Isomorphisms Conjectures

Wolfgang Lück
Münster
Germany
email lueck@math.uni-muenster.de
http://www.math.uni-muenster.de/u/lueck/

Winter term 08/09

Introduction

- These slides cover parts of the course on Isomorphisms Conjectures from the winter term 80/09 but also contain some additional material which will not be presented in the lectures.
- In the actual talks more background information and more examples are given on the blackboard.

Definition (G-CW-complex)

A *G-CW-complex* X is a G-space together with a G-invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n \geq 0} X_n = X$$

such that X carries the colimit topology with respect to this filtration, and X_n is obtained from X_{n-1} for each $n \geq 0$ by attaching equivariant n-dimensional cells, i.e., there exists a G-pushout

$$\begin{tikzcd}
\coprod_{i \in I_n} G/H_i \times S^{n-1} \arrow[r, "\coprod_{i \in I_n} q_i^n"] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_{i \in I_n} G/H_i \times D^n \arrow[r, "\coprod_{i \in I_n} Q_i^n"] & X_n
\end{tikzcd}$$

Example (Simplicial actions)

Let X be a simplicial complex. Suppose that G acts simplicially on X. Then G acts simplicially also on the barycentric subdivision X', and the G-space X' inherits the structure of a G-CW-complex.

Example (Smooth actions)

If G acts properly and smoothly on a smooth manifold M, then M inherits the structure of G-CW-complex.

- A G-CW-complex X is the same as a CW-complex with a G-action such that for any open cell e with $g \cdot e \cap e \neq \emptyset$ we have $gx = x$ for all $x \in e$.

Definition (G-homology theory)

A *G-homology theory* $\mathcal{H}_*$ is a covariant functor from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded abelian groups together with natural transformations

$$\partial_n(X,A)\colon \mathcal{H}_n(X,A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- G-homotopy invariance;
- Long exact sequence of a pair;
- Mayer-Vietoris sequence;
- Disjoint union axiom.

Example (Bredon homology)

Consider any covariant functor

$$M\colon \mathrm{Or}G \to \mathbb{Z}-\mathrm{Modules}.$$

Then there is up to natural equivalence of G-homology theories precisely one G-homology theory $H_*^G(-, M)$, called Bredon homology, with the property that the covariant functor

$$H_n^G\colon \mathrm{Or}G \to \mathbb{Z}-\mathrm{Modules}, \quad G/H \mapsto H_n^G(G/H)$$

is trivial for $n \neq 0$ and naturally equivalent to M for $n = 0$.

- Let M be the constant functor with value the abelian group A. Then we get for every G-CW-complex X

$$H_n^G(X; M) \cong H_n(G\backslash X; A).$$

Definition (Equivariant homology theory)

An *equivariant homology theory* $\mathcal{H}_*^?$ assigns to every group G a G-homology theory $\mathcal{H}_*^G$. These are linked together with the following so called *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a H-CW-pair (X, A) there are for all $n \in \mathbb{Z}$ natural homomorphisms

$$\operatorname{ind}_\alpha\colon \mathcal{H}_n^H(X, A) \quad \to \quad \mathcal{H}_n^G(\operatorname{ind}_\alpha(X, A))$$

satisfying:

- Bijectivity
  If $\ker(\alpha)$ acts freely on X, then $\operatorname{ind}_\alpha$ is a bijection;
- Compatibility with the boundary homomorphisms;
- Functoriality in α;
- Compatibility with conjugation.

Example (Equivariant homology theories)

- Given a $\mathcal{K}_*$ non-equivariant homology theory, put

$$
\begin{array}{rcll}
\mathcal{H}_*^G(X) & := & \mathcal{K}_*(X/G); & \\
\mathcal{H}_*^G(X) & := & \mathcal{K}_*(EG \times_G X) & \text{Borel homology.}
\end{array}
$$

- Equivariant bordism $\Omega_*^?(X)$;
- Equivariant topological K-homology $K_*^?(X)$ in the sense of Kasparov. Recall for $H \subseteq G$ finite

$$
K_n^G(G/H) \cong K_n^H(\mathrm{pt}) \cong \begin{cases} R_{\mathbb{C}}(H) & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}
$$

Example (Bredon homology)

- Consider a covariant functor

$$M\colon \mathrm{Groupoids} \to \mathbb{Z}-\mathrm{Modules}.$$

- Given a G-set S, let $\mathcal{G}^G(S)$ be the associated transport groupoid. The set of objects is S. The set of morphisms from s_1 to s_2 is $\{g \in G \mid gs_1 = s_2\}$.
- Composing M with the covariant functor $\mathcal{G}^G\colon \mathrm{Or}G \to \mathrm{Groupoids}$ yields a covariant functor $M^G\colon \mathrm{Or}G \to \mathbb{Z}-\mathrm{Modules}$.
- Let $H_*^G(X; M^G)$ be the G-homology theory given by the Bredon homology with coefficients in M^G.
- Then the collection H_*^G defines an equivariant homology theory $H_*^?(-; M)$.

Theorem (Equivalences of homology theories)

Let $\mathcal{H}_*^G$ and $\mathcal{K}_*^G$ be G-homology theories. Let $t_*^G \colon \mathcal{H}_*^G \to \mathcal{K}_*^G$ be a transformation of G-homology theories. Suppose that for any subgroup $H \subseteq G$ and $n \in \mathbb{Z}$, the map $t_n^G(G/H) \colon \mathcal{H}_n^G(G/H) \to \mathcal{K}_n^G(G/H)$ is bijective. Then for every G-CW-complex X and $n \in \mathbb{Z}$ the map

$$t_n^G(X) \colon \mathcal{H}_n^G(X) \to \mathcal{K}_n^G(X)$$

is bijective.

Definition (Spectrum)

A *spectrum*

$$\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$$

is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps*

$$\sigma(n)\colon E(n) \wedge S^1 \longrightarrow E(n+1).$$

A *map of spectra*

$$\mathbf{f}\colon \mathbf{E} \to \mathbf{E}'$$

is a sequence of maps $f(n)\colon E(n) \to E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e., $f(n+1) \circ \sigma(n) \; = \; \sigma'(n) \circ (f(n) \wedge \mathrm{id}_{S^1})$ holds for all $n \in \mathbb{Z}$.

- Given a spectrum $\mathbf{E}$, a classical construction in algebraic topology assigns to it a homology theory $H_*(-, \mathbf{E})$ with the property

$$H_n(\mathrm{pt}; \mathbf{E}) = \pi_n(\mathbf{E}).$$

  Put

$$H_n(X; \mathbf{E}) := \pi_n\left(X_+ \wedge \mathbf{E}\right).$$

- The basic example of a spectrum is the sphere spectrum $\mathbf{S}$. Its n-th space is S^n and its n-th structure map is the standard homeomorphism $S^n \wedge S^1 \xrightarrow{\cong} S^{n+1}$.
  Its associated homology theory is stable homotopy $\pi_*^s(-) = H_*(-; \mathbf{S})$.
- This construction can be extended to the equivariant setting as follows.

Theorem (Lück-Reich (2005))

Given a functor $\mathbf{E}\colon \mathrm{Groupoids} \to \mathrm{Spectra}$ *sending equivalences to weak equivalences, there exists an equivariant homology theory* $\mathcal{H}_*^?(-;\mathbf{E})$ *satisfying*

$$\mathcal{H}_n^H(pt) \cong \mathcal{H}_n^G(G/H) \cong \pi_n(\mathbf{E}(H)).$$

Theorem (Equivariant homology theories associated to K and L-theory, Davis-Lück (1998))

Let R be a ring (with involution). There exist covariant functors

$$\begin{aligned} \mathbf{K}_R, \mathbf{L}_R^{\langle -\infty \rangle}, \mathbf{K}_{l^1}^{\mathrm{top}} \colon \mathrm{Groupoids} &\to \mathrm{Spectra}; \\ \mathbf{K}^{\mathrm{top}} \colon \mathrm{Groupoids}^{\mathrm{inj}} &\to \mathrm{Spectra}, \end{aligned}$$

with the following properties:

- *They send equivalences to weak equivalences;*
- *For every group G and all $n \in \mathbb{Z}$ we have:*

$$\begin{aligned} \pi_n(\mathbf{K}_R(G)) &\cong K_n(RG); \\ \pi_n(\mathbf{L}_R^{\langle -\infty \rangle}(G)) &\cong L_n^{\langle -\infty \rangle}(RG); \\ \pi_n(\mathbf{K}^{\mathrm{top}}(G)) &\cong K_n(C_r^*(G)); \\ \pi_n(\mathbf{K}_{l^1}^{\mathrm{top}}(G)) &\cong K_n(l^1(G)). \end{aligned}$$

Example (Equivariant homology theories associated to K and L-theory)

We get equivariant homology theories:

$$\begin{gathered} H_*^?(-;\mathbf{K}_R); \\ H_*^?(-;\mathbf{L}_R^{\langle -\infty\rangle}); \\ H_*^?(-;\mathbf{K}^{\mathrm{top}}); \\ H_*^?(-;\mathbf{K}_{l^1}^{\mathrm{top}}), \end{gathered}$$

satisfying for $H \subseteq G$:

$$\begin{array}{lclcl} H_n^G(G/H;\mathbf{K}_R) & \cong & H_n^H(\mathrm{pt};\mathbf{K}_R) & \cong & K_n(RH); \\ H_n^G(G/H;\mathbf{L}_R^{\langle -\infty\rangle}) & \cong & H_n^H(\mathrm{pt};\mathbf{L}_R^{\langle -\infty\rangle}) & \cong & L_n^{\langle -\infty\rangle}(RH); \\ H_n^G(G/H;\mathbf{K}^{\mathrm{top}}) & \cong & H_n^H(\mathrm{pt};\mathbf{K}^{\mathrm{top}}) & \cong & K_n(C_*^r(H)); \\ H_n^G(G/H;\mathbf{K}_{l^1}^{\mathrm{top}}) & \cong & H_n^H(\mathrm{pt};\mathbf{K}_{l^1}^{\mathrm{top}}) & \cong & K_n(l^1(H)). \end{array}$$

Classifying spaces for families of subgroups

Definition (Family of subgroups)

A *family $\mathcal{F}$ of subgroups* of G is a set of (closed) subgroups of G which is closed under conjugation and taking subgroups.

Examples for $\mathcal{F}$ are:

$$\begin{array}{lcl} \mathcal{TR} & = & \{\text{trivial subgroup}\}; \\ \mathcal{FC}\text{yc} & = & \{\text{finite cyclic subgroups}\}; \\ \mathcal{F}\text{in} & = & \{\text{finite subgroups}\}; \\ \mathcal{C}\text{yc} & = & \{\text{cyclic subgroups}\}; \\ \mathcal{VC}\text{yc} & = & \{\text{virtually cyclic subgroups}\}; \\ \mathcal{ALL} & = & \{\text{all subgroups}\}. \end{array}$$

Definition (Classifying G-CW-complex for a family of subgroups, tom Dieck(1974))

Let $\mathcal{F}$ be a family of subgroups of G. A model for the *classifying G-CW-complex for the family $\mathcal{F}$* is a G-CW-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to X$.

We abbreviate $\underline{E}G := E_{\mathcal{F}\mathrm{in}}(G)$ and call it the *universal G-CW-complex for proper G-actions*.

We abbreviate $\underline{\underline{E}}G := E_{\mathcal{VC}\mathrm{yc}}(G)$.

We also write $EG = E_{\mathcal{TR}}(G)$.

Theorem (Homotopy characterization of $E_{\mathcal{F}}(G)$)

Let $\mathcal{F}$ be a family of subgroups.

- *There exists a model for $E_{\mathcal{F}}(G)$ for any family $\mathcal{F}$;*
- *Two model for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent;*
- *A G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy groups belong to $\mathcal{F}$ and for each $H \in \mathcal{F}$ the H-fixed point set X^H is weakly contractible.*

- We have $EG = \underline{E}G$ if and only if G is torsionfree.
- We have $\underline{E}G = \mathrm{pt}$ if and only if G is finite.
- A model for $\underline{E}D_\infty$ is the real line with the obvious $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 * \mathbb{Z}/2$-action.
  Every model for ED_∞ is infinite dimensional, e.g., the universal covering of $\mathbb{RP}^\infty \vee \mathbb{RP}^\infty$.
- The spaces $\underline{E}G$ are interesting in their own right and have often very nice geometric models which are rather small.
- On the other hand any CW-complex is homotopy equivalent to $G\backslash\underline{E}G$ for some group G (see Leary-Nucinkis (2001)).

- Rips complex for word hyperbolic groups;
- Teichmüller space for mapping class groups;
- Outer space for the group of outer automorphisms of free groups;
- L/K for a connected Lie group L, a maximal compact subgroup $K \subseteq L$ and $G \subseteq L$ a discrete subgroup;
- CAT(0)-spaces with proper isometric G-actions, e.g., simply connected Riemannian manifolds with non-positive sectional curvature or trees.

Example ($SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z})$)

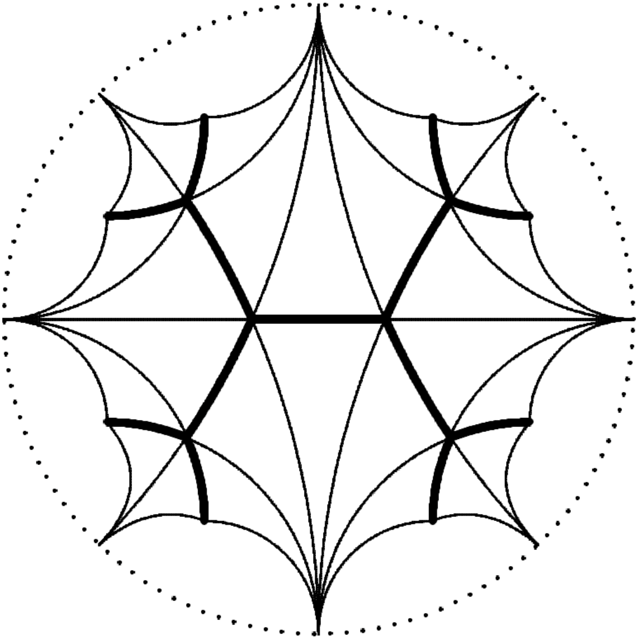
- In order to illustrate some of the general statements above we consider the special example $SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z})$.
- Notice that the notion of classifying space for proper actions $\underline{E}L$ makes also sense for a Lie group L. One demands that $\underline{E}L$ is a L-CW-complex, all isotropy groups are compact and $\underline{E}L^K$ is contractible for any compact subgroup K of L.
- If G is a discrete subgroup of the Lie group L, then the restriction of $\underline{E}L$ to G is a model for $\underline{E}G$.
- All of the results above have an analogue in this more general setting.

Example (continued)

- Let $\mathbb{H}^2$ be the 2-dimensional hyperbolic space. The group $SL_2(\mathbb{R})$ acts by isometric diffeomorphisms on the upper half-plane by Moebius transformations. This action is proper and transitive. The isotropy group of $z = i$ is $SO(2)$. Since $\mathbb{H}^2$ is a simply-connected Riemannian manifold, whose sectional curvature is constant -1, the $SL_2(\mathbb{R})$-space $\mathbb{H}^2$ is a model for $\underline{E}SL_2(\mathbb{R})$.
- The group $SL_2(\mathbb{R})$ is a connected Lie group and $SO(2) \subseteq SL_2(\mathbb{R})$ is a maximal compact subgroup. Hence $SL_2(\mathbb{R})/SO(2)$ is a model for $\underline{E}SL_2(\mathbb{R})$
- Since the $SL_2(\mathbb{R})$-action on $\mathbb{H}^2$ is transitive and $SO(2)$ is the isotropy group at $i \in \mathbb{H}^2$, we see that the $SL_2(\mathbb{R})$-manifolds $SL_2(\mathbb{R})/SO(2)$ and $\mathbb{H}^2$ are $SL_2(\mathbb{R})$-diffeomorphic.

Example (continued)

- Since $SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\mathbb{R})$, the space $\mathbb{H}^2$ with the obvious $SL_2(\mathbb{Z})$-action is a model for $\underline{E}SL_2(\mathbb{Z})$.
- The group $SL_2(\mathbb{Z})$ is isomorphic to the amalgamated product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$. This implies that there is a tree on which $SL_2(\mathbb{Z})$ acts with finite stabilizers. The tree has alternately two and three edges emanating from each vertex. This is a 1-dimensional model for $\underline{E}SL_2(\mathbb{Z})$.
- The tree model and the other model given by $\mathbb{H}^2$ must be $SL_2(\mathbb{Z})$-homotopy equivalent. They can explicitly be related by the following construction.



Example (continued)

- Divide the Poincaré disk into fundamental domains for the $SL_2(\mathbb{Z})$-action. Each fundamental domain is a geodesic triangle with one vertex at infinity, i.e., a vertex on the boundary sphere, and two vertices in the interior. Then the union of the edges, whose end points lie in the interior of the Poincaré disk, is a tree T with $SL_2(\mathbb{Z})$-action which is the tree model above. The tree is a $SL_2(\mathbb{Z})$-equivariant deformation retraction of the Poincaré disk. A retraction is given by moving a point p in the Poincaré disk along a geodesic starting at the vertex at infinity, which belongs to the triangle containing p, through p to the first intersection point of this geodesic with T.

- In the case of the Farrell-Jones Conjecture we will have to deal with $\underline{\underline{E}}G = E_{\mathcal{VC}\mathrm{yc}}(G)$ instead of $\underline{E}G = E_{\mathcal{F}\mathrm{in}}(G)$.
- Unfortunately, $\underline{\underline{E}}G$ is much more complicated than $\underline{E}G$ and we will spend some time later in analyzing how $\underline{\underline{E}}G$ can be obtained from $\underline{E}G$.

