2)}(BG) &= r_p && \text{for } 1 \leq p \leq l; \\ b_p^{(2)}(BG) &= 0 && \text{for } l+1 \leq p; \\ b_p(BG) &= 0 && \text{for } p \geq 1. \end{aligned}$$

Namely, take for appropriate k, l, m_i, n_i

$$G = \mathbb{Z}/k \times *_{i=2}^{l} G_i(m_i, n_i),$$

$$G_i(m_i, n_i) = \mathbb{Z}/n_i \times \left(*_{k=1}^{2m_i+2} \mathbb{Z}/2 \right) \times \left(\prod_{j=1}^{i-1} *_{l=1}^{4} \mathbb{Z}/2 \right).$$

For any sequence n_1, n_2, ... of non-negative integers there is a CW-complex X of finite type such that for $p \geq 1$

$$\begin{aligned} b_p(X) &= n_p; \\ b_p^{(2)}(\widetilde{X}) &= 0. \end{aligned}$$

Namely take $X = B(\mathbb{Z}/2 * \mathbb{Z}/2) \times \bigvee_{p=1}^{\infty} \left(\bigvee_{i=1}^{n_p} S^p \right)$.

Theorem 2.11 Approximation Theorem, L.)
Let X be a free G-CW-complex of finite type. Suppose that G is residually finite, i.e. there is a nested sequence

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots$$

of normal subgroups of finite index with $\cap_{n \geq 1} G_n = \{1\}$. Then for any such sequence $(G_n)_{n \geq 1}$

$$b_p^{(2)}(X; \mathcal{N}(G)) \;=\; \lim_{n \to \infty} \frac{b_p(G_n \backslash X)}{[G : G_n]}.$$

Remark 2.12 Ordinary Betti numbers are not multiplicative under finite coverings, whereas the L^2-Betti numbers are, i.e. for a d-sheeted covering $p \colon X \to Y$ we get

$$b_p^{(2)}(\widetilde{X}; \mathcal{N}(\pi_1(X)) \;=\; d \cdot b_p^{(2)}(\widetilde{Y}; \mathcal{N}(\pi_1(Y)).$$

With the expression $\lim_{n \to \infty} \frac{b_p(G \backslash X)}{[G:G_n]}$ we try to force the Betti numbers to be multiplicative by a limit process.

Theorem 2.11 says that L^2-Betti numbers are asymptotic Betti numbers. It was conjectured by Gromov.

There is another interesting L^2-invariant, the L^2-**torsion**

$$\rho^{(2)}(\widetilde{X}) \in \mathbb{R}. \tag{2.13}$$

It is defined for a finite connected CW-complex X such that $\widetilde{X}$ is L^2-**acyclic** i.e. $b_p^{(2)}(\widetilde{X}) = 0$ for $p \geq 0$. (We ignore questions about determinant class).

Theorem 2.14 (Cellular L^2-torsion for universal coverings, L.)

1. **Homotopy invariance**
   Let $f\colon X \to Y$ be a homotopy equivalence of finite CW-complexes. Let $\tau(f) \in \mathrm{Wh}(\pi_1(Y))$ be its Whitehead torsion. Suppose that $\widetilde{X}$ or $\widetilde{Y}$ is L^2-acyclic. Then both $\widetilde{X}$ and $\widetilde{Y}$ are L^2-acyclic and

   $$\rho^{(2)}(\widetilde{Y}) - \rho^{(2)}(\widetilde{X}) = \Phi^{\pi_1(Y)}(\tau(f)),$$

   where $\Phi^{\pi_1(Y)}\colon \mathrm{Wh}(\pi_1(Y)) \to \mathbb{R}$ is given by the Fuglede-Kadison determinant;

2. **Sum formula**

Consider the cellular pushout of finite CW-complexes

$$\begin{array}{ccc} X_0 & \xrightarrow{j_1} & X_1 \\ {\scriptstyle j_2}\big\downarrow & & \big\downarrow{\scriptstyle i_1} \\ X_2 & \xrightarrow[i_2]{} & X \end{array}$$

Assume $\widetilde{X_0}$, $\widetilde{X_1}$, and $\widetilde{X_2}$ are L^2-acyclic and that for $k = 0,1,2$ the obvious map $i_k \colon X_k \to X$ are π_1-injective.

Then $\widetilde{X}$ is L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{X}) = \rho^{(2)}(\widetilde{X_1}) + \rho^{(2)}(\widetilde{X_2}) - \rho^{(2)}(\widetilde{X_0});$$

3. **Poincaré duality**

Let M be a closed manifold of even dimension such that $\widetilde{M}$ is L^2-acyclic. Then

$$\rho^{(2)}(\widetilde{M}) = 0;$$

4. **Product formula**

   Let X and Y be finite CW-complexes. Suppose that $\widetilde{X}$ is L^2-acyclic. Then $\widetilde{X \times Y}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{X \times Y}) = \chi(Y) \cdot \rho^{(2)}(\widetilde{X});$$

5. **Multiplicativity**

   Let $X \to Y$ be a finite covering of finite CW-complexes with d sheets. Then $\widetilde{X}$ is L^2-acyclic if and only if $\tilde{Y}$ is L^2-acyclic and in this case

$$\rho^{(2)}(\widetilde{X}) = d \cdot \rho^{(2)}(\tilde{Y});$$

Remark 2.15 Notice the formal analogy between the behaviour of $\rho^{(2)}(\widetilde{X})$ and the ordinary Euler characteristic $\chi(X)$.

The next result follows from work of Burghelea-Friedlander-Kappeler-McDonald and Schick-L.

