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Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers II: Applications to Grothendieck groups, L^2-Euler characteristics and Burnside groups

The paper is dedicated to Martin Kneser on the occasion of his seventieth birthday

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Abstract. We continue the study of the generalized dimension function. We detect elements in the Grothendieck group $G_0(\mathbb{C}\Gamma)$ of finitely generated $\mathbb{C}\Gamma$-modules, provided that Γ is amenable. We investigate the class of groups for which the zero-th and first L^2-Betti numbers resp. all L^2-Betti numbers vanish. We study L^2-Euler characteristics and introduce for a discrete group Γ its Burnside group extending the classical notions of Burnside ring and Burnside ring congruences for finite Γ.

Introduction

In part one of this paper [21] we have defined for a finite von Neumann algebra $\mathscr{A}$ and an arbitrary module over $\mathscr{A}$ (just viewed as a ring), its *dimension*

(0.1) $$\dim(M) \in [0, \infty].$$

It extends the classical notion of von Neumann dimension which is a priori defined for finitely generated Hilbert $\mathscr{A}$-modules and thus for finitely generated projective $\mathscr{A}$-modules. This dimension inherits all the useful properties such as Additivity, Cofinality and Continuity and is uniquely characterized by these properties. This allows to define for any topological space with action of a discrete group Γ its p-th L^2-Betti number

(0.2) $$b_p^{(2)}(X; \mathscr{N}(\Gamma)) := \dim\big(H_p^{\Gamma}(X; \mathscr{N}(\Gamma))\big)$$

as the dimension of the homology of the $\mathcal{N}(\Gamma)$-chain complex $\mathcal{N}(\Gamma) \otimes_{Z\Gamma} C_*^{\mathrm{sing}}(X)$, where $C_*^{\mathrm{sing}}(X)$ is the singular chain complex of X and $\mathcal{N}(\Gamma)$ is the von Neumann algebra associated to Γ. For a group Γ we define

$$(0.3) \qquad b_p^{(2)}(\Gamma) = b_p\big(E\Gamma; \mathcal{N}(\Gamma)\big),$$

where $E\Gamma \to B\Gamma$ is the universal Γ-principal bundle. If X is the universal covering of a closed Riemannian manifold, this agrees with the definition of the p-th L^2-Betti number of Atiyah [1] in terms of the heat kernel on X

$$(0.4) \qquad b_p^{(2)}(X) = \lim_{t \to \infty} \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{R}}\big(e^{-t\Delta_p}(x,x)\big)\, d\mathrm{vol}_x \quad \in [0,\infty),$$

where $\mathcal{F}$ is a fundamental domain for the Γ-action $\Gamma = \pi_1(x)$. We have shown in [21] that for any infinite amenable group Γ, any $\mathbb{C}\Gamma$-module M and any Γ-space X

$$(0.5) \qquad \dim\big(\mathrm{Tor}_p^{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma))\big) = 0 \quad \text{for } p \geqq 1,$$

$$(0.6) \qquad b_p^{(2)}\big(X; \mathcal{N}(\Gamma)\big) = \dim\big(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} H_p^{\mathrm{sing}}(X)\big);$$

$$(0.7) \qquad b_p^{(2)}(\Gamma) = 0 \quad \text{for } p \geqq 0,$$

holds. In particular $b_p^{(2)}\big(X; \mathcal{N}(\Gamma)\big)$ depends only on the $\mathbb{Z}\Gamma$-module given by the p-th singular homology $H_p^{\mathrm{sing}}(X)$ and (0.7) is just the theorem of Cheeger and Gromov [6], Theorem 0.2 on page 191. The necessary ingredients of these results for the present paper will be reviewed in Section 1.

Equation (0.5) plays a crucial role in detecting non-trivial elements in the Grothendieck group $G_0(\mathbb{C}\Gamma)$ of finitely generated $\mathbb{C}\Gamma$-modules for amenable groups Γ which will be investigated in Section 2. We will construct for amenable Γ a map

$$(0.8) \qquad G_0(\mathbb{C}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathrm{class}(\Gamma)_{cf},$$

where $\mathrm{class}(\Gamma)_{cf}$ is the complex vector space of functions from the set $\mathrm{con}(\Gamma)_{cf}$ of finite conjugacy classes (γ) of elements in Γ to $\mathbb{C}$ (Lemma 2.3 and Theorem 2.12). This map is related to the Hattori-Stallings rank and the universal center-valued trace and dimension of $\mathcal{N}(\Gamma)$ (Theorem 2.12). In particular we will show that the class of $\mathbb{C}\Gamma$ in $G_0(\mathbb{C}\Gamma)$ generates an infinite cyclic subgroup in $G_0(\mathbb{C}\Gamma)$ if Γ is amenable and is trivial if Γ contains a free group of rank 2 as subgroup (Remark 2.23).

We will investigate for $d = 0, 1, \dots$ and $d = \infty$ the class $\mathcal{B}_d$ of groups Γ for which $b_p\big(E\Gamma; \mathcal{N}(\Gamma)\big) = 0$ for $p \leqq d$ holds (Theorem 3.3) and discuss applications in Section 3 (Theorem 3.2).

We analyse L^2-Euler characteristics and the Burnside group in Section 4 generalizing the classical notions of Burnside ring, Burnside ring congruences and equivariant Euler characteristic for finite groups to infinite groups (Theorem 4.4, Lemma 4.11, Lemma 4.13, Remark 4.14 and Lemma 4.17). In particular the L^2-Euler characteristic extends the notion of virtual Euler characteristic of a group to a larger class of groups and we get some vanishing results (Corollary 4.5).

In Section 5 we analyse the possible values of the L^2-Betti numbers (Theorem 5.2). If there is no bound on the orders of finite subgroups of Γ, then any non-negative real number can be realized as $b_p^{(2)}(X; \mathcal{N}(\Gamma))$ for $p \geqq 3$ and a free Γ-CW-complex X. Otherwise we show for the least common multiple d of the orders of finite subgroups that $d \cdot b_p^{(2)}(X; \mathcal{N}(\Gamma))$ is an integer or infinite for any Γ-space X if this holds for any finite free Γ-CW-complex Y (Theorem 5.2). The last condition holds for instance for elementary amenable groups and free groups Γ by Linnell [14].

1. Review of the generalized dimension function

We review some of the notions and results of the dimension function introduced in [21] for a finite von Neumann algebra $\mathcal{A}$ as far as needed here in order to keep this paper rather self-contained. Recall that the *dual module* M^* of a left $\mathcal{A}$-module is the left $\mathcal{A}$-module $\hom_A(M, \mathcal{A})$, where the $\mathcal{A}$-multiplication is given by $(af)(x) = f(x)a^*$ for $f \in M^*$, $x \in M$ and $a \in \mathcal{A}$. Let K be an $\mathcal{A}$-submodule of the $\mathcal{A}$-module M. Define the *closure of K in M* to be the $\mathcal{A}$-submodule of M

(1.1) $$\bar{K} := \{x \in M \mid f(x) = 0 \text{ for all } f \in M^* \text{ with } K \subset \ker(f)\}.$$

For a finitely generated $\mathcal{A}$-module M define the $\mathcal{A}$-submodule $\mathbf{T}M$ and the $\mathcal{A}$-quotient module $\mathbf{P}M$ by:

(1.2) $$\mathbf{T}M := \{x \in M \mid f(x) = 0 \text{ for all } f \in M^*\};$$

(1.3) $$\mathbf{P}M := M/\mathbf{T}M.$$

The notion of $\mathbf{T}M$ and $\mathbf{P}M$ corresponds in [12] to the torsion part and the projective part.

Theorem 1.4. (1) *$\mathcal{A}$ is semi-hereditary, i.e. any finitely generated submodule of a projective module is projective.*

(2) *If $K \subset M$ is a submodule of the finitely generated $\mathcal{A}$-module M, then $M/\bar{K}$ is finitely generated and projective and $\bar{K}$ is a direct summand in M.*

(3) *If M is a finitely generated $\mathcal{A}$-module, then $\mathbf{P}M$ is finitely generated projective and*

$$M \cong PM \oplus \mathbf{T}M.$$

(4) *The dimension* dim *has the following properties*:

(a) *Continuity.*

If $K \subset M$ is a submodule of the finitely generated $\mathscr{A}$-module M, then:

$$\dim(K) = \dim(\bar{K}).$$

(b) *Cofinality.*

Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of M, i.e. $M = \bigcup_{i \in I} M_i$ and for two indices i and j there is an index k in I satisfying $M_i, M_j \subset M_k$. Then:

$$\dim(M) = \sup\{\dim(M_i) \mid i \in I\}.$$

(c) *Additivity.*

If $0 \to M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \to 0$ is an exact sequence of $\mathscr{A}$-modules, then:

$$\dim(M_1) = \dim(M_0) + \dim(M_2),$$

where $r + s$ for $r, s \in [0, \infty]$ is the ordinary sum of two real numbers if both r and s are not ∞ and is ∞ otherwise.

(d) *Extension Property.*

If M is finitely generated projective, then $\dim(M)$ *is the von Neumann dimension of the finitely generated Hilbert $\mathscr{N}(\Gamma)$-module associated to M* (*see* [21], *Definition* 1.6 *and Theorem* 1.8.

(e) *If M is a finitely generated $\mathscr{A}$-module, then*:

$$\dim(M) = \dim(\mathbf{P}M);$$

$$\dim(\mathbf{T}M) = 0.$$

(f) *The dimension* dim *is uniquely determined by Continuity, Cofinality, Additivity and the Extension Property.* □

Notice that in view of Theorem 1.4 there are some similiarities between the ring $\mathbb{Z}$ of the integers and the ring $\mathscr{A}$ given by a finite von Neumann algebra. If one substitutes in the statements of Theorem 1.4 $\mathscr{A}$ by $\mathbb{Z}$ and requires in the Extension Property that $\dim(M)$ for a finitely generated abelian group is the usual rank, then all statements remain true and $\dim(M)$ becomes the dimension of the rational vector space $M \otimes_{\mathbb{Z}} \mathbb{Q}$. Moreover, $\mathbf{T}M$ of a finitely generated $\mathbb{Z}$-module is just its torsion submodule in the ordinary sense. However, there are two important differences. A finite von Neumann algebra is in general not Noetherian and hence harder to study than the Noetherian ring $\mathbb{Z}$. On the other hand the dimension of a finitely generated projective $\mathscr{A}$-module can be an arbitrary small positive real number and hence the dimension of a countable direct sum of non-trivial finitely generated projective $\mathscr{A}$-modules can be a finite number what can never happen over $\mathbb{Z}$ (see also [21], Remark 2.14).

2. Dimension functions and $G_0(\mathbb{C}\Gamma)$

Let $G_0(\mathbb{G}\Gamma)$ be the abelian group which has as set of generators the isomorphism classes of finitely generated (not necessarily projective) $\mathbb{C}\Gamma$-modules and has for each exact sequence of finitely generated $\mathbb{C}\Gamma$-modules $0 \to M_0 \to M_1 \to M_2 \to 0$ the relation $[M_0] - [M_1] + [M_2] = 0$. Given a finitely generated $\mathbb{C}\Gamma$-module M, the $\mathcal{N}(\Gamma)$-module $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M$ is a finitely generated $\mathcal{N}(\Gamma)$-module. We have defined $\mathbf{T}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M$ and $\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M$ in (1.2) and (1.3). Recall from Theorem 1.4 (3) that $\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M$ is a finitely generated projective $\mathcal{N}(\Gamma)$-module. Define maps

(2.1) $$i : K_0(\mathbb{C}\Gamma) \to G_0(\mathbb{C}\Gamma), \qquad [P] \mapsto [P];$$

(2.2) $$k : K_0(\mathbb{C}\Gamma) \to K_0(\mathcal{N}(\Gamma)), \quad [P] \mapsto [\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} P].$$

Lemma 2.3. *If Γ is amenable, the map*

$$j : G_0(\mathbb{C}\Gamma) \to K_0(\mathcal{N}(\Gamma)), \quad [M] \mapsto [\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M]$$

is a well-defined homomorphism. The composition $j \circ i$ agrees with k for the maps i and k defined in (2.1) *and* (2.2) *above.*

Proof. If $0 \to M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \to 0$ is an exact sequence of finitely generated $\mathbb{C}\Gamma$-modules we have to check in $K_0(\mathcal{N}(\Gamma))$

$$[\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_0] - [\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_1] + [\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_2] = 0.$$

Consider the induced sequence

$$\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_0 \xrightarrow{\bar{i}} \mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_1 \xrightarrow{\bar{p}} \mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_2.$$

Obviously $\bar{p}$ is surjective as p is surjective. We conclude from Theorem 1.4 (1) that $\ker(\bar{i})$ and $\ker(\bar{p})$ are finitely generated projective $\mathcal{N}(\Gamma)$-modules. Theorem 1.4 (4) and (0.5) imply

$$\dim_{\mathcal{N}(\Gamma)}(\ker(\bar{i})) = 0;$$
$$\dim_{\mathcal{N}(\Gamma)}(\operatorname{im}(\bar{i})) = \dim_{\mathcal{N}(\Gamma)}(\ker(\bar{p})).$$

Notice that a finitely generated projective $\mathcal{A}$-module is trivial if and only if its dimension is zero. We conclude from Theorem 1.4 that $\bar{i} : \mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_0 \to \ker(\bar{p})$ is a weak isomorphism, i.e. its kernel is trivial and $\overline{\operatorname{im}(\bar{i})} = \ker(\bar{p})$. Since the functor ν of [21], Theorem 1.8 respects weak exactness and the Polar Decomposition Theorem applied to a weak isomorphism has an isomorphism as unitary part, $\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_0$ and $\ker(\bar{p})$ are isomorphic as $\mathcal{N}(\Gamma)$-modules. Since $\ker(\bar{p}) \oplus \mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_2$ and $\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_1$ are isomorphic, Lemma 2.3 follows. □

If we regard $\hom_{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma))$ as left $\mathcal{N}(\Gamma)$-module by $(af)(x) = f(x) \cdot a^*$ for $a \in \mathcal{N}(\Gamma)$, $f \in \hom_{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma))$ and $x \in M$, we obtain isomorphisms of $\mathcal{N}(\Gamma)$-modules

$$\hom_{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma)) \xrightarrow{\simeq} (\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M)^*;$$
$$(\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M)^* \xrightarrow{\simeq} (\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M)^*.$$

Since for a finitely generated projective $\mathcal{N}(\Gamma)$-module P its dual P^* is isomorphic to P, we conclude for a finitely generated $\mathbb{C}\Gamma$-module M

$$j([M]) = [\hom_{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma))]. \tag{2.4}$$

For a finitely generated projective $\mathcal{N}(\Gamma)$-module P let

$$\dim^u_{\mathcal{N}(\Gamma)}(P) \in \operatorname{cent}(\mathcal{N}(\Gamma)) \tag{2.5}$$

be its *center-valued von Neumann dimension* which is given in terms of the universal center-valued trace $\operatorname{tr}^u_{\mathcal{N}(\Gamma)}$ [13], Theorem 8.2.8 on page 517, Proposition 8.3.10 on page 525 and Theorem 8.4.3 on page 532, [19], section 3. The center-valued von Neumann dimension is additive under direct sums and two finitely generated projective $\mathcal{N}(\Gamma)$-modules P and Q are isomorphic if and only if $\dim^u_{\mathcal{N}(\Gamma)}(P) = \dim^u_{\mathcal{N}(\Gamma)}(Q)$. We obtain an injection

$$\begin{aligned} \dim^u_{\mathcal{N}(\Gamma)}: K_0(\mathcal{N}(\Gamma)) &\to \operatorname{cent}(\mathcal{N}(\Gamma))^+ \\ &= \{a \in \operatorname{cent}(\mathcal{N}(\Gamma)) \mid a = bb^* \text{ for } b \in \mathcal{N}(\Gamma)\}, \end{aligned} \tag{2.6}$$

which is an isomorphism if $\mathcal{N}(\Gamma)$ is of type II, for instance if Γ is finitely generated and does not contain an abelian subgroup of finite index ([19], Corollary 3.2 and Lemma 3.3).

Next we investigate the relationship between $K_0(\mathbb{C}\Gamma)$ and $G_0(\mathbb{C}\Gamma)$ and between $\dim^u_{\mathcal{N}(\Gamma)}$ and the Hattori-Stallings rank. Let $\operatorname{con}(\Gamma)$ be the set of conjugacy classes of elements in Γ. Let $\operatorname{con}(\Gamma)_f$ be the subset of $\operatorname{con}(\Gamma)$ of conjugacy classes (γ) for which each representative γ has finite order. Let $\operatorname{con}(\Gamma)_{cf}$ be the subset of $\operatorname{con}(\Gamma)$ of conjugacy classes (γ) which contain only finitely many elements. We denote by $\operatorname{class}_0(\Gamma)$ and $\operatorname{class}_0(\Gamma)_f$ respectively the complex vector space with the set $\operatorname{con}(\Gamma)$ and $\operatorname{con}(\Gamma)_f$ respectively as basis. We denote by $\operatorname{class}(\Gamma)$ and $\operatorname{class}(\Gamma)_{cf}$ respectively the complex vector space of functions from the set $\operatorname{con}(\Gamma)$ and $\operatorname{con}(\Gamma)_{cf}$ respectively to $\mathbb{C}$. Notice that $\operatorname{class}_0(\Gamma)$ is the complex vector space of class functions from Γ to $\mathbb{C}$ with finite support. Define the *universal $\mathbb{C}\Gamma$-trace* of $\sum_{\gamma \in \Gamma} \lambda_\gamma \gamma \in \mathbb{C}\Gamma$ by

$$\operatorname{tr}^u_{\mathbb{C}\Gamma}\left(\sum_{\gamma \in \Gamma} \lambda_\gamma \gamma\right) := \sum_{\gamma \in \Gamma} \lambda_\gamma \cdot (\gamma) \quad \in \operatorname{class}_0(\Gamma), \tag{2.7}$$

This extends to square matrices in the usual way

$$\operatorname{tr}^u_{\mathbb{C}\Gamma}: M(n, n, \mathbb{C}\Gamma) \to \operatorname{class}_0(\Gamma), \quad A \mapsto \sum_{i=1}^{n} \operatorname{tr}^u_{\mathbb{C}\Gamma}(a_{i,i}). \tag{2.8}$$

Let P be a finitely generated projective $\mathbb{C}\Gamma$-module. Define its *Hattori-Stallings rank* by

$$\operatorname{HS}(P) := \operatorname{tr}^u_{\mathbb{C}\Gamma}(A) \quad \in \operatorname{class}_0(\Gamma), \tag{2.9}$$

where A is any element in $M(n,n,\mathbb{C}\Gamma)$ with $A^2 = A$ such that the image of the map $\mathbb{C}\Gamma^n \to \mathbb{C}\Gamma^n$ given by right multiplication with A is $\mathbb{C}\Gamma$-isomorphic to P. This definition is independent of the choice of A. The Hattori-Stallings rank defines a homomorphism

$$\text{(2.10)} \qquad \mathrm{HS}: K_0(\mathbb{C}\Gamma) \to \mathrm{class}_0(\Gamma), \quad [P] \mapsto \mathrm{HS}(P).$$

Define a homomorphism

$$\text{(2.11)} \qquad \phi: \mathrm{cent}\big(\mathcal{N}(\Gamma)\big) \to \mathrm{class}(\Gamma)_{cf}$$

by assigning to $u \in \mathrm{cent}\big(\mathcal{N}(\Gamma)\big)$

$$\phi(u): \mathrm{con}(\Gamma)_{cf} \to \mathbb{C}, \quad (\delta) \mapsto \mathrm{tr}_{\mathcal{N}(\Gamma)}\left(u \cdot \sum_{\delta' \in (\delta)} (\delta')^{-1}\right).$$

Theorem 2.12. *Suppose that Γ is amenable. Then the following diagram commutes*:

$$\begin{array}{ccccccc} K_0(\mathbb{C}\Gamma) & & & \xrightarrow{\mathrm{HS}} & & & \mathrm{class}_0(\Gamma) \\ {\scriptstyle i}\downarrow & & & & & & {\scriptstyle r}\downarrow \\ G_0(\mathbb{C}\Gamma) & \xrightarrow{j} & K_0\big(\mathcal{N}(\Gamma)\big) & \xrightarrow{\dim^u_{\mathcal{N}(\Gamma)}} & \mathrm{cent}\big(\mathcal{N}(\Gamma)\big) & \xrightarrow{\phi} & \mathrm{class}(\Gamma)_{cf} \end{array}$$

where r is given by restriction and the other maps have been defined in (2.1), *Lemma* 2.3, (2.6), (2.10) *and* (2.11).