Example ($\underline{\underline{E}}\mathbb{Z}^n$)

- A model for $\underline{E}\mathbb{Z}^n$ is $\mathbb{R}^n$ with the free standard $\mathbb{Z}^n$-action.
- If we cross it with $\mathbb{R}$ with the trivial action, we obtain another model for $\underline{E}\mathbb{Z}^n$.
- Let $\{C_k \mid k \in \mathbb{Z}\}$ be the set of infinite cyclic subgroups of $\mathbb{Z}^n$. Then a model for $\underline{\underline{E}}\mathbb{Z}^n$ is obtained from $\mathbb{R}^n \times \mathbb{R}$ if we collapse for every $k \in \mathbb{Z}$ the n-dimensional real vector space $\mathbb{R}^n \times \{k\}$ to the $(n-1)$-dimensional real vector space $\mathbb{R}^n/V_C$, where V_C is the one-dimensional real vector space generated by the C-orbit through the origin.

Conjecture (Isomorphism Conjecture)

Let $\mathcal{H}_*^?$ be an equivariant homology theory. It satisfies the Isomorphism Conjecture for the group G and the family $\mathcal{F}$ if the projection $E_{\mathcal{F}}(G) \to pt$ induces for all $n \in \mathbb{Z}$ a bijection

$$\mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(pt).$$

- The point is to find a as small as possible family $\mathcal{F}$.
- The Isomorphism Conjecture is always true for $\mathcal{F} = \mathcal{ALL}$ since it becomes a trivial statement because of $E_{\mathcal{ALL}}(G) = \mathrm{pt}$.
- The philosophy is to be able to compute the functor of interest for G by knowing it on the values of elements in $\mathcal{F}$.

Example (Farrell-Jones Conjecture)

The Farrell-Jones Conjecture for K-theory or L-theory respectively with coefficients in R is the Isomorphism Conjecture for $\mathcal{H}^?_* = H_*(-;\mathbf{K}_R)$ or $\mathcal{H}^?_* = H_*(-;\mathbf{L}_R^{\langle -\infty \rangle})$ respectively and $\mathcal{F} = \mathcal{VC}\mathrm{yc}$.
In other words, it predicts that the the assembly map

$$H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G),\mathbf{K}_R) \to H_n^G(\mathrm{pt},\mathbf{K}_R) = K_n(RG)$$

or

$$H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G),\mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(\mathrm{pt},\mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Example (Baum-Connes Conjecture)

The Baum-Connes Conjecture is the Isomorphism Conjecture for $\mathcal{H}^?_* = K^?_* = H^?_*(-;\mathbf{K}^{\mathrm{top}})$ and $\mathcal{F} = \mathcal{F}\mathrm{in}$.
In other words it predicts that the the assembly map

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{F}\mathrm{in}}(G), \mathbf{K}^{\mathrm{top}}) \to H_n^G(\mathrm{pt}, \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

Example (Bost Conjecture)

The Baum-Connes Conjecture is the Isomorphism Conjecture for $\mathcal{H}^?_* = K^?_* = H^?_*(-;\mathbf{K}^{\mathrm{top}}_{l^1})$ and $\mathcal{F} = \mathcal{F}\mathrm{in}$.
In other words it predicts that the the assembly map

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{F}\mathrm{in}}(G), \mathbf{K}^{\mathrm{top}}_{l^1}) \to H_n^G(\mathrm{pt}, \mathbf{K}^{\mathrm{top}}_{l^1}) = K_n(l^1(G))$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (Meta-Conjecture)

Let $\mathcal{H}^?_*$ be an equivariant homology theory. Let $\mathcal{C}$ be a class of groups closed under isomorphism and taking subgroups. For a group G let $\mathcal{C}(G)$ be the family of subgroups of G consisting of those subgroups of G which belong to G.

Then the assembly map

$$H_n^G(E_{\mathcal{C}(G)}(G), \mathbf{K}^{\mathrm{top}}) \to H_n^G(pt, \mathbf{K}^{\mathrm{top}})$$

is bijective for all $n \in \mathbb{Z}$.

Changing the family

- Fix an equivariant homology theory $\mathcal{H}_*^?$.

Theorem (Transitivity Principle)

Suppose $\mathcal{F} \subseteq \mathcal{G}$ are two families of subgroups of G. Assume that for every element $H \in \mathcal{G}$ the group H satisfies the Isomorphism Conjecture for $\mathcal{F}|_H = \{K \subseteq H \mid K \in \mathcal{F}\}$.
Then the map

$$\mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{G}}(G))$$

is bijective for all $n \in \mathbb{Z}$.
Moreover, $(G, \mathcal{G})$ satisfies the Isomorphism Conjecture if and only if $(G, \mathcal{F})$ satisfies the Isomorphism Conjecture.

Sketch of proof.

- For a G-CW-complex X with isotropy group in $\mathcal{G}$ consider the natural map induced by the projection

$$s_*^G(X)\colon \mathcal{H}_*^G(X \times E_{\mathcal{F}}(G)) \to \mathcal{H}_*^G(X).$$

- This a natural transformation of G-homology theories defined for G-CW-complexes with isotropy groups in $\mathcal{G}$.
- In order to show that it is a natural equivalence it suffices to show that $s_n^G(G/H)$ is an isomorphism for all $H \in \mathcal{G}$ and $n \in \mathbb{Z}$.

$\square$

Sketch of proof (continued).

- The G-space $G/H \times E_{\mathcal{F}}(G)$ is G-homeomorphic to $G \times_H \operatorname{res}_G^H E_{\mathcal{F}}(G)$ and $\operatorname{res}_G^H E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F}|_H}(H)$.
- Hence by the induction structure $s_n^G(G/H)$ can be identified with the assembly map
$$\mathcal{H}_*^H(E_{\mathcal{F}|_H}(H)) \to \mathcal{H}_*^H(\mathrm{pt}),$$
which is bijective by assumption.
- Now apply this to $X = E_{\mathcal{G}}(G)$ and observe that $E_{\mathcal{G}}(G) \times E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F}}(G)$.

$\square$

Example (Passage from $\mathcal{F}\text{in}$ to $\mathcal{VC}\text{yc}$ for the Baum-Connes Conjecture)

- Consider the Baum-Connes setting, i.e., take $\mathcal{H}_*^? = K_*^?$.
- Consider the families $\mathcal{F}\text{in} \subseteq \mathcal{VC}\text{yc}$.
- For every virtually cyclic group V the Baum-Connes Conjecture is true, i.e.,
$$K_n^G(E_{\mathcal{F}\text{in}}(V)) \to K_n(C_r^*(V))$$
is bijective for $n \in \mathbb{Z}$.
- Hence by the Transitivity principle the following map is bijective for all groups G and all $n \in \mathbb{Z}$
$$K_n^G(\underline{E}G) = K_n^G(E_{\mathcal{F}\text{in}}(G)) \to K_n^G(E_{\mathcal{VC}\text{yc}}(G)).$$

- This explains why in the Baum-Connes setting it is enough to deal with $\mathcal{F}$in instead of $\mathcal{VC}$yc.
- This is not true in the Farrell-Jones setting and causes many extra difficulties there (NIL and UNIL-phenomena).
- This difference is illustrated by the following isomorphisms due to Pimsner-Voiculescu and Bass-Heller-Swan:

$$\begin{aligned} K_n(C_r^*(\mathbb{Z})) &\cong K_n(\mathbb{C}) \oplus K_{n-1}(\mathbb{C}); \\ K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R). \end{aligned}$$

- Due to Matthey-Mislin and Lück the map

$$K_n^G(E_{\mathcal{FC}\mathrm{yc}}(G)) \xrightarrow{\cong} K_n^G(\underline{E}G)$$

is bijective for all $n \in \mathbb{Z}$.

- In general the relative assembly maps

$$\begin{aligned} H_n^G(\underline{E}G;\mathbf{K}_R) &\to H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G);\mathbf{K}_R); \\ H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}) &\to H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G);\mathbf{L}_R^{\langle -\infty\rangle}), \end{aligned}$$

  are not bijective.
- Hence in the Farrell-Jones setting one has to pass to $\mathcal{VC}$yc and cannot use the easier to handle family $\mathcal{F}$in.

Example (Passage from $\mathcal{F}$in to $\mathcal{VC}$yc for the Farrell-Jones Conjecture)

For instance the Bass-Heller Swan decomposition

$$K_{n-1}(R) \oplus K_n(R) \oplus NK_n(R) \oplus NK_n(R)) \xrightarrow{\cong} K_n(R[t, t^{-1}]) \cong K_n(R[\mathbb{Z}])$$

and the isomorphism

$$H_n^{\mathbb{Z}}(\underline{E}\mathbb{Z}; \mathbf{K}_R) = H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_R) = H_n^{\{1\}}(S^1, \mathbf{K}_R) = K_{n-1}(R) \oplus K_n(R)$$

show that

$$H_n^{\mathbb{Z}}(\underline{E}\mathbb{Z}; \mathbf{K}_R) \to H_n^{\mathbb{Z}}(\mathrm{pt}; \mathbf{K}_R) = K_n(R\mathbb{Z})$$

is bijective if and only if $NK_n(R) = 0$.

Conjecture (*K*-theoretic Farrell-Jones Conjecture for regular rings and torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG;\mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (*K*-theoretic Farrell-Jones Conjecture for regular rings containing $\mathbb{Q}$)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R with $\mathbb{Q} \subseteq R$ predicts that the assembly map

$$H_n(\underline{E}G; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- By the Transitivity Principle the general version reduces to the version above if G is torsionfree and R is regular.
- Notice that the version above is close to the Baum-Connes Conjecture and that $\mathbb{C}$ is a regular ring.

- An infinite virtually cyclic group G is called of type I if it admits an epimorphism onto $\mathbb{Z}$ and of type II otherwise. A virtually cyclic group is of type II if and only if admits an epimorphism onto D_∞.
- Let $\mathcal{VC}\mathrm{yc}_I$ or $\mathcal{VC}\mathrm{yc}_{II}$ respectively be the family of subgroups which are either finite or which are virtually cyclic of type I or II respectively.

Theorem (Lück (2004), Quinn (2007), Reich (2007))

The following maps are bijective for all $n \in \mathbb{Z}$

$$\begin{aligned} H_n^G(E_{\mathcal{VC}yc_I}(G);\mathbf{K}_R) &\to H_n^G(E_{\mathcal{VC}yc}(G);\mathbf{K}_R); \\ H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty \rangle}) &\to H_n^G(E_{\mathcal{VC}yc_I}(G);\mathbf{L}_R^{\langle -\infty \rangle}). \end{aligned}$$

Theorem (Cappell (1973), Grunewald (2005), Waldhausen (1978))

- *The following maps are bijective for all $n \in \mathbb{Z}$.*

$$\begin{aligned} H_n^G(\underline{E}G;\mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} &\to H_n^G(E_{\mathcal{VC}yc}(G);\mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q}; \\ H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle})\left[\frac{1}{2}\right] &\to H_n^G(E_{\mathcal{VC}yc}(G);\mathbf{L}_R^{\langle -\infty\rangle})\left[\frac{1}{2}\right]; \end{aligned}$$

- *If R is regular and $\mathbb{Q} \subseteq R$, then for all $n \in \mathbb{Z}$ we get a bijection*

$$H_n^G(\underline{E}G;\mathbf{K}_R) \to H_n^G(E_{\mathcal{VC}yc}(G);\mathbf{K}_R).$$

Theorem (Bartels (2003))

For every $n \in \mathbb{Z}$ the two maps

$$\begin{aligned} H_n^G(\underline{E}G;\mathbf{K}_R) &\to H_n^G(E_{\mathcal{VC}yc}(G);\mathbf{K}_R); \\ H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}) &\to H_n^G(E_{\mathcal{VC}yc}(G);\mathbf{L}_R^{\langle -\infty\rangle}), \end{aligned}$$

are split injective.

- Hence we get (natural) isomorphisms

$$H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G);\mathbf{K}_R) \;\cong\; H_n^G(\underline{E}G;\mathbf{K}_R) \oplus H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G),\underline{E}G;\mathbf{K}_R);$$

$$\begin{aligned} & H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \\ & \qquad \cong H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}) \oplus H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G),\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}). \end{aligned}$$

- The analysis of the terms $H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G),\underline{E}G;\mathbf{K}_R)$ and $H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G),\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle})$ boils down to investigating Nil-terms and UNil-terms in the sense of Waldhausen and Cappell.

Conjecture (*L*-theoretic Farrell-Jones Conjecture for torsionfree groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n\big(BG; \mathbf{L}_R^{\langle -\infty \rangle}\big) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Other versions of Isomorphism Conjectures

- There are functors $\mathcal{P}$ and A which assign to a space X the space of pseudo-isotopies and its A-theory.
- Composing it with the functor sending a groupoid to its classifying space yields functors $\mathbf{P}$ and $\mathbf{A}$ from $\mathcal{G}roupoids$ to $\mathcal{S}pectra$.
- Thus we obtain equivariant homology theories $H_*^?(-;\mathbf{P})$ and $H_*^?(-;\mathbf{A})$. They satisfy $H_n^G(G/H;\mathbf{P}) = \pi_n(\mathcal{P}(BH))$ and $H_n^G(G/H;\mathbf{A}) = \pi_n(A(BH))$.

Conjecture (The Farrell-Jones Conjecture for pseudo-isotopies and A-theory)

The Farrell-Jones Conjecture for pseudo-isotopies and A-theory respectively is the Isomorphism Conjecture for $H_*^?(-;\mathbf{P})$ and $H_*^?(-;\mathbf{A})$ respectively for the family $\mathcal{VC}yc$.

Theorem (Relating pseudo-isotopy and K-theory)

The rational version of the K-theoretic Farrell-Jones Conjecture for coefficients in $\mathbb{Z}$ is equivalent Farrell-Jones Conjecture for Pseudoisotopies. In degree $n \leq 1$ this is even true integrally.

- Pseudo-isotopy and A-theory are important theories. In particular they are closely related to the space of selfhomeomorphisms and the space of selfdiffeomorphisms of closed manifolds.

- There are functors **THH** and **TC** which assign to a ring (or more generally to an S-algebra) a spectrum describing its topological Hochschild homology and its topological cyclic homology.
- These functors play an important role in K-theoretic computations.
- Composing them with the functor sending a groupoid to a kind of group ring yields functors $\mathbf{THH}_R$ and $\mathbf{TC}_R$ from $\mathcal{G}roupoids$ to $\mathcal{S}pectra$.
- Thus we obtain equivariant homology theories $H_*^?(-;\mathbf{THH}_R)$ and $H_*^?(-;\mathbf{TC}_R)$. They satisfy $H_n^G(G/H;\mathbf{THH}_R) = \mathbf{THH}_n(RH)$ and $H_n^G(G/H;\mathbf{TC}_R) = \mathbf{TC}_n(RH)$.

Conjecture (The Farrell-Jones Conjecture for topological Hochschild homology and cyclic homology)

The Farrell-Jones Conjecture for topological Hochschild homology and cyclic homology respectively is the Isomorphism Conjecture for $H_*^?(-;\mathbf{THH})$ and $H_*^?(-;\mathbf{TC})$ respectively for the family $\mathcal{C}yc$ of cyclic subgroups.

Theorem (Lück-Reich-Rognes-Varisco ($\geq$ 2009))

The Farrell-Jones Conjecture for topological Hochschild homology is true for all groups.

- There is a joint project by Lück-Rognes-Reich-Varisco aiming at this conjecture for TC and its application to the Farrell-Jones Conjecture generalizing the results of Bökstedt-Hsiang-Madsen.

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Conjecture (Kadison Conjecture)

The Kaplansky Conjecture says for a torsionfree group G that 0 and 1 are the only idempotents in $C_r^*(G)$.

Conjecture (Projective class groups)

Let R be a regular ring. Suppose that G is torsionfree. Then:

- $K_n(RG) = 0$ *for* $n \leq -1$;
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective;*
- *If R is a principal ideal domain, then* $\widetilde{K}_0(RG) = 0$.

- The vanishing of $\widetilde{K}_0(RG)$ is equivalent to the statement that any finitely generated projective RG-module P is stably free, i.,e., there are $m, n \geq 0$ with $P \oplus RG^m \cong RG^n$;
- Let G be a finitely presented. The vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ is equivalent to the geometric statement that any finitely dominated space X with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex.

Conjecture (Whitehead group)

If G is torsionfree, then the Whitehead group $\mathrm{Wh}(G)$ *vanishes.*

- The vanishing of $\mathrm{Wh}(G)$ is equivalent to the following statement about matrices: Any invertible matrix $A \in GL(\mathbb{Z}G)$ can be reduced by elementary row and column operations, (de-)stabilization and by multiplication of a column or row with an element of the form $\pm g$ to the trivial empty matrix.

Theorem (s-Cobordism Theorem, Barden, Mazur, Stallings, Kirby-Siebenmann)

Let M_0 be a closed (smooth) manifold of dimension $n \geq 5$. Let $(W; M_0, M_1)$ be an h-cobordism over M_0.
Then W is homeomorphic (diffeomorphic) to $M_0 \times [0,1]$ relative M_0 if and only if its Whitehead torsion

$$\tau(W, M_0) \in \mathrm{Wh}(\pi_1(M_0))$$

vanishes.

- Fix $n \geq 6$. Let G be finitely presented. The vanishing of $\mathrm{Wh}(G)$ is equivalent to the following geometric statement: Every compact n-dimensional h-cobordism W with $G \cong \pi_1(W)$ is trivial;

- The s-cobordism theorem is a key ingredient in the surgery program for the classification of closed manifolds due to Browder, Novikov, Sullivan and Wall.
- If $\mathrm{Wh}(G)$ vanishes, every h-cobordism $(W; M_0, M_1)$ of dimension ≥ 6 with $G \cong \pi_1(W)$ is trivial and in particular $M_0 \cong M_1$.

Conjecture (Poincaré Conjecture)

Let M be an n-dimensional topological manifold which is a homotopy sphere, i.e., homotopy equivalent to S^n.
Then M is homeomorphic to S^n.

Theorem (Smale, Freedman, Perelman)

The Poincaré Conjecture is true.

Conjecture (Moody's Induction Conjecture)

- *Let R be a regular ring with $\mathbb{Q} \subseteq R$. Then the map given by induction from finite subgroups of G*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(RH) \to K_0(RG)$$

  is bijective;

- *Let F be a field of characteristic p for a prime number p. Then the map*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(FH)[1/p] \to K_0(FG)[1/p]$$

  is bijective.

- If G is torsionfree, the Induction Conjecture says that everything comes from the trivial subgroup and we rediscover some of the previous conjectures.

Conjecture (Bass Conjecture)

Let R be a commutative integral domain and let G be a group. Let $g \neq 1$ be an element in G. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R.
Then the Bass Conjecture predicts that for every finitely generated projective RG-module P the value of its Hattori-Stallings rank $\mathrm{HS}_{RG}(P)$ at (g) is trivial.

- The Hattori-Stallings rank extends the notion of a character of a representation of a finite group to infinite groups.
- Roughly speaking, the Bass Conjecture extends basic facts of the representation theory of finite groups to infinite groups.
- If G is finite, the Bass Conjecture reduces to the Theorem of Swan.

Conjecture (L^2-torsion)

If X and Y are det-L^2-acyclic finite G-CW-complexes, which are G-homotopy equivalent, then their L^2-torsion agree:

$$\rho^{(2)}(X;\mathcal{N}(G)) = \rho^{(2)}(Y;\mathcal{N}(G)).$$

- The L^2-torsion of a closed Riemannian manifold M is defined in terms of the heat kernel on the universal covering.
- If M is hyperbolic and has odd dimension, its L^2-torsion is up to dimension constant its volume.
- The conjecture above allows to extend the notion of volume to word hyperbolic groups whose L^2-Betti numbers all vanish.

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\operatorname{sign}_x(M,f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M,f), i.e., for every orientation preserving homotopy equivalence of closed oriented manifolds $g\colon M_0 \to M_1$ and homotopy equivalence $f_i\colon M_i \to BG$ with $f_1 \circ g \simeq f_2$ we have

$$\operatorname{sign}_x(M_0,f_0) = \operatorname{sign}_x(M_1,f_1).$$

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism. In particular M and N are homeomorphic.

- This is the topological version of Mostow rigidity. One version of Mostow rigidity says that any homotopy equivalence between hyperbolic closed Riemannian manifolds is homotopic to an isometric diffeomorphism. In particular they are isometrically diffeomorphic if and only if their fundamental groups are isomorphic.
- The Borel Conjecture becomes in general false if one replaces homeomorphism by diffeomorphism. A counterexample is T^n for $n \geq 5$.

- In some sense the Borel Conjecture is opposed to the Poincaré Conjecture. Namely, in the Borel Conjecture the fundamental group can be complicated but there no higher homotopy groups, whereas in the Poincaré Conjecture there is no fundamental group but complicated higher homotopy groups.
- A systematic study of topologically rigid manifolds is presented in a paper by Kreck-Lück (2006), where a kind of interpolation between the Poincaré Conjecture and the Borel Conjecture is studied.
- Thurston's Geometrization Conjecture implies the Borel Conjecture in dimension 3.
- The Borel Conjecture in dimension 1 and 2 is obviously true.

- There is also an existence part of the Borel Conjecture.
- Namely, if X is an aspherical finite Poincaré complex, then X is homotopy equivalent to an ANR-homology manifold.
- One may also hope that X is homotopy equivalent to a closed manifold.
- But then one runs into Quinn's resolutions obstruction which seem to be a completely different story (see Bryant-Ferry-Mio-Weinberger (2006)).
- The question is whether it vanishes for closed aspherical manifolds.

- All the conjectures above follow from the Farrell-Jones Conjecture provided some sometimes some dimension restrictions or conditions about R hold.
- We illustrate this in certain examples.

Lemma

Let R be a regular ring and let G be a torsionfree group such that K-theoretic Farrell-Jones Conjecture holds. Then

- $K_n(RG) = 0$ *for* $n \leq -1$;
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective. In particular* $\widetilde{K}_0(RG)$ *is trivial if and only if* $\widetilde{K}_0(R)$ *is trivial;*
- *The Whitehead group* $\mathrm{Wh}(G)$ *is trivial.*

Recall:

Conjecture (*K*-theoretic Farrell-Jones Conjecture for regular rings and torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n\big(BG;\mathbf{K}_R\big) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG; \mathbf{K}_R)$ whose E^2-term is given by

$$E^2_{p,q} = H_p(BG, K_q(R)).$$

- Since R is regular by assumption, we get $K_q(R) = 0$ for $q \leq -1$.
- Hence the edge homomorphism yields an isomorphism

$$K_0(R) = H_0(\mathrm{pt}, K_0(R)) \xrightarrow{\cong} H_0(BG; \mathbf{K}_R) \cong K_0(RG).$$

- We have $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = \{\pm 1\}$. We get an exact sequence

$$\begin{aligned} 0 \to H_0(BG; \mathbf{K}_{\mathbb{Z}}) = \{\pm 1\} \to H_1(BG; \mathbf{K}_{\mathbb{Z}}) \cong K_1(\mathbb{Z}G) & \\ \to H_1(BG; K_0(\mathbb{Z})) = G/[G,G] \to 1. & \end{aligned}$$

This implies

$$\mathrm{Wh}(G) := K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\} \cong 0.$$

Theorem

If the K-theoretic and the L-theoretic Farrell-Jones Conjecture holds for the group G, then the Borel Conjecture holds for any n-dimensional closed manifold with $\pi_1(M) \cong G$ provided that $n \geq 5$.

Recall:

Conjecture (L-theoretic Farrell-Jones Conjecture for torsionfree groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n\big(BG; \mathbf{L}_R^{\langle -\infty \rangle}\big) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Definition (Structure set)

The *structure set* $S^{top}(M)$ of a manifold M consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold N as source.

Two such homotopy equivalences $f_0 \colon N_0 \to M$ and $f_1 \colon N_1 \to M$ are equivalent if there exists a homeomorphism $g \colon N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

Theorem

The Borel Conjecture holds for a closed manifold M if and only if $\mathcal{S}^{\mathrm{top}}(M)$ consists of one element.

Theorem (Algebraic surgery sequence Ranicki (1992))

There is an exact sequence of abelian groups called algebraic surgery exact sequence for an n-dimensional closed manifold M

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}}$$
$$\mathcal{S}^{\mathrm{top}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

It can be identified with the classical geometric surgery sequence due to Sullivan and Wall in high dimensions.

- $\mathcal{S}^{\mathrm{top}}(M)$ consist of one element if and only if A_{n+1} is surjective and A_n is injective.
- $H_k(M;\mathbf{L}\langle 1\rangle) \to H_k(M;\mathbf{L})$ is bijective for $k \geq n+1$ and injective for $k = n$ if both the K-theoretic and L-theoretic Farrell-Jones Conjecture hold for $G = \pi_1(M)$ and $R = \mathbb{Z}$.

Definition (Bott manifold)

A *Bott manifold* is any simply connected closed Spin-manifold B of dimension 8 whose $\widehat{A}$-genus $\widehat{A}(B)$ is 8.

- We fix such a choice. (The particular choice does not matter.)
- Notice that the index defined in terms of the Dirac operator $\operatorname{ind}_{C_r^*(\{1\};\mathbb{R})}(B) \in KO_8(\mathbb{R}) \cong \mathbb{Z}$ is a generator and the product with this element induces the Bott periodicity isomorphisms $KO_n(C_r^*(G;\mathbb{R})) \xrightarrow{\cong} KO_{n+8}(C_r^*(G;\mathbb{R}))$.
- In particular

$$\operatorname{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) \quad = \quad \operatorname{ind}_{C_r^*(\pi_1(M\times B);\mathbb{R})}(M \times B),$$

if we identify $KO_n(C_r^*(\pi_1(M);\mathbb{R})) = KO_{n+8}(C_r^*(\pi_1(M);\mathbb{R}))$ via Bott periodicity.

- If M carries a Riemannian metric with positive scalar curvature, then the index

$$\operatorname{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) \in KO_n(C_r^*(\pi_1(M);\mathbb{R})),$$

which is defined in terms of the Dirac operator on the universal covering, must vanish by the Bochner-Lichnerowicz formula.

Conjecture ((Stable) Gromov-Lawson-Rosenberg Conjecture)

Let M be a closed connected Spin*-manifold of dimension $n \geq 5$. Then $M \times B^k$ carries for some integer $k \geq 0$ a Riemannian metric with positive scalar curvature if and only if*

$$\operatorname{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) \; = \; 0 \quad \in KO_n(C_r^*(\pi_1(M);\mathbb{R})).$$

Theorem (Stolz (2002))

Suppose that the assembly map for the real version of the Baum-Connes Conjecture

$$H_n^G(\underline{E}G;\mathbf{K}O^{\mathrm{top}}) \to KO_n(C_r^*(G;\mathbb{R}))$$

is injective for the group G.

Then the Stable Gromov-Lawson-Rosenberg Conjecture true for all closed Spin*-manifolds of dimension ≥ 5 with $\pi_1(M) \cong G$.*

- The requirement $\dim(M) \geq 5$ is essential in the Stable Gromov-Lawson-Rosenberg Conjecture, since in dimension four new obstructions, the Seiberg-Witten invariants, occur.
- The unstable version of the Gromov-Lawson-Rosenberg Conjecture says that M carries a Riemannian metric with positive scalar curvature if and only if $\mathrm{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) = 0$.
- Schick(1998) has constructed counterexamples to the unstable version using minimal hypersurface methods due to Schoen and Yau.
- It is not known whether the unstable version is true or false for finite fundamental groups.
- Since the Baum-Connes Conjecture is true for finite groups (for the trivial reason that $\underline{E}G = \mathrm{pt}$ for finite groups G), the Stable Gromov-Lawson Conjecture holds for finite fundamental groups.

The status of the Farrell-Jones Conjecture

Theorem (Bartels-Lück (2009))

Let $\mathcal{FJ}$ be the class of groups for which both the K-theoretic and the L-theoretic Farrell-Jones Conjecture hold with coefficients in any additive G-category (with involution) is true has the following properties:

- *Hyperbolic group and virtually nilpotent groups belongs to $\mathcal{FJ}$;*
- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ belongs to $\mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}(R)$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{F}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *If we demand on the K-theory version only that the assembly map is 1-connected and keep the full L-theory version, then the properties above remain valid and the class $\mathcal{FJ}$ contains also all $\mathrm{CAT}(0)$-groups.*

- Limit groups in the sense of Zela are CAT(0)-groups (Alibegovic-Bestvina (2005)).
- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.
- On examples is the construction of groups with expanders due to Gromov. These yield counterexamples to the Baum-Connes Conjecture with coefficients (see Higson-Lafforgue-Skandalis (2002)).
- However, our results show that these groups do satisfy the Farrell-Jones Conjecture in its most general form and hence also the other conjectures mentioned above.
- Bartels-Echterhoff-Lück(2008) show that the Bost Conjecture with coefficients in C^*-algebras is true for colimits of hyperbolic groups. Thus the failure of the Baum-Connes Conjecture with coefficients comes from the fact that the change of rings map

$$K_0\big(\mathcal{A} \rtimes_{l^1} G\big) \to K_0\big(\mathcal{A} \rtimes_{C_r^*} G\big)$$

is not bijective for all G-C^*-algebras $\mathcal{A}$.

- Mike Davis (1983) has constructed exotic closed aspherical manifolds using Gromov's hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.
- However, in all cases the universal coverings are CAT(0)-spaces and hence the fundamental groups are CAT(0)-groups.
- Hence by our main theorem they satisfy the Farrell-Jones Conjecture and hence the Borel Conjecture in dimension ≥ 5.

- There are still many interesting groups for which the Farrell-Jones Conjecture in its most general form is open. Examples are:
  - Amenable groups;
  - $Sl_n(\mathbb{Z})$ for $n \geq 3$;
  - Mapping class groups;
  - $\mathrm{Out}(F_n)$;
- If one looks for a counterexample, there seems to be no good candidates which do not fall under our main theorems and have some exotic properties which may cause the failure of the Farrell-Jones Conjecture.
- One needs a property which can be used to detect a non-trivial element which is not in the image of the assembly map or is in its kernel.

Inheritance properties under colimits

- Let $\psi\colon H \to G$ be a (not necessarily injective) group homomorphism. Given G-CW-complex Y, let $\psi^* Y$ be the H-CW-complex obtained from Y by restricting the G-action to a H-action via ψ.
  Given H-CW-complex X, let $\psi_* X$ be the G-CW-complex obtained from Y by induction with ψ, i.e., $\psi_* X = G \times_\psi X$.
- Consider a directed system of groups $\{G_i \mid i \in I\}$ with (not necessarily injective) structure maps $\psi_i\colon G_i \to G$ for $i \in I$. Put $G = \operatorname{colim}_{i \in I} G_i$.
- Let X be a G-CW-complex.

- We have the canonical G-map

$$ad\colon (\psi_i)_*\psi_i^*X = G \times_{G_i} X \to X, \quad (g,x) \mapsto gx.$$

- Define a homomorphism

$$t_n^G(X)\colon \operatorname*{colim}_{i \in I} \mathcal{H}_n^{G_i}(\psi_i^*X) \to \mathcal{H}_n^G(X)$$

by the colimit of the system of maps indexed by $i \in I$

$$\mathcal{H}_n^{G_i}(\psi_i^*X) \xrightarrow{\operatorname{ind}_{\psi_i}} \mathcal{H}_n^G\left((\psi_i)_*\psi_i^*X\right) \xrightarrow{\mathcal{H}_n^G(ad)} \mathcal{H}_n^G(X).$$

Definition (Strongly continuous equivariant homology theory)

An equivariant homology theory $\mathcal{H}_*^?$ is called strongly continuous if for every group G and every directed system of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i \in I} G_i$ the map

$$t_n^G(\mathrm{pt}) \colon \operatorname*{colim}_{i \in I} \mathcal{H}_n^{G_i}(\mathrm{pt}) \to \mathcal{H}_n^G(\mathrm{pt})$$

is an isomorphism for every $n \in \mathbb{Z}$.

Theorem (Bartels-Echterhoff-Lück (2008))

Consider a directed system of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i \in I} G_i$. Let X be a G-CW-complex. Suppose that $\mathcal{H}_*^?$ is strongly continuous. Then the homomorphism

$$t_n^G(X)\colon \operatorname*{colim}_{i \in I} \mathcal{H}_n^{G_i}(\psi_i^* X) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

is bijective for every $n \in \mathbb{Z}$.

Idea of proof.

- Show that t_*^G is a transformation of G-homology theories.
- Prove that the strong continuity implies that $t_n^G(G/H)$ is bijective for all $n \in \mathbb{Z}$ and $H \subseteq G$.
- Then a general comparison theorem gives the result.

□

Application to the Bost Conjecture

- The Baum-Connes-Conjecture and the Bost-Conjecture have versions, where one allows coefficients in a G-C^*algebra A

$$\begin{aligned} K_n^G(\underline{E}G; A) &\to K_n(A \rtimes_{C_r^*} G); \\ K_n^G(\underline{E}G; A) &\to K_n(A \rtimes_{l^1} G). \end{aligned}$$

- There is a natural map

$$\iota \colon K_n(A \rtimes_{l^1} G) \to K_n(A \rtimes_{C_r^*} G)$$

map.
The composite of the assembly map appearing in the Bost Conjecture with ι is the assembly map appearing in the Baum-Connes Conjecture.

- We will see that the Bost Conjecture has a better chance to be true than the Baum-Connes Conjecture.
- On the other hand the Baum-Connes Conjecture has a higher potential for applications since it is related to index theory and thus has interesting consequences for instance to the Conjectures due to Bass, Gromov-Lawson-Rosenberg, Novikov, Kadison, Kaplansky.
- These conjecture have been proved for interesting classes of groups. Prominent papers have been written for instance by Connes, Gromov, Higson, Kasparov, Lafforgue, Mineyev, Skandalis, Yu, Weinberger and others.

Lemma (Davis-Lück(1998))

There are equivariant homology theories $\mathcal{H}_*^?(-;C_r^*)$ and $\mathcal{H}_*^?(-;l^1)$ defined for all equivariant CW-complexes with the following properties:

- *If $H \subseteq G$ is a (not necessarily finite) subgroup, then*

$$\begin{array}{ccccc} \mathcal{H}_n^G(G/H;C_r^*) & \cong & \mathcal{H}_n^H(pt;C_r^*) & \cong & K_n(C_r^*(H)); \\ \mathcal{H}_n^G(G/H;l^1) & \cong & \mathcal{H}_n^H(pt;l^1) & \cong & K_n(l^1(H)); \end{array}$$

- *$\mathcal{H}_*^?(-,l^1)$ is strongly continuous;*
- *Both $\mathcal{H}_*^?(-;C_r^*)$ and $\mathcal{H}_*^?(-;l^1)$ agree for proper equivariant CW-complexes with equivariant topological K-theory $K_*^?$ in the sense of Kasparov.*

- One ingredient in the proof of the strong continuity of $\mathcal{H}_*^?(-;l^1)$ is to show
$$\operatorname*{colim}_{i \in I} K_n(l^1(G_i)) \cong K_n(l^1(G)).$$
- This statement does not make sense for the reduced group C^*-algebra since it is not functorial under arbitrary group homomorphisms.
- For instance, $C_r^*(\mathbb{Z} * \mathbb{Z})$ is a simple C^*-algebra and hence no epimorphism $C_r^*(\mathbb{Z} * \mathbb{Z}) \to C_r^*(\{1\})$ exists.
- Hence $\mathcal{H}_*^?(-;C_r^*)$ is not strongly continuous.

Theorem (Inheritance under colimits for the Bost Conjecture, Bartels-Echterhoff-Lück (2008))

Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i \in I} G_i$ and (not necessarily injective) structure maps $\psi_i \colon G_i \to G$. Suppose that the Bost Conjecture with C^*-coefficients holds for all groups G_i.
Then the Bost Conjecture with C^*-coefficients holds for G.

Theorem (Lafforgue (2002))

The Bost Conjecture holds with C^*-coefficients holds for all hyperbolic groups.

Corollary

Let $\{G_i \mid i \in I\}$ be a directed system of hyperbolic groups with (not necessarily injective structure maps).
Then the Bost Conjecture holds with C^*-coefficients holds for $\operatorname{colim}_{i \in I} G_i$.

- Certain groups with expanders yield counterexamples to the surjectivity of the assembly map appearing Baum-Connes Conjecture with coefficients by a construction due to Higson-Lafforgue-Skandalis (2002).
- These implies that the map $K_n(A \rtimes_{l^1} G) \to K_n(A \rtimes_r G)$ is not surjective in general.
- The main critical point concerning the Baum-Connes Conjecture is that the reduced group C^*-algebra of a group lacks certain functorial properties which are present on the left side of the assembly map. This is not true if one deals with $l^1(G)$ or groups rings RG.

- The counterexamples above raised the hope that one may find counterexamples to the conjectures due to Baum-Connes, Borel, Bost, Farrell-Jones, Novikov.
- The results above due to Bartels-Echterhoff-Lück (2008) and unpublished work by Bartels-Lück (2009) prove all these conjectures (with coefficients) except the Baum-Connes Conjecture for colimits of hyperbolic groups.
- There is no counterexample to the Baum-Connes Conjecture (without coefficients) in the literature.

Equivariant Chern character

Theorem (Dold (1962))

Let $\mathcal{H}_*$ be a generalized homology theory with values in Λ-modules for $\mathbb{Q} \subseteq \Lambda$.
Then there exists for every $n \in \mathbb{Z}$ and every CW-complex X a natural isomorphism

$$\bigoplus_{p+q=n} H_p(X;\Lambda) \otimes_\Lambda \mathcal{H}_q(pt) \xrightarrow{\cong} \mathcal{H}_n(X).$$

- This means that the Atiyah-Hirzebruch spectral sequence collapses in the strongest sense.
- The assumption $\mathbb{Q} \subseteq \Lambda$ is necessary.

Dolds' Chern character for a CW-complex X is given by the following composite

$$\mathrm{ch}_n \colon \bigoplus_{p+q=n} H_p(X;\mathcal{H}_q(*)) \xleftarrow{\alpha} \bigoplus_{p+q=n} H_p(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{H}_q(*)$$
$$\xleftarrow[\cong]{\bigoplus_{p+q=n} \mathrm{hur}\otimes \mathrm{id}} \bigoplus_{p+q=n} \pi_p^s(X_+,*) \otimes_{\mathbb{Z}} \mathcal{H}_q(*)$$
$$\xrightarrow{\bigoplus_{p+q=n} D_{p,q}} \mathcal{H}_n(X).$$

- We want to extend this to the equivariant setting.
- This requires an extra structure on the coefficients of an equivariant homology theory $\mathcal{H}^?_*$.
- We define a covariant functor called induction

$$\mathsf{ind}\colon \mathrm{FGINJ} \to \Lambda\text{-}\mathrm{Mod}$$

from the category FGINJ of finite groups with injective group homomorphisms as morphisms to the category of Λ-modules as follows. It sends G to $\mathcal{H}^G_n(\mathrm{pt})$ and an injection of finite groups $\alpha\colon H \to G$ to the morphism given by the induction structure

$$\mathcal{H}^H_n(\mathrm{pt}) \xrightarrow{\mathrm{ind}_\alpha} \mathcal{H}^G_n(\mathrm{ind}_\alpha \mathrm{pt}) \xrightarrow{\mathcal{H}^G_n(\mathrm{pr})} \mathcal{H}^G_n(\mathrm{pt}).$$

Definition (Mackey extension)

We say that $\mathcal{H}_*^?$ has a Mackey extension if for every $n \in \mathbb{Z}$ there is a contravariant functor called restriction

$$\text{res}\colon \text{FGI} \to \Lambda\text{- Mod}$$

such that these two functors ind and res agree on objects and satisfy the double coset formula, i.e., we have for two subgroups $H, K \subset G$ of the finite group G

$$\text{res}_G^K \circ \text{ind}_H^G \;=\; \sum_{KgH \in K\backslash G/H} \text{ind}_{c(g)\colon H\cap g^{-1}Kg \to K} \circ \text{res}_H^{H\cap g^{-1}Kg},$$

where $c(g)$ is conjugation with g, i.e., $c(g)(h) = ghg^{-1}$.

- In every case we will consider such a Mackey extension does exist and is given by an actual restriction.
- For instance for $H_0^?(-;\mathbf{K}^{\mathrm{top}})$ induction is the functor complex representation ring $R_{\mathbb{C}}$ with respect to induction of representations. The restriction part is given by the restriction of representations.

Theorem (Lück (2002))

Let $\mathcal{H}_*^?$ be a proper equivariant homology theory with values in Λ-modules for $\mathbb{Q} \subseteq \Lambda$. Suppose that $\mathcal{H}_*^?$ has a Mackey extension. Let I be the set of conjugacy classes (H) of finite subgroups H of G.
Then there is for every group G, every proper G-CW-complex X and every $n \in \mathbb{Z}$ a natural isomorphism called equivariant Chern character

$$\mathrm{ch}_n^G \colon \bigoplus_{p+q=n} \bigoplus_{(H)\in I} H_p(C_GH\backslash X^H;\Lambda) \otimes_{\Lambda[W_GH]} S_H\left(\mathcal{H}_q^H(*)\right) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

- C_GH is the centralizer and N_GH the normalizer of $H \subseteq G$;
- $W_GH := N_GH/H \cdot C_GH$ (This is always a finite group);
- $S_H\left(\mathcal{H}_q^H(*)\right) := \mathrm{cok}\left(\bigoplus_{\substack{K\subset H\\ K\neq H}} \mathrm{ind}_K^H \colon \bigoplus_{\substack{K\subset H\\ K\neq H}} \mathcal{H}_q^K(*) \to \mathcal{H}_q^H(*)\right).$
- $\mathrm{ch}_*^?$ is an equivalence of equivariant homology theories.

Example (Extensions of $\mathbb{Z}^n$ by $\mathbb{Z}/p$ for a prime p)

- Consider an extension of groups $1 \to \mathbb{Z}^n \xrightarrow{i} \Gamma \xrightarrow{q} \mathbb{Z}/p \to 1$ for a prime number p.
- Since $\mathbb{Z}^n$ is abelian, there is a well-defined conjugation action

$$\rho\colon \mathbb{Z}/p \to \operatorname{aut}(\mathbb{Z}^n).$$

  We denote by $\mathbb{Z}^n_\rho$ the associated $\mathbb{Z}[\mathbb{Z}/p]$-module.
- There are bijections

$$\begin{array}{rcll} H^1(\mathbb{Z}/p;\mathbb{Z}^n_\rho) & \cong & (\mathbb{Z}/p)^k & \text{for some } k; \\ H^1(\mathbb{Z}/p;\mathbb{Z}^n_\rho) & \xrightarrow{\cong} & \mathcal{P} := \{(P) \mid P \subset \Gamma, 1 < |P| < \infty\}. & \end{array}$$

  If we fix a generator $s \in \mathbb{Z}/p$, the last bijection sends the class $\overline{u} \in H^1(\mathbb{Z}/p;\mathbb{Z}^n_\rho)$ of $u \in \ker(\sum_{k=0}^{p-1} \rho^k \colon \mathbb{Z}^n \to \mathbb{Z}^n)$ to the conjugacy class of the subgroup of order p generated by us.

Example (continued)

- We get for every non-trivial finite subgroup $P \subseteq \Gamma$

$$P \cong \mathbb{Z}/p$$

and

$$N_\Gamma P = C_\Gamma P = (\mathbb{Z}^n)^{\mathbb{Z}/p} \times P.$$

- Proof: Fix a generator $t \in P$. Then we get for $a \in (\mathbb{Z}^n)^{\mathbb{Z}/p}$

$$ata^{-1} = tt^{-1}ata^{-1} = taa^{-1} = t$$

This implies

$$(\mathbb{Z}^n)^{\mathbb{Z}/p} \times P \subseteq C_\Gamma P \subseteq N_\Gamma P.$$

It remains to show $N_\Gamma P \subseteq (\mathbb{Z}^n)^{\mathbb{Z}/p} \times P$.

Example (continued)

- Every element in Γ is of the form at^i for some $i \in \{0, 1, 2, \ldots p-1\}$ and some $a \in A$.
  Suppose $at^i \in N_\Gamma H$.
  Then $at^i t(at^i)^{-1} = ata^{-1} = t^j$ holds for some $j \in \{0, 1, 2, \ldots p-1\}$.
  From $q(t) = q(ata^{-1}) = q(t^j) = q(t)^j$ we conclude $j = 1$.
  This implies $ata^{-1} = t$ and hence $t^{-1}at = a$.
  We conclude $a \in A^{\mathbb{Z}/p}$ and hence $at^i \in A^{\mathbb{Z}/p} \times H$.

Example (continued)

- In particular we get for every non-trivial finite subgroup $P \subseteq \Gamma$

$$\begin{aligned} W_\Gamma P &= \{1\}; \\ S_P\left(\mathcal{H}_q^P(*)\right) &= \operatorname{cok}\left(\operatorname{ind}_{\{1\}}^{\mathbb{Z}/p} \colon \mathcal{H}_q^{\{1\}}(\mathrm{pt}) \to \mathcal{H}_q^{\mathbb{Z}/p}(\mathrm{pt})\right). \end{aligned}$$

- Hence the Chern character boils down to an isomorphism

$$\left(\bigoplus_{p+q=n} H_p(\Gamma\backslash X;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathcal{H}_q^{\{1\}}(\mathrm{pt})\right) \oplus \left(\bigoplus_{p+q=n} \bigoplus_{(P)\in\mathcal{P}}\right.$$
$$\left. H_p\left(((\mathbb{Z}^n)^{\mathbb{Z}/p} \times P)\backslash X^P;\mathbb{Q}\right) \otimes_{\mathbb{Q}} \operatorname{cok}\left(\operatorname{ind}_{\{1\}}^{\mathbb{Z}/p} \colon \mathcal{H}_q^{\{1\}}(\mathrm{pt}) \to \mathcal{H}_q^{\mathbb{Z}/p}(\mathrm{pt})\right)\right)$$
$$\xrightarrow{\cong} \mathcal{H}_n^\Gamma(X).$$

Example (continued)

- In the special case $X = \underline{E}\Gamma$ we get

$$\left(\bigoplus_{p+q=n} H_p(\Gamma\backslash\underline{E}\Gamma;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathcal{H}_q^{\{1\}}(\mathrm{pt})\right) \oplus \left(\bigoplus_{p+q=n}\bigoplus_{(P)\in\mathcal{P}} H_p\left(B(\mathbb{Z}^n)^{\mathbb{Z}/p};\mathbb{Q}\right) \otimes_{\mathbb{Q}} \operatorname{cok}\left(\operatorname{ind}_{\{1\}}^{\mathbb{Z}/p} \colon \mathcal{H}_q^{\{1\}}(\mathrm{pt}) \to \mathcal{H}_q^{\mathbb{Z}/p}(\mathrm{pt})\right)\right) \xrightarrow{\cong} \mathcal{H}_n^{\Gamma}(\underline{E}\Gamma).$$

Example (continued)

- If $\mathcal{H}^{\Gamma}_*$ is rationalized equivariant topological K-theory $K^{\Gamma}_* \otimes_{\mathbb{Z}} \mathbb{Q}$, we get

$$\left(\bigoplus_{p \in \mathbb{Z}} H_{n+2p}(\Gamma \backslash \underline{E}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}\right) \oplus \left(\bigoplus_{p \in \mathbb{Z}} \bigoplus_{(P) \in \mathcal{P}} \right.$$

$$\left. H_{n+2p}\left(\left(B(\mathbb{Z}^n)^{\mathbb{Z}/p}; \mathbb{Q}\right) \otimes_{\mathbb{Q}} \operatorname{cok}\left(\operatorname{ind}_{\{1\}}^{\mathbb{Z}/p} \colon R_{\mathbb{C}}(\{1\}) \to R_{\mathbb{C}}(\mathbb{Z}/p)\right)\right)\right)$$

$$\xrightarrow{\cong} K_n^{\Gamma}(\underline{E}\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Example (continued)

- Let e be the number determined by

$$(\mathbb{Z}^n)^{\mathbb{Z}/p} = \mathbb{Z}^e$$

- Let k be the natural number determined by

$$H^1(\mathbb{Z}/p; \mathbb{Z}^n_\rho) \cong (\mathbb{Z}/p)^k.$$

- Then

$$|\mathcal{P}| = p^k.$$

- We have

$$\operatorname{cok}\left(\operatorname{ind}_{\{1\}}^{\mathbb{Z}/p} \colon R_{\mathbb{C}}(\{1\}) \to R_{\mathbb{C}}(\mathbb{Z}/p)\right) \cong \mathbb{Z}^{p-1}.$$

Example (continued)

- The Baum-Connes conjecture is known to be true for Γ.
- Hence we get

$$K_n(C_r^*(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \left(\bigoplus_{p \in \mathbb{Z}} H_{n+2p}(\Gamma \backslash \underline{E}\Gamma; \mathbb{Q}) \right) \oplus \mathbb{Q}^{d_n}$$

for

$$d_n = (p-1) \cdot p^k \cdot \left(\sum_{\substack{k \in \{0,1,\ldots,e\} \\ k \equiv n \mod 2}} \binom{n}{k} \right).$$

Example (continued)

- Furthermore assume that $\mathbb{Z}/p$ acts freely on $\mathbb{Z}^n$ outside the origin, or, equivalently, $e = 0$.
- Then the $\mathbb{Z}[\mathbb{Z}/p]$-module $\mathbb{Z}^n_\rho$ is actually a module over the Dedekind domain $\mathbb{Z}(\exp(2\pi i/p))$ and hence decomposes as a direct sum of ideals. The number of ideals turns out to be the number k.

$$\mathbb{Z}^n_\rho \cong I_1 \oplus I_2 \oplus \ldots I_k.$$

Example (continued)

- Since $I_1 \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Z}(\exp(2\pi i p)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}(\exp(2\pi i p)) \cong \mathbb{Q}^{p-1}$, we conclude

$$\begin{aligned} \mathbb{Z}^n_\rho \otimes_{\mathbb{Z}} \mathbb{Q} &\cong \mathbb{Q}(\exp(2\pi i p))^k; \\ n &= (p-1) \cdot k. \end{aligned}$$

- We also conclude

$$H_m(B\mathbb{Z}^n; \mathbb{Q})^{\mathbb{Z}/p} \cong \left(\Lambda^m \left(\mathbb{Q}(\exp(2\pi i p))^k\right)\right)^{\mathbb{Z}/p},$$

where Λ^m means the m-th exterior power of a $\mathbb{Q}$-module.

Example (continued)

- In this special case we obtain

$$K_n(C_r^*(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{Q}$$
$$\cong \begin{cases} \left(\bigoplus_{p\in\mathbb{Z}} H_{n+2p}(\Gamma\backslash\underline{E}\Gamma;\mathbb{Q})\right) \oplus \mathbb{Q}^{(p-1)\cdot p^{\frac{n}{p-1}}} & n \text{ even};\\ \bigoplus_{p\in\mathbb{Z}} H_{n+2p}(\Gamma\backslash\underline{E}\Gamma;\mathbb{Q}) & n \text{ odd}.\end{cases}$$

- It remains to investigate

$$H_m(\Gamma\backslash\underline{E}\Gamma;\mathbb{Q}) \cong H_m(B\mathbb{Z}^n;\mathbb{Q}) \otimes_{\mathbb{Q}[\mathbb{Z}/p]} \mathbb{Q}$$
$$\cong H_m(B\mathbb{Z}^n;\mathbb{Q})^{\mathbb{Z}/p} \cong \left(\Lambda^m\left(\mathbb{Q}(\exp(2\pi i p))^k\right)\right)^{\mathbb{Z}/p}.$$

Example (continued)

- Put
$$r_m \quad := \quad \mathrm{rk}_{\mathbb{Q}}\left(\left(\Lambda^m\left(\mathbb{Q}(\exp(2\pi i p))^k\right)\right)^{\mathbb{Z}/p}\right).$$

- We have
$$\begin{aligned} r_0 &= 1; \\ r_m &= 0 \quad \text{for } m \geq p; \\ r_m &= \frac{1}{p}\cdot\left(\binom{p-1}{m} + (-1)^m(p-1)\right) \quad \text{for } 2 \leq m < p. \end{aligned}$$

Example (continued)

- We obtain

$$H_m(\Gamma\backslash\underline{E}\Gamma;\mathbb{Q}) \cong \Big(\Lambda^m\big(\mathbb{Q}(\exp(2\pi i p))^k\big)\Big)^{\mathbb{Z}/p} \cong \mathbb{Q}^{r_m};$$

- This implies

$$K_n(C_r^*(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \begin{cases} \mathbb{Q}^{(p-1)p^{\frac{n}{p-1}} + \sum_{l\in\mathbb{Z}} r_{2l}} & n \text{ even;} \\ \mathbb{Q}^{\sum_{l\in\mathbb{Z}} r_{2l+1}} & n \text{ odd,} \end{cases}$$

provided that the conjugation action of $\mathbb{Z}/p$ on $\mathbb{Z}^n$ is free outside the origin.

Theorem (Atiyah-Segal (1968))

Let G be a finite group.
Then there are isomorphisms of abelian groups

$$\begin{aligned} K^0(BG) &\cong \operatorname{Rep}_{\mathbb{C}}(G)\widehat{_{\mathbb{I}_G}} \\ &\cong \mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(G) \otimes_{\mathbb{Z}} \mathbb{Z}\widehat{_p} \cong \mathbb{Z} \times \prod_{p \text{ prime}} (\mathbb{Z}\widehat{_p})^{r(p)}; \\ K^1(BG) &\cong 0. \end{aligned}$$

- For a prime p denote by $r(p) = |\operatorname{con}_p(G)|$ the number of conjugacy classes (g) of elements $g \neq 1$ in G of p-power order and by G_p the p-Sylow subgroup.
- $\mathbb{I}_G$ is the augmentation ideal of $\operatorname{Rep}_{\mathbb{C}}(G)$.
- Let $\mathbb{I}_p(G)$ be the image of the restriction homomorphism $\mathbb{I}(G) \to \mathbb{I}(G_p)$.

Theorem (Lück (2007))

Let G be a discrete group. Denote by $K^*(BG)$ the topological (complex) K-theory of its classifying space BG. Suppose that there is a cocompact G-CW-model for the classifying space $\underline{E}G$ for proper G-actions. Then there is a $\mathbb{Q}$-isomorphism

$$\overline{\mathrm{ch}}_G^n \colon K^n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong}$$
$$\left(\prod_{i \in \mathbb{Z}} H^{2i+n}(BG;\mathbb{Q})\right) \times \prod_{p \text{ prime}} \prod_{(g) \in \mathrm{con}_p(G)} \left(\prod_{i \in \mathbb{Z}} H^{2i+n}(BC_G\langle g\rangle;\widehat{\mathbb{Q}_p})\right),$$

- The multiplicative structure can also be determined.

Example ($SL_3(\mathbb{Z})$)

- It is well-known that its rational cohomology satisfies $\widetilde{H}^n(BSL_3(\mathbb{Z});\mathbb{Q}) = 0$ for all $n \in \mathbb{Z}$.
- Actually, by a result of Soule (1978) the quotient space $SL_3(\mathbb{Z})\backslash \underline{E}SL_3(\mathbb{Z})$ is contractible and compact.
- From the classification of finite subgroups of $SL_3(\mathbb{Z})$ we see that $SL_3(\mathbb{Z})$ contains up to conjugacy two elements of order 2, two elements of order 4 and two elements of order 3 and no further conjugacy classes of non-trivial elements of prime power order.
- The rational homology of each of the centralizers of elements in $\operatorname{con}_2(G)$ and $\operatorname{con}_3(G)$ agrees with the one of the trivial group.
- Hence we get

$$\begin{aligned} K^0(BSL_3(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong \mathbb{Q} \times (\widehat{\mathbb{Q}_2})^4 \times (\widehat{\mathbb{Q}_3})^2; \\ K^1(BSL_3(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong 0. \end{aligned}$$

Example (Continued)

- The identification of $K^0(BSL_3(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q}$ above is compatible with the multiplicative structures.
- Actually the computation using Brown-Petersen cohomology and the Conner-Floyd relation by Tezuka-Yagita (1992) gives the integral computation

$$\begin{array}{rcl} K^0(BSL_3(\mathbb{Z})) & \cong & \mathbb{Z} \times (\widehat{\mathbb{Z}_2})^4 \times (\widehat{\mathbb{Z}_3})^2; \\ K^1(BSL_3(\mathbb{Z})) & \cong & 0. \end{array}$$

- Soule (1978) has computed the integral cohomology of $SL_3(\mathbb{Z})$.

Theorem (Lück (2002))

Let G be a group. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. There is a commutative diagram

$$\begin{tikzcd}
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) \arrow[r] \arrow[d] & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[d] \\
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) \arrow[r] & K_n^{\mathrm{top}}(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C}
\end{tikzcd}$$

- The vertical arrows come from the obvious change of rings and of K-theory maps.
- The horizontal arrows can be identified with the assembly maps occurring in the Farrell-Jones Conjecture and the Baum-Connes Conjecture by the equivariant Chern character.
- Splitting principle.

An improved equivariant Chern character for equivariant topological K-theory

Theorem (Artin's Theorem)

Let G be finite. Then the map

$$\bigoplus_{C \subset G} \operatorname{ind}_C^G \colon \bigoplus_{C \subset G} R_{\mathbb{C}}(C) \to R_{\mathbb{C}}(G)$$

is surjective after inverting $|G|$, where $C \subset G$ runs through the cyclic subgroups of G.

- Let C be a finite cyclic group.
- The Artin defect is the cokernel of the map

$$\bigoplus_{D \subset C, D \neq C} \operatorname{ind}_D^C \colon \bigoplus_{D \subset C, D \neq C} R_{\mathbb{C}}(D) \to R_{\mathbb{C}}(C).$$

- For an appropriate idempotent $\theta_C \in R_{\mathbb{Q}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right]$ the Artin defect is after inverting the order of $|C|$ canonically isomorphic to

$$\theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right].$$

Theorem (Lück (2002))

Let X be a proper G-CW-complex. Let $\mathbb{Z} \subseteq \Lambda^G \subset \mathbb{Q}$ be the subring of $\mathbb{Q}$ obtained by inverting the orders of all the finite subgroups of G.
Then there is a natural isomorphism

$$\operatorname{ch}^G \colon \bigoplus_{(C)} K_n(C_G C \backslash X^C) \otimes_{\mathbb{Z}[W_G C]} \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\cong} K_n^G(X) \otimes_{\mathbb{Z}} \Lambda^G,$$

where (C) runs through the conjugacy classes of finite cyclic subgroups and $W_G C = N_G C / C \cdot C_G C$.

Example (Improvement of Artin's Theorem)

Consider the special case where G is finite and $X = \mathrm{pt}$ Then we get an improvement of Artin's theorem, namely,

$$\bigoplus_{(C)} \mathbb{Z} \otimes_{\mathbb{Z}[W_G C]} \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|G|}\right] \xrightarrow{\cong} R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|G|}\right]$$

Example ($X = \underline{E}G$)

In the special case $X = \underline{E}G$ we get an isomorphism

$$\bigoplus_{(C)} K_n(BC_G C) \otimes_{\mathbb{Z}[W_G C]} \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\cong} K_n^G(\underline{E}G) \otimes_{\mathbb{Z}} \Lambda^G,$$

Conjecture (Trace Conjecture for G)

The image of the trace map

$$K_0(C_r^*(G)) \xrightarrow{\mathrm{tr}} \mathbb{R}$$

is the additive subgroup of $\mathbb{R}$ *generated by* $\{\frac{1}{|H|} \mid H \subset G, |H| < \infty\}$.

Lemma

Let G be torsionfree. Then the Baum-Connes Conjecture for G implies the Trace Conjecture for G.

Proof.

The following diagram commutes because of the L^2-index theorem due to Atiyah(1974).

$$\begin{tikzcd}
K_0^G(EG) \arrow[r] \arrow[d, "\cong"] & K_0(C_r^*(G)) \arrow[r, "\mathrm{tr}"] & \mathbb{R} \\
K_0(BG) \arrow[r] & K_0(\mathrm{pt}) \arrow[r, "\cong"] & \mathbb{Z} \arrow[u]
\end{tikzcd}$$

□

Theorem (Roy (1999))

The Trace Conjecture is false in general.

Proof.

- Define an algebraic smooth variety

$$M = \{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 \mid z_0^{15} + z_1^{15} + z_2^{15} + z_3^{15} = 0\}.$$

- The group $G = \mathbb{Z}/3 \times \mathbb{Z}/3$ acts on it by

$$\begin{aligned} [z_0, z_1, z_2, z_3] &\mapsto [\exp(2\pi i/3) \cdot z_0, z_1, z_2, z_3] \\ [z_0, z_1, z_2, z_3] &\mapsto [z_0, z_3, z_1, z_2] \end{aligned}$$

$\square$

Proof (continued).

- One obtains

$$\begin{aligned} M^G &= \emptyset; \\ \operatorname{sign}(M) &= -1105; \\ \pi_1(M) &= \{1\}. \end{aligned}$$

- An equivariant version of a construction due to Davis and Januszkiewicz yields
  - A closed oriented aspherical manifold N with G-action;
  - A G-map $f : N \to M$ of degree one;
  - An isomorphism $f^*TM \cong TN$.

$\square$

Proof (continued).

- There is an extension of groups

$$1 \to \pi = \pi_1(N) \to \Gamma \xrightarrow{\cong} G \to 1$$

  and a Γ-action on $\widetilde{N}$ extending the π-action on $\widetilde{N}$ and covering the G-action on N.

- We compute using the Hirzebruch signature formula

$$\begin{aligned}\operatorname{sign}(N) &= \langle \mathcal{L}(N), [N] \rangle = \langle f^* \mathcal{L}(M), [N] \rangle \\ &= \langle \mathcal{L}(M), f_*([N]) \rangle = \langle \mathcal{L}(M), [M]) \rangle = \operatorname{sign}(M).\end{aligned}$$

□

Proof (continued).

- Next we prove that any finite subgroup $H \subset \Gamma$ satisfies

$$|H| \in \{1, 3\}.$$

- Since $\widetilde{N}$ turns out to be a CAT(0)-space, any finite subgroup $H \subset \Gamma$ has a fixed point by a result of Bruhat and Tits. This implies

$$\widetilde{N}^H \neq \emptyset \Rightarrow N^{p(H)} \neq \emptyset \Rightarrow M^{p(H)} \neq \emptyset \Rightarrow p(H) \neq G.$$

  Since $\pi_1(N)$ is torsionfree, $p|_H : H \to p(H)$ is bijective.

$\square$

Proof (continued).

- On $\widetilde{N}$ we have the signature operator $\widetilde{S}$. We claim that the composite

$$K_0^\Gamma(\underline{E}\Gamma) \xrightarrow{\text{asmb}} K_0(C_r^*(\Gamma)) \to K_0(\mathcal{N}(\Gamma)) \xrightarrow{\text{tr}_{\mathcal{N}(\Gamma)}} \mathbb{R}$$

sends $[\widetilde{N}, \widetilde{S}]$ to

$$\frac{1}{[\Gamma : \pi]} \cdot \text{sign}(N) \;=\; \frac{-1105}{9}.$$

- The Trace Conjecture for Γ says

$$\frac{-1105}{9} \in \{r \in \mathbb{R} \mid 3 \cdot r \in \mathbb{Z}\}.$$

This is not true (by some very deep number theoretic considerations).

□

Conjecture (Modified Trace Conjecture)

Let $\Lambda^G \subset \mathbb{Q}$ be the subring of $\mathbb{Q}$ obtained from $\mathbb{Z}$ by inverting the orders of finite subgroups of G. Then the image of the trace map

$$K_0(C_r^*(G)) \xrightarrow{\operatorname{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

is contained in Λ^G.

Theorem (Lück (2002))

The image of the composite

$$K_0^G(\underline{E}G) \xrightarrow{\text{asmb}} K_0(C_r^*(G)) \xrightarrow{\text{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

is contained in Λ^G*.*
In particular the Baum-Connes Conjecture implies the Modified Trace Conjecture.

- Problem: What is the image of the trace map in terms of G?
- Take $X = \underline{E}G$. Elements in $K_*(\underline{E}G)$ are given by elliptic G-operators P over cocompact proper G-manifolds with Riemannian metrics.
- Problem: What is the concrete preimage of its class under ch_*^G?
- One term could be the index of P^C on M^C giving an element in $K_0(C_GC\backslash\underline{E}^C)$ which is $K_0(BC_GC)$ after tensoring with Λ^G.
- Another term could come from the normal data of M^C in M which yields an element in $\theta_C \cdot R_{\mathbb{C}}(C)$.
- The failure of the Trace Conjecture shows that this is more complicated than one anticipates. The answer to the question above would lead to a kind of orbifold L^2-index theorem whose possible denominators, however, are not of the expected shape $\frac{n}{|H|}$ for $H \subseteq G$ finite.

Equivariant spectral sequences

Theorem (Equivariant Atiyah-Hirzebruch spectral sequence)

Let $\mathcal{H}_*^G$ be a G-homology theory. Then there is for every G-CW-complex X a spectral sequence, the so called equivariant Atiyah-Hirzebruch spectral sequence, which converges to $\mathcal{H}_{p+q}^G(X)$ and whose E^2-term is given by the Bredon homology

$$E^2_{p,q} = H_p^G(X; \mathcal{H}_q^G)$$

of X with coefficients in the covariant functor

$$\mathcal{H}_q^G \colon \mathrm{Or}G \to \mathbb{Z} - \mathrm{Modules}, \quad G/H \mapsto \mathcal{H}_q^G(G/H).$$

- The existence of the equivariant Chern character can be rephrased as the assertion that rationally the equivariant Atiyah-Hirzebruch spectral sequence collapses.
- There is another more elaborate spectral sequence, the p-chain spectral sequence of Davis-Lück (2003).
- The complexity of the equivariant Atiyah-Hirzebruch spectral sequence growth with the dimension, whereas the complexity of the p-chain spectral sequence growth with the maximal length of chains of finite isotropy groups $H_0 \subsetneq H_1 \subsetneq H_2 \ldots \subsetneq H_l$.

- Let R be a a regular ring with $\mathbb{Q} \subseteq R$.
- Then the K-theoretic Farrell-Jones Conjecture reduces to the claim that the assembly map

$$H_n^G(\underline{E}G; \mathbf{K}_R) \to K_n(RG)$$

is bijective for $n \in \mathbb{Z}$.

- Since $K_q(RH) = 0$ for finite $H \subseteq G$, this implies

$$K_n(RG) = 0 \quad \text{for } n \leq -1$$

and

$$H_0^G(\underline{E}G; K_0(R?)) = \operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}\mathrm{in}}(G)} K_0(RH) \xrightarrow{\cong} K_0(RG)$$

is bijective.

Groups with special maximal finite subgroups

- Let G be a discrete group. Let $\mathcal{MF}\mathrm{in}$ be the subset of $\mathcal{F}\mathrm{in}$ consisting of elements in $\mathcal{F}\mathrm{in}$ which are maximal in $\mathcal{F}\mathrm{in}$.
- Assume that G satisfies the following assertions:
  - (M) Every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup;
  - (NM) $M \in \mathcal{MF}\mathrm{in}, M \neq \{1\} \ \Rightarrow \ N_G M = M$.
- Here are some examples of groups G which satisfy conditions (M) and (NM):
  - Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$;
  - Fuchsian groups;
  - One-relator groups G.

- For such a group there is a nice model for $\underline{E}G$ with as few non-free cells as possible. Let $\{(M_i) \mid i \in I\}$ be the set of conjugacy classes of maximal finite subgroups of $M_i \subseteq G$. By attaching free G-cells we get an inclusion of G-CW-complexes $j_1 \colon \coprod_{i \in I} G \times_{M_i} EM_i \to EG$.
- Define $\underline{E}G$ as the G-pushout

$$\begin{array}{ccc} \coprod_{i \in I} G \times_{M_i} EM_i & \xrightarrow{j_1} & EG \\ \downarrow u_1 & & \downarrow f_1 \\ \coprod_{i \in I} G/M_i & \xrightarrow{k_1} & \underline{E}G \end{array}$$

where u_1 is the obvious G-map obtained by collapsing each EM_i to a point.

- Next we explain why $\underline{E}G$ is a model for the classifying space for proper actions of G.
- Its isotropy groups are all finite. We have to show for $H \subseteq G$ finite that $\underline{E}G^H$ contractible.
- We begin with the case $H \neq \{1\}$. Because of conditions (M) and (NM) there is precisely one index $i_0 \in I$ such that H is subconjugated to M_{i_0} and is not subconjugated to M_i for $i \neq i_0$. We get

$$\left(\coprod_{i \in I} G/M_i\right)^H \;=\; (G/M_{i_0})^H \;=\; \mathrm{pt}.$$

  Hence $\underline{E}G^H = \mathrm{pt}$.
- It remains to treat $H = \{1\}$. Since u_1 is a non-equivariant homotopy equivalence and j_1 is a cofibration, f_1 is a non-equivariant homotopy equivalence. Hence $\underline{E}G$ is contractible.

- Consider the pushout obtained from the G-pushout above by dividing the G-action

$$\begin{tikzcd} \coprod_{i \in I} BM_i \arrow[r] \arrow[d] & BG \arrow[d] \\ \coprod_{i \in I} \mathrm{pt} \arrow[r] & G\backslash \underline{E}G \end{tikzcd}$$

- The associated Mayer-Vietoris sequence yields

$$\ldots \to \widetilde{H}_{p+1}(G\backslash \underline{E}G) \to \bigoplus_{i \in I} \widetilde{H}_p(BM_i) \to \widetilde{H}_p(BG) \\ \to \widetilde{H}_p(G\backslash \underline{E}G) \to \ldots$$

- In particular we obtain an isomorphism for $p \geq \dim(\underline{E}G) + 1$

$$\bigoplus_{i \in I} H_p(BM_i) \xrightarrow{\cong} H_p(BG).$$

Theorem

Let G be a discrete group which satisfies the conditions (M) and (NM) above.
Then there is an isomorphism

$$K_1^G(\underline{E}G) \xrightarrow{\cong} K_1(G\backslash\underline{E}G),$$

and a short exact sequence

$$0 \to \bigoplus_{i \in I} \widetilde{R}_{\mathbb{C}}(M_i) \to K_0(\underline{E}G) \to K_0(G\backslash\underline{E}G) \to 0.$$

which splits of we invert the orders of the finite subgroups of G.

- If the Baum-Connes Conjecture is true for G, then

$$K_n(C_r^*(G)) \cong K_n^G(\underline{E}G).$$

Example (One-relator groups)

- Let $G = \langle s_1, s_2, \ldots s_g \mid r \rangle$ be a finitely generated one-relator-group.
- The Baum-Connes Conjecture is known to be true for G.
- If G is torsionfree, the presentation complex associated to the presentation above is a model for BG and we get

$$K_n(C_r^*(G)) \cong K_n(BG) \cong \begin{cases} H_0(BG) \oplus H_2(BG) & n \text{ even;} \\ H_1(BG) & n \text{ odd.} \end{cases}$$

- Now suppose that G is not torsionfree.

Example (continued)

- Let F be the free group with basis $\{q_1, q_2, \ldots, q_g\}$. Then r is an element in F. There exists an element $s \in F$ and an integer $m \geq 2$ such that $r = s^m$, the cyclic subgroup C generated by the class $\overline{s} \in Q$ represented by s has order m, any finite subgroup of G is subconjugated to C and for any $g \in G$ the implication $g^{-1}Cg \cap C \neq 1 \Rightarrow g \in C$ holds.
- Hence G satisfies (M) and (NM).
- There is an explicit two-dimensional model for $\underline{E}G$ with one 0-cell $G/C \times D^0$, g 1-cells $G \times D^1$ and one free 2-cell $G \times D^2$.

Example (continued)

- We conclude for $n \geq 3$

$$H_n(BC) \cong H_n(BG).$$

- We obtain for odd n

$$K_n(C_r^*(G)) \cong K_1(G\backslash \underline{E}G) \cong H_1(G\backslash \underline{E}G).$$

- We obtain for even n

$$K_n(C_r^*(G)) \cong \widetilde{R}_{\mathbb{C}}(C) \oplus H_0(G\backslash \underline{E}G) \oplus H_2(G\backslash \underline{E}G).$$

(Co-)homology for extensions of $\mathbb{Z}^n$ by $\mathbb{Z}/p$ for an odd prime p

- Fix an extension
$$1 \to \mathbb{Z}^n \xrightarrow{i} \Gamma \xrightarrow{q} \mathbb{Z}/p \to 1$$
for an odd prime p such that the conjugation action of $\mathbb{Z}/p$ on $\mathbb{Z}^n$ is free outside the origin.
- Define natural numbers for $m, j, k \in \mathbb{Z}$, $m, k, j \geq 0$
$$\begin{aligned} r_m &:= \operatorname{rk}_{\mathbb{Z}}\left(\left(\Lambda^m(\mathbb{Z}[\zeta]^k)\right)^{\mathbb{Z}/p}\right); \\ a_j &:= \left|\{(\ell_1, \ldots, \ell_k) \in \mathbb{Z}^k \mid \ell_1 + \cdots + \ell_k = j, 0 \leq \ell_i \leq p-1\}\right|; \\ s_m &:= \sum_{j=0}^{m-1} a_j. \end{aligned}$$

- We have

$$\begin{aligned} r_0 &= 1; \\ r_m &= 0 \quad \text{for } m \geq p; \\ r_m &= \frac{1}{p} \cdot \left(\binom{p-1}{m} + (-1)^m \cdot (p-1) \right) \quad \text{for } 2 \leq m < p. \end{aligned}$$

- If $k = 1$, we get

$$\begin{aligned} a_j &= 1 && \text{for} \quad 0 \leq j \leq p-1; \\ a_j &= 0 && \text{for} \quad p \leq j; \\ s_m &= m && \text{for} \quad 0 \leq j \leq p; \\ s_m &= p && \text{for} \quad p \leq j. \end{aligned}$$

Theorem (Cohomology of $B\Gamma$ and $\underline{B}\Gamma$)

- *There is an integer k which is uniquely determined by $n = (p-1) \cdot k$;*
- *For $m \geq 0$ the map*

$$\iota^m \colon H^m(B\Gamma) \to H^m(\mathbb{Z}^n)^{\mathbb{Z}/p}$$

  is surjective and its kernel is isomorphic to $(\mathbb{Z}/p)^{s_m}$ if m is even and trivial, if m is odd;
- *For $m \geq 0$ we get*

$$H^m(B\mathbb{Z}^n)^{\mathbb{Z}/p} \cong \mathbb{Z}^{r_m};$$

Theorem (continued)

- *For $m \geq 0$ we get*

$$H^m(B\Gamma) \cong \begin{cases} \mathbb{Z}^{r_m} \oplus (\mathbb{Z}/p)^{s_m} & m \text{ even;} \\ \mathbb{Z}^{r_m} & m \text{ odd;} \end{cases}$$

- *For $m \geq 0$ we get for $\underline{B}\Gamma := \Gamma \backslash \underline{E}\Gamma$*

$$H^m(\underline{B}\Gamma) \cong \begin{cases} \mathbb{Z}^{r_m} & m \text{ even;} \\ \mathbb{Z}^{r_m} \oplus (\mathbb{Z}/p)^{p^k - s_m} & m \text{ odd}, m \geq 3; \\ \mathbb{Z}^{r_1} = 0 & m = 1. \end{cases}$$

- We want to give some information about its proof. This needs some preparation.
- Let
$$N = t^0 + t + \cdots + t^{p-1} \in \mathbb{Z}[\mathbb{Z}/p]$$
be the *norm element*.
Denote by $I(\mathbb{Z}/p)$ the *augmentation ideal*, i.e., the kernel of the augmentation homomorphism $\mathbb{Z}[\mathbb{Z}/p] \to \mathbb{Z}$.
Let $\zeta = e^{2\pi i/p} \in \mathbb{C}$ be a primitive p-th root of unity.
- We have isomorphism of $\mathbb{Z}[\mathbb{Z}/p]$-modules
$$\mathbb{Z}[\mathbb{Z}/p]/N \cong I(\mathbb{Z}/p) \cong \mathbb{Z}[\zeta].$$
- $\mathbb{Z}[\zeta]$ is a Dedekind domain

Example ($\mathbb{Z}^n_\rho = I(\mathbb{Z}/p)$)

If we take for ρ the one given by the $(p-1) \times (p-1)$ matrix

$$\rho = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -1 \\ 1 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 1 & \cdots & 0 & 0 & -1 \\ \hdotsfor{6} \\ 0 & 0 & \cdots & 1 & 0 & -1 \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}$$

then $\mathbb{Z}^{p-1}_\rho$ is $\mathbb{Z}[\mathbb{Z}/p]$-isomorphic to $\mathbb{Z}[\mathbb{Z}/p]/N \cong I(\mathbb{Z}/p) \cong \mathbb{Z}[\zeta]$.

Lemma

1. *We have*
$$\mathbb{Z}_\rho^n \cong I_1 \oplus \cdots \oplus I_k,$$
where the I_j are non-zero ideals of $\mathbb{Z}[\zeta]$ and $n = k(p-1)$;
2. *The map π splits as a group homomorphism. Conversely the semidirect product of $\mathbb{Z}/p$ acting on a finite direct sum of nonzero ideals of $\mathbb{Z}[\zeta]$ by multiplication by ζ satisfies the conditions placed on Γ above;*
3. *Each non-trivial finite subgroup P of Γ is isomorphic to $\mathbb{Z}/p$ and its Weyl group $W_\Gamma P := N_\Gamma P/P$ is trivial;*

4. There are bijections

$$\begin{array}{rcl} H^1(\mathbb{Z}/p;\mathbb{Z}^n_\rho) & \xrightarrow{\cong} & \operatorname{cok}(\rho - \operatorname{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n); \\ H^1(\mathbb{Z}/p;\mathbb{Z}^n_\rho) & \cong & (\mathbb{Z}/p)^k; \\ \operatorname{cok}\bigl(\rho - \operatorname{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n\bigr) & \xrightarrow{\cong} & \mathcal{P} := \{(P) \mid P \subset \Gamma, 1 < |P| < \infty\}. \end{array}$$

If we fix a generator $s \in \mathbb{Z}/p$, the last bijection sends the element $\overline{u} \in \mathbb{Z}^n_\rho/(1-s)\mathbb{Z}^n_\rho$ the subgroup of order p generated by us;

5. We have $|\mathcal{P}| = p^k$;
6. $[\Gamma,\Gamma] = \operatorname{im}(\rho - \operatorname{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n)$;
7. $\Gamma/[\Gamma,\Gamma] \cong \operatorname{cok}(\rho - \operatorname{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n) \oplus \mathbb{Z}/p = (\mathbb{Z}/p)^{k+1}$.

Proof.

We only give the proof of assertion (1)

- Let $u \in \mathbb{Z}^n_\rho$. Then $N \cdot u$ is fixed by conjugation with $t \in \mathbb{Z}/p$ and hence is zero by assumption.
- Thus $\mathbb{Z}^n_\rho$ is a finitely generated module over the Dedekind domain $\mathbb{Z}[\mathbb{Z}/p]/\Sigma = \mathbb{Z}[\zeta]$.
- Any finitely generated torsion-free module over a Dedekind domain is isomorphic to a direct sum of nonzero ideals.
- Since $I_j \otimes \mathbb{Q} \cong \mathbb{Q}[\zeta]$, we see $\operatorname{rk}_{\mathbb{Z}}(I_j) = p - 1$.

$\square$

- Next will analyze the Hochschild-Serre Spectral sequence associated to the extension
$$1 \to \mathbb{Z}^n \xrightarrow{i} \Gamma \xrightarrow{q} \mathbb{Z}/p \to 1;$$
- Recall that its E_2-term is
$$E_2^{i,j} = H^i(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho))$$
and it converges to $H^{i+j}(\Gamma)$.
- Recall that $\widehat{H}^*(G;M)$ denotes the Tate cohomology of a finite group G with coefficients in the $\mathbb{Z}G$-module M.
- We have
$$\begin{array}{rcll} \widehat{H}^i(G;M) & = & H^i(G;M) & \text{for } i \geq 1; \\ \widehat{H}^i(G;M) & = & H_{-i-1}(G;M) & \text{for } i \leq -2. \end{array}$$
- There is an exact sequence
$$0 \to \widehat{H}^{-1}(G;M) \to M \otimes_{\mathbb{Z}G} \mathbb{Z} \xrightarrow{N} M^G \to \widehat{H}^0(G;M) \to 0.$$

Lemma

1. $$\widehat{H}^i(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)) \cong \bigoplus_{\substack{\ell_1+\cdots+\ell_k=j \\ 0\le\ell_q\le p-1}} \widehat{H}^{i+j}(\mathbb{Z}/p) = \begin{cases} (\mathbb{Z}/p)^{a_j} & i+j \text{ even;} \\ 0 & i+j \text{ odd;} \end{cases}$$

2. *The Hochschild-Serre spectral sequence associated to the extension*

$$1 \to \mathbb{Z}^n \xrightarrow{i} \Gamma \xrightarrow{q} \mathbb{Z}/p \to 1$$

collapses.

Proof.

- We begin with assertion (1).
- We obtain a $\mathbb{Z}[\mathbb{Z}/p]$-isomorphism

$$H^j(\mathbb{Z}^n_\rho) \quad \cong \quad \Lambda^j \mathbb{Z}^n_{\rho^*}. \tag{0.1}$$

- Given an ideal $I \subseteq \mathbb{Z}[\zeta]$, the inclusion of I into $\mathbb{Z}[\zeta]$ induces an isomorphism of $\mathbb{Z}_{(p)}[\zeta]$-modules

$$I \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \xrightarrow{\cong} \mathbb{Z}[\zeta] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[\zeta].$$

- One deduces from this the existence of an isomorphism

$$\widehat{H}^i(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)) \cong \widehat{H}^i\big(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)_{(p)}\big) \cong \widehat{H}^i(\mathbb{Z}/p; \Lambda^j \mathbb{Z}[\zeta]^k).$$

$\square$

Proof continued.

- Since

$$\Lambda^*\Big(\bigoplus_k \mathbb{Z}[\zeta]\Big) = \bigotimes_k \Lambda^*(\mathbb{Z}[\zeta])$$

and $\Lambda^l(\mathbb{Z}[\zeta]) = 0$ for $l \geq p$, we get

$$\Lambda^j(\mathbb{Z}[\zeta]^k)) = \bigoplus_{\substack{\ell_1+\cdots+\ell_k=j\\ 0\leq\ell_q\leq p-1}} \Lambda^{\ell_1}\mathbb{Z}[\zeta] \otimes \cdots \otimes \Lambda^{\ell_k}\mathbb{Z}[\zeta].$$

- Therefore we obtain an isomorphism

$$\widehat{H}^i(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)) \cong \bigoplus_{\substack{\ell_1+\cdots+\ell_k=j\\ 0\leq\ell_q\leq p-1}} \widehat{H}^i\left(\mathbb{Z}/p; \Lambda^{\ell_1}\mathbb{Z}[\zeta] \otimes \cdots \otimes \Lambda^{\ell_k}\mathbb{Z}[\zeta]\right).$$

$\square$

Proof continued.

- Next we proof assertion (2).
- Next we want to show that the differentials $d_r^{i,j}$ are zero for all $r \geq 2$ and i, j.
- By the checkerboard pattern of the E_2-term it suffices to show for $r \geq 2$ and that the differentials $d_r^{0,j}$ are trivial for $r \geq 2$ and all odd $j \geq 1$.
- This is equivalent to show that for every odd $j \geq 1$ the edge homomorphism

$$\iota^j \colon H^j(\Gamma) \to H^j(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} = E_2^{0,j}$$

is surjective.

□

Proof continued.

- Let
$$\mathrm{trf}^j\colon H^j(\mathbb{Z}^n) \to H^j(\Gamma)$$
be the transfer map associated to ι.
- The composite $\iota^j \circ \mathrm{trf}^j$ agrees with the norm map $N\colon H^j(\mathbb{Z}^n_\rho) \to H^j(\mathbb{Z}^n_\rho)$ given by multiplication with the norm element N. Its image is contained in $H^j(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$.
- Hence it suffices to show that the cokernel of $N\colon H^j(\mathbb{Z}^n_\rho) \to H^j(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$ is trivial.
- Since this cokernel is isomorphic to $\widehat{H^0}(\mathbb{Z}/p, H^j(\mathbb{Z}^n_\rho))$, the claim follows from assertion (1).

$\square$

Proof continued.

- It remains to show that all extensions are trivial.
  Since the composite

$$H^{i+j}(\Gamma) \xrightarrow{\mathrm{trf}^{i+j}} H^{i+j}(\mathbb{Z}^n) \xrightarrow{\iota^{i+j}} H^{i+j}(\Gamma)$$

  is multiplication with p, the torsion in $H^{i+j}(\Gamma)$ has exponent p.
- Since $p \cdot E_\infty^{i,j} = p \cdot E_2^{i,j} = 0$ for $i > 0$, all extensions are trivial and

$$H^m\Gamma \cong \bigoplus_{i+j=m} E_\infty^{i,j} = \bigoplus_{i+j=m} E_2^{i,j}.$$

□

Proof continued.

- Hence it suffices to show for $l_1, \ldots, l_k$ in $\{0, 1, \ldots, (p-1)\}$

$$\widehat{H}^i \left(\mathbb{Z}/p; \Lambda^{\ell_1}\mathbb{Z}[\zeta] \otimes \cdots \otimes \Lambda^{\ell_k}\mathbb{Z}[\zeta] \right) \cong \widehat{H}^{i+\sum_{a=1}^k \ell_a}(\mathbb{Z}/p).$$

- This is done by induction over over $j = \sum_{a=1}^k l_a$.
- The main ingredient is an exact sequence of $\mathbb{Z}[\mathbb{Z}/p]$-modules with a free $\mathbb{Z}[\mathbb{Z}/p]$-module in the middle

$$\begin{aligned} 1 \to \Lambda^{l_1-1}\mathbb{Z}[\zeta] \otimes \Lambda^{\ell_2}\mathbb{Z}[\zeta] \otimes \cdots \otimes \Lambda^{\ell_k}\mathbb{Z}[\zeta] & \\ \to \Lambda^{l_1}\mathbb{Z}[\mathbb{Z}/p] \otimes \Lambda^{\ell_2}\mathbb{Z}[\zeta] \otimes \cdots \otimes \Lambda^{\ell_k}\mathbb{Z}[\zeta] & \\ \to \Lambda^{l_1}\mathbb{Z}[\zeta] \otimes \Lambda^{\ell_2}\mathbb{Z}[\zeta] \otimes \cdots \otimes \Lambda^{\ell_k}\mathbb{Z}[\zeta] \to 1. & \end{aligned}$$

□

- The first four assertion of Theorem about the cohomology of $B\Gamma$ and $\underline{B}\Gamma$ are direct consequences of the last two lemmas.
- Hence it remains to prove the last assertion.
- Before we do this, we look at an example about what we have achieved so far.

Example

For $k = 1$ and $p = 5$ for $E_2^{i,j} = E_\infty^{i,j}$

$$
\begin{array}{ccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & \cdots \\
\mathbb{Z} & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \cdots \\
0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & \cdots \\
\mathbb{Z}^2 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \cdots \\
0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & \cdots \\
\mathbb{Z} & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \cdots
\end{array}
$$

Example (Example continued)

Moreover

$$H^n(\Gamma) = \begin{cases} \mathbb{Z} & n = 0; \\ 0 & n \geq 1, n \text{ odd}; \\ \mathbb{Z}^2 \oplus (\mathbb{Z}/5)^2 & n = 2; \\ \mathbb{Z} \oplus (\mathbb{Z}/5)^4 & n = 4; \\ (\mathbb{Z}/5)^5 & n \geq 6, n \text{ even}. \end{cases}$$

- We have earlier constructed the cellular pushout

$$\begin{array}{ccc} \coprod_{(P)\in\mathcal{P}} BP & \xrightarrow{j_0} & B\Gamma \\ \Big\downarrow \coprod_{(P)\in\mathcal{P}} \overline{\mathrm{pr}}_P & & \Big\downarrow \overline{f} \\ \coprod_{(P)\in\mathcal{P}} \mathrm{pt} & \xrightarrow{j_1} & \underline{B}\Gamma \end{array} \tag{0.2}$$

- It yields for $m \geq 0$ the long exact sequence of reduced integral cohomology groups

$$0 \to \widetilde{H}^{2m}(\underline{B}\Gamma) \xrightarrow{\overline{f}^*} \widetilde{H}^{2m}(\Gamma) \xrightarrow{\varphi^{2m}} \bigoplus_{(P)\in\mathcal{P}} \widetilde{H}^{2m}(P)$$
$$\xrightarrow{\delta^{2m}} \widetilde{H}^{2m+1}(\underline{B}\Gamma) \xrightarrow{\overline{f}^*} \widetilde{H}^{2m+1}(\Gamma) \to 0 \tag{0.3}$$

where ϕ^m is the map induced by the various inclusions $P \subset G$ for $(P) \in \mathcal{P}$.

- In order to compute this sequence we need the following result whose proof is based on a spectral sequence argument.

Lemma

Let $K^{2m} \subseteq H^{2m}(\Gamma)$ *be the kernel of*

$$\varphi^{2m} \colon \widetilde{H}^{2m}(\Gamma) \to \bigoplus_{(P)\in\mathcal{P}} \widetilde{H}^{2m}(P).$$

Let $\beta \in H^2(\mathbb{Z}/p) \cong \mathbb{Z}/p$ *be a generator.*
Let L^{2m} *be the kernel of*

$$- \cup \pi^*(\beta)^n \colon H^{2m}(\Gamma) \to H^{2m+2n}(\Gamma).$$

Then for $m \geq 1$*:*

1. We have $K^{2m} = L^{2m}$;
2. The intersection L^{2m} with the kernel of the epimorphism

$$H^{2m}(\Gamma) \to H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$$

is trivial;
3. The image of L^{2m} under the epimorphism

$$H^{2m}(\Gamma) \to H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} = H^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho))$$

is the kernel of the projection

$$H^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho)) \to \widehat{H}^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho));$$

4. We have $K^{2m} \cong \mathbb{Z}^{r_m}$;
5. The quotient $H^{2m}(\Gamma)/K^{2m}$ is isomorphic to

$$\ker\left(H^{2m}(\Gamma) \to H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}\right) \oplus \widehat{H}^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho)) \cong (\mathbb{Z}/p)^{s_{2m+1}}.$$

Proof.

- We give only the proof of the first assertion.
- The following diagram computes

$$\begin{tikzcd} H^{2m}(\Gamma) \arrow[r, "\varphi^{2m}"] \arrow[d, "-\cup\pi^*(\beta)^n"] & \bigoplus_{(P)\in\mathcal{P}} H^{2m}(P) \arrow[d, "-\cup\phi^{2m+2n}(\pi^*(\beta)^n)"] \\ H^{2m+2n}(\Gamma) \arrow[r, "\varphi^{2m+2n}"] & \bigoplus_{(P)\in\mathcal{P}} H^{2m+2n}(P) \end{tikzcd}$$

- Since $\dim(\underline{B}\Gamma) \leq n$, we have $H^{i+2n}(\underline{B}\Gamma) = 0$ for $i \geq 1$.
- Hence the lower horizontal arrow is bijective by a long excat sequence from above. The right vertical arrow is bijective.

$\square$

- We conclude from the already proved fourth assertion of the Theorem about the cohomology of $B\Gamma$ and $\underline{B}\Gamma$. and the Lemma above:

Corollary

For $m \geq 1$ the long exact sequence

$$0 \to \widetilde{H}^{2m}(\underline{B}\Gamma) \xrightarrow{\overline{f}^*} \widetilde{H}^{2m}(\Gamma) \xrightarrow{\varphi^{2m}} \bigoplus_{(P)\in\mathcal{P}} \widetilde{H}^{2m}(P)$$
$$\xrightarrow{\delta^{2m}} \widetilde{H}^{2m+1}(\underline{B}\Gamma) \xrightarrow{\overline{f}^*} \widetilde{H}^{2m+1}(\Gamma) \to 0 \quad (0.4)$$

can be identified with

$$0 \to \mathbb{Z}^{r_{2m}} \to \mathbb{Z}^{r_{2m}} \oplus (\mathbb{Z}/p)^{s_{2m}} \to (\mathbb{Z}/p)^{p^k} \to \mathbb{Z}^{r_{2m+1}} \oplus (\mathbb{Z}/p)^{p^k - s_{2m+1}} \to \mathbb{Z}^{r_{2m+1}} \longrightarrow$$

- The corollary above implies fifth assertion of the Theorem about the cohomology of $B\Gamma$ and $\underline{B}\Gamma$.

Theorem (Homology of $B\Gamma$ and $\underline{B}\Gamma$)

- *For $m \geq 0$ we get*

$$H_m(B\Gamma) \cong \begin{cases} \mathbb{Z}^{r_m} \oplus (\mathbb{Z}/p)^{s_{m+1}} & m \text{ odd;} \\ \mathbb{Z}^{r_m} & m \text{ even;} \end{cases}$$

- *For $m \geq 0$ we get*

$$H_m(\underline{B}\Gamma) \cong \begin{cases} \mathbb{Z}^{r_m} & m \text{ odd;} \\ \mathbb{Z}^{r_m} \oplus (\mathbb{Z}/p)^{p^k - s_{m+1}} & m \text{ even}, m \geq 2; \\ \mathbb{Z} & m = 0; \end{cases}$$

Proof.

Universal coefficient theorem $\square$

Topological K-theory for extensions of $\mathbb{Z}^n$ by $\mathbb{Z}/p$ for an odd prime p

We fix the same data before, namely:

- Fix an extension
$$1 \to \mathbb{Z}^n \xrightarrow{i} \Gamma \xrightarrow{q} \mathbb{Z}/p \to 1$$
for an odd prime p such that the conjugation action of $\mathbb{Z}/p$ on $\mathbb{Z}^n$ is free outside the origin.
- Define natural numbers for $m, j, k \in \mathbb{Z}, m, k, j \geq 0$

$$\begin{aligned}
r_m &:= \operatorname{rk}_{\mathbb{Z}}\left(\left(\Lambda^m(\mathbb{Z}[\zeta]^k\right)^{\mathbb{Z}/p}\right); \\
a_j &:= \left|\{(\ell_1, \ldots, \ell_k) \in \mathbb{Z}^k \mid \ell_1 + \cdots + \ell_k = j, 0 \leq \ell_i \leq p-1\}\right|; \\
s_m &:= \sum_{j=0}^{m-1} a_j.
\end{aligned}$$

Theorem (K-cohomology of $B\Gamma$ and $\underline{B}\Gamma$)

- *There is an exact sequence of abelian groups*

$$0 \to (\widehat{\mathbb{Z}_p})^{(p-1)p^k} \to K^0(B\Gamma) \to K^0(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} \to 0$$

which splits.
We have

$$K^0(B\Gamma) \cong \mathbb{Z}^{\sum_{l \in \mathbb{Z}} r_{2l}} \oplus (\widehat{\mathbb{Z}_p})^{(p-1)p^k};$$

- *The inclusion* $\iota\colon \mathbb{Z}^n \to \Gamma$ *induces an isomorphism*

$$K^1(B\Gamma) \xrightarrow{\cong} K^1(B\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}.$$

Theorem (continued)

- *We have*
$$K^1(B\Gamma) \cong \mathbb{Z}^{\sum_{l \in \mathbb{Z}} r_{2l+1}};$$
- *We have*
$$K^0(\underline{B}\Gamma) \cong \mathbb{Z}^{\sum_{l \in \mathbb{Z}} r_{2l}};$$
- *We have*
$$K^1(\underline{B}\Gamma) \cong \mathbb{Z}^{\sum_{l \in \mathbb{Z}} r_{2l+1}} \oplus T^1;$$
for an (unknown) finite abelian p-group T^1;
- *The map $K^1(\underline{B}\Gamma) \to K^1(B\Gamma)$ induces an isomorphism*
$$K^1(\underline{B}\Gamma)/p - \mathrm{tors} \xrightarrow{\cong} K^1(B\Gamma)$$
It kernel is isomorphic to T^1.

Theorem (K-homology of $B\Gamma$ and $\underline{B}\Gamma$)

- *The inclusion* $\iota\colon \mathbb{Z}^n \to \Gamma$ *induces an isomorphism*

$$K_0(\mathbb{Z}^n_\rho) \otimes_{\mathbb{Z}[\mathbb{Z}/p]} \mathbb{Z} \xrightarrow{\cong} K_0(B\Gamma).$$

We have

$$K_0(B\Gamma) \cong \mathbb{Z}^{\sum_{l \in \mathbb{Z}} r_{2l+1}};$$

- *There is a short exact sequence of abelian groups*

$$0 \to (\mathbb{Z}/p^\infty)^{(p-1)p^k} \to K_1(B\Gamma) \to K_1(\underline{B}\Gamma) \to 0,$$

which splits.

Theorem (continued)

- *We have*
$$K_1(B\Gamma) \cong \mathbb{Z}^{\sum_{l \in \mathbb{Z}} r_{2l+1}} \oplus (\mathbb{Z}/p^\infty)^{(p-1)p^k};$$

- *We have*
$$K_n(\underline{B}\Gamma) \cong \begin{cases} \mathbb{Z}^{\sum_{l \in \mathbb{Z}} r_{2l}} \oplus T^1 & \textit{if } n \textit{ is even;} \\ \mathbb{Z}^{\sum_{l \in \mathbb{Z}} r_{2l+1}} & \textit{if } n \textit{ is odd.} \end{cases}$$

- We have already explained earlier that there is a bijection

$$H^1(\mathbb{Z}/p;\mathbb{Z}^n_\rho) \xrightarrow{\cong} \mathcal{P} := \{(P) \mid P \subset \Gamma, 1 < |P| < \infty\}.$$

  and

$$|\mathcal{P}| = p^{\frac{n}{p-1}}.$$

- The Baum-Connes Conjecture is known to be true for G.

- Hence we get from a previous theorem an isomorphism

$$K_1^G(\underline{E}\Gamma) \xrightarrow{\cong} K_1(\Gamma\backslash\underline{E}\Gamma),$$

and a short exact sequence

$$0 \to \bigoplus_{(P)\in\mathcal{P}} \widetilde{R}_{\mathbb{C}}(P) \to K_0(\underline{E}\Gamma) \to K_0(G\backslash\underline{E}\Gamma) \to 0.$$

which splits of we invert p.
- We have

$$\widetilde{R}_{\mathbb{C}}(M_i) \cong \mathbb{Z}^{p-1}.$$

In particular we get

$$\bigoplus_{(P)\in\mathcal{P}} \widetilde{R}_{\mathbb{C}}(P) \cong \mathbb{Z}^{(p-1)\cdot p^{\frac{n}{p-1}}}.$$

- We have already computed

$$K_n(\Gamma\backslash \underline{E}\Gamma) \quad\cong\quad \begin{cases} \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l}} \oplus T^1 & \text{if } n \text{ is even;}\\ \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l+1}} & \text{if } n \text{ is odd,}\end{cases}$$

  where T^1 is an unknown finite abelian p-group.

- But a miracle occurs, namely, that $K_n^\Gamma(\underline{E}\Gamma)$ contains no torsion and the relevant extension problem can be solved although we do not know T^1.
- Namely, we get

$$K_n(C_r^*(\Gamma)) \cong \begin{cases} \mathbb{Z}^{(p-1)p^{\frac{n}{p-1}}+\sum_{l\in\mathbb{Z}} r_{2l}} & \text{if } n \text{ is even;}\\ \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l+1}} & \text{if } n \text{ is odd.}\end{cases}$$

Theorem (Equivariant K-cohomology of $\underline{E}\Gamma$)

- *The canonical maps*

$$\begin{array}{rcl} K_\Gamma^1(\underline{E}\Gamma) & \xrightarrow{\cong} & K^1(B\Gamma); \\ K^1(B\Gamma) & \xrightarrow{\cong} & K^1(B\mathbb{Z}_\rho^n)^{\mathbb{Z}/p}; \end{array}$$

  are bijections;

- *We have the miracle*

$$K_n(C_r^*(\Gamma)) \cong K_\Gamma^n(\underline{E}\Gamma) \cong \begin{cases} \mathbb{Z}^{(p-1)p^k + \sum_{l\in\mathbb{Z}} r_{2l}} & \text{if } n \text{ is even}; \\ \mathbb{Z}^{\sum_{l\in\mathbb{Z}} r_{2l+1}} & \text{if } n \text{ is odd}; \end{cases}$$

- *There is an exact sequence (for unknown T^1)*

$$0 \to K^0(\underline{B}\Gamma) \to K_\Gamma^0(\underline{E}\Gamma) \to \bigoplus_{(P)\in\mathcal{P}} \mathbb{I}_{\mathbb{C}}(P) \to T^1 \to 0.$$

- Later we will deal with algebraic K and L-theory what is more complicated.
- We see that for computations of group homology or of K- and L-groups of group rings and group C^*-algebras it is important to understand the spaces $G\backslash \underline{E}G$.
- Often geometric input ensures that $G\backslash \underline{E}G$ is a fairly small CW-complex.
- On the other hand recall the result due to Leary-Nucinkis (2001) that for any CW-complex X there exists a group G with $X \simeq G\backslash \underline{E}G$.

Passing from $\mathcal{F}$in to $\mathcal{VC}$yc

- Recall that $\mathcal{F}$in is the family of finite subgroups and $\mathcal{VC}$yc the family of virtually cyclic subgroups.
- Define an equivalence relation $\sim$ on $\mathcal{VC}\text{yc} \setminus \mathcal{F}\text{in}$ by

$$V_1 \sim V_2 \Longleftrightarrow |V_1 \cap V_2| = \infty.$$

  Let $[\mathcal{VC}\text{yc} \setminus \mathcal{F}\text{in}]$ be the set of equivalence classes.
- Define for a virtually cyclic subgroup $V \subseteq G$ a subgroup of G by

$$N_G[V] = \{g \in G \mid g^{-1}Vg \sim V\}.$$

- This is the isotropy group of the class $[V] \in [\mathcal{VC}\text{yc} \setminus \mathcal{F}\text{in}]$ under the conjugation action. It contains the normalizer $N_G V$ but is in general larger.

- Define for $V \in \mathcal{VC}\text{yc} \setminus \mathcal{F}\text{in}$ a family of subgroups of G

$$\mathcal{G}[V] := \{K \subseteq N_G[H] \mid K \in \mathcal{VC}\text{yc} \setminus \mathcal{F}\text{in}, [K] = [H]\} \cup (\mathcal{F}\text{in} \cap N_G[V]).$$

Theorem (Constructing $\underline{\underline{E}}G$ from $\underline{E}G$, Lück-Weiermann (2007))

Let I be a complete system of representatives $[V]$ of the G-orbits in $[\mathcal{VC}yc \setminus \mathcal{F}in]$ under the G-action coming from conjugation. Then there exists a the cellular G-pushout

$$\begin{array}{ccc} \coprod_{V \in I} G \times_{N_G[V]} E_{\mathcal{F}in \cap N_G[V]}(N_G[V]) & \xrightarrow{i} & \underline{E}G \\ \Big\downarrow \coprod_{V \in I} \mathrm{id}_G \times_{N_G[V]} f_V & & \Big\downarrow \\ \coprod_{[H] \in I} G \times_{N_G[V]} E_{\mathcal{G}[V]}(N_G[V]) & \longrightarrow & \underline{\underline{E}}G \end{array}$$

- Consider families of subgroups $\mathcal{F} \subseteq \mathcal{G}$.
- We shall say that G satisfies $(M_{\mathcal{F}\subseteq\mathcal{G}})$ if every subgroup $H \in \mathcal{G} \setminus \mathcal{F}$ is contained in a unique $H_{\max} \in \mathcal{G} \setminus \mathcal{F}$ which is maximal in $\mathcal{G} \setminus \mathcal{F}$, i.e., $H_{\max} \subseteq K$ for $K \in \mathcal{G} \setminus \mathcal{F}$ implies $H_{\max} = K$.
- We shall say that G satisfies $(NM_{\mathcal{F}\subseteq\mathcal{G}})$ if M satisfies $(M_{\mathcal{F}\subseteq\mathcal{G}})$ and every maximal element $M \in \mathcal{G} \setminus \mathcal{F}$ equals its normalizer, i.e., $N_G M = M$ or, equivalently, $W_G M = \{1\}$.

Corollary

Suppose that G satisfies $(M_{\mathcal{F}in \subseteq \mathcal{VC}yc})$. Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of subgroups in $\mathcal{VC}yc \setminus \mathcal{F}in$ which are maximal in $\mathcal{VC}yc \setminus \mathcal{F}in$.
Then there is a cellular G-pushout

$$\begin{array}{ccc}
\coprod_{M \in \mathcal{M}} G \times_{N_G M} \underline{E} N_G M & \xrightarrow{i} & \underline{E}G \\
\Big\downarrow {\scriptstyle \coprod_{M \in \mathcal{M}} \mathrm{id}_G \times_{N_G M} f_{[H]}} & & \Big\downarrow \\
\coprod_{M \in \mathcal{M}} G \times_{N_G M} E W_G M & \longrightarrow & \underline{\underline{E}}G
\end{array}$$

Corollary

Let G be a group satisfying $(NM_{\mathcal{F}in \subseteq \mathcal{VC}yc})$. Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of subgroups in $\mathcal{VC}yc \setminus \mathcal{F}in$ which are maximal in $\mathcal{VC}yc \setminus \mathcal{F}in$.
Then there is a cellular G-pushout

$$\begin{array}{ccc} \coprod_{M \in \mathcal{M}} G \times_M \underline{E}M & \xrightarrow{i} & \underline{E}G \\ \downarrow{\scriptstyle \coprod_{M \in \mathcal{M}} p} & & \downarrow \\ \coprod_{M \in \mathcal{M}} G/M & \longrightarrow & \underline{\underline{E}}G \end{array}$$

where i is an inclusion of G-CW-complexes and p is the obvious projection.

- Let $\mathcal{H}_*^?$ be an equivariant homology theory.
- Suppose that G satisfies $(M_{\mathcal{F}\text{in}\subseteq\mathcal{VC}\text{yc}})$ or $(NM_{\mathcal{F}\text{in}\subseteq\mathcal{VC}\text{yc}})$. Then we obtain from the two corollaries above isomorphisms

$$\begin{aligned}
\bigoplus_{M\in\mathcal{M}} \mathcal{H}_n^{N_GM}(\underline{E}N_GM \to EW_GM) &\xrightarrow{\cong} \mathcal{H}_n^G(\underline{E}G \to \underline{\underline{E}}G); \\
\bigoplus_{M\in\mathcal{M}} \mathcal{H}_n^{N_GM}(\underline{E}M \to \mathrm{pt}) &\xrightarrow{\cong} \mathcal{H}_n^G(\underline{E}G \to \underline{\underline{E}}G).
\end{aligned}$$

- Recall that for every $n \in \mathbb{Z}$ the maps

$$\begin{aligned}
H_n^G(\underline{E}G; \mathbf{K}_R) &\to H_n^G(\underline{\underline{E}}G; \mathbf{K}_R); \\
H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty\rangle}) &\to H_n^G(\underline{\underline{E}}G; \mathbf{L}_R^{\langle -\infty\rangle}),
\end{aligned}$$

are split injective.

- Suppose that G satisfies $(M_{\mathcal{F}\mathrm{in}\subseteq\mathcal{VC}\mathrm{yc}})$ or $(NM_{\mathcal{F}\mathrm{in}\subseteq\mathcal{VC}\mathrm{yc}})$. Then we get isomorphisms

$$\mathcal{H}_n^G(\underline{E}G;\mathbf{K}_R) \oplus \bigoplus_{M\in\mathcal{M}} \mathcal{H}_n^{N_GM}(\underline{E}N_GM \to EW_GM;\mathbf{K}_R) \xrightarrow{\cong} K_n(RG)$$

$$\mathcal{H}_n^G(\underline{E}G;\mathbf{K}_R) \oplus \bigoplus_{M\in\mathcal{M}} \mathcal{H}_n^{M}(\underline{E}M \to \mathrm{pt};\mathbf{K}_R) \xrightarrow{\cong} K_n(RG),$$

and

$$\mathcal{H}_n^G(\underline{E}G;\mathbf{L}_R^{\langle-\infty\rangle}) \oplus \bigoplus_{M\in\mathcal{M}} \mathcal{H}_n^{N_GM}(\underline{E}N_GM \to EW_GM;\mathbf{L}_R^{\langle-\infty\rangle}) \xrightarrow{\cong} L_n^{\langle-\infty\rangle}(RG)$$

$$\mathcal{H}_n^G(\underline{E}G;\mathbf{L}_R^{\langle-\infty\rangle}) \oplus \bigoplus_{M\in\mathcal{M}} \mathcal{H}_n^{M}(\underline{E}M \to \mathrm{pt};\mathbf{L}_R^{\langle-\infty\rangle}) \xrightarrow{\cong} L_n^{\langle-\infty\rangle}(RG),$$

provided that the K-theoretic and the L-theoretic Farrell-Jones Conjectures are true for G.

Theorem

Suppose that G is torsionfree and satisfies $(NM_{\mathcal{F}in \subseteq \mathcal{VC}yc})$. Assume that the K-theoretic and the L-theoretic Farrell-Jones Conjecture are true for G. Then we obtain isomorphisms

$$H_n^G(BG;\mathbf{K}_R) \oplus \bigoplus_{M \in \mathcal{M}} \left(NK_n(R) \oplus NK_n(R)\right) \xrightarrow{\cong} K_n(RG),$$

and

$$H_n^G(BG;\mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG).$$

- The *n-th Whitehead group* for $n \in \mathbb{Z}$ is defined to be

$$\mathrm{Wh}_n(G;R) \quad := \quad H_n^G\big(EG \to \mathrm{pt}; \mathbf{K}_R\big).$$

- It fits into a long exact sequence

$$\cdots \to H_n\big(BG;\mathbf{K}_R\big) \to K_n(RG) \to \mathrm{Wh}_n(G;R) \\ \to H_{n-1}\big(BG;\mathbf{K}_R\big) \to K_{n-1}(RG) \to \cdots,$$

- If $R = \mathbb{Z}$, then $\mathrm{Wh}_1(G;\mathbb{Z})$ is the classical Whitehead group $\mathrm{Wh}(G)$.

- Whitehead groups are connected with Waldhausen's A-theory, pseudoisotopy and spaces of h-cobordisms.
- If G is torsionfree and R is regular, the Farrell-Jones Conjecture predicts

$$\mathrm{Wh}_n(G;R) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Theorem (Whitehead groups)

Suppose that G satisfies $(NM_{\mathcal{TR}\subseteq\mathcal{F}in})$ and $(NM_{\mathcal{F}in\subseteq\mathcal{VC}yc})$. Assume that the K-theoretic and the L-theoretic Farrell-Jones Conjecture are true for G. Let I and J respectively be complete set of representatives of the conjugacy classes of maximal finite subgroups and maximal infinite virtually cyclic subgroups respectively.
Then we obtain isomorphisms

$$\left(\bigoplus_{F\in I}\mathrm{Wh}(F;R)\right) \oplus \left(\bigoplus_{V\in I} H_n^V(\underline{E}V \to pt;\mathbf{K}_R)\right) \xrightarrow{\cong} \mathrm{Wh}(G;R).$$

Theorem (Groups satisfying ($NM_{\mathcal{F}\text{in}\subseteq\mathcal{VC}\text{yc}}$), Lück-Weiermann (2007))

Suppose that the group G satisfies the following two conditions:

- *Every infinite cyclic subgroup $C \subseteq G$ has finite index $[C_G C : C]$ in its centralizer;*
- *Every ascending chain $H_1 \subseteq H_2 \subseteq \dots$ of finite subgroups of G becomes stationary, i.e., there is an $n_0 \in \mathbb{N}$ such that $H_n = H_{n_0}$ for all $n \geq n_0$.*

Then G satisfies ($NM_{\mathcal{F}in\subseteq\mathcal{VC}yc}$).

Hyperbolic groups

- Hyperbolic groups satisfy the assumptions appearing in the last theorem.
- Both the K-theoretic and the L-theoretic Farrell-Jones Conjecture are known to be true for hyperbolic groups.
- Hence we obtain from a theorem above.

Theorem (Hyperbolic groups)

- *Let G be a hyperbolic group. Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal infinite virtual cyclic subgroups of G.*
- *For $n \in \mathbb{Z}$ there is an isomorphism*

$$H_n^G(\underline{E}G; \mathbf{K}_R) \oplus \bigoplus_{V \in \mathcal{M}} \mathcal{H}_n^V(\underline{E}V \to pt; \mathbf{K}_R) \xrightarrow{\cong} K_n(RG);$$

- *For $n \in \mathbb{Z}$ there is an isomorphism*

$$H_n(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) \oplus \bigoplus_{V \in \mathcal{M}} \mathcal{H}_n^V(\underline{E}V \to pt; \mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG).$$

Theorem (Torsionfree hyperbolic groups)

If G is a torsionfree hyperbolic group, then we get isomorphisms

$$H_n(BG;\mathbf{K}_R) \oplus \left(\bigoplus_{\substack{(C), C\subseteq G, C\neq 1\\ C \text{ maximal cyclic}}} \left(NK_n(R) \oplus NK_n(R)\right) \right) \xrightarrow{\cong} K_n(RG),$$

and

$$H_n(BG;\mathbf{L}_R^{\langle -\infty\rangle}) \xrightarrow{\cong} L_n^{\langle -\infty\rangle}(RG).$$

- The Baum-Connes Conjecture and the Bost Conjecture are also known to be true for hyperbolic groups and reduce therefore for obvious reasons for a torsionfree hyperbolic group to

$$K_n(BG) \cong K_n(C_r^*(G)) \cong K_n(l^1(G)).$$

Finitely generated free abelian groups

Theorem (The algebraic K- and L-theory of $\mathbb{Z}^d$)

Let $d \geq 1$ be an integer. Let $\mathcal{MICY}$ be the set of maximal infinite cyclic subgroups of $\mathbb{Z}^d$. Then we obtain isomorphisms

$$\begin{aligned}
\mathrm{Wh}_n(\mathbb{Z}^d; R) &\cong \bigoplus_{C \in \mathcal{MICY}} \bigoplus_{i=0}^{d-1} \big(NK_{n-i}(R) \oplus NK_{n-i}(R)\big)^{\binom{d-1}{i}}; \\
K_n(R[\mathbb{Z}^d]) &\cong \left(\bigoplus_{i=0}^{d} K_{n-i}(R)^{\binom{d}{i}}\right) \oplus \mathrm{Wh}_n(\mathbb{Z}^d; R); \\
L_n^{\langle -\infty \rangle}(R[\mathbb{Z}^d]) &\cong \bigoplus_{i=0}^{d} L_{n-i}^{\langle -\infty \rangle}(R)^{\binom{d}{i}}.
\end{aligned}$$

Example ($\mathbb{Z}^d \times G$)

- Let G be a group.
- Then we conclude from the Theorem above

$$\begin{aligned} K_n(R[G \times \mathbb{Z}^d]) &\cong K_n(RG[\mathbb{Z}^d]) \\ &\cong \bigoplus_{i=0}^{d} K_{n-i}(RG)^{\binom{d}{i}} \oplus \bigoplus_{C \in \mathcal{MICY}(\mathbb{Z}^d)} \bigoplus_{i=0}^{d-1} \\ &\qquad \left(NK_{n-i}(RG) \oplus NK_{n-i}(RG)\right)^{\binom{d-1}{i}}, \end{aligned}$$

where $\mathcal{MICY}(\mathbb{Z}^d)$ is the set of maximal infinite cyclic subgroups of $\mathbb{Z}^d$.

Example (continued)

- Since

$$H_n\big(B(G \times \mathbb{Z}^d); \mathbf{K}_R\big) \cong \bigoplus_{i=0}^{d} H_n\big(BG; \mathbf{K}_R\big)^{\binom{d}{i}}$$

this implies

$$\mathrm{Wh}_n(G \times \mathbb{Z}^d; R) \cong \bigoplus_{i=0}^{d} \mathrm{Wh}_{n-i}(G; R)^{\binom{d}{i}}$$

$$\oplus \bigoplus_{C \in \mathcal{MICY}(\mathbb{Z}^d)} \bigoplus_{i=0}^{d-1} \big(NK_{n-i}(RG) \oplus NK_{n-i}(RG)\big)^{\binom{d-1}{i}}.$$

Finite extensions of $\mathbb{Z}^n$

- Consider an extension
$$1 \to \mathbb{Z}^n \to G \to Q \to 1$$
for a finite group Q such that the conjugation action of Q on $\mathbb{Z}^n$ is free outside the origin.
- Let $\mathcal{MICY}(\mathbb{Z}^n)$ be the set of maximal infinite cyclic subgroups of $\mathbb{Z}^n$.
- The Q-conjugation operation $\mathbb{Z}^n$ induces a Q-operation on $\mathcal{MICY}(A)$.
- Fix a subset
$$I \quad \subseteq \quad \mathcal{MICY}(\mathbb{Z})$$
such that the intersection of every Q-orbit in $\mathcal{MICY}(IZ^n)$ with I consists of precisely one element.

- For $C \in I$ let

$$Q_C \quad \subseteq \quad Q$$

be the isotropy group of $C \in \mathcal{MICY}(\mathbb{Z})$ under the Q-operation.

- Put

$$\begin{aligned} I_1 &= \bigl\{C \in I \mid Q_C = \{1\}\bigr\}; \\ I_2 &= \bigl\{C \in I \mid Q_C = \mathbb{Z}/2\bigr\}. \end{aligned}$$

- Let J be a complete system of representatives of maximal non-trivial finite subgroups of G.

- Define the periodic structure group

$$\mathcal{S}_n^{\mathrm{per},\langle -\infty\rangle}(G;R) \quad := \quad H_n^G\big(EG \to \mathrm{pt}; \mathbf{L}_R^{\langle -\infty\rangle}\big).$$

- Notice that they fit into the periodic version of the long exact surgery sequence with decoration $\langle -\infty\rangle$

$$\cdots \to H_n\big(BG;\mathbf{L}^{\langle -\infty\rangle}(R)\big) \to L_n^{\langle -\infty\rangle}(RG) \to \mathcal{S}_n^{\mathrm{per},\langle -\infty\rangle}(G;R) \\ \to H_{n-1}\big(BG;\mathbf{L}^{\langle -\infty\rangle}(R)\big) \to L_{n-1}^{\langle -\infty\rangle}(RG) \to \cdots ,$$

- This periodic surgery sequence (with a different decoration) for $R = \mathbb{Z}$ appears in the classification of ANR-homology manifolds (see Bryant-Ferry-Mio-Weinberger) and is related to the algebraic surgery exact sequence due to Ranicki and thus to the classical surgery sequence due to Browder-Novikov-Sullivan-Wall.

Theorem

Under the conditions above we obtain for $n \in \mathbb{Z}$ isomorphisms

$$\left(\bigoplus_{F \in J} \mathrm{Wh}_n(F;R)\right) \oplus \left(\bigoplus_{C \in I_1} \bigoplus_{i=0}^{d-1} \left(NK_{n-i}(R) \oplus NK_{n-i}(R)\right)^{\binom{d-1}{i}}\right) \oplus \left(\bigoplus_{C \in I_2} \bigoplus_{i=0}^{d-1} NK_{n-i}(R)\right) \xrightarrow{\cong} \mathrm{Wh}_n(G;R).$$

and

$$\left(\bigoplus_{F \in J} \mathcal{S}_n^{\mathrm{per},\langle -\infty\rangle}(F;R)\right) \oplus \left(\bigoplus_{C \in I_2} \bigoplus_{H \in J_C} \mathrm{UNil}_n^{\langle -\infty\rangle}(D_\infty;R)\right) \xrightarrow{\cong} \mathcal{S}_n^{\mathrm{per},\langle -\infty\rangle}(G;R).$$

Fibered Isomorphism Conjectures

- Fix an equivariant homology theory $\mathcal{H}_*^?$.
- Given a group homomorphism $\phi\colon K \to G$ and a family $\mathcal{F}$ of subgroups of G, define the family $\phi^*\mathcal{F}$ of subgroups of K to be the set of subgroups $H \subset K$ with $\phi(H) \in \mathcal{F}$.

Definition (Fibered Isomorphism Conjecture)

A group G together with a family of subgroups $\mathcal{F}$ of G satisfies the *Fibered Isomorphism Conjecture (for $\mathcal{H}_*^?$)* if for each group homomorphism $\phi\colon K \to G$ the group K satisfies the Isomorphism Conjecture with respect to the family $\phi^*\mathcal{F}$, i.e., the projection $E_{\phi^*\mathcal{F}}(K) \to \mathrm{pt}$ induces for all $n \in \mathbb{Z}$ a bijection

$$\mathcal{H}_n^K(E_{\phi^*\mathcal{F}}(K)) \to \mathcal{H}_n^K(\mathrm{pt}).$$

Theorem (Transitivity Principle)

Suppose $\mathcal{F} \subseteq \mathcal{G}$ are two families of subgroups of G. Assume that for every element $H \in \mathcal{G}$ the group H satisfies the (Fibered) Isomorphism Conjecture for $\mathcal{F}|_H$.
Then $(G, \mathcal{G})$ satisfies the (Fibered) Isomorphism Conjecture if and only if $(G, \mathcal{F})$ satisfies the (Fibered) Isomorphism Conjecture

Lemma

Let $\phi\colon K \to G$ be a group homomorphism and let $\mathcal{F}$ be a family of subgroups of G. If G satisfies the Fibered Isomorphism Conjecture for the family $\mathcal{F}$, then K satisfies the Fibered Isomorphism Conjecture for the family $\phi^*\mathcal{F}$.

Proof.

If $\psi\colon L \to K$ is a group homomorphism, then $\psi^*(\phi^*\mathcal{F}) = (\phi \circ \psi)^*\mathcal{F}$. □

Lemma

Let G be a group and let $\mathcal{F} \subset \mathcal{G}$ be families of subgroups of G. Suppose that G satisfies the Fibered Isomorphism Conjecture for the family $\mathcal{F}$. Then G satisfies the Fibered Isomorphism Conjecture for the family $\mathcal{G}$.

Proof.

We want to use Transitivity Principle. Therefore we have to show for each subgroup K in $\mathcal{G}$ that it satisfies the Fibered Isomorphism Conjecture for $\mathcal{F} \cap K$. If $i\colon K \to G$ is the inclusion, then $i^*\mathcal{F} = \mathcal{F} \cap K$. Now apply a previous lemma. $\square$

Theorem

Let $\mathcal{F}$ be a family of subgroups of G. Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i \colon G_i \to G$. Suppose that $\mathcal{H}^?_*$ is strongly continuous and for every $i \in I$ the Fibered Isomorphism Conjecture holds for G_i and $\psi_i^* \mathcal{F}$.
Then the Fibered Isomorphism Conjecture holds for G and $\mathcal{F}$.

Proof.

This follows from the following commutative square, whose horizontal arrows are bijective because of the last Lemma, and the identification $\psi_i^* E_{\mathcal{F}}(G) = E_{\psi_i^*\mathcal{F}}(G_i)$

$$\begin{tikzcd}
\operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(E_{\psi_i^*\mathcal{F}}(G_i)) \arrow[r, "t_n^G(E_{\mathcal{F}}(G))", "\cong"'] \arrow[d] & \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \arrow[d] \\
\operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(\mathrm{pt}) \arrow[r, "t_n^G(\mathrm{pt})", "\cong"'] & \mathcal{H}_n^G(\mathrm{pt})
\end{tikzcd}$$

□

- Fix a class of groups $\mathcal{C}$ closed under isomorphisms, taking subgroups and taking quotients, e.g., the class of finite groups or the class of virtually cyclic groups.
- For a group G let $\mathcal{C}(G)$ be the family of subgroups of G which belong to $\mathcal{C}$.

Theorem (Inheritance of the Fibered Isomorphism Conjecture under colimits)

Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i \colon G_i \to G$. Suppose that $\mathcal{H}^?_*$ is strongly continuous and that the Fibered Isomorphism Conjecture is true for G_i and $\mathcal{C}(G_i)$ for all $i \in I$.

Then the Fibered Isomorphism Conjecture is true for G and $\mathcal{C}(G)$.

Proof.

- Since $\mathcal{C}$ is closed under quotients, we have $\mathcal{C}(G_i) \subseteq \psi_i^*\mathcal{C}(G)$ for all $i \in I$.
- By a previous Lemma we conclude that the Fibered Isomorphism Conjecture is true for G_i and $\psi_i^*\mathcal{C}(G)$ for all $i \in I$.
- Now the claim follows from the last Theorem.

$\square$

- