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Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers I: Foundations

The paper is dedicated to Martin Kneser on the occasion of his seventieth birthday

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Abstract. We define for arbitrary modules over a finite von Neumann algebra $\mathscr{A}$ a dimension taking values in $[0, \infty]$ which extends the classical notion of von Neumann dimension for finitely generated projective $\mathscr{A}$-modules and inherits all its useful properties such as Additivity, Cofinality and Continuity. This allows to define L^2-Betti numbers for arbitrary topological spaces with an action of a discrete group Γ extending the well-known definition for regular coverings of compact manifolds. We show for an amenable group Γ that the p-th L^2-Betti number depends only on the $\mathbb{C}\Gamma$-module given by the p-th singular homology.

0. Introduction

Let us recall the original definition of L^2-Betti numbers by Atiyah [2]. Let $\bar{M} \to M$ be a regular covering of a closed Riemannian manifold M with Γ as group of deck transformations. We lift the Riemannian metric to a Γ-invariant Riemannian metric on $\bar{M}$. Let $L^2\Omega^p(\bar{M})$ be the Hilbert space completion of the space $C_0^\infty\Omega^p(\bar{M})$ of smooth $\mathbb{R}$-valued p-forms on $\bar{M}$ with compact support and the standard L^2-pre-Hilbert structure. The Laplace operator Δ_p is essentially selfadjoint in $L^2\Omega^p(\bar{M})$. Let $\Delta_p = \int \lambda\, dE_\lambda^p$ be the spectral decomposition with right-continuous spectral family $\{E_\lambda^p \mid \lambda \in \mathbb{R}\}$. Let $E_\lambda^p(\bar{x}, \bar{y})$ be the Schwartz kernel of E_λ^p. Since $E_\lambda^p(\bar{x}, \bar{x})$ is an endomorphism of a finite-dimensional real vector space, its trace $\operatorname{tr}_{\mathbb{R}}\big(E_\lambda^p(\bar{x}, \bar{x})\big) \in \mathbb{R}$ is defined. Let $\mathscr{F}$ be a fundamental domain for the Γ-action on $\bar{M}$. Define the *analytic L^2-Betti number* by

$$(0.1) \qquad b_p^{(2)}(\bar{M}) := \int_{\mathscr{F}} \operatorname{tr}_{\mathbb{R}}\big(E_0^p(\bar{x}, \bar{x})\big)\, d\mathrm{vol}_{\bar{x}} \quad \in [0, \infty).$$

By means of a Laplace transformation this can also be expressed in terms of the heat kernel $e^{-t\Delta_p}(\bar{x}, \bar{y})$ on $\bar{M}$ by

$$(0.2) \qquad b_p^{(2)}(\bar{M}) := \lim_{t \to \infty} \int_{\mathscr{F}} \operatorname{tr}_{\mathbb{R}}\big(e^{-t\Delta_p}(\bar{x}, \bar{x})\big)\, d\mathrm{vol}_{\bar{x}} \quad \in [0, \infty).$$

The p-th L^2-Betti number measures the size of the space of smooth harmonic L^2-integrable p-forms on $\bar{M}$ and vanishes precisely if there is no such non-trivial form. For a survey on L^2-Betti numbers and related invariants like Novikov-Shubin invariants and L^2-torsion and their applications and relations to geometry, spectral theory, group theory and K-theory we refer for instance to [14], section 8, [18], [22] and [27]. In this paper, however, we will not deal with the analytic side, but take an algebraic point of view.

The L^2-Betti numbers can also be defined in an algebraic manner. Farber [12], [13] has shown that the category of finitely generated Hilbert $\mathscr{A}$-modules for a finite von Neumann algebra $\mathscr{A}$ can be embedded in an appropriate abelian category and that one can treat L^2-homology from a homological algebraic point of view. Farber gives as an application for instance an improvement of the Morse inequalites of Novikov and Shubin [25], [26] in terms of L^2-Betti numbers by taking the minimal number of generators into account. An equivalent more algebra oriented approach is developed in [21] where it is shown that the category of finitely generated projective modules over $\mathscr{A}$, viewed just as a ring, is equivalent to the category of finitely generated Hilbert $\mathscr{A}$-modules and that the category of finitely presented $\mathscr{A}$-modules is an abelian category. This allows to define for a finitely generated projective $\mathscr{A}$-module P *its von Neumann dimension*

$$(0.3) \qquad \dim_{\mathscr{A}}(P) \in [0, \infty)$$

by using the classical definition for finitely generated Hilbert $\mathscr{A}$-modules in terms of the von Neumann trace of a projector. This will be reviewed in Section 1.

In Section 2 we will prove the main technical result of this paper that this dimension can be extended to arbitrary $\mathscr{A}$-modules if one allows that the value may be infinite (what fortunately does not happen in a lot of interesting situations). Moreover, this extension inherits all good properties from the original definition for finitely generated Hilbert $\mathscr{A}$-modules such as Additivity, Cofinality and Continuity and is uniquely determined by these properties. More precisely, we will introduce

Definition 0.4. Define for an $\mathscr{A}$-module M

$$\dim'(M) := \sup\{\dim(P) \mid P \subset M \text{ finitely generated projective } \mathscr{A}\text{-submodule}\} \in [0, \infty]. \quad \square$$

Recall that the *dual module* M^* of a left $\mathscr{A}$-module is the left $\mathscr{A}$-module $\hom_{\mathscr{A}}(M, \mathscr{A})$ where the $\mathscr{A}$-multiplication is given by $(af)(x) = f(x)a^*$ for $f \in M^*$, $x \in M$ and $a \in \mathscr{A}$.

Definition 0.5. Let K be an $\mathscr{A}$-submodule of the $\mathscr{A}$-module M. Define the *closure of K in M* to be the $\mathscr{A}$-submodule of M

$$\bar{K} := \{x \in M \mid f(x) = 0 \text{ for all } f \in M^* \text{ with } K \subset \ker(f)\}.$$

For a finitely generated $\mathscr{A}$-module M define the $\mathscr{A}$-submodule $\mathbf{T}M$ and the $\mathscr{A}$-quotient module $\mathbf{P}M$ by:

$$\mathbf{T}M := \{x \in M \mid f(x) = 0 \text{ for all } f \in M^*\};$$

$$\mathbf{P}M := M/\mathbf{T}M. \quad \square$$

The notion of $\mathbf{T}M$ and $\mathbf{P}M$ corresponds in [12] to the torsion part and the projective part. Notice that $\mathbf{T}M$ is the closure of the trivial submodule in M. It is the same as the kernel of the canonical map $i(M) : M \to (M)^*$ which sends $x \in M$ to the map $M^* \to \mathscr{A}$, $f \mapsto f(x)$. We will prove for a finite von Neumann algebra $\mathscr{A}$ in Section 2

Theorem 0.6. (1) *$\mathscr{A}$ is semi-hereditary, i.e. any finitely generated submodule of a projective module is projective.*

(2) *If $K \subset M$ is a submodule of the finitely generated $\mathscr{A}$-module M, then $M/\bar{K}$ is finitely generated and projective and $\bar{K}$ is a direct summand in M.*

(3) *If M is a finitely generated $\mathscr{A}$-module, then $\mathbf{P}M$ is finitely generated projective and*

$$M \cong \mathbf{P}M \oplus \mathbf{T}M.$$

(4) *The dimension* $\dim'$ *has the following properties*:

(a) *Continuity.*

If $K \subset M$ is a submodule of the finitely generated $\mathscr{A}$-module M, then:

$$\dim'(K) = \dim'(\bar{K}).$$

(b) *Cofinality.*

Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of M, i.e. $M = \bigcup_{i \in I} M_i$ and for two indices i and j there is an index k in I satisfying $M_i, M_j \subset M_k$. Then:

$$\dim'(M) = \sup\{\dim'(M_i) \mid i \in I\}.$$

(c) *Additivity.*

If $0 \longrightarrow M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \longrightarrow 0$ is an exact sequence of $\mathscr{A}$-modules, then:

$$\dim'(M_1) = \dim'(M_0) + \dim'(M_2),$$

where $r + s$ for $r, s \in [0, \infty]$ is the ordinary sum of two real numbers if both r and s are not ∞ and is ∞ otherwise.

(d) *Extension Property.*

If M is finitely generated projective, then:

$$\dim'(M) = \dim(M).$$

(e) *If M is a finitely generated $\mathscr{A}$-module, then*:

$$\dim'(M) = \dim(\mathbf{P}M);$$

$$\dim'(\mathbf{T}M) = 0.$$

(f) *The dimension* dim′ *is uniquely determined by Continuity, Cofinality, Additivity and the Extension Property.* □

In the sequel we write dim instead of dim′. In Section 3 we will show for an inclusion $i : \Delta \to \Gamma$ that the dimension function is compatible with induction with the induced ring homomorphism $i : \mathcal{N}(\Delta) \to \mathcal{N}(\Gamma)$ and that $\mathcal{N}(\Gamma)$ is faithfully flat over $\mathcal{N}(\Delta)$ (Theorem 3.3). This is important if one wants to relate the L^2-Betti numbers of a regular covering to the ones of the universal covering. We will prove that Γ is non-amenable if and only if $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}$ is trivial (Lemma 3.4.2). This generalizes the result of Brooks [5], Remark 4.11.

In Section 4 we use this generalized dimension function to define for a (discrete) group Γ and a Γ-space X its *p-th L^2-Betti number* by

(0.7) $$b_p^{(2)}(X; \mathcal{N}(\Gamma)) := \dim_{\mathcal{N}(\Gamma)}\big(H_p^\Gamma(X; \mathcal{N}(\Gamma))\big) \quad \in [0, \infty],$$

where $H_p^\Gamma(X; \mathcal{N}(\Gamma))$ denotes the $\mathcal{N}(\Gamma)$-module given by the singular homology of X with coefficients in the $\mathcal{N}(\Gamma)$-$\mathbb{Z}\Gamma$-bimodule $\mathcal{N}(\Gamma)$ (Definition 4.1). This definition agrees with Atiyah's definition 0.1 if X is the total space and Γ the group of deck transformations of a regular covering of a closed Riemannian manifold. We will compare our definition also with the one of Cheeger and Gromov [7], section 2, Remark 4.12. In particular we can define for an arbitrary (discrete) group Γ its p-th L^2-Betti number

(0.8) $$b_p^{(2)}(\Gamma) := b_p^{(2)}(E\Gamma; \mathcal{N}(\Gamma)) \quad \in [0, \infty],$$

where $E\Gamma \to B\Gamma$ is the universal Γ-principle bundle. These generalizations inherit all the useful properties from the original versions and it pays off to have them at hand in this generality. For instance if one is only interested in the L^2-Betti numbers of a group Γ for which $B\Gamma$ is a CW-complex of finite type and hence the original (simplicial) definition does apply, it is important to have the more general definition available because such a group Γ may contain an interesting normal subgroup Δ which is not even finitely generated. A typical situation is when Γ contains a normal infinite amenable subgroup Δ. Then all the L^2-Betti numbers of $B\Gamma$ are trivial by a result of Cheeger and Gromov [7], Theorem 0.2 on page 191. This result was the main motivation for our attempt to construct the extensions of dimension and of L^2-Betti numbers described above.

In Section 5 we will get the theorem of Cheeger and Gromov mentioned above as a corollary of the following result. If Γ is amenable and M is a $\mathbb{C}\Gamma$-module, then

(0.9) $$\dim_{\mathcal{N}(\Gamma)}\big(\mathrm{Tor}_p^{\mathbb{C}\Gamma}(\mathcal{N}(\Gamma), \mathcal{M})\big) = 0 \quad \text{for } p \geqq 1,$$

where we consider $\mathcal{N}(\Gamma)$ as an $\mathcal{N}(\Gamma)$-$\mathbb{C}\Gamma$-bimodule (Theorem 5.1). We get from (0.9) by a spectral sequence argument that the L^2-Betti numbers of a Γ-space X depend only on its singular homology with complex coefficients viewed as $\mathbb{C}\Gamma$-module, namely

(0.10) $$b_p^{(2)}\big(X;\mathcal{N}(\Gamma)\big)=\dim_{\mathcal{N}(\Gamma)}\big(\mathcal{N}(\Gamma)\otimes_{\mathbb{C}\Gamma}H_p^{\text{sing}}(X;\mathbb{C})\big),$$

provided that Γ is amenable (Theorem 5.11). The result of Cheeger and Gromov mentioned above follows from (0.10) since the singular homology of $E\Gamma$ is trivial in all dimensions except for dimension 0 where it is $\mathbb{C}$.

We will discuss applications of this generalized dimension function to the Grothendieck group $G_0(\mathbb{C}\Gamma)$ of finitely generated $\mathbb{C}\Gamma$-modules and L^2-Euler characteristics and the Burnside group in part II of the paper [23].

1. Review of von Neumann dimension

In this section we recall some basic facts about finitely generated Hilbert-modules and finitely generated projective modules over a finite von Neumann algebra. We fix for the sequel

Notation 1.1. Let $\mathcal{A}$ be a finite von Neumann algebra and $\operatorname{tr}:\mathcal{A}\to\mathbb{C}$ be a normal finite faithful trace. Denote by Γ an (arbitrary) discrete group. Let $\mathcal{N}(\Gamma)$ be the group von Neumann algebra with the standard trace $\operatorname{tr}_{\mathcal{N}(\Gamma)}$.

Module means always left-module and group actions on spaces are from the left unless explicitly stated differently. We will always work in the category of compactly generated spaces (see [29] and [31], I.4). □

Next we recall our main example for $\mathcal{A}$ and tr, namely *the group von Neumann algebra* $\mathcal{N}(\Gamma)$ with *the standard trace*. The reader who is not familiar with the general concept of finite von Neumann algebras may always think of this example. Let $l^2(\Gamma)$ be the Hilbert space of formal sums $\sum_{\gamma\in\Gamma}\gamma_\lambda\cdot\gamma$ with complex coefficients λ_γ which are square-summable, i.e. $\sum_{\gamma\in\Gamma}|\lambda_\gamma|^2<\infty$. Define *the group von Neumann algebra* and *the standard trace* by

(1.2) $$\mathcal{N}(\Gamma):=\mathcal{B}\big(l^2(\Gamma),l^2(\Gamma)\big)^\Gamma;$$

(1.3) $$\operatorname{tr}_{\mathcal{N}(\Gamma)}(a):=\langle a(e),e\rangle_{l^2(\Gamma)};$$

where $\mathcal{B}\big(l^2(\Gamma),l^2(\Gamma)\big)^\Gamma$ is the space of bounded Γ-equivariant operators from $l^2(\Gamma)$ to itself, $a\in\mathcal{N}(\Gamma)$ and $e\in\Gamma\subset l^2(\Gamma)$ is the unit element. The given trace on $\mathcal{A}$ extends to a trace on square-matrices over $\mathcal{A}$ in the usual way

(1.4) $$\operatorname{tr}:M(n,n,\mathcal{A})\to\mathbb{C},\quad A\mapsto\sum_{i=1}^n\operatorname{tr}(A_{i,i}).$$

Taking adjoints induces the structure of a *ring with involution* on $\mathcal{A}$, i.e. we obtain a map $*:\mathcal{A}\to\mathcal{A}$, $a\mapsto a^*$, which satisfies $(a+b)^*=a^*+b^*$, $(ab)^*=b^*a^*$ and $(a^*)^*=a$ and $1^*=1$ for all $a,b\in\mathcal{A}$. This involution induces an involution on matrices

(1.5) $$*:M(m,n,\mathcal{A})\to M(n,m,\mathcal{A}),\quad A=(A_{i,j})\mapsto A^*=(A_{j,i}^*).$$

Definition 1.6. Let P be a finitely generated projective $\mathscr{A}$-module. Let $A \in M(n, n, \mathscr{A})$ be a matrix such that $A = A^*$, $A^2 = A$ and the image of the $\mathscr{A}$-linear map $A : \mathscr{A}^n \to \mathscr{A}^n$ induced by right multiplication with A is $\mathscr{A}$-isomorphic to P. Define the *von Neumann dimension* of

$$\dim(P) = \dim_{\mathscr{A}}(P) := \operatorname{tr}_{\mathscr{A}}(A) \quad \in [0, \infty). \quad \square$$

It is not hard to check that this definition is independent of the choice of A and depends only on the isomorphism class of P. Moreover the dimension is faithful, i.e. $\dim(P) = 0$ implies $P = 0$, is additive under direct sums and satisfies $\dim(\mathscr{A}^n) = n$.

We recall that we have defined $\bar{K}$, $\mathbf{T}M$ and $\mathbf{P}M$ for $K \subset M$ in Definition 0.5. A sequence $L \xrightarrow{f} M \xrightarrow{g} N$ of $\mathscr{A}$-modules is *weakly exact* resp. *exact at* M if $\overline{\operatorname{im}(f)} = \ker(g)$ resp. $\operatorname{im}(f) = \ker(g)$ holds. A morphism $f: M \to N$ of $\mathscr{A}$-modules is called a *weak isomorphism* if its kernel is trivial and the closure of its image is N.

Next we explain how these concepts above correspond to their analogues for finitely generated Hilbert $\mathscr{A}$-modules. Let $l^2(\mathscr{A})$ be the Hilbert space completion of $\mathscr{A}$ which is viewed as a pre-Hilbert space by the inner product $\langle a, b\rangle = \operatorname{tr}(ab^*)$. A *finitely generated Hilbert* $\mathscr{A}$*-module* V is a Hilbert space V together with a left operation of $\mathscr{A}$ by $\mathbb{C}$-linear maps such that there exists a unitary $\mathscr{A}$-embedding of V in $\bigoplus_{i=1}^{n} l^2(\mathscr{A})$ for some n. A morphism of finitely generated Hilbert $\mathscr{A}$-modules is a bounded $\mathscr{A}$-equivariant operator. Denote by {fin. gen. Hilb. $\mathscr{A}$-mod.} the category of finitely generated Hilbert $\mathscr{A}$-modules. A sequence $U \xrightarrow{f} V \xrightarrow{g} W$ of finitely generated Hilbert $\mathscr{A}$-modules is *exact resp. weakly exact at* V if $\operatorname{im}(f) = \ker(g)$ resp. $\overline{\operatorname{im}(f)} = \ker(g)$ holds. A morphism $f: V \to W$ is a *weak isomorphism* if its kernel is trivial and its image is dense. For a survey on finite von Neumann algebras and Hilbert $\mathscr{A}$-modules we refer for instance to [19], section 1, [24], section 1.

The right regular representation $\mathscr{A} \to \mathscr{B}(l^2(\mathscr{A}), l^2(\mathscr{A}))^{\mathscr{A}}$ from $\mathscr{A}$ into the space of bounded $\mathscr{A}$-equivariant operators from $l^2(\mathscr{A})$ to itself sends $a \in \mathscr{A}$ to the extension of the map $\mathscr{A} \to \mathscr{A}$, $b \mapsto ba^*$. It is known to be bijective [9], Theorem 1 in I.5.2 on page 80, Theorem 2 in I.6.2 on page 99. Hence we obtain a bijection

$$\nu : M(m, n, \mathscr{A}) \to \mathscr{B}(l^2(\mathscr{A})^m, l^2(\mathscr{A})^n)^{\mathscr{A}}, \tag{1.7}$$

which is compatible with the $\mathbb{C}$-vector space structures, the involutions and composition.

The details of the following theorem and its proof can be found in [21], section 2. It is essentially a consequence of (1.7) and the construction of the idempotent completion of a category. It allows us to forget the Hilbert-module-structures and simply work with the von Neumann algebra as a plain ring. An equivalent approach is given by Farber [12], [13] and is identified with the one here in [21], Theorem 0.9. An *inner product* on a finitely generated projective $\mathscr{A}$-module P is a map $\mu : P \times P \to \mathscr{A}$ which is linear in the first variable, symmetric in the sense $\mu(x, y) = \mu(y, x)^*$ and positive in the sense $\mu(x, x) > 0 \Leftrightarrow x \neq 0$ such that the induced map $P \to P^*$ sending $y \in P$ to $\mu(-, y)$ is bijective.

Theorem 1.8. (1) *There is a functor*

$$\nu : \{\textit{fin. gen. proj. } \mathscr{A}\textit{-mod. with inner prod.}\} \to \{\textit{fin. gen. Hilb. } \mathscr{A}\textit{-mod.}\}$$

which is an equivalence of $\mathbb{C}$*-categories with involutions.*

(2) *Any finitely generated projective* $\mathscr{A}$*-module has an inner product. Two finitely generated projective* $\mathscr{A}$*-modules with inner product are unitarily* $\mathscr{A}$*-isomorphic if and only if the underlying* $\mathscr{A}$*-modules are* $\mathscr{A}$*-isomorphic.*

(3) *Let* ν^{-1} *be an inverse of* ν *which is well-defined up to unitary natural equivalence. The composition of* ν^{-1} *with the forgetful functor induces an equivalence of* $\mathbb{C}$*-categories*

$$\{\textit{fin. gen. Hilb. } \mathscr{A}\textit{-mod.}\} \to \{\textit{fin. gen. proj. } \mathscr{A}\textit{-mod.}\}\,.$$

(4) ν *and* ν^{-1} *preserve weak exactness and exactness.* □

Of course Definition 1.6 of $\dim(P)$ for a finitely generated projective $\mathscr{A}$-module agrees with the usual von Neumann dimension of the associated Hilbert $\mathscr{A}$-module $\nu(P)$ after any choice of inner product on P.

2. The generalized dimension function

In this section we give the proof of Theorem 0.6 and investigate the behaviour of the dimension under colimits. We recall that we have introduced $\dim'(M)$ for an arbitrary $\mathscr{A}$-module M in Definition 0.4 and $\bar{K}$, $\mathbf{T}M$ and $\mathbf{P}M$ for $K \subset M$ in Definition 0.5. We begin with the proof of Theorem 0.6.

Proof. (1) is proven in [21], Corollary 2.4 for finite von Neumann algebras. However, Pardo pointed out to us that any von Neumann algebra is semi-hereditary. This follows from the facts that any von Neumann algebra is a Baer $*$-ring and hence in particular a Rickart C^*-algebra [4], Definition 1, Definition 2 and Proposition 9 in Chapter 1.4, and that a C^*-algebra is semi-hereditary if and only if it is Rickart [1], Corollary 3.7 on page 270.

(2) and (4)(a) in the special case that $M = P$ for a finitely generated projective $\mathscr{A}$-module P.

Let $\mathscr{P} = \{P_i \mid i \in I\}$ be the directed system of finitely generated projective $\mathscr{A}$-submodules of K. Notice that $\mathscr{P}$ is indeed directed by inclusion since the submodule of P generated by two finitely generated projective submodules is again finitely generated and hence by (1) finitely generated projective. Let $j_i : P_i \to P$ be the inclusion. Equip P and each P_i with a fixed inner product and let $\mathrm{pr}_i : \nu(P) \to \nu(P)$ be the orthogonal projection satisfying $\mathrm{im}(\mathrm{pr}_i = \overline{\mathrm{im}(\nu(j_i))}$ and $\mathrm{pr} : \nu(P) \to \nu(P)$ be the orthogonal projection satisfying $\mathrm{im}(\mathrm{pr}) = \overline{\bigcup_{i \in I} \mathrm{im}(\mathrm{pr}_i)}$. Next we show

$$\mathrm{im}(\nu^{-1}(\mathrm{pr})) = \bar{K}. \tag{2.1}$$

Let $f: P \to \mathscr{A}$ be an $\mathscr{A}$-map with $K \subset \ker(f)$. Then $f \circ j_i = 0$ and therefore $\nu(f) \circ \nu(j_i) = 0$ for all $i \in I$. We get $\operatorname{im}(\mathrm{pr}_i) \subset \ker(\nu(f))$ for all $i \in I$. Because the kernel of $\nu(f)$ is closed we conclude $\operatorname{im}(\mathrm{pr}) \subset \ker(\nu(f))$. This shows $\operatorname{im}(\nu^{-1}(\mathrm{pr})) \subset \ker(f)$ and hence $\operatorname{im}(\nu^{-1}(\mathrm{pr})) \subset \bar{K}$. As $K \subset \ker(\mathrm{id} - \nu^{-1}(\mathrm{pr})) = \operatorname{im}(\nu^{-1}(\mathrm{pr}))$, we conclude $\bar{K} \subset \operatorname{im}(\nu^{-1}(\mathrm{pr}))$. This finishes the proof of (2.1) and of (2) in the special case $M = P$.

Next we prove

$$\dim'(K) = \dim(\bar{K}). \tag{2.2}$$

The inclusion j_i induces a weak isomorphism $\nu(P_i) \to \operatorname{im}(\mathrm{pr}_i)$ of finitely generated Hilbert $\mathscr{A}$-modules. If we apply the Polar Decomposition Theorem to it we obtain a unitary $\mathscr{A}$-isomorphism from $\nu(P_i)$ to $\operatorname{im}(\mathrm{pr}_i)$. This implies $\dim(P_i) = \operatorname{tr}(\mathrm{pr}_i)$. Therefore it remains to prove

$$\operatorname{tr}(\mathrm{pr}) := \sup\{\operatorname{tr}(\mathrm{pr}_i) \mid i \in I\}. \tag{2.3}$$

As tr is normal, it suffices to show for $x \in \nu(P)$ that the net $\{\mathrm{pr}_i(x) \mid i \in I\}$ converges to $\mathrm{pr}(x)$. Let $\varepsilon > 0$ be given. Choose $i(\varepsilon) \in I$ and $x_{i(\varepsilon)} \in \operatorname{im}(\mathrm{pr}_{i(\varepsilon)})$ with $\|\mathrm{pr}(x) - x_{i(\varepsilon)}\| \leqq \varepsilon/2$. We conclude for all $i \geqq i(\varepsilon)$

$$\begin{aligned}
\|\mathrm{pr}(x) - \mathrm{pr}_i(x)\| &\leqq \|\mathrm{pr}(x) - \mathrm{pr}_{i(\varepsilon)}(x)\| \\
&\leqq \|\mathrm{pr}(x) - \mathrm{pr}_{i(\varepsilon)}(x_{i(\varepsilon)})\| + \|\mathrm{pr}_{i(\varepsilon)}(x_{i(\varepsilon)}) - \mathrm{pr}_{i(\varepsilon)}(x)\| \\
&\leqq \|\mathrm{pr}(x) - x_{i(\varepsilon)}\| + \|\mathrm{pr}_{i(\varepsilon)}(x_{i(\varepsilon)} - \mathrm{pr}(x))\| \\
&\leqq \|\mathrm{pr}(x) - x_{i(\varepsilon)}\| + \|\mathrm{pr}_{i(\varepsilon)}\| \cdot \|x_{i(\varepsilon)} - \mathrm{pr}(x)\| \\
&\leqq 2 \cdot \|\mathrm{pr}(x) - x_{i(\varepsilon)}\| \\
&\leqq \varepsilon.
\end{aligned}$$

Now (2.3) and hence (2.2) follow. In particular we get from (2.2) for any finitely generated projective submodule Q_0 of a finitely generated projective $\mathscr{A}$-module Q

$$\dim(Q_0) \leqq \dim(Q), \tag{2.4}$$

since by definition $\dim(Q_0) \leqq \dim'(Q_0)$ and $\dim(\bar{Q}_0) \leqq \dim(Q)$ follows from additivity of dim under direct sums and that we have already proven that $\bar{Q}_0$ is a direct summand in Q. This implies for a finitely generated projective $\mathscr{A}$-module Q

$$\dim(Q) = \dim'(Q). \tag{2.5}$$

Now (2.2) and (2.5) imply (4)(a) in the special case that $M = P$ for a finitely generated projective $\mathscr{A}$-module P.

(4)(d) has been already proven in 2.5.

(4)(b) If $P \subset M$ is a finitely generated projective submodule, then there is an index $i \in I$ with $P \subset M_i$ by cofinality.

(4)(c) Let $P \subset M_2$ be a finitely generated projective submodule. We obtain an exact sequence $0 \to M_0 \to p^{-1}(P) \to P \to 0$. Since $p^{-1}(P) \cong M_0 \oplus P$, we conclude

$$\dim'(M_0) + \dim(P) \leqq \dim'\big(p^{-1}(P)\big) \leqq \dim'(M_1).$$

Since this holds for all finitely generated projective sumodules $P \subset M_2$, we get

$$\dim'(M_0) + \dim'(M_2) \leqq \dim'(M_1). \tag{2.6}$$

Let $Q \subset M_1$ be finitely generated projective. Let $\overline{i(M_0) \cap Q}$ be the closure of $i(M_0) \cap Q$ in Q. We obtain exact sequences

$$\begin{aligned} 0 \to i(M_0) \cap Q \to Q \to & \quad p(Q) \quad \to 0; \\ 0 \to \overline{i(M_0) \cap Q} \to Q \to & \; Q/\overline{i(M_0) \cap Q} \to 0. \end{aligned}$$

By the special case of (2) which we have already proven above $\overline{i(M_0) \cap Q}$ is a direct summand in Q. We conclude

$$\dim(Q) = \dim\big(\overline{i(M_0) \cap Q}\big) + \dim\big(Q/\overline{i(M_0) \cap Q}\big).$$

From the special case (4)(a) we have already proven above, (4)(d) and the fact that there is an epimorphism from $p(Q)$ onto the finitely generated projective $\mathscr{A}$-module $Q/\overline{i(M_0) \cap Q}$, we conclude

$$\begin{aligned} \dim\big(\overline{i(M_0) \cap Q}\big) &= \dim'\big(i(M_0) \cap Q\big); \\ \dim\big(Q/\overline{i(M_0) \cap Q}\big) &\leqq \dim'\big(p(Q)\big). \end{aligned}$$

Since obviously $\dim'(M) \leqq \dim'(N)$ holds for $\mathscr{A}$-modules M and N with $M \subset N$, we get

$$\begin{aligned} \dim(Q) &= \dim\big(\overline{i(M_0) \cap Q}\big) + \dim\big(Q/\overline{i(M_0) \cap Q}\big) \\ &\leqq \dim'\big(i(M_0) \cap Q\big) + \dim'\big(p(Q)\big) \\ &\leqq \dim'(M_0) + \dim'(M_2). \end{aligned}$$

Since this holds for all finitely generated projective submodules $Q \subset M_1$, we get

$$\dim'(M_1) \leqq \dim'(M_0) + \dim'(M_2). \tag{2.7}$$

Now (4)(c) follows from (2.6) and (2.7).

(2) and (4)(a) Choose a finitely generated free $\mathscr{A}$-module F together with an epimorphism $q: F \to M$. One easily checks that $q^{-1}(\bar{K})$ is $\overline{q^{-1}(K)}$ and that $F/q^{-1}(\bar{K})$ and $M/\bar{K}$ are isomorphic. From the special case of (2) and (4)(a) which we have already proven above we conclude that $F/\overline{q^{-1}(K)}$ and hence $M/\bar{K}$ are finitely generated projective and

$$\dim'\big(q^{-1}(K)\big) = \dim'\big(\overline{q^{-1}(K)}\big) = \dim'\big(q^{-1}(\bar{K})\big).$$

If L is the kernel of q, we conclude from Additivity

$$\dim'\big(q^{-1}(\bar{K})\big) = \dim'(L) + \dim'(\bar{K});$$
$$\dim'\big(q^{-1}(K)\big) = \dim'(L) + \dim'(K).$$

Now (2) and (4)(a) follow in general.

(3) follows from (2), as $\overline{\{0\}} = \mathbf{T}M$ and $M/\mathbf{T}M = \mathbf{P}M$ by definition.

(4)(e) From (2), (4)(c) and (4)(d) we get: $\dim'(M) = \dim'(\mathbf{T}M) + \dim(\mathbf{P}M)$. If we apply (4)(a) to $\{0\} \subset M$ we get $\dim'(\mathbf{T}M) = 0$ because of $\overline{\{0\}} = \mathbf{T}M$.

(4)(f) Let dim″ be another function satisfying Continuity, Cofinality, Additivity and the Extension Property. We want to show for an $\mathscr{A}$-module M

$$\dim''(M) = \dim'(M).$$

Since (4)(e) is a consequence of Continuity, Additivity and the Extension Property alone, this is obvious provided M is finitely generated. Since the system of finitely generated submodules of a module is cofinal, the claim follows from Cofinality. This finishes the proof of Theorem 0.6. □

Notation 2.8. In view of Theorem 0.6 we will not distinguish between dim′ and dim in the sequel. □

Next we investigate the behaviour of dimension under colimits indexed by a directed set. We mention that colimit is sometimes called in the literature also inductive limit or direct limit. The harder case of inverse limits which is not needed in this paper will be treated at a different place (see also [7], Appendix).

Theorem 2.9. *Let I be a category such that between two objects there is at most one morphism and for two objects i_1 and i_2 there is an object i_0 with $i_1 \leqq i_0$ and $i_2 \leqq i_0$ where we write $i \leqq k$ for two objects i and k if and only if there is a morphism from i to k. Let M_i be a covariant functor from I to the category of $\mathscr{A}$-modules. For $i \leqq j$ let $\phi_{i,j}: M_i \to M_j$ be the associated morphism of $\mathscr{A}$-modules. For $i \in I$ let $\psi_i: M_i \to \operatorname{colim}_I M_i$ be the canonical morphism of $\mathscr{A}$-modules. Then*:

(1) *We get for the dimension of the $\mathscr{A}$-module given by the colimit* $\operatorname{colim}_I M_i$

$$\dim(\operatorname{colim}_I M_i) = \sup\{\dim\big(\operatorname{im}(\psi_i)\big) \mid i \in I\}.$$

(2) *Suppose for each $i \in I$ that there is $i_0 \in I$ with $i \leqq i_0$ such that* $\dim\big(\operatorname{im}(\phi_{i,i_0})\big) < \infty$ *holds. Then*:

$$\dim(\operatorname{colim}_I M_i) = \sup\{\inf\{\dim\big(\operatorname{im}(\phi_{i,j}: M_i \to M_j)\big) \mid j \in I, i \leqq j\} \mid i \in I\}.$$

Proof. (1) Recall that $\operatorname{colim}_I M_i$ can be constructed as $\coprod_{i \in I} M_i/\sim$ for the equivalence relation for which $x \in M_i \sim y \in M_j$ holds precisely if there is $k \in I$ with $i \leqq k$ and $j \leqq k$ with the property $\phi_{i,k}(x) = \phi_{j,k}(y)$. With this description one easily checks

$$\operatorname{colim}_I M_i = \bigcup_{i \in I} \operatorname{im}(\psi_i : M_i \to \operatorname{colim}_I M_i).$$

Now apply Cofinality of dim (see Theorem 0.6(4)).

(2) It remains to show for $i \in I$

$$\dim\big(\operatorname{im}(\psi_i)\big) = \inf\{\dim\big(\operatorname{im}(\phi_{i,j} : M_i \to M_j)\big) \,|\, j \in I, i \leqq j\}. \tag{2.10}$$

By assumption there is $i_0 \in I$ with $i \leqq i_0$ such that $\dim\big(\operatorname{im}(\phi_{i,i_0})\big)$ is finite. Let $K_{i_0,j}$ be the kernel of the map $\operatorname{im}(\phi_{i,i_0}) \to \operatorname{im}(\phi_{i,j})$ induced by $\phi_{i_0,j}$ for $i_0 \leqq j$ and K_{i_0} be the kernel of the map $\operatorname{im}(\phi_{i,i_0}) \to \operatorname{im}(\psi_i)$ induced by ψ_{i_0}. Then $K_{i_0} = \bigcup_{j \in I, i_0 \leqq j} K_{i_0,j}$ and hence by Cofinality (see Theorem 0.6(4))

$$\dim(K_{i_0}) = \sup\{\dim(K_{i_0,j}) \,|\, j \in I, i_0 \leqq j\}.$$

Since $\dim\big(\operatorname{im}(\phi_{i,i_0})\big)$ is finite, we get from Additivity (see Theorem 0.6(4))

$$\begin{aligned} \dim\big(\operatorname{im}(\psi_i)\big) &= \dim\big(\operatorname{im}\big(\psi_{i_0}|_{\operatorname{im}(\phi_{i,i_0})} : \operatorname{im}(\phi_{i,i_0}) \to \operatorname{colim}_I M_i\big)\big) \\ &= \dim\big(\operatorname{im}(\phi_{i,i_0})\big) - \dim(K_{i_0}) \\ &= \dim\big(\operatorname{im}(\phi_{i,i_0})\big) - \sup\{\dim(K_{i_0,j}) \,|\, j \in I, i_0 \leqq j\} \\ &= \inf\{\dim\big(\operatorname{im}(\phi_{i,i_0})\big) - \dim(K_{i_0,j}) \,|\, j \in I, i_0 \leqq j\} \\ &= \inf\{\dim\big(\operatorname{im}\big(\phi_{i_0,j}|_{\operatorname{im}(\phi_{i,i_0})} : \operatorname{im}(\phi_{i,i_0}) \to \operatorname{im}(\phi_{i,j})\big)\big) \,|\, j \in I, i_0 \leqq j\} \\ &= \inf\{\dim\big(\operatorname{im}(\phi_{i,j})\big) \,|\, j \in I, i_0 \leqq j\}. \end{aligned} \tag{2.11}$$