Proof. One has to show for an element $A \in M(n,n,\mathbb{C}\Gamma)$ and $\delta \in \Gamma$ such that (δ) is finite

$$\text{(2.13)} \qquad \mathrm{tr}^u_{\mathbb{C}\Gamma}(A)(\delta) = \big(\phi \circ \mathrm{tr}^u_{\mathcal{N}(\Gamma)}(A)\big)(\delta).$$

It suffices to show for $\gamma \in \Gamma$ and $\delta \in \Gamma$ such that (δ) is finite

$$\text{(2.14)} \qquad \mathrm{tr}^u_{\mathbb{C}\Gamma}(\gamma)(\delta) = \big(\phi \circ \mathrm{tr}^u_{\mathcal{N}(\Gamma)}(\gamma)\big)(\delta).$$

The universal center-valued von Neumann trace satisfies for $\gamma \in \Gamma$

$$\mathrm{tr}^u_{\mathcal{N}(\Gamma)} = \begin{cases} |(\gamma)|^{-1} \cdot \sum_{\gamma' \in (\gamma)} \gamma' & \text{if } (\gamma) \text{ is finite}, \\ 0 & \text{otherwise}. \end{cases}$$

This follows from the facts that $\mathrm{tr}^u_{\mathcal{N}(\Gamma)}$ is ultraweakly continuous and the identity on the center of $\mathcal{N}(\Gamma)$ and that $\mathrm{tr}^u_{\mathcal{N}(\Gamma)}\left(\frac{1}{n} \cdot \sum_{i=1}^{n} \delta_i\right) = \mathrm{tr}^u_{\mathcal{N}(\Gamma)}(\delta)$ holds for elements $\delta_1, \delta_2, \ldots, \delta_n$ in (δ). Notice that $\mathrm{tr}_{\mathcal{N}(\Gamma)}(\delta)$ is 1 if $\delta = 1$ and 0 otherwise and that $\mathrm{tr}^u_{\mathbb{C}\Gamma}(\gamma)(\delta) = 0$ if (δ) is finite and (γ) is infinite. Hence (2.14) and thus (2.13) follow from the computation for $\gamma, \delta \in \Gamma$ such that (γ) and (δ) are finite

$$\begin{aligned}
\phi\big(|(\gamma)|^{-1}\cdot \sum_{\gamma'\in(\gamma)}\gamma'\big)(\delta) &= \operatorname{tr}_{\mathcal{N}(\Gamma)}\Bigg(\bigg(|(\gamma)|^{-1}\cdot \sum_{\gamma'\in(\gamma)}\gamma'\bigg)\cdot\bigg(\sum_{\delta'\in(\delta)}(\delta')^{-1}\bigg)\Bigg)\\
&= \sum_{\gamma'\in(\gamma)}\sum_{\delta'\in(\delta)}|(\gamma)|^{-1}\cdot \operatorname{tr}_{\mathcal{N}(\Gamma)}\big(\gamma'\cdot(\delta')^{-1}\big)\\
&= \sum_{\gamma'\in(\gamma),\,\delta'\in(\delta),\,\gamma'=\delta'}|(\gamma)|^{-1}\\
&= \begin{cases}1 & \text{if } (\gamma)=(\delta),\\ 0 & \text{otherwise}\end{cases}\\
&= \operatorname{tr}^{u}_{\mathbb{C}\Gamma}(\gamma)(\delta)\,.
\end{aligned}$$

This finishes the proof of Theorem 2.12. □

Lemma 2.15. *Let Γ be a discrete group. Then there is a commutative diagram whose the left vertical arrow is an isomorphism*:

$$\begin{array}{ccc}
\big(\operatorname{colim}_{\operatorname{Or}(\Gamma,\mathcal{FIN})}K_0(\mathbb{C}H)\big)\otimes_{\mathbb{Z}}\mathbb{C} & \xrightarrow{\ l\ } & K_0(\mathbb{C}\Gamma)\otimes_{\mathbb{Z}}\mathbb{C}\\
h\big\downarrow\cong & & \big\downarrow\operatorname{HS}\\
\operatorname{class}_0(\Gamma)_f & \xrightarrow[e]{} & \operatorname{class}_0(\Gamma)\,.
\end{array}$$

Proof. Firstly we explain the maps in the square. The colimit is taken for the covariant functor

$$\operatorname{Or}(G,\mathcal{FIN})\to \mathrm{ABEL},\quad G/H\mapsto K_0(\mathbb{C}H)$$

to the category of abelian groups which is given by induction. Here $\operatorname{Or}(G,\mathcal{FIN})$ is the full subcategory of the orbit category $\operatorname{Or}(G)$ consisting of objects G/H with finite H. The map l is induced by the universal property of the colimit and the various maps $K_0(\mathbb{C}H)\to K_0(\mathbb{C}\Gamma)$ induced by the inclusions of finite subgroups H of Γ in Γ. The map e is given by the inclusion $\operatorname{con}(\Gamma)_f\to\operatorname{con}(\Gamma)$.

Define for a group homomorphism $\psi:\Gamma\to\Gamma'$ a map $\psi_*:\operatorname{con}(\Gamma)\to\operatorname{con}(\Gamma')$ by sending (h) to $\big(\psi(h)\big)$. It induces a homomorphism $\psi_*:\operatorname{class}_0(\Gamma)\to\operatorname{class}_0(\Gamma')$. One easily checks that the following diagram commutes:

$$\begin{array}{ccc}
K_0(\mathbb{C}\Gamma) & \xrightarrow{\psi_*} & K_0(\mathbb{C}\Gamma')\\
\operatorname{HS}\big\downarrow & & \operatorname{HS}\big\downarrow\\
\operatorname{class}_0(\Gamma) & \xrightarrow{\psi_*} & \operatorname{class}_0(\Gamma')\,.
\end{array}\tag{2.16}$$

There is a canonical isomorphism

$$f_1:\big(\operatorname{colim}_{\operatorname{Or}(\Gamma,\mathcal{FIN})}K_0(\mathbb{C}H)\big)\otimes_{\mathbb{Z}}\mathbb{C}\xrightarrow{\cong}\operatorname{colim}_{\operatorname{Or}(\Gamma,\mathcal{FIN})}K_0(\mathbb{C}H)\otimes_{\mathbb{Z}}\mathbb{C}\,.\tag{2.17}$$

The Hattori-Stallings ranks for the various finite subgroups H of Γ induce an isomorphism

$$f_2:\operatorname{colim}_{\operatorname{Or}(\Gamma,\mathcal{FIN})}K_0(\mathbb{C}H)\otimes_{\mathbb{Z}}\mathbb{C}\xrightarrow{\cong}\operatorname{colim}_{\operatorname{Or}(\Gamma,\mathcal{FIN})}\operatorname{class}(H)\,.\tag{2.18}$$

Let $f_3' : \operatorname{colim}_{\mathrm{Or}(\Gamma, \mathscr{FIN})} \operatorname{con}(H) \to \operatorname{con}(\Gamma)_f$ be the map induced by the inclusions of the finite subgroups H of Γ. Define a map $f_4' : \operatorname{con}(\Gamma)_f \to \operatorname{colim}_{\mathrm{Or}(\Gamma, \mathscr{FIN})} \operatorname{con}(H)$ by sending $(\gamma) \in \operatorname{con}(\Gamma)_f$ to the image of $(\gamma) \in \operatorname{con}(\langle\gamma\rangle)$ under the canonical structure map from $\operatorname{con}(\langle\gamma\rangle)$ to $\operatorname{colim}_{\mathrm{Or}(\Gamma, \mathscr{FIN})} \operatorname{con}(H)$, where $\langle\gamma\rangle$ is the finite cyclic subgroup generated by γ. One easily checks that this is independent of the choice of the representative γ in (γ) and that f_3' and f_4' are inverse to one another. The bijection f_3' induces an isomorphism

$$f_3 : \operatorname{colim}_{\mathrm{Or}(\Gamma, \mathscr{FIN})} \operatorname{class}(H) \to \operatorname{class}(\Gamma)_f, \tag{2.19}$$

because colimit and the functor sending a set to the complex vector space with this set as basis commute. Now the isomorphism h is defined as the composition of the isomorphisms f_1 from (2.17), f_2 from (2.18) and f_3 from (2.19). It remains to check that the square in Lemma 2.15 commutes. This follows from the commutativity of (2.16). This finishes the proof of Lemma 2.15. □

Corollary 2.20. *Suppose that Γ is amenable. Then the image of the composition*

$$G_0(\mathbb{C}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{j \otimes_{\mathbb{Z}} \mathbb{C}} K_0(\mathscr{N}(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\dim^u_{\mathscr{N}(\Gamma)}} \operatorname{cent}(\mathscr{N}(\Gamma)) \xrightarrow{\phi} \operatorname{class}(\Gamma)_{cf}$$

contains the complex vector space $\operatorname{class}_0(\Gamma)_{f,cf}$ *with* $\operatorname{con}(\Gamma)_f \cap \operatorname{con}(\Gamma)_{cf}$ *as basis.* □

Remark 2.21. There is the conjecture that the canonical map

$$\operatorname{colim}_{\mathrm{Or}(\Gamma, \mathscr{FIN})} K_0(\mathbb{C}H) \xrightarrow{\cong} K_0(\mathbb{C}\Gamma)$$

is bijective for all groups Γ. In particular this would imply by Lemma 2.15 that the Hattori-Stallings rank induces an isomorphism

$$\mathrm{HS} : K_0(\mathscr{C}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \operatorname{class}_0(\Gamma)_f. \quad \square$$

Theorem 2.22. (1) *The map*

$$l : (\operatorname{colim}_{\mathrm{Or}(\Gamma, \mathscr{FIN})} K_0(\mathbb{C}H)) \otimes_{\mathbb{Z}} \mathbb{C} \to K_0(\mathbb{C}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C}$$

is injective.