Theorem 2.16 *Let M be a compact irreducible 3-manifold with infinite fundamental group such that its boundary is empty or a disjoint union of incompressible tori. Suppose that M satisfies Thurston's Geometrization Conjecture. Let M_1, M_2, ..., M_r be the hyperbolic pieces in the Jaco-Shalen-Johannson-Thurston splitting along incompressible embedded tori. Then*

$$\rho^{(2)}(\widetilde{M}) \ = \ \frac{-1}{6\pi} \cdot \sum_{i=1}^{r} \text{vol}(M_i).$$

Theorem 2.17 (Hess-Schick)
The L^2-torsion $\rho^{(2)}(\widetilde{M})$ for a closed hyperbolic manifold M of odd dimension is up to a (computable) non-zero dimension constant equal to $\text{vol}(M)$*.*

Definition 2.18 (L^2-torsion of group automorphisms)
*Let $f \colon G \to G$ be a group automorphism. Suppose that there is a finite CW-model for BG. Define the L^2-**torsion** of f by*

$$\rho^{(2)}(f \colon G \to G) \ := \rho^{(2)}(B(\widetilde{G \rtimes_f} \mathbb{Z})) \quad \in \ \mathbb{R}.$$

Next we present the basic properties of this invariant. Notice that its behaviour is similar to the Euler characteristic $\chi(G) := \chi(BG)$.

Theorem 2.19 (L^2-torsion of group automorphisms, L.).
Suppose that all groups appearing below have finite classifying spaces.

1. **Amalgamated Products**
   Suppose that G is the amalgamated product $G_1 *_{G_0} G_2$ for subgroups $G_i \subset G$ and the automorphism $f \colon G \to G$ is the amalgamated product $f_1 *_{f_0} f_2$ for automorphisms $f_i \colon G_i \to G_i$. Then

   $$\rho^{(2)}(f) \ = \ \rho^{(2)}(f_1) + \rho^{(2)}(f_2) - \rho^{(2)}(f_0);$$

2. **Trace property**

   Let $f\colon G \to H$ and $g\colon H \to G$ be isomorphisms of groups. Then

$$\rho^{(2)}(f \circ g) \;=\; \rho^{(2)}(g \circ f).$$

   In particular $\rho^{(2)}(f)$ is invariant under conjugation with automorphisms;

3. **Additivity**

   Suppose that the following diagram of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 1 \\ & & {\scriptstyle f_1}\big\downarrow & & {\scriptstyle f_2}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow & & \\ 1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 1 \end{array}$$

   commutes, has exact rows and its vertical arrows are automorphisms. Then

$$\rho^{(2)}(f_2) \;=\; \chi(BG_3) \cdot \rho^{(2)}(f_1);$$

4. **Multiplicativity**

   Let $f\colon G \to G$ be an automorphism of a group. Then for all integers $n \geq 1$

$$\rho^{(2)}(f^n) \;=\; n \cdot \rho^{(2)}(f);$$

5. **Subgroups of finite index**

   Suppose that G contains a subgroup G_0 of finite index $[G : G_0]$. Let $f\colon G \to G$ be an automorphism with $f(G_0) = G_0$. Then

$$\rho^{(2)}(f) \;=\; \frac{1}{[G : G_0]} \cdot \rho^{(2)}(f|_{G_0});$$

6. **Dependence on L^2-homology**

   Let $f\colon G \to G$ be an automorphism of a group G. Then $\rho^{(2)}(f)$ depends only on the map $H_p^{(2)}(\widetilde{Bf})\colon H_p^{(2)}(\widetilde{BG}) \to H_p^{(2)}(\widetilde{BG})$ induced by f on the L^2-homology of the universal covering of BG.

7. **Vanishing results**

   We have $\rho^{(2)}(f) = 0$ if G satisfies one of the following conditions:

   (a) All L^2-Betti numbers of the universal covering of BG vanish;

   (b) G contains an amenable infinite normal subgroup.

Remark 2.20 Let $f\colon S \to S$ be an automorphisms of a compact connected orientable surface. Let M_1, M_1, ..., M_r be the hyperbolic pieces in the JSJT-splitting. If S is S^2, D^2, or T^2, then $\rho^{(2)}(f) = 0$. Otherwise we get

$$\rho^{(2)}(\pi_1(f)\colon \pi_1(S) \to \pi_1(S)) = \frac{-1}{6\pi} \cdot \sum_{i=1}^{r} \mathrm{vol}(M_i).$$

Suppose f is irreducible. Then f is pseudo-Anosov if and only if $\rho^{(2)}(f) < 0$ and f is periodic if and only if $\rho^{(2)}(f) = 0$.

Question 2.21 *Does for G a finitely generated free group or a surface group $\rho^{(2)}(f)$ determine the conjugacy class of f up to finite ambiguity?*

Theorem 2.22 (L^2-Betti numbers and fibrations, L.). *Let $F \to E \to B$ be a fibration of connected CW-complexes of finite type. Suppose that $\pi_1(F) \to \pi_1(E)$ is injective and $b_p^{(2)}(\widetilde{F}) = 0$ for all $p \geq 0$. Then $b_p^{(2)}(\widetilde{E}) = 0$ for all $p \geq 0$.*

Theorem 2.23 (Proportionality Principle for L^2-Betti numbers, Cheeger-Gromov) *Let M be a simply connected Riemannian manifold. Then there are constants $B_p^{(2)}(M)$ for $p \geq 0$ depending only on the Riemannian manifold M such that for any discrete group G with a free proper cocompact action on M by isometries the following holds*

$$b_p^{(2)}(M; \mathcal{N}(G)) = B_p^{(2)}(M) \cdot \mathrm{vol}(G \backslash M).$$

The analogous statement holds for the L^2-torsion

3. The Generalized Dimension function

In this section we present a purely algebraic approach to L^2-Betti numbers.

Remark 3.1 Recall that by definition

$$\begin{aligned} \mathcal{N}(G) &:= \mathcal{B}(l^2(G), l^2(G))^G \\ &= \operatorname{mor}_{\mathcal{N}(G)}(l^2(G), l^2(G)). \end{aligned}$$

This induces a bijection of $\mathbb{C}$-vector spaces

$$M(m, n, \mathcal{N}(G)) \xrightarrow{\cong} \operatorname{mor}_{\mathcal{N}(G)}(l^2(G)^m, l^2(G)^n).$$

It is compatible with multiplication of matrices and composition of morphisms. This extends to finitely generated Hilbert $\mathcal{N}(G)$-modules and finitely projective $\mathcal{N}(G)$-modules.