Given $j_0 \in J$ with $i \leqq j_0$, there is $j \in I$ with $i_0 \leqq j$ and $j_0 \leqq j$ and hence with

$$\dim\big(\operatorname{im}(\phi_{i,j_0})\big) \geqq \dim\big(\operatorname{im}(\phi_{i,j})\big).$$

This implies

$$\inf\{\dim\big(\operatorname{im}(\phi_{i,j})\big) \,|\, j \in J, i \leqq j\} = \inf\{\dim\big(\operatorname{im}(\phi_{i,j})\big) \,|\, j \in J, i_0 \leqq j\}. \tag{2.12}$$

Now (2.10) follows from (2.11) and (2.12). This finishes the proof of Theorem 2.9. □

Examples 2.13. The condition in Theorem 2.9(2) that for each $i \in I$ there is $i_0 \in I$ with $i \leqq i_0$ with $\dim\big(\operatorname{im}(\phi_{i,i_0})\big) < \infty$ is necessary as the following example shows. Take $I = \mathbb{N}$. Define $M_j = \bigoplus_{n=j}^{\infty} \mathscr{A}$ and $\phi_{j,k} : \bigoplus_{m=j}^{\infty} \mathscr{A} \to \bigoplus_{m=k}^{\infty} \mathscr{A}$ to be the projection. Then $\dim\big(\operatorname{im}(\phi_{j,k})\big) = \infty$ for all $j \leqq k$, but $\operatorname{colim}_I M_i$ is trivial and hence has dimension zero. □

Remark 2.14. From an axiomatic point of view we have only needed the following basic properties of $\mathscr{A}$. Namely, let R be an associative ring with unit which has the following properties:

1. There is a dimension function dim which assigns to any finitely generated projective R-module P an element

$$\dim(P) \in [0, \infty)$$

such that $\dim(P \oplus Q) = \dim(P) + \dim(Q)$ holds and $\dim(P)$ depends only on the isomorphism class of P.

2. If $K \subset P$ is a submodule of the finitely generated projective $\mathscr{A}$-module P, then $\bar{K}$ is a direct summand in P. Moreover

$$\dim(\bar{K}) = \sup\{\dim(P) \mid P \subset K \text{ finitely generated projective } R\text{-submodule}\}.$$

Then with Definition 0.4, Theorem 0.6 carries over to R. One has essentially to copy the part of the proof which begins with 2.4.

An easy example where these axioms are satisfied is the case where R is a principal ideal domain and dim is the usual rank of a finitely generated free R-module. Then the extended dimension for an R-module M is just the dimension of the F-vector space $F \otimes_R M$ for F the quotient field of R. Notice that the case of a von Neumann algebra $R = \mathscr{A}$ is harder since $\mathscr{A}$ is not noetherian in general.

In Definition 0.5 we have defined $\mathbf{T}M$ and $\mathbf{P}M$ only for finitely generated $\mathscr{A}$-modules M although the definition makes sense in general. The reason is that the following definition for arbitrary $\mathscr{A}$-modules seems to be more appropriate

(2.15) $$\mathbf{T}M := \bigcup \{N \subset M \mid \dim(N) = 0\};$$

(2.16) $$\mathbf{P}M := M/\mathbf{T}M.$$

One easily checks using Theorem 0.6(4) that $\mathbf{T}M$ is the largest submodule of M with trivial dimension and that these definitions (2.15) and (2.16) agree with Definition 0.5 if M is finitely generated. One can show by example that they do not agree if one applies them to arbitrary $\mathscr{A}$-modules. In the case of a principal ideal domain R the torsion submodule of an R-module M is just $\mathbf{T}M$ in the sense of definition (2.15). □

3. Induction for group von Neumann algebras

Next we investigate how the dimension behaves under induction. Let $i: \Delta \to \Gamma$ be an inclusion of groups. We claim that associated to i there is a ring homomorphism of the group von Neumann algebras, also denoted by

(3.1) $$i: \mathscr{N}(\Delta) \to \mathscr{N}(\Gamma).$$

Recall that $\mathcal{N}(\Delta)$ is the same as the ring $\mathcal{B}(l^2(\Delta), l^2(\Delta))^{\Delta}$ of bounded Δ-equivariant operators $f: l^2(\Delta) \to l^2(\Delta)$. Notice that $\mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta)$ can be viewed as a dense subspace of $l^2(\Gamma)$ and that f defines a $\mathbb{C}\Gamma$-homomorphism $\mathrm{id} \otimes_{\mathbb{C}\Delta} f: \mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta) \to \mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta)$ which is bounded with respect to the pre-Hilbert structure induced on $\mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta)$ from $l^2(\Gamma)$. Hence $\mathrm{id} \otimes_{\mathbb{C}\Delta} f$ extends to a Γ-equivariant bounded operator $i(f): l^2(\Gamma) \to l^2(\Gamma)$.

Given an $\mathcal{N}(\Delta)$-module M, define *the induction with* i to be the $\mathcal{N}(\Gamma)$-module

$$i_*(M) := \mathcal{N}(\Gamma) \otimes_{\mathcal{N}(\Delta)} M. \tag{3.2}$$

Obviously i_* is a covariant functor from the category of $\mathcal{N}(\Delta)$-modules to the category of $\mathcal{N}(\Gamma)$-modules, preserves direct sums and the properties finitely generated and projective and sends $\mathcal{N}(\Delta)$ to $\mathcal{N}(\Gamma)$.

Theorem 3.3. *Let $i: \Delta \to \Gamma$ be an injective group homomorphisms. Then*:

(1) *Induction with i is a faithfully flat functor from the category of $\mathcal{N}(\Delta)$-modules to the category of $\mathcal{N}(\Gamma)$-modules, i.e. a sequence of $\mathcal{N}(\Delta)$-modules $M_0 \to M_1 \to M_2$ is exact at M_1 if and only if the induced sequence of $\mathcal{N}(\Gamma)$-modules $i_* M_0 \to i_* M_1 \to i_* M_2$ is exact at $i_* M_1$.*

(2) *For any $\mathcal{N}(\Delta)$-module M we have*:

$$\dim_{\mathcal{N}(\Delta)}(M) = \dim_{\mathcal{N}(\Gamma)}(i_* M).$$

Proof. The proof consists of the following steps.

Step 1. $\dim_{\mathcal{N}(\Delta)}(M) = \dim_{\mathcal{N}(\Gamma)}(i_*(M))$, provided M is a finitely generated projective $\mathcal{N}(\Delta)$-module.

Let $A \in M(n, n, \mathcal{N}(\Delta))$ be a matrix such that $A = A^*$, $A^2 = A$ and the image of the $\mathcal{N}(\Delta)$-linear map $A: \mathcal{N}(\Delta)^n \to \mathcal{N}(\Delta)^n$ induced by right multiplication with A is $\mathcal{N}(\Delta)$-isomorphic to M. Let $i(A)$ be the matrix in $M(n, n, \mathcal{N}(\Gamma))$ obtained from A by applying i to each entry. Then $i(A) = i(A)^* = i(A)^2 = i(A)$ and the image of the $\mathcal{N}(\Gamma)$-linear map $i(A): \mathcal{N}(\Gamma)^n \to \mathcal{N}(\Gamma)^n$ induced by right multiplication with $i(A)$ is $\mathcal{N}(\Gamma)$-isomorphic to $i_* M$. Hence we get from Definition 1.6

$$\dim_{\mathcal{N}(\Delta)}(M) = \mathrm{tr}_{\mathcal{N}(\Delta)}(A);$$
$$\dim_{\mathcal{N}(\Gamma)}(i_* M) = \mathrm{tr}_{\mathcal{N}(\Gamma)}(i(A)).$$

Therefore it suffices to show $\mathrm{tr}_{\mathcal{N}(\Gamma)}(i(a)) = \mathrm{tr}_{\mathcal{N}(\Delta)}(a)$ for $a \in \mathcal{N}(\Delta)$. This is an easy consequence of the Definition 1.3 of the standard trace.

Step 2. If M is finitely presented $\mathcal{N}(\Delta)$-module, then

$$\dim_{\mathcal{N}(\Delta)}(M) = \dim_{\mathcal{N}(\Gamma)}(i_*(M));$$
$$\mathrm{Tor}_1^{\mathcal{N}(\Delta)}(\mathcal{N}(\Gamma), M) = 0.$$

Since M is finitely presented, it splits as $M = \mathbf{T}M \oplus \mathbf{P}M$ where $\mathbf{P}M$ is finitely generated projective and there is an exact sequence $0 \to \mathcal{N}(\Delta)^n \xrightarrow{f} \mathcal{N}(\Delta)^n \to \mathbf{T}M \to 0$ with $f^* = f$ [21], Theorem 1.2, Lemma 3.4. If we apply the right exact functor induction with i to it, we get an exact sequence $\mathcal{N}(\Gamma)^n \xrightarrow{i_* f} \mathcal{N}(\Gamma)^n \to i_* \mathbf{T}M \to 0$ with $(i_* f)^* = i_* f$. Because of Step 1, Additivity (see Theorem 0.6(4)) and the definition of Tor it suffices to show that $i_* f$ is injective. Let v be the functor introduced in Theorem 1.8 or [21], section 2. Then $i\big(v(f)\big)$ is $v(i_* f)$. Because v respects weak exactness (see Theorem 1.8 or [21], Lemma 2.3) $v(f)$ has dense image since $\mathcal{N}(\Delta)^n \xrightarrow{i_* f} \mathcal{N}(\Delta)^n \to 0$ is weakly exact. Then one easily checks that $v(i_* f) = i_*\big(v(f)\big)$ has dense image since $\mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta)$ is a dense subspace of $l^2(\Gamma)$. Since the kernel of a bounded operator of Hilbert spaces is the orthogonal complement of the image of its adjoint and $v(i_* f)$ is selfadjoint, $v(i_* f)$ is injective. Since v^{-1} respects exactness (see Theorem 1.8 or [21], Lemma 2.3) $i_* f$ is injective.

Step 3. $\operatorname{Tor}_1^{\mathcal{N}(\Delta)}\big(\mathcal{N}(\Gamma), M\big) = 0$ provided, M is a finitely generated $\mathcal{N}(\Delta)$-module.

Choose an exact sequence $0 \to K \xrightarrow{g} P \to M \to 0$ such that P is a finitely generated projective $\mathcal{N}(\Delta)$-module. The associated long exact sequence of Tor-groups shows that $\operatorname{Tor}_1^{\mathcal{N}(\Delta)}\big(\mathcal{N}(\Gamma), M\big)$ is trivial if and only if $i_* g : i_* K \to i_* P$ is injective. For each element x in $i_* K$ there is a finitely generated submodule $K' \subset K$ such that x lies in the image of the map $i_* K' \to i_* K$ induced by the inclusion. Hence it suffices to show for any finitely generated submodule $K' \subset P$ that the inclusion induces an injection $i_* K' \to i_* P$. This follows since Step 2 applied to the finitely presented module P/K shows $\operatorname{Tor}_1^{\mathcal{N}(\Delta)}\big(\mathcal{N}(\Gamma), P/K\big) = 0$.

Step 4. i_* is an exact functor.

By standard homological algebra we have to show that $\operatorname{Tor}^1_{\mathcal{N}(\Delta)}\big(\mathcal{N}(\Gamma), M\big) = 0$ is trivial for all $\mathcal{N}(\Delta)$-modules M. Notice that M is the colimit of the directed system of its finitely generated submodules (directed by inclusion) and that the functor Tor commutes in both variables with colimits over directed systems [6], Proposition VI.1.3 on page 107. Now the claim follows from Step 3.

Step 5. Let $\{M_i \mid i \in I\}$ be the directed system of finitely generated submodules of the $\mathcal{N}(\Delta)$-module M. Then

$$\dim_{\mathcal{N}(\Delta)}(M) = \sup\{\dim_{\mathcal{N}(\Delta)}(M_i) \mid i \in I\};$$
$$\dim_{\mathcal{N}(\Gamma)}(i_* M) = \sup\{\dim_{\mathcal{N}(\Gamma)}(i_* M_i) \mid i \in I\}.$$

Because of Step 4 we can view $i_* M_i$ as a submodule of $i_* M$. Now apply Cofinality (see Theorem 0.6(4).

Step 6. The second assertion of Theorem 3.3 is true.

Because of Step 5 it suffices to prove the claim in the case that M is finitely generated because any module is the colimit of the directed system of its finitely generated submodules. Choose an exact sequence $0 \to K \xrightarrow{g} P \to M \to 0$ such that P is a finitely generated projective $\mathcal{N}(\Delta)$-module. Because of Step 4 and Additivity (see Theorem 0.6) we get

$$\dim_{\mathcal{N}(\Delta)}(M) = \dim_{\mathcal{N}(\Delta)}(P) - \dim_{\mathcal{N}(\Delta)}(K);$$

$$\dim_{\mathcal{N}(\Gamma)}(i_* M) = \dim_{\mathcal{N}(\Gamma)}(i_* P) - \dim_{\mathcal{N}(\Gamma)}(i_* K).$$

Because of Step 1 it remains to prove

$$\dim_{\mathcal{N}(\Delta)}(K) = \dim_{\mathcal{N}(\Gamma)}(i_* K).$$

Because of Step 5 it suffices to treat the case where $K \subset P$ is finitely generated. Since $\mathcal{N}(\Delta)$ is semi-hereditary (see Theorem 0.6(1)) K is finitely generated projective and the claim follows from Step 1.

Step 7. The first assertion of Theorem 3.3 is true.

Because we know already from Step 4 that i_* is exact, it remains to prove for an $\mathcal{N}(\Delta)$-module M

$$i_* M = 0 \quad \Leftrightarrow \quad M = 0.$$

Suppose $i_* M = 0$. In order to show $M = 0$ we have to prove for any $\mathcal{N}(\Delta)$-map $f: \mathcal{N}(\Delta) \to M$ that it is trivial. Let K be the kernel of f. Because i_* is exact by Step 4 and $i_* M = 0$ by assumption, the inclusion induces an isomorphism $i_* K \to i_* \mathcal{N}(\Delta)$. Since $i_* K$ is a finitely generated $\mathcal{N}(\Gamma)$-module and i_* is exact by Step 4, there is a finitely generated submodule $K' \subset K$ such that the inclusion induces an isomorphism $i_* K' \to i_* \mathcal{N}(\Delta)$. Let $\mathcal{N}(\Delta)^m \to K'$ be an epimorphism. Let $g: \mathcal{N}(\Delta)^m \to \mathcal{N}(\Delta)$ be the obvious composition. Because i_* is exact by Step 4 the induced map $i_* g: i_* \mathcal{N}(\Delta)^m \to i_* \mathcal{N}(\Delta)$ is surjective. Hence it remains to prove that g itself is surjective because then $K' = \mathcal{N}(\Delta)$ and the map $f: \mathcal{N}(\Delta) \to M$ is trivial. Since the functors v^{-1} and v of Theorem 1.8 are exact we have to show for a Δ-equivariant bounded operator $h: l^2(\Delta)^m \to l^2(\Delta)$ that h is surjective if $i(h): l^2(\Gamma)^m \to l^2(\Gamma)$ is surjective. Let $\{E_\lambda \mid \lambda \geqq 0\}$ be the spectral family of the positive operator $h \circ h^*$. Then $\{i(E_\lambda) \mid \lambda \geqq 0\}$ is the spectral family of the positive operator $i(h) \circ i(h)^*$. Notice that h resp. $i(h)$ is surjective if and only if $E_\lambda = 0$ resp. $i(E_\lambda) = 0$ for some $\lambda > 0$. Because $E_\lambda = 0$ resp. $i(E_\lambda) = 0$ is equivalent to $\dim_{\mathcal{N}(\Delta)}(\operatorname{im}(v^{-1}(E_\lambda))) = 0$ resp. $\dim_{\mathcal{N}(\Gamma)}(\operatorname{im}(v^{-1}(i(E_\lambda)))) = 0$ and $\operatorname{im}(v^{-1}(i(E_\lambda))) = i_* \operatorname{im}(v^{-1}(E_\lambda))$, the claim follows from Step 6. This finishes the proof of Theorem 3.3. □

The proof of Theorem 3.3 would be obvious if we would know that $\mathcal{N}(\Gamma)$ viewed as an $\mathcal{N}(\Delta)$-module is projective. Notice that this is a stronger statement than proven in Theorem 3.3. One would have to show that the higher Ext-groups instead of the Tor-groups vanish to get this stronger statement. However, the proof for the Tor-groups does not go through directly since the Ext-groups are not compatible with colimits.

Lemma 3.4. *Let $H \subset \Gamma$ be a subgroup. Then*:

(1) $\dim(\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}[\Gamma/H]) = |H|^{-1}$, *where $|H|^{-1}$ is defined to be zero if H is infinite.*

(2) $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}[\Gamma/H]$ *is trivial if and only if H is non-amenable.*

(3) *If Γ is infinite and V is a $\mathbb{C}\Gamma$-module which is finite-dimensional over $\mathbb{C}$, then*

$$\dim(\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} V) = 0.$$

Proof. (3) Since V is finitely generated as $\mathbb{C}\Gamma$-module $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} V$ is a finitely generated $\mathcal{N}(\Gamma)$-module. Because of Theorem 0.6 it suffices to show that there is no $\mathcal{N}(\Gamma)$-homomorphism from $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} V$ to $\mathcal{N}(\Gamma)$. This is equivalent to the claim that there is no $\mathbb{C}\Gamma$-homomorphism from V to $\mathcal{N}(\Gamma)$. Since the map $\mathcal{N}(\Gamma) \to l^2(\Gamma)$ given by evaluation at the unit element $e \in \Gamma \subset l^2(\Gamma)$ is Γ-equivariant and injective it suffices to show that $l^2(\Gamma)$ contains no Γ-invariant linear subspace W which is finite-dimensional as complex vector space. Since any finite-dimensional topological vector space is complete, W is a Hilbert $\mathcal{N}(\Gamma)$-submodule. Let $\mathrm{pr}: l^2(\Gamma) \to l^2(\Gamma)$ be an orthogonal Γ-equivariant projection onto W. Then we get for any $\gamma \in \Gamma$

$$\dim(W) = \langle \mathrm{pr}(\gamma), \gamma \rangle . \tag{3.5}$$

Let $\{v_1, v_2, \ldots, v_r\}$ be an orthonormal basis for the Hilbert subspace $W \subset l^2(\Gamma)$. For $\gamma \in \Gamma$ we write $\mathrm{pr}(\gamma) = \sum_{i=1}^{r} \lambda_i(\gamma) \cdot v_i$. We get from $\|\mathrm{pr}(\gamma)\|^2 \leqq 1$

$$|\lambda_i(\gamma)| \leqq 1 . \tag{3.6}$$

Given $\varepsilon > 0$, we can choose $\gamma(\varepsilon)$ satisfying

$$\langle v_i, \gamma(\varepsilon) \rangle \leqq r^{-1} \cdot \varepsilon \quad \text{for } i = 1, 2, \ldots, r. \tag{3.7}$$

Now (3.6) and (3.7) imply

$$\langle \mathrm{pr}(\gamma(\varepsilon)), \gamma(\varepsilon) \rangle \leqq \varepsilon . \tag{3.8}$$

Since (3.8) holds for all $\varepsilon > 0$, we conclude $\dim(W) = 0$ and hence $W = 0$ from equation (3.5).

(1) and (2) If $i: H \to \Gamma$ is the inclusion, then $i_*(\mathcal{N}(H) \otimes_{\mathbb{C}H} \mathbb{C})$ and $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}[\Gamma/H]$ are isomorphic as $\mathcal{N}(\Gamma)$-modules. Because of Theorem 3.3 it remains to treat the special case $\Gamma = H$ for the first two assertions. The first assertion follows from the third for infinite Γ and is obvious for finite Γ. Next we prove the second assertion.

Let S be a set of generators of Γ. Then $\bigoplus_{s \in S} \mathbb{C}\Gamma \xrightarrow{\bigoplus_{s \in S} r_{s-1}} \mathbb{C}\Gamma \xrightarrow{\varepsilon} \mathbb{C} \longrightarrow 0$ is exact where $\varepsilon(\sum_{\gamma \in \Gamma} \lambda_\gamma \cdot \gamma) = \sum_{\gamma \in \Gamma} \lambda_\gamma$ and r_u denotes right multiplication with $u \in \mathbb{C}\Gamma$. We obtain an exact sequence $\bigoplus_{s \in S} \mathcal{N}(\Gamma) \xrightarrow{\bigoplus_{s \in S} r_{s-1}} \mathcal{N}(\Gamma) \xrightarrow{\varepsilon} \mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C} \to 0$ since the tensor product is right exact. Hence $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}$ is trivial if and only if $\bigoplus_{s \in S} \mathcal{N}(\Gamma) \xrightarrow{\bigoplus_{s \in S} r_{s-1}} \mathcal{N}(\Gamma)$ is surjective. This is equivalent to the existence of a finite subset $T \subset S$ such that

$$\bigoplus_{s \in T} \mathcal{N}(\Gamma) \xrightarrow{\bigoplus_{s \in T} r_{s-1}} \mathcal{N}(\Gamma)$$

is surjective. Let $\Delta \subset \Gamma$ be the subgroup generated by T. Then the map above is induction with the inclusion of $\Delta \subset \Gamma$ applied to $\bigoplus_{t \in T} \mathcal{N}(\Delta) \xrightarrow{\bigoplus_{t \in T} r_{t-1}} \mathcal{N}(\Delta)$. Hence we conclude from Theorem 3.3(1) that $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}$ is trivial if $\mathcal{N}(\Delta) \otimes_{\mathbb{C}\Delta} \mathbb{C}$ is trivial for some finitely generated subgroup $\Delta \subset \Gamma$. Since Γ is amenable if and only if each of its finitely generated subgroups is amenable [28], Proposition 0.16 on page 14, we can assume without loss of generality that Γ is finitely generated, i.e. S is finite. We can also assume that S is symmetric, i.e. $s \in S$ implies $s^{-1} \in S$.

Because the functor v of Theorem 1.8 is exact, $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}$ is trivial if and only if the operator $f\colon \bigoplus_{s\in S} l^2(\Gamma) \xrightarrow{\bigoplus_{s\in S} r_{s-1}} l^2(\Gamma)$ is surjective. This is equivalent to the bijectivity of the operator

$$\frac{1}{2\cdot|S|} f\circ f^*\colon l^2(\Gamma) \xrightarrow{\mathrm{id} - \sum_{s\in S} \frac{1}{|S|}\cdot r_s} l^2(\Gamma)\,.$$

It is bijective if and only if the spectral radius of the operator $l^2(\Gamma) \xrightarrow{\sum_{s\in S} \frac{1}{|S|}\cdot r_s} l^2(\Gamma)$ is different from 1. Since this operator is convolution with a probability distribution whose support contains S, namely

$$P\colon \Gamma \to [0,1], \quad \gamma \mapsto \begin{cases} |S|^{-1}, & \gamma \in S,\\ 0, & \gamma \notin S, \end{cases}$$

the spectral radius is 1 precisely if Γ is amenable [16]. This finishes the proof of Lemma 3.4. □

4. L^2-invariants for arbitrary Γ-spaces

In this section we extend the notion of L^2-Betti numbers for regular coverings of CW-complexes of finite type (i.e. with finite skeletons) with Γ as group of deck transformations to (compactly generated) topological spaces with action of a (discrete) group Γ. We will continue with using Notation 1.1.

Definition 4.1. Let X be a (left) Γ-space and V be a $\mathcal{A}$-$\mathbb{Z}\Gamma$-bimodule. Let $H_p^\Gamma(X;V)$ be *the singular homology of X with coefficients in V*, i.e. the $\mathcal{A}$-module given by the homology of the $\mathcal{A}$-chain complex $V \otimes_{\mathbb{Z}\Gamma} C_*^{\mathrm{sing}}(X)$, where $C_*^{\mathrm{sing}}(X)$ denotes the singular $\mathbb{Z}\Gamma$-chain complex of X. Define the *p-th L^2-Betti number of X with coefficients in V* by

$$b_p^{(2)}(X;V) \coloneqq \dim_{\mathcal{A}}\big(H_p^\Gamma(X;V)\big) \quad \in [0,\infty]\,,$$

and the *p-th L^2-Betti number of the group Γ* by

$$b_p^{(2)}(\Gamma) \coloneqq b_p^{(2)}\big(E\Gamma;\mathcal{N}(\Gamma)\big)\,. \quad \square$$

Next we compare cellular and singular chain complexes and show that it does not matter whether we use singular or cellular chain complexes in the case that X is a Γ-CW-complex. For basic definitions and facts about Γ-CW-complexes we refer for instance to [8], sections II.1 and II.2, [20], sections 1 and 2.

Lemma 4.2. *Let X be a Γ-CW-complex. Then there is a up to $\mathbb{Z}\Gamma$-homotopy unique and in X natural $\mathbb{Z}\Gamma$-chain homotopy equivalence*

$$f(X): C_*^{\mathrm{cell}}(X) \to C_*^{\mathrm{sing}}(X).$$

In particular we get for any $\mathscr{A}$-$\mathbb{Z}\Gamma$-bimodule V an in X and V natural isomorphism

$$H_p\big(V \otimes_{\mathbb{Z}\Gamma} C_*^{\mathrm{cell}}(X)\big) \xrightarrow{\cong} H_p\big(V \otimes_{\mathbb{Z}\Gamma} C_*^{\mathrm{sing}}(X)\big). \quad \square$$

Proof. Obviously the second assertion follows from the first assertion which is proven as follows.

Let Y be a CW-complex with cellular $\mathbb{Z}$-chain complex C_*^{cell} and singular $\mathbb{Z}$-chain complex C_*^{sing}. We define a third (intermediate) $\mathbb{Z}$-chain complex $C_*^{\mathrm{inte}}(Y)$ as the subcomplex of C_*^{sing} whose n-th chain module is the kernel of

$$C_n^{\mathrm{sing}}(Y_n) \xrightarrow{c_n^{\mathrm{sing}}} C_{n-1}^{\mathrm{sing}}(Y_n) \longrightarrow C_{n-1}^{\mathrm{sing}}(Y_n, Y_{n-1}).$$

There are an in Y natural inclusion and an in Y natural epimorphism of $\mathbb{Z}$-chain complexes

$$i(Y): C_*^{\mathrm{inte}}(Y) \to C_*^{\mathrm{sing}}(Y); \tag{4.3}$$

$$p(Y): C_*^{\mathrm{inte}}(Y) \to C_*^{\mathrm{cell}}(Y); \tag{4.4}$$

which induce isomorphisms on homology [20], page 263.

If Γ acts freely on the Γ-CW-complex X, then C_*^{cell} and C_*^{sing} are free $\mathbb{Z}\Gamma$-chain complexes and we get a $\mathbb{Z}\Gamma$-chain homotopy equivalence well-defined up to $\mathbb{Z}\Gamma$-homotopy from the fundamental theorem of homological algebra and the fact that the chain maps (4.3) and (4.4) induce isomorphisms on homology. In the general case one has to go to the orbit category $\mathrm{Or}(\Gamma)$ and apply module theory over this category instead of over $\mathbb{Z}\Gamma$.

The orbit category $\mathrm{Or}(\Gamma)$ has as objects homogenous spaces and as morphisms Γ-maps. The Γ-CW-complex X defines a contravariant functor

$$\underline{X}: \mathrm{Or}(\Gamma) \to \{CW-\mathrm{COMPLEXES}\}, \quad \Gamma/H \mapsto \mathrm{map}(\Gamma/H, X)^\Gamma = X^H.$$

Its composition with the functor C_*^{cell}, C_*^{inte} resp. C_*^{sing} from the category of CW-complexes to the category of chain complexes yields $\mathbb{Z}\mathrm{Or}(\Gamma)$-chain complexes, i.e. contravariant functors

$$C_*^{\mathrm{cell}}(\underline{X}): \mathrm{Or}(\Gamma) \to \{\mathbb{Z}-\text{CHAIN-COMPLEXES}\};$$

$$C_*^{\mathrm{inte}}(\underline{X}): \mathrm{Or}(\Gamma) \to \{\mathbb{Z}-\text{CHAIN-COMPLEXES}\};$$

$$C_*^{\mathrm{sing}}(\underline{X}): \mathrm{Or}(\Gamma) \to \{\mathbb{Z}-\text{CHAIN-COMPLEXES}\}.$$

We obtain natural transformations from the natural chain maps (4.3) and (4.4)

$$i(\underline{X}): C_*^{\mathrm{inte}}(\underline{X}) \to C_*^{\mathrm{sing}}(\underline{X}); \tag{4.5}$$

(4.6) $$p(\underline{X}) : C_*^{\text{inte}}(\underline{X}) \to C_*^{\text{cell}}(\underline{X});$$

which induce isomorphisms on homology. Hence it suffices to show that $C_*^{\text{cell}}(\underline{X})$ and $C_*^{\text{sing}}(\underline{X})$ are free and hence projective in the sense of [20], Definition 9.17 because then we obtain a homotopy equivalence of $\mathbb{Z}\mathrm{Or}(\Gamma)$-chain complexes from $C_*^{\text{cell}}(X)$ to $C_*^{\text{sing}}(X)$ [20], Lemma 11.3 whose evaluation at $\Gamma/1$ is the desired $\mathbb{Z}\Gamma$-chain homotopy equivalence. The proofs that these two chain complexes are free are simple versions of the arguments in [20], Lemma 13.2. Notice that in [20] the Γ-CW-complex is required to be proper, but this condition is needed there only because there Γ is assumed to be a Lie group and universal coverings are built in, and can be dropped in the discrete case. □

Remark 4.7. Originally the L^2-Betti numbers of a regular covering $\bar{M} \to M$ of a closed Riemannian manifold with group of deck transformations Γ were defined by Atiyah [2] in terms of the heat kernel as explained in (0.2) in the introduction. It follows from the L^2-Hodge-deRham theorem [10] that this analytic definition agrees with the combinatorial definition of $b_p^{(2)}(\bar{X})$ in terms of the associated cellular L^2-chain complex and the von Neumann dimension of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules for a triangulation X of M. Because of Lemma 4.2 this combinatorial definition agrees with the Definition 4.1.

Analogously to the case of L^2-Betti numbers we will extend the notion of Novikov-Shubin invariants for regular coverings of compact Riemannian manifolds to arbitrary Γ-spaces and prove that they are positive for the universal covering of an aspherical closed manifold with elementary-amenable fundamental group in another paper. □

The next results are well-known in the case where X is a regular covering of a CW-complex of finite type. We call a map $g : Y \to Z$ *homologically n-connected for* $n \geqq 1$ if the map induced on singular homology with complex coefficients $g_* : H_k^{\text{sing}}(Y; \mathbb{C}) \to H_k^{\text{sing}}(Y; \mathbb{C})$ is bijective for $k < n$ and surjective for $k = n$. The map g is called a *weak homology equivalence* if it is n-connected for all $n \geqq 1$.

Lemma 4.8. *Let $f : X \to Y$ be a Γ-map and let V be a $\mathcal{A}$-$\mathbb{Z}\Gamma$-bimodule.*

(1) *Suppose for $n \geqq 1$ that for each subgroup $H \subset \Gamma$ the induced map $f^H : X^H \to Y^H$ is homologically n-connected. Then the map induced by f*

$$f_* : H_p^\Gamma(X; V) \to H_p^\Gamma(Y; V)$$

is bijective for $p < n$ and surjective for $p = n$ and we get

$$b_p^{(2)}(X; V) = b_p^{(2)}(Y; V) \quad \text{for } p < n;$$

$$b_p^{(2)}(X; V) \geqq b_p^{(2)}(Y; V) \quad \text{for } p = n.$$

(2) *Suppose such that for each subgroup $H \subset \Gamma$ the induced map $f^H : X^H \to Y^H$ is a weak homology equivalence. Then for all $p \geqq 0$ the map induced by f*

$$f_* : H_p^\Gamma(X; V) \to H_p^\Gamma(Y; V)$$

is bijective and we get

$$b_p^{(2)}(X;V) = b_p^{(2)}(Y;V). \quad \square$$

Proof. We give only the proof of the second assertion, the one of the first assertion is an elementary modification. The map f induces a homotopy equivalence of $\mathbb{Z}\mathrm{Or}(\Gamma)$-chain complexes $C_*^{\mathrm{sing}}(\underline{f}) : C_*^{\mathrm{sing}}(\underline{X}) \to C_*^{\mathrm{sing}}(\underline{Y})$ in the notation of the proof of Lemma 4.2 since the singular $\mathbb{Z}\mathrm{Or}(\Gamma)$-chain complexes of $\underline{X}$ and $\underline{Y}$ are free in the sense of [20], Definition 9.17 (see [20], Lemma 11.3). Its evaluation at $\Gamma/1$ is a $\mathbb{Z}\Gamma$-chain equivalence. Hence f induces a chain equivalence

$$V \otimes_{\mathbb{Z}\Gamma} C_*^{\mathrm{sing}}(f) : V \otimes_{\mathbb{Z}\Gamma} C_*^{\mathrm{sing}}(X) \to V \otimes_{\mathbb{Z}\Gamma} C_*^{\mathrm{sing}}(Y)$$

and Lemma 4.8 follows. $\square$

We get as a direct consequence from Theorem 3.3

Theorem 4.9. *Let $i : \Delta \to \Gamma$ be an inclusion of groups and let X be a Δ-space. Then*

$$H_p^\Gamma\big(\Gamma \times_\Delta X; \mathcal{N}(\Gamma)\big) = i_* H_p^\Delta\big(X; \mathcal{N}(\Delta)\big);$$
$$b_p^{(2)}\big(\Gamma \times_\Delta X; \mathcal{N}(\Gamma)\big) = b_p^{(2)}\big(X; \mathcal{N}(\Delta)\big). \quad \square$$

Theorem 4.10. *Let X be a path-connected Γ-space. Then*:

(1) *There is an isomorphism of $\mathcal{N}(\Gamma)$-modules $H_0^\Gamma\big(X; \mathcal{N}(\Gamma)\big) \cong \mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathbb{C}$.*

(2) $b_0^{(2)}\big(X; \mathcal{N}(\Gamma)\big) = |\Gamma|^{-1}$, *where $|\Gamma|^{-1}$ is defined to be zero if the order $|\Gamma|$ of Γ is infinite.*

(3) $H_0^\Gamma\big(X; \mathcal{N}(\Gamma)\big)$ *is trivial if and only if Γ is non-amenable.*

Proof. The first assertion follows from the fact that $C_1^{\mathrm{sing}}(X) \to C_0^{\mathrm{sing}}(X) \to \mathbb{Z} \to 0$ is an exact sequence of $\mathbb{Z}\Gamma$-modules and the tensor product is right exact. The other two assertions follow from Lemma 3.4. $\square$

Remark 4.11. Let $\tilde{M} \to M$ be the universal covering of a closed Riemannian manifold with fundamental group π. Brooks [5] has shown that the analytic Laplace operator Δ_0 on $\tilde{M}$ in dimension zero has zero not in its spectrum if and only if π is non-amenable. Now Δ_0 has zero not in its spectrum if and only if $H_0^\pi\big(\tilde{M}, \mathcal{N}(\pi)\big)$ is trivial because of [21], paragraph after Definition 3.11, Theorem 6.1, and the fact that the analytic and combinatorial spectral density function are dilatationally equivalent [11]. Hence Theorem 4.10 generalizes the result of Brooks. Notice that both Brook's and our proof use [16]. Compare also with the result [15], Corollary III.2.4 on page 188, that a group Γ is non-amenable if and only if $H^1\big(\Gamma, l^2(\Gamma)\big)$ is Hausdorff. $\square$

Remark 4.12. Next we compare our approach with the one in [7], section 2. We begin with the case of a countable simplicial complex X with free simplicial Γ-action. Then for any exhaustion $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$ by Γ-equivariant simplicial subcomplexes for which X/Γ is compact, the p-th L^2-Betti number in the sense and notation of [7], 2.8 on page 198 is given by

$$b_p^{(2)}(X:\Gamma) = \lim_{j\to\infty} \lim_{k\to\infty} \dim_{\mathcal{N}(\Gamma)}\bigl(\operatorname{im}\bigl(\bar{H}_{(2)}^p(X_k:\Gamma)\bigr) \xrightarrow{i_{j,k}^*} \bar{H}_{(2)}^p(X_j:\Gamma)\bigr),$$

where $i_{j,k}: X_j \to X_k$ is the inclusion for $j \leqq k$. We get from [21], Lemma 1.3, and Lemma 4.2

$$\begin{aligned}&\dim_{\mathcal{N}(\Gamma)}\bigl(\operatorname{im}\bigl(\bar{H}_{(2)}^p(X_k:\Gamma)\bigr) \xrightarrow{i_{j,k}^*} \bar{H}_{(2)}^p(X_j:\Gamma)\bigr)\\ &= \dim_{\mathcal{N}(\Gamma)}\bigl(\operatorname{im}\bigl(\bar{H}_p^\Gamma(X_j;\mathcal{N}(\Gamma))\bigr) \xrightarrow{(i_{j,k})_*} \bar{H}_p^\Gamma(X_k;\mathcal{N}(\Gamma))\bigr).\end{aligned}$$

Hence we conclude from Theorem 2.9 that the definitions in [7], 2.8 on page 198, and in 4.1 agree:

$$b_p^{(2)}(X:\Gamma) = b_p^{(2)}\bigl(X;\mathcal{N}(\Gamma)\bigr). \tag{4.13}$$

If Γ is countable and X is a countable simplicial complex with simplicial Γ-action, then by [7], Proposition 2.2 on page 198, and by (4.13)

$$b_p^{(2)}(X:\Gamma) = b_p^{(2)}(E\Gamma \times X:\Gamma); \tag{4.14}$$

$$b_p^{(2)}(X:\Gamma) = b_p^{(2)}\bigl(E\Gamma \times X;\mathcal{N}(\Gamma)\bigr). \tag{4.15}$$

Cheeger and Gromov [7], Section 2, define L^2-cohomology and L^2-Betti numbers of a Γ-space X by considering the category whose objects are Γ-maps $f: Y \to X$ for a simplicial complex Y with cocompact free simplicial Γ-action and then using inverse limits to extend the classic notions for finite free Γ-CW-complexes such as Y to X. Our approach avoids the technical difficulties concerning inverse limits and is closer to standard notions, the only non-standard part is the verification of the properties of the extended dimension function (Theorem 0.6 and Theorem 3.3).