(2) *If Γ is virtually polycyclic, then we obtain isomorphisms*

$$\mathrm{HS} : K_0(\mathbb{C}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \operatorname{class}_0(\Gamma)_f;$$
$$i : K_0(\mathbb{C}\Gamma) \xrightarrow{\cong} G_0(\mathbb{C}\Gamma).$$

Proof. (1) follows directly from Lemma 2.15.

(2) Moody has shown [23] that the obvious map $\bigoplus_{H \in \mathscr{FIN}} G_0(\mathbb{C}H) \to G_0(\mathbb{C}\Gamma)$ given by induction is surjective. Since Γ is virtually polycyclic the complex group ring $\mathbb{C}\Gamma$ is regular, i.e. noetherian and any $\mathbb{C}\Gamma$-module has a finite-dimensional projective resolution [24], Theorem 8.2.2 and Theorem 8.2.20. Now (1) and Lemma 2.15 prove the claim. □

Theorem 2.22 (2) has already been proven in [5].

Remark 2.23. In particular we get from Theorem 2.12 that the map

$$\iota : \mathbb{Z} \to G_0(\mathbb{C}\Gamma), \quad n \mapsto [\mathbb{C}\Gamma^n]$$

is injective, provided that Γ is amenable. It is likely that this property characterizes amenable groups. At least we can show for a group Γ which contains the free group F_2 in two letters as subgroup, that ι is trivial by the following argument.

Induction with the inclusion $F_2 \to \Gamma$ induces a homomorphism $G_0(\mathbb{C}F_2) \to G_0(\mathbb{C}\Gamma)$ which sends $[\mathbb{C}F_2]$ to $[\mathbb{C}\Gamma]$. Hence it suffices to show $[\mathbb{C}F_2] = 0$ in $G_0(\mathbb{C}F_2)$. The cellular chain complex of the universal covering of $S^1 \vee S^1$ yields an exact sequence of $\mathbb{C}\Gamma$-modules $0 \to (\mathbb{C}F_2)^2 \to \mathbb{C}F_2 \to \mathbb{C} \to 0$, where $\mathbb{C}$ is equipped with the trivial F_2-action. Hence it suffices to show $[\mathbb{C}] = 0$ in $G_0(\mathbb{C}F_2)$. Choose an epimorphism $f : F_2 \to \mathbb{Z}$. Restriction with f defines a homomorphism $G_0(\mathbb{C}\mathbb{Z}) \to G_0(\mathbb{C}F_2)$. It sends $\mathbb{C}$ viewed as trivial $\mathbb{C}\mathbb{Z}$-module to $\mathbb{C}$ viewed as trivial $\mathbb{C}F_2$-module. Hence it remains to show $[\mathbb{C}] = 0$ in $G_0(\mathbb{C}\mathbb{Z})$. This follows from the exact sequence $0 \to \mathbb{C}\mathbb{Z} \xrightarrow{s-1} \mathbb{C}\mathbb{Z} \to \mathbb{C} \to 0$ for s a generator of $\mathbb{Z}$. □

3. Groups with vanishing L^2-Betti numbers

In this section we investigate the following class of groups:

Definition 3.1. Define the class of groups

$$\mathscr{B}_d := \{\Gamma \mid b_p^{(2)}(\Gamma) = 0 \text{ for } 0 \leqq p \leqq d\};$$

$$\mathscr{B}_\infty := \{\Gamma \mid b_p^{(2)}(\Gamma) = 0 \text{ for } 0 \leqq p\}. \quad \square$$

Notice that $\mathscr{B}_0$ is the class of infinite groups by [21], Theorem 4.10. Definition 3.1 is motivated among other things by Corollary 4.5 and the following result.

Theorem 3.2. *Let $1 \to \Delta \to \Gamma \to \pi \to 1$ be an exact sequence of groups. Suppose that Γ is finitely presented and one of the following conditions is satisfied*:

(1) $|\Delta| = \infty$, $b_1^{(2)}(\Delta) < \infty$ *and π contains an element of infinite order or contains finite subgroups of arbitrary large order.*

(2) *The ordinary first Betti number of Δ satisfies $b_1(\Delta) < \infty$ and π belongs to $\mathscr{B}_1$.*

Then:

(1) *Let M be a closed oriented* 4*-manifold with Γ as fundamental group. Then*

$$|\operatorname{sign}(M)| \leqq \chi(M).$$

(2) *Let* $\operatorname{def}(\Gamma)$ *be the deficiency, i.e. the maximum* $g(P)-r(P)$ *for all presentations* P *where* $g(P)$ *is the number of generators and* $r(P)$ *the number of relations. Then*

$$\operatorname{def}(\Gamma) \leqq 1 .$$

Proof. If the first condition is satisfied, then Γ belongs to $\mathscr{B}_1$ by Theorem 3.3 (5). Now apply [18], Theorem 5.1 and Theorem 6.1 on page 212.

Suppose that the second condition is satisfied. Let $p: \bar{M} \to M$ be the regular covering associated to Δ. There is a universal coefficient spectral sequence converging to $H^{\pi}_{p+q}(\bar{M}; \mathcal{N}(\pi))$ with $E^2_{p,q} = \operatorname{Tor}^{\mathbb{C}\pi}_p(H_q(\bar{M}; \mathbb{C}), \mathcal{N}(\pi))$ [27], Theorem 5.6.4 on page 143. Since $H_q(\bar{M}; \mathbb{C})$ is $\mathbb{C}$ with the trivial π-action for $q=0$ and finite-dimensional as complex vector space by assumption for $q=1$, Theorem 1.4 (4) and [21], Lemma 3.4.3 imply $\dim(E^2_{p,q}) = 0$ for $p+q=1$ and hence $b^{(2)}_1(\bar{M}; \mathcal{N}(\pi)) = 0$. The arguments in [18], Theorem 5.1 and Theorem 6.1 on page 212 for the universal covering of M apply also to $\bar{M}$. □

The idea to take another covering than the universal covering in the proof of Theorem 3.2 is taken from [10], Corollary 5.2 on page 391. More information about results like Theorem 3.2 can be found in [11].

Theorem 3.3. *Let* d *be a non-negative integer or* $d = \infty$. *Then*:

(1) *The class* $\mathscr{B}_\infty$ *contains all infinite amenable groups.*

(2) *If* Γ *contains a normal subgroup* Δ *with* $\Delta \in \mathscr{B}_d$, *then* $\Gamma \in \mathscr{B}_d$.

(3) *If* Γ *is the union of a directed system of subgroups* $\{\Gamma_i \mid i \in I\}$ *such that each* Γ_i *belongs to* $\mathscr{B}_d$, *then* $\Gamma \in \mathscr{B}_d$.

(4) *Let* $1 \to \Delta \xrightarrow{i} \Gamma \xrightarrow{p} \pi \to 1$ *be an exact sequence of groups such that* $b^{(2)}_p(\Delta)$ *is finite for all* $p \leqq d$. *Suppose that* $B\pi$ *has finite* d-*skeleton and that there is an injective endomorphism* $j: \pi \to \pi$ *whose image has finite index, but is not equal to* π (*for example* $\pi = \mathbb{Z}^n$). *Then* $\Gamma \in \mathscr{B}_d$.

(5) *Let* $1 \to \Delta \xrightarrow{i} \Gamma \xrightarrow{p} \pi \to 1$ *be an exact sequence of groups such that* $\Delta \in \mathscr{B}_{d-1}$, $b^{(2)}_d(\Delta) < \infty$ *and* π *contains an element of infinite order or a finite subgroup of arbitrary large order. Then* $\Gamma \in \mathscr{B}_d$.

(6) *Suppose that there are groups* Γ_1 *and* Γ_2 *and group homomorphisms* $\phi_i: \Gamma_0 \to \Gamma_i$ *for* $i=1,2$ *such that* ϕ_0 *and* ϕ_1 *are injective,* Γ_0 *belongs to* $\mathscr{B}_{d-1}$, Γ_1 *and* Γ_2 *belong to* $\mathscr{B}_d$ *and* Γ *is the amalgamated product* $\Gamma_1 *_{\Gamma_0} \Gamma_2$ *with respect to* ϕ_1 *and* ϕ_2. *Then* Γ *belongs to* $\mathscr{B}_d$.

Proof. (1) follows from 0.7 or [6], Theorem 0.2 on page 191.

(2) We obtain a fibration $B\Delta \to B\Gamma \to B\pi$ for $\pi = \Gamma/\Delta$. There is the Leray-Serre spectral sequence converging to

$$H^{\Gamma}_{p+q}(E\Gamma, \mathcal{N}(\Gamma)) \quad \text{with} \quad E^1_{p,q} = H^{\Delta}_q(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{Z}\pi} C_p(E\pi)$$

for an appropriate $\mathbb{Z}\pi$-action on $H_q^{\triangle}(E\Delta;\mathcal{N}(\Gamma))$ coming from the fiber transport. Because of Additivity (see Theorem 1.4 (4)) it suffices to show for $p+q \leqq d$

$$\dim_{\mathcal{N}(\Gamma)}(E^1_{p,q}) = 0\,. \tag{3.4}$$

Since $b_q^{(2)}(\Delta) = \dim_{\mathcal{N}(\Gamma)}\big(H_q^{\triangle}(E\Delta;\mathcal{N}(\Gamma))\big)$ by [21], Theorem 4.9, and $C_p(E\pi)$ is a direct sum of copies of $\mathbb{Z}\pi$, Cofinality (see Theorem 1.4 (4)) proves (3.4).

(3) Using for instance the bar-resolution model for $E\Gamma$, one gets that $E\Gamma$ is the colimit of a directed system of subspaces of the form $E\Gamma_i \times_{\Gamma_i} \Gamma$ directed by I. Hence

$$H_p^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma)) = \operatorname{colim}_{i\in I} H_p^{\Gamma}(E\Gamma_i \times_{\Gamma_i} \Gamma;\mathcal{N}(\Gamma))\,.$$

Since $\dim_{\mathcal{N}(\Gamma)}\big(H_p^{\Gamma}(E\Gamma_i \times_{\Gamma_i} \Gamma;\mathcal{N}(\Gamma))\big) = b_p^{(2)}(\Gamma_i)$ by [21], Theorem 4.9 the claim follows from [21], Theorem 2.9.

(4) Fix an integer $n \geqq 1$. Put $\Gamma' = p^{-1}(\operatorname{im}(j^n))$. If k is the index of $\operatorname{im}(j)$ in π, then k^n is the index of $\operatorname{im}(j^n)$ in π and of Γ' in Γ and we conclude

$$b_p^{(2)}(\Gamma) = \frac{b_p^{(2)}(\Gamma')}{k^n}\,. \tag{3.5}$$

Since $\operatorname{im}(j^n)$ is isomorphic to π, we have an exact sequence $1 \to \Delta \to \Gamma' \to \pi \to 1$. Let i_p be the number of p-cells in $B\pi$. We get from the Leray-Serre spectral sequence and Additivity (see Theorem 1.4)

$$b_p^{(2)}(\Gamma') \leqq \sum_{n=0}^{p} b_q^{(2)}(\Delta)\cdot i_{p-q}\,. \tag{3.6}$$

Equations (3.5) and (3.6) imply

$$b_p^{(2)}(\Gamma) = \frac{\sum_{q=0}^{p} b_q^{(2)}(\Delta)\cdot i_{p-q}}{k^n}\,. \tag{3.7}$$

Since $k > 1$ and (3.7) holds for all $n \geqq 1$ and $\sum_{q=0}^{p} b_q^{(2)}(\Delta)\cdot i_{p-q}$ is finite for $p \leqq d$ by assumption, the claim follows.

(5) Using the spectral sequence which converges to $H_{p+q}^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma))$ and has an E^2-term $E^2_{p,q} = H_q^{\pi}\big(E\pi; H_q^{\triangle}(E\Delta;\mathcal{N}(\Gamma))\big)$ the proof of assertion (5) is reduced to the proof of

$$\dim_{\mathcal{N}(\Gamma)}\Big(H_0^{\pi}\big(E\pi; H_d^{\triangle}(E\Delta;\mathcal{N}(\Gamma))\big)\Big) = 0\,, \tag{3.8}$$

since $\dim_{\mathcal{N}(\Gamma)}\big(H_q^{\triangle}(E\Delta;\mathcal{N}(\Gamma))\big) = b_q^{(2)}(\Delta)$ by [21], Theorem 4.9 and hence vanishes for $q < d$ by assumption. Let $\pi' \subset \pi$ be a subgroup (not necessarily normal). Let $\Gamma' \subset \Gamma$ be the preimage of π' under the canonical projection $\Gamma \to \pi$. Then we obtain an exact sequence $1 \to \Delta \to \Gamma' \to 1$. We have

$$H_0^{\pi'}(E\pi'; H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma'))) = H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma')) \otimes_{\mathbb{C}[\pi']} \mathbb{C};$$
$$H_0^{\pi}(E\pi; H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma))) = H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{C}[\pi]} \mathbb{C}.$$

Since $H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{C}[\pi]} \mathbb{C}$ is a quotient of $H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{C}[\pi']} \mathbb{C}$ we conclude from Additivity (see Theorem 1.4) and from [21], Theorem 4.9

$$\dim_{\mathcal{N}(\Gamma)}(H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{C}[\pi]} \mathbb{C}) \leqq \dim_{\mathcal{N}(\Gamma)}(H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{C}[\pi']} \mathbb{C});$$
$$\dim_{\mathcal{N}(\Gamma')}(H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma')) \otimes_{\mathbb{C}[\pi']} \mathbb{C}) = \dim_{\mathcal{N}(\Gamma)}(H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{C}[\pi']} \mathbb{C}).$$

This implies

$$\dim_{\mathcal{N}(\Gamma)}(H_0^{\pi}(E\pi; H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma)))) \leqq \dim_{\mathcal{N}(\Gamma')}(H_0^{\pi'}(E\pi'; H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma')))).$$

Hence (3.8) would follow if we can find for each $\varepsilon < 0$ a subgroup $\pi' \subset \pi$ satisfying

$$\dim_{\mathcal{N}(\Gamma')}(H_0^{\pi'}(E\pi'; H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma')))) \leqq \varepsilon. \tag{3.9}$$

We begin with the case where π' is $\mathbb{Z}$. From assertion 4 we conclude

$$\dim_{\mathcal{N}(\Gamma')}(H_p^{\Gamma'}(E\Gamma'; \mathcal{N}(\Gamma'))) = 0 \quad \text{for } p \leqq d. \tag{3.10}$$

The Leray-Serre spectral sequence associated to $1 \to \Delta \to \Gamma' \to \mathbb{Z} \to 1$ has an E^2-term which satisfies $E^2_{p,q} = 0$ for $q \neq 0, 1$ since $B\mathbb{Z}$ has the 1-dimensional model S^1. Since it converges to $H_{p+q}^{\Gamma'}(E\Gamma'; \mathcal{N}(\Gamma'))$, we conclude (3.9) for $\varepsilon = 0$ from (3.10) and Additivity (see Theorem 1.4). Now suppose π' is finite. Then we get

$$\begin{aligned}\dim_{\mathcal{N}(\Gamma')}(H_0^{\pi'}(E\pi'; H_d^{\triangle}(E\Delta; \mathcal{N}(\Gamma')))) &= \dim_{\mathcal{N}(\Gamma')}(H_d^{\Gamma'}(E\Gamma'; \mathcal{N}(\Gamma')))\\ &= b_d^{(2)}(\Gamma')\\ &= \frac{b_d^{(2)}(\Delta)}{|\pi'|}.\end{aligned}$$

If we can find π' with arbitrary large $|\pi'|$ we get (3.9).

(6) One easily checks using the Seifert-van Kampen Theorem, that there is a Γ-push out

$$\begin{array}{ccc}\Gamma \times_{\Gamma_0} E\Gamma_0 & \longrightarrow & \Gamma \times_{\Gamma_1} E\Gamma_1\\ \downarrow & & \downarrow\\ \Gamma \times_{\Gamma_2} E\Gamma_2 & \longrightarrow & E\Gamma\,.\end{array}$$

We conclude from [21], Theorem 4.9: $\dim(H_p^{\Gamma}(E\Gamma_i \times_{\Gamma_i} \Gamma; \mathcal{N}(\Gamma))) = b_p^{(2)}(\Gamma_i)$. Now the claim follows from Additivity (see Theorem 1.4 (4)) and the long exact homology sequence for $H_*^{\Gamma}(-, \mathcal{N}(\Gamma))$. This finishes the proof of Theorem 3.3. □

So far we have no example with negative answer to the following question and can give an affirmative answer in some special cases.

Question 3.11. Let $1 \to \Delta \to \Gamma \to \pi \to 1$ be an exact sequence such that $b_p^{(2)}(\Delta) < \infty$ for all $p \geqq 0$ and π belongs to $\mathscr{B}_\infty$. Does then Γ belong to $\mathscr{B}_\infty$? Using Theorem 3.3 one can prove this for instance if π is elementary amenable.

More generally, if $F \to E \to B$ is a fibration such that $b_p^{(2)}(\tilde{F}; \mathscr{N}(\pi_1(E))) < \infty$ and $b_p^{(2)}(\tilde{B}; \mathscr{N}(\pi_1(B))) = 0$ holds for $p \geqq 0$, does then $b_p^{(2)}(\tilde{E}; \mathscr{N}(\pi_1(E))) = 0$ hold for $p \geqq 0$? □

Remark 3.12. Compact 3-manifolds whose fundamental groups belong to $\mathscr{B}_\infty$ are characterized in [16], Proposition 6.5 on page 54. The generalized Singer-Conjecture says that for an aspherical closed manifold M all the L^2-Betti numbers of its universal covering vanish possibly except in the middle dimension. In particular it implies that the fundamental group of an aspherical closed odd-dimensional manifold belongs to $\mathscr{B}_\infty$. Thompson's group F belongs to $\mathscr{B}_\infty$ [19], Theorem 0.8. More information about the class $\mathscr{B}_1$ is given in [3]. □

4. L^2-Euler characteristics and the Burnside group

In this section we extend some of the results [6] about L^2-Euler characteristics and investigate the Burnside group of a discrete group Γ. This extends the classical notions of the Burnside ring, Burnside ring congruences and equivariant Euler characteristics for finite groups.

If X is a Γ-CW-complex, denote by $I(X)$ the set of its equivariant cells. For a cell $c \in I(X)$ let (Γ_c) be the conjugacy class of subgroups of Γ given by its orbit type and let $\dim(c)$ be its dimension. Denote by $|\Gamma_c|^{-1}$ the inverse of the order of any representative of (Γ_c), where $|\Gamma_c|^{-1}$ is to be understood to be zero if the order is infinite.

Definition 4.1. Let X be a (left) Γ-space and V be a $\mathscr{A}$-$\mathbb{Z}\Gamma$-module. Define

$$\begin{aligned} h(X;V) &:= \sum_{p \geqq 0} b_p^{(2)}(X;V) \quad \in [0,\infty]; \\ \chi^{(2)}(X;V) &:= \sum_{p \geqq 0} (-1)^p \cdot b_p^{(2)}(X;V) \quad \in \mathbb{R}, \quad \text{if } h(X;V) < \infty; \\ m(X) &:= \sum_{c \in I(X)} |\Gamma_c|^{-1} \in [0,\infty], \qquad \text{if } X \text{ is a } \Gamma\text{-}CW\text{-complex}. \quad \square \end{aligned}$$

The condition $h(X;V) < \infty$ ensures that the sum defining $\chi^{(2)}(X;V)$ converges and that $\chi^{(2)}(X;V)$ satisfies the usual additivity formula, i.e. for a Γ-CW-complex X with Γ-CW-subcomplexes X_0, X_1 and X_2 satisfying $X = X_1 \cup X_2$, $X_0 = X_1 \cap X_2$ and $h(X_k;V) < \infty$ for $k = 0,1,2$ one has

$$h(X;V) < \infty; \tag{4.2}$$

$$\chi^{(2)}(X;V) = \chi^{(2)}(X_1;V) + \chi^{(2)}(X_2;V) - \chi^{(2)}(X_0;V). \tag{4.3}$$

The next theorem generalizes [6], Theorem 0.3 on page 191.

Theorem 4.4. *Let and Y be Γ-CW-complexes such that $m(X) < \infty$ and $m(Y) < \infty$ holds. Then*:

(1)

$$h(X; \mathcal{N}(\Gamma)) < \infty;$$

$$\sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |\Gamma_c|^{-1} = \chi^{(2)}(X; \mathcal{N}(\Gamma)).$$

(2) *Suppose that Γ is amenable. Then*

$$\sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |\Gamma_c|^{-1} = \sum_{p \geqq 0} (-1)^p \cdot \dim \left(\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} H_p(X; \mathbb{C})\right),$$

where $H_p(X; \mathbb{C})$ is the cellular or the singular homology of X with complex coefficients. In particular $\sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |\Gamma_c|^{-1}$ depends only on the $\mathbb{C}\Gamma$-isomorphism class of the $\mathbb{C}\Gamma$-modules $H_n(X; \mathbb{C})$ for all $n \geqq 0$.

(3) *If for all $c \in I(X)$ the group Γ_c is finite or belongs to the class $\mathcal{B}_\infty$, then*

$$b_p^{(2)}(X; \mathcal{N}(\Gamma)) = b_p^{(2)}(E\Gamma \times X; \mathcal{N}(\Gamma)) \quad \text{for } p \geqq 0;$$

$$\chi^{(2)}(X; \mathcal{N}(\Gamma)) = \chi^{(2)}(E\Gamma \times X; \mathcal{N}(\Gamma));$$

$$\sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |\Gamma_c|^{-1} = \chi^{(2)}(E\Gamma \times X; \mathcal{N}(\Gamma)).$$

(4) *Suppose that $f: X \to Y$ is a Γ-equivariant map, such that the induced map $H_p(f; \mathbb{C})$ on the singular or cellular homology with complex coefficients is bijective. Suppose that for all $c \in I(X)$ and $c \in I(Y)$ the group Γ_c is finite or belongs to the class $\mathcal{B}_\infty$. Then*

$$\sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |\Gamma_c|^{-1} = \sum_{c \in I(Y)} (-1)^{\dim(c)} \cdot |\Gamma_c|^{-1}.$$

Proof. (1) Additivity and Cofinality (see Theorem 1.4) and [21], Lemma 3.4 (1) imply

$$\dim(C_p(X; \mathcal{N}(\Gamma))) = \sum_{c \in I(X), \dim(c) = p} |\Gamma_c|^{-1};$$

$$\dim(H_p^\Gamma(X; \mathcal{N}(\Gamma))) \leqq \dim(C_p(X; \mathcal{N}(\Gamma)));$$

$$\sum_{p \geqq 0} (-1)^p \cdot \dim(C_p(X; \mathcal{N}(\Gamma))) = \chi^{(2)}(X; \mathcal{N}(\Gamma)).$$

(2) follows from the first assertion and (0.6).

(3) Because of the first equation it suffices to prove that the dimension of the kernel and the cokernel of the map induced by the projection

$$\mathrm{pr}_*: H_p^\Gamma(E\Gamma \times X; \mathcal{N}(\Gamma)) \to H_p^\Gamma(X; \mathcal{N}(\Gamma))$$

are trivial. Notice that X is the colimit of its finite Γ-subcomplexes. Since $H_p^\Gamma(-,\mathcal{N}(\Gamma))$ is compatible with colimits and colimit preserves exact sequences, we can assume by [21], Theorem 2.9, and Additivity (see Theorem 1.4) that X itself is finite. By induction over the number of equivariant cells, the long exact homology sequence and Additivity (see Theorem 1.4) the claim reduces to the case where X is of the shape Γ/H. Because of [21], Theorem 4.9 it suffices to prove for the map $\mathrm{pr}_*: H_p^H(EH;\mathcal{N}(H)) \to H_p^H(\{*\};\mathcal{N}(H))$ that its kernel and cokernel have trivial dimension, provided that H is finite or belongs to $\mathcal{B}_\infty$. This is obvious for finite H and follows for $H \in \mathcal{B}_\infty$ from the definition of $\mathcal{B}_\infty$ and [21], Theorem 4.10.

(4) Since $E\Gamma \times X$ is free and the map $\mathrm{id} \times f: E\Gamma \times X \to E\Gamma \times Y$ induces an isomorphism on singular homology it induces an isomorphism

$$H_p^\Gamma(E\Gamma \times X;\mathcal{N}(\Gamma)) \to H_p^\Gamma(E\Gamma \times Y;\mathcal{N}(\Gamma))$$

and we conclude $\chi^2(E\Gamma \times X;\mathcal{N}(\Gamma)) = \chi^{(2)}(E\Gamma \times Y;\mathcal{N}(\Gamma))$. Now assertion (4) follows from assertion (3). This finishes the proof of Theorem 4.4. □

As explained in [6], Proposition 0.4 on page 192, the L^2-Euler characteristic extends the notion of the virtual Euler characteristic which is due to Wall [25]. Information about this notion can be found for instance in [4], chaper IX. The next result generalizes [6], Corollary 0.6 on page 193.

Corollary 4.5. *Let Γ be a group belonging to $\mathcal{B}_\infty$. Then $\chi^2(E\Gamma;\mathcal{N}(\Gamma))$ is defined and vanishes. If its virtual Euler characteristic $\chi_{\mathrm{virt}}(\Gamma)$ is defined, then it vanishes. In particular $\chi(B\Gamma)$ vanishes if $B\Gamma$ can be choosen to be a finite CW-complex.* □

Next we introduce the Burnside group and the equivariant Euler characteristic. The elementary proof the following lemma is left to the reader.

Lemma 4.6. *Let H and K be subgroups of Γ. Let NK be the normalizer of K in Γ and WK be NK/K. Then*:

(1) $\Gamma/H^K = \{g^{-1}Kg \subset H\}$.

(2) *The map*

$$\phi: \Gamma/H^K \to \mathrm{consub}(H), \quad gH \mapsto g^{-1}Kg$$

induces an injection

$$WK\backslash(\Gamma/H^K) \to \mathrm{consub}(H),$$

where consub(H) *is the set of conjugacy classes in H of subgroups of H.*

(3) *The WK-isotropy group of $gH \in \Gamma/H^K$ is $(gHg^{-1} \cap NK)/K \subset NK/K = WK$.*

(4) *If H is finite, then Γ/H^K is a finite union of WK-orbits of the shape WK/L for finite subgroups $L \subset WK$.* □

Definition 4.7. Define the *Burnside group* $A(\Gamma)$ by the Grothendieck group of the abelian monoid under disjoint union of Γ-isomorphism classes of proper cocompact Γ-sets S, i.e. Γ-sets S for which the isotropy group of each element in S and the quotient $\Gamma \backslash S$ are finite. □

Notice that $A(\Gamma)$ is the free abelian group generated by Γ-isomorphism classes of orbits Γ/H for finite subgroups $H \subset \Gamma$ and that Γ/H and Γ/K are Γ-isomorphic if and only if H and K are conjugate in Γ. If Γ is a finite group, $A(\Gamma)$ is the classical Burnside ring [8], section 5, [9], chapter IV. If Γ is infinite, then the cartesian product of two proper cocompact Γ-sets with the diagonal action is not cocompact any more so that the cartesian product does not induce a ring structure on $A(\Gamma)$. At least there is a bilinear map induced by the cartesian product $A(\Gamma_1) \otimes A(\Gamma_2) \to A(\Gamma_1 \times \Gamma_2)$.

Definition 4.8. Let X be a proper finite Γ-CW-complex. Define its *equivariant Euler characteristic*

$$\chi^\Gamma(X) := \sum_{c \in I(X)} (-1)^{\dim(c)} \cdot [\Gamma/\Gamma_c] \in A(\Gamma). \quad \square$$

An *additive invariant* (A, a) for proper finite Γ-CW-complexes X consists of an abelian group A and a function a which assigns to any proper finite Γ-CW-complex X an element $a(X) \in A$ such that the following three conditions hold, (i) if X and Y are Γ-homotopy equivalent, then $a(X) = a(Y)$, (ii) if X_0, X_1 and X_2 are Γ-CW-subcomplexes of X with $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$, then $a(X) = a(X_1) + a(X_2) - a(X_0)$, and (iii) $a(\emptyset) = 0$. We call an additive invariant (U, u) *universal*, if for any additive invariant (A, a) there is precisely one homomorphism $\psi: U \to A$ such that $\psi(u(X)) = a(X)$ holds for all proper finite Γ-CW-complexes. One easily checks using induction over the number of equivariant cells

Lemma 4.9. $(A(\Gamma), \chi^\Gamma)$ *is the universal additive invariant for finite proper Γ-CW-complexes.* □

Definition 4.10. Define for a finite subgroup $K \subset \Gamma$ the L^2-*character map*

$$\mathrm{ch}_K^\Gamma: A(\Gamma) \to \mathbb{Q}, \quad [S] \mapsto \sum_{i=1}^{r} |L_i|^{-1}$$

if $WK/L_1, WK/L_2, \ldots, WK/L_r$ are the WK-orbits of S^K. Define the *global L^2-character map* by

$$\mathrm{ch}^\Gamma := \prod_{(K)} \mathrm{ch}_K^\Gamma: A(\Gamma) \to \prod_{(K)} \mathbb{Q}$$

where (K) runs over the conjugacy classes of finite subgroups of Γ. □

Lemma 4.11. *Let X be a finite proper Γ-CW-complex and $K \subset \Gamma$ be a finite subgroup. Then X^K is a finite proper WK-CW-complex and*

$$\chi^{(2)}(X^K; \mathcal{N}(WK)) = \mathrm{ch}_K^\Gamma(\chi^\Gamma(X)).$$

Proof. The WK-space X^K is a finite proper WK-CW-complex because for finite $H \subset \Gamma$ the WK-set Γ/H^K is proper and cocompact by Lemma 4.6. Since the assignment which associates to a finite proper Γ-CW-complex X the element $\chi^{(2)}(X^K; \mathcal{N}(WK))$ in $\mathbb{Q}$ is an additive invariant, it suffices by Lemma 4.9 to check the claim for $X = \Gamma/H$ for finite $H \subset \Gamma$. Then the claim follows from the fact that $\chi^{(2)}(WK/L; \mathcal{N}(WK)) = |L|^{-1}$ holds for finite $L \subset WK$. □

Notice that one gets from Lemma 4.6 the following explicit formula for the value of $\mathrm{ch}_K^\Gamma(\Gamma/H)$. Namely, define

$$\mathcal{L}_K(H) := \{(L) \in \mathrm{consub}(H) \mid L \text{ conjugated to } K \text{ in } \Gamma\}.$$

For $(L) \in \mathcal{L}_K(H)$ choose $L \in (L)$ and $g \in L$ with $g^{-1}Kg = L$. Then

$$g(H \cap NL)g^{-1} = gHg^{-1} \cap NK;$$

$$|(gHg^{-1} \cap NK)/K|^{-1} = \frac{|K|}{|H \cap NL|}.$$

This implies

$$\mathrm{ch}_K^\Gamma(\Gamma/H) = \sum_{(L) \in \mathcal{L}_K(H)} \frac{|K|}{|H \cap NL|}. \tag{4.12}$$

Lemma 4.13. *The global L^2-character map of Definition* 4.10 *induces a map denoted by*

$$\mathrm{ch}^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q} : A(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \to \prod_{(K)} \mathbb{Q}.$$

It is injective. If Γ has only finitely many conjugacy classes of finite subgroups, then it is bijective.