Theorem 3.2 (Modules over $\mathcal{N}(G)$ and Hilbert $\mathcal{N}(G)$-modules)
We obtain an equivalence of $\mathbb{C}$-categories

$$\begin{aligned} \nu \colon & \{\textit{fin. gen. proj. } \mathcal{N}(G)\textit{-mod.}\} \\ & \to \{\textit{fin. gen. Hilb. } \mathcal{N}(G)\textit{-mod.}\}. \end{aligned}$$

Definition 3.3 *Let R be a ring. Let M be a R-submodule of N. Define the* **closure** *of M in N to be the R-submodule of N*

$$\overline{M} = \{x \in N \mid f(x) = 0 \text{ for all } f \in N^* \text{ with } M \subset \ker(f)\}.$$

For a R-module M define the R-submodule $\mathbf{T}M$ and the R-quotient module $\mathbf{P}M$ by:

$$\begin{aligned} \mathbf{T}M &:= \{x \in M \mid f(x) = 0 \text{ for all } f \in M^*\}; \\ \mathbf{P}M &:= M/\mathbf{T}M. \end{aligned}$$

We call a sequence of R-modules $L \xrightarrow{i} M \xrightarrow{q} N$ **weakly exact** *if $\overline{\operatorname{im}(i)} = \ker(q)$.*

Notice that $\mathbf{T}M$ is the closure of the trivial submodule in M. It can also be described as the kernel of the canonical map

$$i(M) \colon M \to (M^*)^*$$

which sends $x \in M$ to the map $M^* \to R$ $f \mapsto f(x)^*$. Notice that $\mathbf{TP}M = 0$ and that $\mathbf{P}M = 0$ is equivalent to $M^* = 0$.

Example 3.4 *Let $R = \mathbb{Z}$. Let M be a finitely generated $\mathbb{Z}$-module and $K \subset M$. Then*

$$\begin{aligned}
\overline{K} &= \{x \in M \mid n \cdot x \in K \text{ for some } n \in \mathbb{Z}\}; \\
\mathbf{T}M &:= \operatorname{tors}(M); \\
\mathbf{P}M &= M/\operatorname{tors}(M).
\end{aligned}$$

A sequence $M_0 \to M_1 \to M_2$ of finitely generated $\mathbb{Z}$-modules is weakly exact if and only if it is exact after applying $\mathbb{Q} \otimes_{\mathbb{Z}} -$.

Definition 3.5 *Let P be a finitely generated projective $\mathcal{N}(G)$-module. Choose a matrix $A \in M_n(\mathcal{N}(G))$ with $A^2 = A$ such that the image of $r_A \colon \mathcal{N}(G)^n \to \mathcal{N}(G)^n$ is $\mathcal{N}(G)$-isomorphic to P. Define*

$$\dim_{\mathcal{N}(G)}(P) := \operatorname{tr}_{\mathcal{N}(G)}(A) \quad [0, \infty).$$

Lemma 3.6 1. *The functors ν and ν^{-1} preserve exact sequences and weakly exact sequences;*

2. *If P is a finitely generated projective $\mathcal{N}(G)$-module, then*

$$\dim_{\mathcal{N}(G)}(P) = \dim_{\mathcal{N}(G)}(\nu(P)).$$

Remark 3.7 $\mathcal{N}(G)$ is Noetherian if and only if G is finite. It contains zero-divisors if G is non-trivial.

Definition 3.8 *A ring R is called* **semi-hereditary** *if any finitely generated submodule of a projective module is projective.*

Lemma 3.9 $\mathcal{N}(G)$ *is semihereditary.*

Proof: It suffices to prove for a finitely generated $\mathcal{N}(G)$-submodule $M \subset \mathcal{N}(G)^n$ that it is projective. Choose a $\mathcal{N}(G)$-map $f\colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ whose image is M. Let $\nu(f)\colon l^2(G)^m \to l^2(G)^n$ be the morphism corresponding to f under ν. Choose a projection $\mathrm{pr}\colon l^2(G)^m \to l^2(G)^m$ with image $\ker(\nu(f))$. Then

$$l^2(G)^m \xrightarrow{\mathrm{pr}} l^2(G)^m \xrightarrow{\nu^{-1}(f)} l^2(G)^n$$

is exact. Hence

$$\mathcal{N}(G)^m \xrightarrow{\nu^{-1}(\mathrm{pr})} \mathcal{N}(G)^m \xrightarrow{f} \mathcal{N}(G)^n$$

is exact and $\nu^{-1}(\mathrm{pr})^2 = \nu^{-1}(\mathrm{pr})$. Hence $\ker(f) \subset \mathcal{N}(G)^m$ is a direct summand and $M = \mathrm{im}(f)$ is projective. $\square$

Remark 3.10 The following results and definitions can be understood by the slogan that $\mathcal{N}(G)$ behaves like $\mathbb{Z}$ if one forgets that $\mathbb{Z}$ is Noetherian and has no-zero-divisors. In this sense all properties of $\mathbb{Z}$ carry over to $\mathcal{N}(G)$.

Lemma 3.11 *Let M be a finitely generated $\mathcal{N}(G)$-module. Then*

1. *Let $K \subset M$ be a submodule. Then $\overline{K} \subset M$ is a direct summand and $M/\overline{K}$ is finitely generated projective;*

2. *$\mathbf{P}M$ is a finitely generated projective $\mathcal{N}(G)$-module and we get a splitting*
$$M \cong \mathbf{T}M \oplus \mathbf{P}M;$$

3. *If M is finitely presented, then there is an exact sequence*
$$0 \to \mathcal{N}(G)^n \to \mathcal{N}(G)^n \to \mathbf{T}M \to 0.$$

Theorem 3.12 (Dimension function for arbitrary $\mathcal{N}(G)$-modules, L.)

There is precisely one dimension function

$$\dim\colon \{\mathcal{N}(G)-\textit{modules}\} \;\to\; [0,\infty]$$

which has the following properties;

1. **Extension Property**
   If M is a finitely generated projective R-module, then $\dim(M)$ agrees with the previously defined notion;

2. **Additivity**
   If $0 \to M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \to 0$ is an exact sequence of R-modules, then

   $$\dim(M_1) \;=\; \dim(M_0) + \dim(M_2);$$

3. **Cofinality**
   Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of M, i.e. $M = \cup_{i \in I} M_i$ and for two indices i and j there is an

index k in I satisfying $M_i, M_j \subset M_k$. Then

$$\dim(M) = \sup\{\dim(M_i) \mid i \in I\};$$

4. **Continuity**
   If $K \subset M$ is a submodule of the finitely generated R-module M, then

$$\dim(K) = \dim(\overline{K});$$

5. **Dimension and Torsion**
   If M is a finitely generated R-module, then

$$\begin{aligned} \dim(M) &= \dim(\mathbf{P}M); \\ \dim(\mathbf{T}M) &= 0. \end{aligned}$$

Proof: We give the proof of uniqueness which leads to the definition of dim. Any $\mathcal{N}(G)$-module M is the colimit over the directed system of its finitely generated submodules $\{M_i \mid i \in I\}$. Hence by Cofinality

$$\dim(M) = \sup\{\dim(M_i) \mid i \in I\}.$$

We get for each M_i from Additivity

$$\dim(M_i) = \dim(\mathbf{P}M_i).$$

Hence we get

$$\dim(M) = \sup\{\dim(P) \mid P \subset M \text{ finitely generated projective}\}.\square$$

Definition 3.13 *Let X be a (left) G-space. Its* **homology with coefficients in $\mathcal{N}(G)$** *is*

$$H_p^G(X;\mathcal{N}(G)) = H_p\left(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^{\text{sing}}(X)\right).$$

Define the **p-th L^2-Betti number of X** *by*

$$b_p^{(2)}(X;\mathcal{N}(G)) := \dim_{\mathcal{N}(G)}\left(H_p^G(X;\mathcal{N}(G))\right) \in [0,\infty].$$

Lemma 3.14 *Let X be a free G-CW-complex of finite type. Then Definition 3.13 of L^2-Betti numbers $b_p^{(2)}(X;\mathcal{N}(G))$ agrees with the previous one.*

Definition 3.15 *The* **p-th L^2-Betti number** *of a group G is*

$$b_p^{(2)}(G) := b_p^{(2)}(EG, \mathcal{N}(G)).$$

Theorem 3.16 **L^2-Betti numbers for arbitrary spaces, L.)**

1. **Homotopy invariance**
   Let $f\colon X \to Y$ be a G-map. Suppose such that for each subgroup $H \subset G$ the induced map $f^H\colon X^H \to Y^H$ is a homology equivalence (for singular homology with $\mathbb{C}$-coefficients). Then for all $p \geq 0$

   $$b_p^{(2)}(X) = b_p^{(2)}(Y) \qquad \text{for } p \geq 0;$$

2. **Independence of equivariant cells with infinite isotropy**
   Let X be a G-CW-complex. Let $X[\infty]$ be the G-CW-subcomplex consisting of those points whose isotropy subgroups are infinite. Then we get for all $p \geq 0$

   $$b_p^{(2)}(X; \mathcal{N}(G)) = b_p^{(2)}(X, X[\infty]; \mathcal{N}(G));$$

3. **Künneth formula**
Let X be a G-space and Y be a H-space. Then $X \times Y$ is a $G \times H$-space and we get for all $n \geq 0$

$$b_n^{(2)}(X \times Y) \;=\; \sum_{p+q=n} b_p^{(2)}(X) \cdot b_q^{(2)}(Y),$$

where $0 \cdot \infty := 0$, $r \cdot \infty := \infty$ for $r \in (0, \infty]$ and $r + \infty = \infty$ for $r \in [0, \infty]$;

4. **Induction**
Let $H \subseteq G$ be a subgroup. Then

$$b_p^{(2)}(G \times_H X; \mathcal{N}(G)) \;=\; b_p^{(2)}(X; \mathcal{N}(H));$$

5. **Restriction**
Let $H \subset G$ be a subgroup of finite index $[G : H]$. Let X be a G-space. Then

$$\begin{aligned} &b_p^{(2)}(\mathrm{res}(X); \mathcal{N}(H)) \\ &\qquad\qquad = \; [G : H] \cdot b_p^{(2)}(X; \mathcal{N}(G)); \end{aligned}$$

6. **Zero-th homology and L^2-Betti number**
For a path-connected G-space X

$$b_0^{(2)}(X; \mathcal{N}(G)) = |G|^{-1}.$$

Definition 3.17 *A group G is called* **amenable** *if there is a (left) G-invariant linear operator $\mu\colon l^\infty(G,\mathbb{R}) \to \mathbb{R}$ with $\mu(1) = 1$ which satisfies*

$$\inf\{f(g) \mid g \in G\} \leq \mu(f) \leq \sup\{f(g) \mid g \in G\} \quad \text{for all } f \in l^\infty(G,\mathbb{R}).$$

The **class of elementary amenable** *groups is defined as the smallest class of groups, which contains all finite and all abelian groups and is closed under i.) taking subgroups, ii) taking quotient groups iii.) under extensions and iv.) under directed unions.*

Remark 3.18 The class of amenable groups contains the class of elementary amenable groups. A group which contains $\mathbb{Z} * \mathbb{Z}$ is not amenable.