5. Amenable groups

In this section we investigate amenable groups. For information about amenable groups we refer for instance to [28]. The main technical result of this section is the next lemma whose proof uses ideas of the proof of [7], Lemma 3.1 on page 203.

Theorem 5.1. *Let Γ be amenable and M be a $\mathbb{C}\Gamma$-module. Then*

$$\dim_{\mathcal{N}(\Gamma)}\bigl(\operatorname{Tor}_p^{\mathbb{C}\Gamma}\bigl(\mathcal{N}(\Gamma), M\bigr)\bigr) = 0 \quad \text{for } p \geqq 1,$$

where we consider $\mathcal{N}(\Gamma)$ as an $\mathcal{N}(\Gamma)$-$\mathbb{C}\Gamma$-bimodule.

Proof. *Step* 1. If M is a finitely presented $\mathbb{C}\Gamma$-module, then

$$\dim_{\mathcal{N}(\Gamma)}\bigl(\operatorname{Tor}_1^{\mathbb{C}\Gamma}\bigl(\mathcal{N}(\Gamma), M\bigr)\bigr) = 0.$$

Choose a finite presentation

$$\bigoplus_{i=1}^{m} \mathbb{C}\Gamma \xrightarrow{f} \bigoplus_{i=1}^{n} \mathbb{C}\Gamma \xrightarrow{p} M \longrightarrow 0 .$$

For an element $u = \sum_{\gamma} \lambda_\gamma \cdot \gamma$ in $l^2(\Gamma)$ define its *support*

$$\operatorname{supp}(u) := \{\gamma \in \Gamma \mid \lambda_\gamma \neq 0\} \quad \subset \Gamma .$$

Let $B \in M(m, n, \mathbb{C}\Gamma)$ be the matrix describing f, i.e. the component $f_{i,j} : \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ is given by right multiplication with $b_{i,j}$. Define the finite subset S by

$$S := \{\gamma \mid \gamma \text{ or } \gamma^{-1} \in \bigcup_{i,j} \operatorname{supp}(b_{i,j})\} .$$

Let $f^{(2)} : \bigoplus_{i=1}^{m} l^2(\Gamma) \to \bigoplus_{i=1}^{n} l^2(\Gamma)$ be the bounded Γ-equivariant operator induced by f. Denote by K the Γ-invariant linear subspace of $\bigoplus_{i=1}^{m} l^2(\Gamma)$ which is the image of the kernel of f under the canonical inclusion $k : \bigoplus_{i=1}^{m} \mathbb{C}\Gamma \to \bigoplus_{i=1}^{m} l^2(\Gamma)$. Next we show for the closure $\bar{K}$ of K

$$\bar{K} = \ker(f^{(2)}) . \tag{5.2}$$

Let $\mathrm{pr} : \bigoplus_{i=1}^{m} l^2(\Gamma) \to \bigoplus_{i=1}^{m} l^2(\Gamma)$ be the orthogonal projection onto the closed Γ-invariant subspace $\bar{K}^\perp \cap \ker(f^{(2)})$. The von Neumann dimension of $\operatorname{im}(\mathrm{pr})$ is zero if and only if pr itself is zero. Hence (5.2) will follow if we can prove

$$\operatorname{tr}_{\mathcal{N}(\Gamma)}(\mathrm{pr}) = 0 . \tag{5.3}$$

Let $\varepsilon > 0$ be given. Since Γ is amenable, there is a finite non-empty subset $A \subset \Gamma$ satisfying [3], Theorem F.6.8 on page 308,

$$\frac{|\partial_S A|}{|A|} \leqq \varepsilon , \tag{5.4}$$

where $\partial_S A$ is defined by $\{a \in A \mid \text{there is } s \in S \text{ with } as \notin A\}$. Define

$$\Delta := \{\gamma \in \Gamma \mid \gamma \in \partial_S A \text{ or } \gamma s \in \partial_S A \text{ for some } s \in S\} = \partial_S A \cup \Big(\bigcup_{s \in S} (\partial_S A) s\Big) .$$

Let $\mathrm{pr}_A : l^2(\Gamma) \to l^2(\Gamma)$ be the projection sending $\sum_{\gamma \in \Gamma} \lambda_\gamma \cdot \gamma$ to $\sum_{\gamma \in A} \lambda_\gamma \cdot \gamma$. Define $\mathrm{pr}_\Delta : l^2(\Gamma) \to l^2(\Gamma)$ analogously. Next we show for $s \in S$ and $u \in l^2(\Gamma)$

$$\mathrm{pr}_A \circ r_s(u) = r_s \circ \mathrm{pr}_A(u), \quad \text{if } \mathrm{pr}_\Delta(u) = 0, \tag{5.5}$$

where $r_s : l^2(\Gamma) \to l^2(\Gamma)$ is right multiplication with s. Since $s \in S$ implies $s^{-1} \in S$, we get the following equality of subsets of Γ:

$$\{\gamma \in \Gamma \mid \gamma s \in A, \gamma \notin \Delta\} = \{\gamma \in \Gamma \mid \gamma \in A, \gamma \notin \Delta\}.$$

Now (5.5) follows from the following calculation for $u = \sum_{\gamma \in \Gamma, \gamma \notin \Delta} \lambda_\gamma \cdot \gamma \in l^2(\Gamma)$:

$$\begin{aligned} \mathrm{pr}_A \circ r_s(u) &= \sum_{\gamma s \in A, \gamma \notin \Delta} \lambda_\gamma \cdot \gamma s \\ &= \sum_{\gamma \in A, \gamma \notin \Delta} \lambda_\gamma \cdot \gamma s \\ &= \Big(\sum_{\gamma \in A, \gamma \notin \Delta} \lambda_\gamma \cdot \gamma \Big) \cdot s \\ &= r_s \circ \mathrm{pr}_A(u). \end{aligned}$$

We have defined S such that each entry in the matrix B describing f is a linear combination of elements in S. Hence (5.5) implies

$$\Big(\bigoplus_{j=1}^{n} \mathrm{pr}_A\Big) \circ f^{(2)}(u) = f^{(2)} \circ \Big(\bigoplus_{i=1}^{m} \mathrm{pr}_A\Big)(u), \quad \text{if } \mathrm{pr}_A(u_i) = 0 \text{ for } i = 1, 2, \ldots, m.$$

Notice that the image of $\bigoplus_{i=1}^{m} \mathrm{pr}_A$ lies in $\bigoplus_{i=1}^{m} \mathbb{C}\Gamma$. We conclude

$$\bigoplus_{i=1}^{m} \mathrm{pr}_A(u) \in K, \quad \text{if } u \in \ker(f^{(2)}),\ \mathrm{pr}_\Delta(u_i) = 0 \text{ for } i = 1, 2, \ldots, m.$$

This shows

$$\Big(\mathrm{pr} \circ \bigoplus_{i=1}^{m} \mathrm{pr}_A\Big)\Big(\ker(f^{(2)}) \cap \bigoplus_{i=1}^{m} \ker(\mathrm{pr}_\Delta)\Big) = 0.$$

Since $\ker(\mathrm{pr}_\Delta)$ has complex codimension $|\Delta|$ in $l^2(\Gamma)$ and $|\Delta| \leqq (|S| + 1) \cdot |\partial_S A|$, we conclude for the complex dimension $\dim_\mathbb{C}$ of complex vector spaces

$$\dim_\mathbb{C}\Big(\Big(\mathrm{pr} \circ \bigoplus_{i=1}^{m} \mathrm{pr}_A\Big)\big(\ker(f^{(2)})\big)\Big) \leqq m \cdot (|S| + 1) \cdot |\partial_S A|. \tag{5.6}$$

Since $\mathrm{pr} \circ \mathrm{pr}_A$ is an endomorphism of Hilbert spaces with finite-dimensional image, it is trace-class and its trace $\mathrm{tr}_\mathbb{C}(\mathrm{pr} \circ \mathrm{pr}_A)$ is defined. We get

$$\mathrm{tr}_{\mathcal{N}(\Gamma)}(\mathrm{pr}) \leqq \frac{\mathrm{tr}_\mathbb{C}\big(\mathrm{pr} \circ \bigoplus_{i=1}^{m} \mathrm{pr}_A\big)}{|A|} \tag{5.7}$$

from the following computation for $e \in \Gamma \subset l^2(\Gamma)$ the unit element

$$\begin{aligned}
\operatorname{tr}_{\mathcal{N}(\Gamma)}(\mathrm{pr}) &= \sum_{i=1}^{m} \langle \mathrm{pr}_{i,i}(e), e \rangle \\
&= \frac{1}{|A|} \sum_{i=1}^{m} |A| \cdot \langle \mathrm{pr}_{i,i}(e), e \rangle \\
&= \frac{1}{|A|} \sum_{i=1}^{m} \sum_{\gamma \in A} \langle \mathrm{pr}_{i,i}(\gamma), \gamma \rangle \\
&= \frac{1}{|A|} \sum_{i=1}^{m} \sum_{\gamma \in A} \langle \mathrm{pr}_{i,i} \circ \mathrm{pr}_A(\gamma), \gamma \rangle \\
&= \frac{1}{|A|} \sum_{i=1}^{m} \sum_{\gamma \in \Gamma} \langle \mathrm{pr}_{i,i} \circ \mathrm{pr}_A(\gamma), \gamma \rangle \\
&= \frac{1}{|A|} \sum_{i=1}^{m} \operatorname{tr}_{\mathbb{C}}(\mathrm{pr}_{i,i} \circ \mathrm{pr}_A) \\
&= \frac{1}{|A|} \operatorname{tr}_{\mathbb{C}}\big(\mathrm{pr} \circ (\sum_{i=1}^{m} \mathrm{pr}_A)\big).
\end{aligned}$$

If H is a Hilbert space and $f \colon H \to H$ is a bounded operator with finite-dimensional image, then $\operatorname{tr}_{\mathbb{C}}(f) \leqq \|f\| \cdot \dim_{\mathbb{C}}\big(f(\operatorname{im}(f))\big)$. Since the image of pr is contained in $\ker(f^{(2)})$ and pr and pr_A have operator norm 1, we conclude

$$\operatorname{tr}_{\mathbb{C}}(\mathrm{pr} \circ \bigoplus_{i=1}^{m} \mathrm{pr}_A) \leqq \dim_{\mathbb{C}}\big((\mathrm{pr} \circ \bigoplus_{i=1}^{m} \mathrm{pr}_A)\big(\ker(f^{(2)})\big)\big). \tag{5.8}$$

Equations (5.4), (5.6), (5.7) and (5.8) imply

$$\operatorname{tr}_{\mathcal{N}(\Gamma)}(\mathrm{pr}) \leqq m \cdot (|S| + 1) \cdot \varepsilon .$$

Since this holds for all $\varepsilon > 0$, we get (5.3) and hence (5.2) is true.

Let $\mathrm{pr}_{\bar{K}} \colon \bigoplus_{i=1}^{m} l^2(\Gamma) \to \bigoplus_{i=1}^{m} l^2(\Gamma)$ be the projection onto $\bar{K}$. Let $i \colon \ker(f) \to \bigoplus_{i=1}^{m} \mathbb{C}\Gamma$ be the inclusion. It induces a map

$$j \colon \mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \ker(f) \xrightarrow{\mathrm{id} \otimes_{\mathbb{C}\Gamma} i} \mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \bigoplus_{i=1}^{m} \mathbb{C}\Gamma \xrightarrow{\cong} \bigoplus_{i=1}^{m} \mathcal{N}(\Gamma).$$

Next we want to show

$$\operatorname{im}\big(v^{-1}(\mathrm{pr}_{\bar{K}})\big) = \overline{\operatorname{im}(j)}. \tag{5.9}$$

Let $x \in \ker(f)$. Then

$$\big(\mathrm{id} - v^{-1}(\mathrm{pr}_{\bar{K}})\big) \circ j(1 \otimes x) = (\mathrm{id} - \mathrm{pr}_{\bar{K}}) \circ k \circ i(x), \tag{5.10}$$

where $k: \bigoplus_{i=1}^{m} \mathbb{C}\Gamma \to \bigoplus_{i=1}^{m} l^2(\Gamma)$ is the inclusion. Since $(\mathrm{id} - \mathrm{pr}_{\bar{K}})$ is trivial on K we get $(\mathrm{id} - \mathrm{pr}_{\bar{K}}) \circ k \circ i = 0$. Now we conclude from (5.10) that $\mathrm{im}(j) \subset \ker(\mathrm{id} - v^{-1}(\mathrm{pr}_{\bar{K}}))$ and hence $\mathrm{im}(j) \subset \mathrm{im}(v^{-1}(\mathrm{pr}_{\bar{K}}))$ holds. This shows $\overline{\mathrm{im}(j)} \subset \mathrm{im}(v^{-1}(\mathrm{pr}_{\bar{K}}))$. It remains to prove for any $\mathcal{N}(\Gamma)$-map $g: \bigoplus_{i=1}^{m} \mathcal{N}(\Gamma) \to \bigoplus_{i=1}^{m} \mathcal{N}(\Gamma)$ with $\mathrm{im}(j) \subset \ker(g)$ that $g \circ v^{-1}(\mathrm{pr}_{\bar{K}})$ is trivial. Obviously $K \subset \ker(v(g))$. Since $\ker(v(g))$ is a closed subspace, we get $\bar{K} \subset \ker(v(g))$. We conclude $v(g) \circ \mathrm{pr}_{\bar{K}} = 0$ and hence $g \circ v^{-1}(\mathrm{pr}_{\bar{K}}) = 0$. This finishes the proof of (5.9).

Since v^{-1} preserves exactness by Theorem 1.8 and $\mathrm{id} \otimes_{\mathbb{C}\Gamma} f = v^{-1}(f^{(2)})$, we conclude from (5.2) and (5.9) that the sequence

$$\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \ker(f) \xrightarrow{\mathrm{id} \otimes_{\mathbb{C}\Gamma} i} \mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \bigoplus_{i=1}^{m} \mathbb{C}\Gamma \xrightarrow{\mathrm{id} \otimes_{\mathbb{C}\Gamma} f} \mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \bigoplus_{i=1}^{m} \mathbb{C}\Gamma$$

is weakly exact. Continuity of the dimension function (see Theorem 0.6(4)) implies

$$\dim_{\mathcal{N}(\Gamma)}(\ker(\mathrm{id} \otimes_{\mathbb{C}\Gamma} f)/\mathrm{im}(\mathrm{id} \otimes_{\mathbb{C}\Gamma} i)) = 0\,.$$

Since $\mathrm{Tor}_1^{\mathbb{C}\Gamma}(\mathcal{N}(\Gamma), M) = \ker(\mathrm{id} \otimes_{\mathbb{C}\Gamma} f)/\mathrm{im}(\mathrm{id} \otimes_{\mathbb{C}\Gamma} i)$ holds, Step 1 follows.

Step 2. If M is a $\mathbb{C}\Gamma$-module, then $\dim_{\mathcal{N}(\Gamma)}(\mathrm{Tor}_1^{\mathbb{C}\Gamma}(\mathcal{N}(\Gamma), M)) = 0$.

Obviously M is the union of its finitely generated submodules. Any finitely generated module M is a colimit over a directed system of finitely presented modules, namely, choose an epimorphism from a finitely generated free module F to M with kernel K. Since K is the union of its finitely generated submodules, M is the colimit of the directed system F/L where L runs over the finitely generated submodules of K. The functor Tor commutes in both variables with colimits over directed system [6], Proposition VI.1.3 on page 107. Now the claim follows from Step 1 and Theorem 2.9.

Step 3. If M is a $\mathbb{C}\Gamma$-module, then $\dim_{\mathcal{N}(\Gamma)}(\mathrm{Tor}_p^{\mathbb{C}\Gamma}(\mathcal{N}(\Gamma), M)) = 0$ for all $p \geqq 1$.

We use induction over $p \geqq 1$. The induction begin is already done in Step 2. Choose an exact sequence $0 \to N \to F \to M \to 0$ of $\mathcal{N}(\Gamma)$-modules such that F is free. Then we obtain an isomorphism $\mathrm{Tor}_p^{\mathbb{C}\Gamma}(\mathcal{N}(\Gamma), M) \cong \mathrm{Tor}_{p-1}^{\mathbb{C}\Gamma}(\mathcal{N}(\Gamma), N)$ and the induction step follows. This finishes the proof of Theorem 5.1. □

Theorem 5.11. *Let Γ be an amenable group and X a Γ-space. Then*

$$b_p^{(2)}(X; \mathcal{N}(\Gamma)) = \dim_{\mathcal{N}(\Gamma)}(\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} H_p^{\mathrm{sing}}(X; \mathbb{C}))$$

where $H_p^{\mathrm{sing}}(X; \mathbb{C})$ is the $\mathbb{C}\Gamma$-module given by the singular homology of X with complex coefficients. In particular $b_p^{(2)}(X; \mathcal{N}(\Gamma))$ depends only on the $\mathbb{C}\Gamma$-module $H_p^{\mathrm{sing}}(X; \mathbb{C})$.

Proof. We have to show for a $\mathbb{C}\Gamma$-chain complex C_* with $C_p = 0$ for $p < 0$

$$\text{(5.12)} \qquad \dim\big(H_p(\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} C_*)\big) = \dim\big(\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} H_p(C_*)\big).$$

We begin with the case where C_* is projective. Then there is a universal coefficient spectral sequence converging to $H_{p+q}(\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} C_*)$ [30], Theorem 5.6.4 on page 143, whose E^2-term is $E^2_{p,q} = \mathrm{Tor}_p^{\mathbb{C}\Gamma}(\mathcal{N}(\Gamma), H_q(C_*))$. Now Additivity of $\dim_{\mathcal{N}(\Gamma)}$ (see Theorem 0.6(4)) together with Theorem 5.1 imply (5.12) if C_* is projective.

Next we prove (5.12) in the case where C_* is acyclic. One reduces the claim to two-dimensional C_* and then checks this special case using long exact Tor-sequences, Additivity (see Theorem 0.6(4)) and Theorem 5.1.

In the general case one chooses a projective $\mathbb{C}\Gamma$-chain complex P_* together with a $\mathbb{C}\Gamma$-chain map $f_*: P_* \to C_*$ which induces an isomorphism on homology. Since the mapping cylinder is $\mathbb{C}\Gamma$-chain homotopy equivalent to C_*, the mapping cone of f_* is acyclic and hence (5.12) is true for P_* and the mapping cone, we get (5.12) for C_* from Additivity (see Theorem 0.6(4)). This finishes the proof of Theorem 5.11. □

We obtain as an immediate corollary from Theorem 4.10 and Theorem 5.11 (cf. [7], Theorem 0.2 on page 191)

Corollary 5.13. *If Γ is infinite amenable, then for all $p \geqq 0$*

$$b_p^{(2)}(\Gamma) = 0. \quad \square$$

Remark 5.14. It is likely that Theorem 5.1 characterizes amenable groups. Namely, if Γ contains a free group F of rank two, then Γ is non-amenable and Theorem 5.1 becomes false because Theorem 3.3 implies

$$\begin{aligned}
\dim_{\mathcal{N}(\Gamma)}\big(\mathrm{Tor}_1^{\mathbb{C}\Gamma}(\mathcal{N}(\Gamma), \mathbb{C}[\Gamma/F])\big) &= \dim_{\mathcal{N}(\Gamma)}\big(\mathrm{Tor}_1^{\mathbb{C}F}(\mathcal{N}(\Gamma), \mathbb{C})\big) \\
&= \dim_{\mathcal{N}(\Gamma)}\big(\mathcal{N}(\Gamma) \otimes_{\mathcal{N}(F)} \mathrm{Tor}_1^{\mathbb{C}F}(\mathcal{N}(F), \mathbb{C})\big) \\
&= \dim_{\mathcal{N}(F)}\big(\mathrm{Tor}_1^{\mathbb{C}F}(\mathcal{N}(F), \mathbb{C})\big) \\
&= b_1^{(2)}(F) \\
&= -\chi(BF) \\
&= 1. \quad \square
\end{aligned}$$

Remark 5.15. In view of Theorem 5.1 the question arises when $\mathcal{N}(\Gamma)$ is flat over $\mathbb{C}\Gamma$. Except for virtually cyclic groups, i.e. groups which are finite or contain an infinite cyclic subgroup of finite index, we know no examples of finitely presented groups Γ such that $\mathcal{N}(\Gamma)$ is flat over $\mathbb{C}\Gamma$. If $\mathcal{N}(\Gamma)$ is flat over $\mathbb{C}\Gamma$, then $\mathcal{N}(\Delta)$ is flat over $\mathbb{C}\Delta$ for any subgroup $\Delta \subset \Gamma$ by Theorem 3.3(1). Moreover, $\mathcal{N}(\Gamma)$ is flat over $\mathbb{C}\Gamma$ if and only if $\mathcal{N}(\Delta)$ is flat over $\mathbb{C}\Delta$ for any finitely generated subgroup $\Delta \subset \Gamma$. This follows from Theorem 3.3(1) and the facts that the functor Tor commutes in both variables with colimit over directed systems [6], Proposition VI.1.3 on page 107, any $\mathbb{C}\Gamma$-module is the colimit of its finitely generated submodules, any finitely generated $\mathbb{C}\Gamma$-module is the colimit of a directed system of finitely presented $\mathbb{C}\Gamma$-modules and any finitely presented $\mathbb{C}\Gamma$-submodule is obtained by induction from a finitely presented $\mathbb{C}\Delta$-module for a finitely generated subgroup $\Delta \subset \Gamma$. It is not hard to check that $\mathcal{N}(\Gamma)$ is flat over $\mathbb{C}\Gamma$, if $\mathcal{N}(\Delta)$ is flat over $\mathbb{C}\Delta$ for some subgroup $\Delta \subset \Gamma$ of finite index and that $\mathcal{N}(\mathbb{Z})$ is flat over $\mathbb{C}\mathbb{Z}$ using the fact that $\mathbb{C}G$ is semi-simple for finite G and $\mathbb{C}\mathbb{Z}$ is a principal ideal domain. In particular $\mathcal{N}(\Gamma)$ is flat over $\mathbb{C}\Gamma$ if Γ is virtually cyclic.

Now suppose that $\mathcal{N}(\Gamma)$ is flat over $\mathbb{C}\Gamma$. If $B\Gamma$ is a CW-complex of finite type, then $b_p^{(2)}\big(E\Gamma; \mathcal{N}(\Gamma)\big) = 0$ for $p \geqq 1$ and the p-th Novikov-Shubin invariant satisfies $\alpha_p(B\Gamma) = \infty^+$ for $p \geqq 2$. This implies for instance that Γ does not contain a subgroup which is isomorphic to $\mathbb{Z} * \mathbb{Z}$ (Remark 5.14) or $\mathbb{Z} \times \mathbb{Z}$ [17], Proposition 39 on page 494. If Γ is non-amenable and $B\Gamma$ is a finite CW-complex, then $B\Gamma$ is a counterexample to the zero-in-the-spectrum-conjecture [18], [22], section 11. □

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