Proof. Consider an element $\sum_{i=1}^{n} r_i \cdot [\Gamma/H_i]$ in the kernel of $\mathrm{ch}^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. We show by induction over n that the element must be trivial. The begin $n = 0$ is trivial, the induction step done as follows. We can choose the numeration such that H_i subconjugated to H_j implies $i \geqq j$. We get from (4.12)

$$\mathrm{ch}_K^\Gamma(\Gamma/H) = 1, \quad \text{if } H = K;$$

$$\mathrm{ch}_K^\Gamma(\Gamma/H) = 0, \quad \text{if } K \text{ is not subconjugated to } H \text{ in } \Gamma.$$

This implies

$$\mathrm{ch}_{H_1}^\Gamma\left(\sum_{i=1}^{n} r_i \cdot [\Gamma/H_i]\right) = r_1$$

and hence $r_1 = 0$. Hence the global L^2-character map is injective. If Γ has only finitely many conjugacy classes of finite subgroups, then the source and target of $\mathrm{ch}^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ are rational vector spaces of the same finite dimension and $\mathrm{ch}^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ must be bijective. □

Remark 4.14. Suppose that there are only finitely many conjugacy classes $(H_1), (H_2), \ldots, (H_r)$ of finite subgroups in Γ. Without loss of generality we can assume that

H_i subconjugated to H_j implies $i \geqq j$. With respect to the obvious ordered basis for the source and target the map $\mathrm{ch}^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ is described by an upper triangular matrix A with ones on the diagonal. One can get an explicit inverse A^{-1} which again has ones on the diagonal. This leads to a characterization of the image of $A(\Gamma)$ under the global L^2-character map χ^Γ. Namely, an element in $\eta \in \prod_{i=1}^{r} \mathbb{Q}$ lies in $\mathrm{ch}^\Gamma(A(\Gamma))$ if and only if the following *Burnside integrality conditions* are satisfied:

$$A^{-1}\eta \in \prod_{i=1}^{r} \mathbb{Z}\,. \tag{4.15}$$

Now suppose that Γ is finite. Then the global L^2-character map is related to the classical character map by the factor $|WK|^{-1}$, i.e. we have for each subgroup K of Γ and any finite Γ-set S

$$\mathrm{ch}_K^\Gamma(S) = |WK|^{-1} \cdot |S^K|\,. \tag{4.16}$$

One easily checks that under the identification (4.16) the integrality conditions (4.15) correspond to the classical Burnside ring congruences for finite groups [8], section 5.8, [9], section IV. 5. □

Let $E(\Gamma, \mathcal{FIN})$ be the classifying Γ-space for the family $\mathcal{FIN}$ of finite subgroups. This Γ-CW-complex is characterized up to Γ-homotopy by the property that its H-fixed point set is contractible if $H \subset \Gamma$ is finite and empty otherwise. It is also called the classifying space for proper Γ-spaces and denoted by $\underline{E}\Gamma$ in the literature. For more information about $E(\Gamma, \mathcal{FIN})$ we refer for instance to [2], [7], section 7, [9], section I. 6,

Lemma 4.17. *Suppose that there is a model for $E(\Gamma, \mathcal{FIN})$ which is a finite Γ-CW-complex. Then there are only finitely many conjugacy classes of finite subgroups and for a finite subgroup $K \subset \Gamma$*

$$\mathrm{ch}_K^\Gamma\big(\chi^\Gamma\big(E(\Gamma, \mathcal{FIN})\big)\big) = \chi^{(2)}(WK)\,.$$

If Γ is amenable, then we get for a finite subgroup $K \subset \Gamma$

$$\mathrm{ch}_K^\Gamma\big(\chi^\Gamma\big(E(\Gamma, \mathcal{FIN})\big)\big) = |WK|^{-1}\,,$$

where $|WK|^{-1}$ is to be understood as 0 *for infinite WK.*

Proof. We get from Lemma 4.11 since $E(\Gamma, \mathcal{FIN})^K$ is a model for $E(WK, \mathcal{FIN})$

$$\mathrm{ch}_K^\Gamma\big(\chi^\Gamma\big(E(\Gamma, \mathcal{FIN})\big)\big) = \chi^{(2)}\big(E(WK, \mathcal{FIN}); \mathcal{N}(WK)\big)\,.$$

Now apply Theorem 4.4 (3) and (0.7). □

Example 4.18. Let $1 \to \mathbb{Z}^n \to \Gamma \to \mathbb{Z}/p \to 1$ be an extension of groups for $n > 1$ and a prime number p. Then $E(\Gamma, \mathcal{FIN})$ can be choosen as a finite Γ-CW-complex because only the following cases can occur. If Γ contains a finite subgroup, then Γ is a semi-direct product of $\mathbb{Z}^n$ and $\mathbb{Z}/p$ and one can construct a finite Γ-CW-complex as model for

$E(\Gamma, \mathscr{FIN})$ with $\mathbb{R}^n$ as underlying space. If the group Γ contains no finite subgroup, one shows inductively over n that there is a finite model for $B\Gamma$. In the induction step use the fact that Γ can be written as an extension of a group for which the induction hypothesis applies and $\mathbb{Z}$. We want to compute $\chi^\Gamma\big(E(\Gamma, \mathscr{FIN})\big)$. The conjugation action of Γ on the normal subgroup $\mathbb{Z}^n$ factorizes through the projection $\Gamma \to \mathbb{Z}/p$ into an operation ϱ of $\mathbb{Z}/p$ onto $\mathbb{Z}^n$. If this operation has a non-trivial fixed point, then WH is infinite for any finite subgroup H of Γ and we conclude from Lemma 4.13 and Theorem 4.17 that

$$\chi^\Gamma\big(E(\Gamma, \mathscr{FIN})\big) = 0\,.$$

Now suppose that this operation ϱ has no non-trivial fixed points. Let H_0 be the trivial subgroup and $H_1, H_2, \ldots, H_r$ be a complete set of representatives of the conjugacy classes of finite subgroups. Each H_i is isomorphic to $\mathbb{Z}/p$. One easily checks that there is a bijection

$$H^1(\mathbb{Z}/p; \mathbb{Z}^n_\varrho) \to \{(H) \mid H \subset \Gamma, 1 < |H| < \infty\}$$

and in particular $r \geqq 1$, where $\mathbb{Z}^n_\varrho$ denotes the $\mathbb{Z}[\mathbb{Z}/p]$-module given by $\mathbb{Z}^n$ and ϱ. We compute using (4.12)

$$\begin{aligned}
\mathrm{ch}^\Gamma_{H_0}(!/\ H_0) &= 1\,; \\
\mathrm{ch}^\Gamma_{H_0}(\Gamma/H_j) &= \frac{1}{p}, \quad j = 1, 2, \ldots, r\,; \\
\mathrm{ch}^\Gamma_{H_i}(\Gamma/H_j) &= 1\,, \quad i = j, \quad i,j = 1, 2, \ldots, r\,; \\
\mathrm{ch}^\Gamma_{H_i}(\Gamma/H_j) &= 0\,, \quad i \neq j, \quad i,j = 1, 2, \ldots, r\,;
\end{aligned}$$

We conclude

$$\chi^\Gamma\big(E(\Gamma, \mathscr{FIN})\big) = -\frac{r}{p} \cdot [\Gamma/H_0] + \sum_{i=1}^{r} [\Gamma/H_i]\,.$$

The integrality conditions of (4.15) become in this case

$$\begin{aligned}
\eta_0 - \frac{1}{p} \cdot \sum_{i=1}^{r} \eta_i &\in \mathbb{Z}\,; \\
\eta_i &\in \mathbb{Z}\,, \quad i = 1, 2, \ldots, r\,. \quad \square
\end{aligned}$$

5. Values of L^2-Betti numbers

In this section we investigate the possible values of L^2-Betti numbers.

Conjecture 5.1. *Let Γ be a group and let X be a free finite Γ-CW-complex. Then*

$$b_p^{(2)}\big(X; \mathscr{N}(\Gamma)\big) \in \mathbb{Q}\,.$$

If d is a positive integer such that the order of any finite subgroup of Γ divides d, then

$$d \cdot b_p^{(2)}\big(X; \mathscr{N}(\Gamma)\big) \in \mathbb{Z}\,. \quad \square$$

The significance of Conjecture 5.1 and its relation to a question of Atiyah [1], page 72 about the rationality of the analytic L^2-Betti numbers of (0.4) are explained in [20], Section 2. Let $\mathscr{C}$ be the smallest class of groups which contains all free groups, is closed under directed unions and satisfies $G \in \mathscr{C}$ whenever G contains a normal subgroup H such that H belongs to $\mathscr{C}$ and G/H is elementary amenable. Conjecture 5.1 has been proven for groups $\Gamma \in \mathbb{C}$ by Linnell [14], provided that there is an upper bound on the order of finite subgroups of Γ.

Theorem 5.2. (1) *Let Γ be a group such that there is no bound on the order of finite subgroups. Then*:

(a) *Given $\beta \in [0, \infty]$, there is a countably generated projective $\mathbb{Z}\Gamma$-module P satisfying*

$$\dim_{\mathscr{N}(\Gamma)}\big(\mathscr{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} P\big) = \beta\,.$$

(b) *Given a sequence $\beta_3, \beta_4, \ldots$ of elements in $[0, \infty]$, there is a free Γ-CW-complex X satisfying*

$$b_p^{(2)}\big(X; \mathscr{N}(\Gamma)\big) = \beta_p \quad \textit{for } p \geqq 3\,.$$

If Γ is countably presented, one can arrange that X has countably many Γ-equivariant cells.

(2) *Let Γ be a group such that there is a bound on the order of finite subgroups. Let d be the least common multiple of the orders of finite subgroups of Γ. Suppose that Conjecture* 5.1 *holds for Γ. Then we get for any Γ-space X and $p \geqq 0$*

$$d \cdot b_p^{(2)}\big(X; \mathscr{N}(\Gamma)\big) \in \mathbb{Z} \cup \{\infty\}\,.$$

(3) *Given a sequence of elements $\beta_1, \beta_2, \ldots, [0, \infty]$, there is a countable group Γ with $b_p^{(2)}(\Gamma) = \beta_p$ for $p \geqq 1$. If β_1 is rational, Γ can be chosen to be finitely generated.*

Proof. (1) (a) Since there is no bound on the order of finite subgroups, we can find a sequence of finite subgroups $H_1, H_2, \ldots$ of Γ such that $\beta = \sum_{i=1}^{\infty} |H_i|^{-1}$. Then

$$P = \bigoplus_{i=1}^{\infty} \mathbb{C}[\Gamma/H_i]$$

is the desired module by [21], Lemma 3.4 (1) and Additivity and Cofinality (see Theorem 1.4 (4).

(1) (b) By assertion (1) (a) we can choose a sequence of countably generated projective $\mathbb{C}\Gamma$-modules $P_3, P_4, \ldots$ such that for $p \geqq 3$

$$\dim_{\mathscr{N}(\Gamma)}\big(\mathscr{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} P_p\big) = \beta_p\,. \tag{5.3}$$

Next we construct inductively a nested sequence $X_2 \subset X_3 \subset \ldots$ of Γ-CW-complexes together with Γ-retractions $r_p \colon X_p \to X_{p-1}$ for $p \geqq 3$ such that X_2 is the 2-skeleton of a model for $E\Gamma$, X_p is obtained from X_{p-1} by attaching countably many free Γ-equivariant p-cells and $p+1$-cells and

(5.4)
$$H_n(X_p, X_{p-1}) = \begin{cases} P_p, & n = p, \\ 0, & n \neq p. \end{cases}$$

The Eilenberg-swindle yields a split-exact sequence $0 \to C_{p+1} \xrightarrow{c_{p+1}} C_p \to P_p \to 0$ of $\mathbb{C}\Gamma$-modules such that C_{p+1} and C_p are countably generated free $\mathbb{Z}\Gamma$-modules with a basis. Now one attaches for each element of the basis of C_p trivially a free Γ-equivariant p-cell to X. Then one attaches for each element of the basis of C_{p+1} a free Γ-equivariant $p+1$-cell to X, where the attaching maps are choosen such that the cellular $\mathbb{C}\Gamma$-chain complex of (Y, X) is just the $\mathbb{Z}\Gamma$-chain complex which is concentrated in dimension $p+1$ and p and given there by $C_{p+1} \xrightarrow{c_{p+1}} C_p$. Details of the construction of the Γ-CW-complexes X_p and Γ-retractions r_p can be found in [22], Theorem 2.2, page 201, [26]. Now define $Y = \operatorname{colim}_{p \to \infty} X_p$. One easily checks for $p \geqq 3$

$$H_p^\Gamma(Y; \mathcal{N}(\Gamma)) = \mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} P_p;$$
$$b_p^{(2)}(Y; \mathcal{N}(\Gamma)) = \beta_p.$$

(2) Let $f: Y \to X$ be a Γ-CW-approximation of X [17], page 35, i.e. a Γ-CW-complex Y with a Γ-map f such that f^H is a weak homotopy equivalence and hence a weak homology equivalence [28], Theorem IV. 7.15 on page 182 for $H \subset \Gamma$. We get from [21], Lemma 4.8 (2) that $b_p^{(2)}(Y; \mathcal{N}(\Gamma)) = b_p^{(2)}(X; \mathcal{N}(\Gamma))$ holds for all $p \geqq 0$. Hence we can assume without loss of generality that X is a Γ-CW-complex.

Conjecture 5.1 is equivalent to the statement that for any finitely presented $\mathbb{Z}\Gamma$-module M

(5.5)
$$d \cdot \dim_{\mathcal{N}(\Gamma)}(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} M) \in \mathbb{Z}.$$

This follows essentially from [20], Lemma 2.2. Now let D_* be any $\mathbb{Z}\Gamma$-chain complex such that D_p is isomorphic to $\bigoplus_{i=1}^{r} \mathbb{Z}[\Gamma/H_i]$ for some non-negative integer r and finite subgroups H_i. Then the cokernel of each of the differentials d_p is a finitely presented $\mathbb{Z}\Gamma$-module and (5.5) yields for all $p \geqq 0$ using Additivity (see Theorem 1.4 (4))

(5.6)
$$d \cdot \dim_{\mathcal{N}(\Gamma)}(\operatorname{cok}(\mathrm{id}_{\mathcal{N}(\Gamma)} \otimes_{\mathbb{Z}\Gamma} d_p)) \in \mathbb{Z};$$
$$d \cdot \dim_{\mathcal{N}(\Gamma)}(\operatorname{im}(\mathrm{id}_{\mathcal{N}(\Gamma)} \otimes_{\mathbb{Z}\Gamma} d_p)) \in \mathbb{Z};$$
$$d \cdot \dim_{\mathcal{N}(\Gamma)}(\ker(\mathrm{id}_{\mathcal{N}(\Gamma)} \otimes_{\mathbb{Z}\Gamma} d_p)) \in \mathbb{Z};$$
$$d \cdot \dim_{\mathcal{N}(\Gamma)}(H_p(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} D_*)) \in \mathbb{Z}.$$

Let X^∞ be the Γ-CW-subcomplex of X consisting of points whose isotropy groups are infinite. The sequence $0 \to C_*(X^\infty) \to C_*(X) \to C_*(X, X^\infty) \to 0$ of cellular $\mathbb{Z}\Gamma$-chain complexes is exact. Since it is $\mathbb{Z}\Gamma$-split exact in each dimension, the sequence obtained by tensoring with $\mathcal{N}(\Gamma)$ is still exact. The associated long exact homology sequence, Additivity (see Theorem 1.4 (4)) [21], Lemma 3.4 (1) imply for $p \geqq 0$

(5.7)
$$\dim_{\mathcal{N}(\Gamma)}(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_p(X^\infty)) = 0;$$

(5.8) $$b_p^{(2)}(X;\mathcal{N}(\Gamma)) = \dim_{\mathcal{N}(\Gamma)}\big(H_p(\mathcal{N}(\Gamma)\otimes_{\mathbb{Z}\Gamma} C_*(X,X^\infty))\big).$$

Notice that $C_p(X,X^\infty)$ is a sum of $\mathbb{Z}\Gamma$-modules of the shape $\mathbb{Z}[\Gamma/H]$ for finite groups $H\subset\Gamma$. Hence $C_*(X,X^\infty)$ is a colimit over a directed set I of subcomplexes $D_*[i]$ (directed by inclusion) such that each $D_p[i]$ is isomorphic to $\bigoplus_{i=1}^{r}\mathbb{Z}[\Gamma/H_i]$ for some non-negative integer r and finite subgroups H_i. Since homology commutes with colimits we conclude from (5.8) and [21], Theorem 2.9 (2)

$$b_p^{(2)}(X:\mathcal{N}(\Gamma)) = \sup\{\inf\{\dim_{\mathcal{N}(\Gamma)}(\operatorname{im}(H_p(\mathcal{N}(\Gamma)\otimes_{\mathbb{Z}\Gamma} D_*[i]) \to H_p(\mathcal{N}(\Gamma)\otimes_{\mathbb{Z}\Gamma} D_*[j])))\mid j\in I, i\leqq j\}\mid i\in I\}.$$

Since the set $\{r\in\mathbb{R}\mid d\cdot r\in\mathbb{Z}\}$ is discrete in $\mathbb{R}$, it suffices to show for each inclusion $\iota: D_*[i]\to D_*[j]$ and all $p\geqq 0$

(5.9) $$d\cdot\dim_{\mathcal{N}(\Gamma)}\big(\operatorname{im}(\iota_*: H_p(\mathcal{N}(\Gamma)\otimes_{\mathbb{Z}\Gamma} D_*[i]) \to H_p(\mathcal{N}(\Gamma)\otimes_{\mathbb{Z}\Gamma} D_*[j]))\big)\in\mathbb{Z}.$$

Let F_* be any acyclic $\mathcal{N}(\Gamma)$-chain complex with $F_p=0$ for $p<0$ such that $d\cdot\dim(F_p)\in\mathbb{Z}$ holds for all $p\geqq 0$. Then we get $d\cdot\dim(\operatorname{im}(f_p: F_p\to F_{p-1}))\in\mathbb{Z}$ for all $p\geqq 0$ since we have the short exact sequences $0\to\operatorname{im}(f_{p+1})\to F_p\to\operatorname{im}(f_p)\to 0$ and $\operatorname{im}(f_1)=F_0$. Hence we obtain (5.9) from (5.6) and the conclusion above for the case where F is the long exact homology sequence of the pair $(\operatorname{cyl}(\iota), D_*[i])$ since there is a $\mathbb{Z}\Gamma$-chain homotopy equivalence from the mapping cylinder $\operatorname{cyl}(\iota)$ to $D_*[j]$ whose composition with the inclusion of $D_*[i]$ in $\operatorname{cyl}(\iota)$ *is* ι.

(3) is proven in [6], section 4. This finishes the proof of Theorem 5.2. □

Remark 5.10. The group $\Gamma=\prod_{i=1}^{\infty}\mathbb{Z}*\mathbb{Z}$ satisfies $H_p^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma))=0$ for all $p\geqq 0$. This is interesting in connection with the zero-in-the-spectrum conjecture ([15], [20], section 11). □

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