Theorem 3.19 (Dimension-flatness of $\mathcal{N}(G)$ over $\mathbb{C}G$ for amenable G), L.)
Let G be amenable and M be a $\mathbb{C}G$-module. Then for $p \geq 1$

$$\dim_{\mathcal{N}(G)}\left(\mathrm{Tor}_p^{\mathbb{C}G}(\mathcal{N}(G), M)\right) = 0.$$

Theorem 3.20 *Let G be an amenable group and X be a G-space. Then*

$$\begin{aligned} & b_p^{(2)}(X; \mathcal{N}(G)) \\ & \quad = \dim_{\mathcal{N}(G)}\left(\mathcal{N}(G) \otimes_{\mathbb{C}G} H_p(X; \mathbb{C})\right). \end{aligned}$$

Corollary 3.21 (Cheeger-Gromov)
Let G be a group which contains an infinite normal amenable subgroup. Then for $p \geq 0$

$$b_p^{(2)}(G; \mathcal{N}(G)) = 0.$$

If there is a finite model for BG, then

$$\chi(G) := \chi(BG) = 0.$$

Proof: If G is amenable, this follows from $H_p(EG; \mathbb{C}) = 0$ for $p \geq 1$. In the general case use a spectral sequence argument. $\square$

Definition 3.22 *Let R be an (associative) ring (with unit). Define its* **projective class group** $K_0(R)$ *to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for any exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules. Define $G_0(R)$ analogously but replacing finitely generated projective by finitely generated.*

Theorem 3.23 (L.)
Let G be an amenable group. Then we get a well-defined map

$$\begin{aligned}\dim \colon G_0(\mathbb{C}G) &\to \mathbb{R},\\ [M] &\mapsto \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{C}G} M).\end{aligned}$$

In particular $[\mathbb{C}G]$ generates an infinite cyclic subgroup in $G_0(\mathbb{C}G)$.

Lemma 3.24 *If G contains $\mathbb{Z} * \mathbb{Z}$ as subgroup, then*

$$[\mathbb{C}G] = 0 \quad \in G_0(\mathbb{C}G).$$

Conjecture 3.25 *G is amenable if and only if*

$$[\mathbb{C}G] \neq 0 \quad \in G_0(\mathbb{C}G).$$

Remark 3.26 Elek has generalized Theorem 3.23 to arbitrary fields as coefficients instead of $\mathbb{C}$ by defining dimension functions also in this context.

Definition 3.27 *Let G be a finitely presented group. Define its* **deficiency** $\operatorname{def}(G)$ *to be the maximum $g(P) - r(P)$, where P runs over all presentations P of G and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation P.*

Example 3.28 The free group F_g has the obvious presentation $\langle s_1, s_2, \ldots s_g \mid \emptyset \rangle$ and its deficiency is realized by this presentation, namely $\operatorname{def}(F_g) = g$.

If G is a finite group, $\operatorname{def}(G) \leq 0$ by Lemma 3.30 as $b_0^{(2)}(G) = |G|^{-1}$ and $b_1^{(2)}(G) = 0$.

The deficiency of a cyclic group $\mathbb{Z}/n$ is 0, the obvious presentation $\langle s \mid s^n \rangle$ realizes the deficiency.

The deficiency of $\mathbb{Z}/n \times \mathbb{Z}/n$ is -1, the obvious presentation $\langle s, t \mid s^n, t^n, [s, t] \rangle$ realizes the deficiency.

Example 3.29 The deficiency is not additive under free products by the following example due to Hog, Lustig and Metzler(1985). The group $(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$ has the obvious presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = t_0^2 = [s_0, t_0] = s_1^3 \\ = t_1^3 = [s_1, t_1] = 1 \rangle$$

One may think that its deficiency is -2. However, it turns out that its deficiency is -1 realized by the following presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = 1, [s_0, t_0] = t_0^2, s_1^3 = 1, \\ [s_1, t_1] = t_1^3, t_0^2 = t_1^3 \rangle.$$

Lemma 3.30 *Let G be a finitely presented group. Then*

$$\mathrm{def}(G) \leq 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Proof We have to show for any presentation P that

$$g(P)-r(P) \leq 1-b_0^{(2)}(G)+b_1^{(2)}(G)-b_2^{(2)}(G).$$

Let X be a CW-complex realizing P. Then

$$\begin{aligned} \chi(X) &= 1 - g(P) + r(P) \\ &= b_0^{(2)}(\widetilde{X}) + b_1^{(2)}(\widetilde{X}) - b_2^{(2)}(\widetilde{X}). \end{aligned}$$

Since the classifying map $X \to BG$ is 2-connected, we get

$$\begin{aligned} b_p^{(2)}(\widetilde{X}) &= b_p^{(2)}(G) \quad \text{for } p = 0,1; \\ b_2^{(2)}(\widetilde{X}) &\geq b_2^{(2)}(G). \square \end{aligned}$$

Theorem 3.31 (Deficiency and extensions, L.)
Let $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ *be an exact sequence of infinite groups. Suppose that* G *is finitely presented and one of the following conditions is satisfied.*

1. $b_1^{(2)}(H) < \infty$;

2. *The ordinary first Betti number of* H *satisfies* $b_1(H) < \infty$ *and* $b_1^{(2)}(K) = 0$.

Then:

(i) $\mathrm{def}(G) \leq 1$;

(ii) Let M *be a closed oriented 4-manifold with* G *as fundamental group. Then*

$$|\operatorname{sign}(M)| \ \leq \ \chi(M).$$

Let F be Thompson's group. It is the group of orientation preserving dyadic PL-automorphisms of $[0,1]$ where dyadic means that all slopes are integral powers of 2 and the break points are contained in $\mathbb{Z}[1/2]$. It has the presentation

$$F = \langle x_0, x_1, x_2, \ldots \mid x_i^{-1} x_n x_i = x_{n+1} \text{ for } i < n \rangle.$$

It has a model of finite type for BG but no finite-dimensional model since $\mathbb{Z}^n$ is a subgroup of F for all $n \geq 1$. It does not contain $\mathbb{Z} * \mathbb{Z}$ as subgroup and is not elementary amenable. It is not known whether it is amenable.

Theorem 3.32 (L.)
The L^2-Betti numbers of Thompson's group $b_p^{(2)}(BF)$ vanish for all $p \geq 0$.

Theorem 3.33 (L^2-Betti numbers and S^1-actions, L.)
Let X be a connected S^1-CW-complex. Suppose that for one orbit S^1/H (and hence for all orbits) the inclusion into X induces a map on π_1 with infinite image. (In particular the S^1-action has no fixed points.) Let $\widetilde{X}$ be the universal covering of X with the canonical $\pi_1(X)$-action. Then we get for all $p \geq 0$

$$b_p^{(2)}(\widetilde{X}) = 0.$$

Theorem 3.34 (L^2-Betti numbers and fibrations, L.)
Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of connected CW-complexes. Suppose that $\pi_1(F) \to \pi_1(E)$ is injective. Suppose for a given integer $d \geq 1$ that $b_p^{(2)}(\widetilde{F}) = 0$ for $p \leq d-1$ and $b_d^{(2)}(\widetilde{F}) < \infty$ holds. Suppose that $\pi_1(B)$ contains an element of infinite order or finite subgroups of arbitrary large order. Then $b_p^{(2)}(\widetilde{E}) = 0$ for $p \leq d$.

4. Survey on Further Results and Conjectures

Given a group G, let $\mathcal{FIN}(G)$ be the set of finite subgroups of G. Denote by

$$\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z} \ \subset \ \mathbb{Q}$$

the additive subgroup of $\mathbb{R}$ generated by the set of rational numbers $\{\frac{1}{|H|} \mid H \in \mathcal{FIN}(G)\}$.

Conjecture 4.1 (Atiyah Conjecture) *A group G satisfies the* **Atiyah Conjecture** *if for any matrix $A \in M(m,n,\mathbb{Q}G)$ the von Neumann dimension of the kernel of the G-equivariant bounded operator $r_A^{(2)}\colon l^2(G)^m \to l^2(G)^n$, $x \mapsto xA$ satisfies*

$$\dim_{\mathcal{N}(G)}\left(\ker\left(r_A^{(2)}\colon l^2(G)^m \to l^2(G)^n\right)\right) \in \ \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}.$$

Remark 4.2 If G is torsionfree, then

$$\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z} = \mathbb{Z}.$$

Lemma 4.3 *Let G be a group. Then the following statements are equivalent:*

1. *For any cocompact free proper G-manifold M without boundary we have*
$$b_p^{(2)}(M;\mathcal{N}(G)) \;\in\; \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z};$$

2. *For any cocompact free proper G-CW-complex X we have*
$$b_p^{(2)}(X;\mathcal{N}(G)) \;\in\; \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z};$$

3. *For any finitely presented $\mathbb{Q}G$-module M*
$$\dim_{\mathcal{N}(G)}(\mathcal{N}(G)\otimes_{\mathbb{Q}G}M) \;\in\; \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z};$$

4. *The Atiyah Conjecture 4.1 is true for G.*

Remark 4.4 Atiyah asked originally the following question. Let $G \to \overline{M} \to M$ be a G-covering of a closed Riemannian manifold M. Is then

$$b_p^{(2)}(M) \;=\; \lim_{t\to\infty} \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{C}}(e^{-t\Delta_p}(x,x))\; dvol$$

a rational number?

Remark 4.5 The Farrell-Jones Conjecture for $K_0(\mathbb{Q}G)$ says that the canonical map

$$\mathrm{colim}_{H\subseteq G,|H|<\infty} K_0(\mathbb{Q}H) \to K_0(\mathbb{Q}G)$$

is bijective. Surjectivity of this map implies for any finitely generated projective $\mathbb{Q}G$-module M that

$$\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{Q}G} M) \in \frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}.$$

The Atiyah-Conjecture requires this for all finitely presented $\mathbb{C}G$-modules.

Conjecture 4.6 Kaplanski Conjecture)
The Kaplanski Conjecture for a torsionfree group G and a field F says that the group ring FG has no non-trivial zero-divisors.

Lemma 4.7 *The Kaplanski Conjecture holds for G and the field $\mathbb{Q}$ if the Atiyah Conjecture 4.1 holds for G.*

Proof: Let $x \in \mathbb{Q}G$ be a zero-divisor. Let $r_x^{(2)} \colon l^2(G) \to l^2(G)$ be given by right multiplication with x. We get

$$0 < \dim_{\mathcal{N}(G)}\left(\ker(r_x^{(2)})\right) \leq 1$$

Since by assumption $\dim_{\mathcal{N}(G)}\left(\ker(r_x^{(2)})\right) \in \mathbb{Z}$, we conclude

$$\dim_{\mathcal{N}(G)}\left(\ker(r_x^{(2)})\right) = 1.$$

Since $\ker(r_x^{(2)})$ is closed in $l^2(G)$, we conclude $\ker(r_x^{(2)}) = l^2(G)$ and hence $x = 0$.
$\square$

Definition 4.8 *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed union and extensions with elementary amenable quotients.*

Theorem 4.9 (Linnell)
Let G be a group such that there is an upper bound on the orders of finite subgroups and G belongs to $\mathcal{C}$. Then the Atiyah Conjecture 4.1 holds for G.

Remark 4.10 Schick has enlarged the class of torsionfree groups, for which the Atiyah Conjecture is true, considerably by approximations techniques. For instance the Atiyah Conjecture is known for all torsionfree groups which are residually elementary amenable torsionfree.

The **lamplighter group** L is defined by the semidirect product

$$L := \oplus_{n\in\mathbb{Z}}\mathbb{Z}/2 \rtimes \mathbb{Z}$$

with respect to the shift automorphism of $\oplus_{n\in\mathbb{Z}}\mathbb{Z}/2$, which sends $(x_n)_{n\in\mathbb{Z}}$ to $(x_{n-1})_{n\in\mathbb{Z}}$. Let $e_0 \in \oplus_{n\in\mathbb{Z}}\mathbb{Z}/2$ be the element whose entries are all zero except the entry at 0. Denote by $t \in \mathbb{Z}$ the standard generator of $\mathbb{Z}$.Then $\{e_0t, t\}$ is a set of generators for L. The associate **Markov operator** $M\colon l^2(G) \to l^2(G)$ is given by right multiplication with $\frac{1}{4} \cdot (e_0t + t + (e_0t)^{-1} + t^{-1})$. It is related to the Laplace operator $\Delta_0\colon l^2(G) \to l^2(G)$ of the Cayley graph of G by $\Delta_0 = 4 \cdot \mathrm{id} - 4 \cdot M$.

Theorem 4.11 Grigorchuk-Linnell-Schick-Żuk

The von Neumann dimension of the kernel of the Markov operator M of the lamplighter group L associated to the set of generators $\{e_0t, t\}$ is $1/3$. In particular L does not satisfy the Atiyah Conjecture 4.1.

Remark 4.12 No counterexample to the Atiyah Conjecture 4.1 is known if one replaces $\frac{1}{|\mathcal{FIN}(G)|}\mathbb{Z}$ by $\mathbb{Q}$ or if one assumes that there is a bound on the orders of finite subgroups of G.

Conjecture 4.13 (Singer Conjecture)
If M is an aspherical closed manifold, then

$$b_p^{(2)}(\widetilde{M}) \;=\; 0 \quad \textit{if } 2p \neq \dim(M).$$

If M is a closed Riemannian manifold with negative sectional curvature, then

$$b_p^{(2)}(\widetilde{M}) \begin{cases} = 0 & \textit{if } 2p \neq \dim(M); \\ > 0 & \textit{if } 2p = \dim(M). \end{cases}$$

Because of the Euler-Poincaré formula

$$\chi(M) = \sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(\widetilde{M})$$

the Singer Conjecture 4.13 implies the following conjecture provided that M is aspherical or has negative sectional curvature.

Conjecture 4.14 (Hopf Conjecture) *If M is an aspherical closed manifold of even dimension, then*

$$(-1)^{\dim(M)/2} \cdot \chi(M) \geq 0.$$

If M is a closed Riemannian manifold of even dimension with sectional curvature $\sec(M)$*, then for $\epsilon = (-1)^{\dim(M)/2}$*

$$\begin{array}{rclcrcl}
\epsilon \cdot \chi(M) & > & 0 & \textit{if} & \sec(M) & < & 0; \\
\epsilon \cdot \chi(M) & \geq & 0 & \textit{if} & \sec(M) & \leq & 0; \\
\chi(M) & = & 0 & \textit{if} & \sec(M) & = & 0; \\
\chi(M) & \geq & 0 & \textit{if} & \sec(M) & \geq & 0; \\
\chi(M) & > & 0 & \textit{if} & \sec(M) & > & 0.
\end{array}$$

Remark 4.15 Under certain negative pinching conditions the Singer Conjecture has been proved by Ballmann-Brüning, Donelly-Xavier, Jost-Xin. Direct computations show that the Singer Conjecture 4.13 holds for a closed Riemannian manifold M if $\dim(M) \leq 3$ (assuming Thurston's Geometrization) or if M is a locally symmetric space or if M carries an S^1-action.

Definition 4.16 *A* **Kähler hyperbolic manifold** *is a closed connected Kähler manifold M whose fundamental form ω is $\widetilde{d}$(bounded), i.e. its lift $\widetilde{\omega} \in \Omega^2(\widetilde{M})$ to the universal covering can be written as $d(\eta)$ holds for some bounded 1-form $\eta \in \Omega^1(\widetilde{M})$.*

Theorem 4.17 (Gromov)
Let M be a closed Kähler hyperbolic manifold of complex dimension m and real dimension $n = 2m$. Then

$$\begin{array}{rcll} b_p^{(2)}(\widetilde{M}) & = & 0 & \text{if } p \neq m; \\ b_m^{(2)}(\widetilde{M}) & > & 0; & \\ (-1)^m \cdot \chi(M) & > & 0; & \end{array}$$

Remark 4.18 Let M be a closed Kähler manifold. It is Kähler hyperbolic if it admits some Riemannian metric with negative sectional curvature, or, if, generally $\pi_1(M)$ is word-hyperbolic and $\pi_2(M)$ is trivial. A consequence of Theorem 4.17 is that any Kähler hyperbolic manifold manifold is a projective algebraic variety

Let X be a topological space and let $C_*^{\mathrm{sing}}(X;\mathbb{R})$ be its singular chain complex with real coefficients. Let $S_p(X)$ be the set of all singular p-simplices. Then $C_p(X;\mathbb{R})$ is the real vector space with $S_p(X)$ as basis. Define the L^1-**norm** of an element $x \in C_p(X)$, which is given by the (finite) sum $\sum_{\sigma \in S_p(X)} \lambda_\sigma \cdot \sigma$, by

$$||x||_1 \; := \; \sum_\sigma |\lambda_\sigma|.$$

Define the L^1-**seminorm** of an element y in the p-th singular homology $H_p^{\mathrm{sing}}(X;\mathbb{R}) := H_p(C_*^{\mathrm{sing}}(X;\mathbb{R}))$ by

$$\begin{aligned} ||y||_1 \; := \; \inf\{||x||_1 \mid x \in C_p^{\mathrm{sing}}(X;\mathbb{R}), \\ \partial_p(x) = 0, y = [x]\}. \end{aligned}$$

Notice that $||y||_1$ defines only a semi-norm on $H_p^{\mathrm{sing}}(X;\mathbb{R})$, it is possible that $||y||_1 = 0$ but $y \neq 0$. The next definition is due to Gromov and Thurston.

Definition 4.19 *Let M be a closed connected orientable manifold of dimension n. Define its* **simplicial volume** *to be the non-negative real number*

$$||M|| := ||j([M])||_1 \quad \in [0, \infty)$$

for any choice of fundamental class $[M] \in H_n^{\mathrm{sing}}(M, \mathbb{Z})$ and $j\colon H_n^{\mathrm{sing}}(M;\mathbb{Z}) \to H_n^{\mathrm{sing}}(M;\mathbb{R})$ the change of coefficients map associated to the inclusion $\mathbb{Z} \to \mathbb{R}$.

Theorem 4.20 (Simplical volume of hyperbolic manifolds)
Let M be a closed hyperbolic orientable manifold of dimension n. Let v_n be the volume of the regular ideal simplex in $\overline{\mathbb{H}^n}$. Then

$$||M|| = \frac{\mathrm{vol}(M)}{v_n}.$$

Example 4.21 We have $||S^2|| = ||T^2|| = 0$. Let F_g be the closed connected orientable surface of genus $g \geq 1$. Then

$$||F_g|| = 2 \cdot |\chi(F_g)| = 4g - 4.$$

Definition 4.22 *Let M be a smooth manifold. Define its* **minimal volume** $\operatorname{minvol}(M)$ *to be the infimum over all volumes* $\operatorname{vol}(M, g)$*, where g runs though all complete Riemannian metrics on M, for which the sectional curvature satisfies* $|\sec(M, g)| \leq 1$*.*

Example 4.23 Obviously any closed flat Riemannian manifold has vanishing minimal volume. Hence we get

$$\operatorname{minvol}(T^n) = ||T^n|| = 0.$$

Let F_g be the closed orientable surface of genus $g \geq 2$, then

$$\begin{aligned} \operatorname{minvol}(F_g) &= 2\pi \cdot |\chi(F_g)| = 2\pi \cdot |2 - 2g| \\ &= \pi \cdot ||F_g|| \end{aligned}$$

essentially by the Gauss-Bonnet formula.

Notice that $||S^2|| = 0$ and $\operatorname{minvol}(S^2) \neq 0$.

We have

$$\begin{aligned} \operatorname{minvol}(\mathbb{R}^2) &= 2\pi(1 + \sqrt{2}); \\ \operatorname{minvol}(\mathbb{R}^n) &= 0 \qquad \text{for } n \geq 3. \end{aligned}$$

Theorem 4.24 (Gromov-Thurston)
Let M be a closed connected orientable Riemannian manifold of dimension n. Then

$$||M|| \leq (n-1)^n \cdot n! \cdot \text{minvol}(M).$$

Conjecture 4.25 (Simplical volume and L^2-invariants)
Let M be an aspherical closed oriented manifold of dimension ≥ 1. Suppose that its simplicial volume $||M||$ vanishes. Then

$$\begin{aligned} b_p^{(2)}(\widetilde{M}) &= 0 \quad \textit{for } p \geq 0; \\ \rho^{(2)}(\widetilde{M}) &= 0. \end{aligned}$$

Example 4.26 Let M be an aspherical closed orientable manifold. Then Conjecture 4.25 is true in the following cases:

- **S^1-actions**
  If M carries an S^1-action, then
  $$\begin{aligned} \operatorname{minvol}(M) &= 0; \\ ||M|| &= 0; \\ b_p^{(2)}(\widetilde{M}) &= 0 \quad \text{for all } p \geq 0; \end{aligned}$$

- **Amenable normal non-trivial subgroups**
  Let $H \subset \pi_1(M)$ be a normal infinite subgroup. If H is amenable, then
  $$\begin{aligned} ||M|| &= 0; \\ b_p^{(2)}(\widetilde{M}) &= 0 \quad \text{for all } p \geq 0; \end{aligned}$$
  If H is elementary amenable, then
  $$\rho^{(2)}(\widetilde{M}) = 0;$$

- **Selfmaps of degree $\neq -1, 0, 1$**
  If there is a selfmap $f \colon M \to M$ of degree $\deg(f)$ different from -1, 0, and 1, then
  $$||M|| = 0.$$

.

If any normal subgroup of finite index of $\pi_1(M)$ is Hopfian and there is a self-map $f\colon M \to M$ of degree $\deg(f)$ different from -1, 0, and 1, then for all $p \geq 0$

$$b_p^{(2)}(\widetilde{M}) = 0;$$

- **Dimension 3**
  Suppose that M has dimension 3 (and satisfies Thurston's Geometrization Conjecture). Then $||M|| = 0$ implies

$$\begin{aligned} b_p^{(2)}(\widetilde{M}) &= 0 \quad \text{for all } p \geq 0; \\ \rho^{(2)}(\widetilde{M}) &= 0; \end{aligned}$$

- **Vanishing of mininal volume**
  If the minimal volume minvol(M) of M is zero, then

$$\begin{aligned} ||M|| &= 0; \\ b_p^{(2)}(\widetilde{M}) &= 0 \quad \text{for all } p \geq 0. \end{aligned}$$

Definition 4.27 *Two countable groups* G_0 *and* G_1 *are called* **measure equivalent** *if there exist commuting measure-preserving free actions of* G_0 *and* G_1 *on some infinite Lebesgue measure space* (Ω, m) *such that the actions of both* G_0 *and* G_1 *admit finite measure fundamental domains.*

Remark 4.28 The notion of measure equivalence can be viewed as the measure theoretic analog of the metric notion of quasi-isometric groups. Namely, two finitely generated groups G_0 and G_1 are quasi-isometric if and only if there exist commuting proper (continuous) actions of G_0 and G_1 on some locally compact space such that each action has a compact fundamental domain

Remark 4.29 There are groups which are quasi-isometric but not measure equivalent, and vice versa. Put $G_n = (F_3 \times F_3) * F_n$. Then G_m and G_n are quasi-isometric for $n \geq 2$. Since $b_1^{(2)}(G_n) = n$ and $b_2^{(2)}(G_n) = 4$, they are measure equivalent if and only if $m = n$. The groups $\mathbb{Z}$ and $\mathbb{Z}^2$ have different growth rate and are hence not quasi-isometric. Since both are infinite amenable, they are measure equivalent by a result of Ornstein and Weiss.

Theorem 4.30 (L^2-Betti numbers and measure equivalence, Gaboriau)
Let G_0 and G_1 be two countable groups which are measure equivalent. Then there is a constant $C > 0$ such that for all $p \geq 0$

$$b_p^{(2)}(G_0) \;=\; C \cdot b_p^{(2)}(G_1).$$

Remark 4.31 The corresponding result of Gaboriau for quasi-isometry instead of measure equivalence is false. But the vanishing of the p-th L^2-Betti numbers is a quasi-isometry invariant by a result of Pansu.

Remark 4.32 Since any infinite amenable group of measure equivalent to $\mathbb{Z}$ by a result of Ornstein and Weiss and the L^2-Betti numbers of $\mathbb{Z}$ vanish, we rediscover the result that the L^2-Betti numbers of an infinite amenable group vanish.

Conjecture 4.33 Measure equivalence and L^2-torsion)
Let G_0 and G_1 be two countable groups which are measure equivalent. Suppose that there are finite models for BG_0 and BG_1 and $b_p^{(2)}(G_0) = b_p^{(2)}(G_1) = 0$ for $p \geq 0$. Then

$$\rho^{(2)}(EG_0) = 0 \Leftrightarrow \rho^{(2)}(EG_1) = 0.$$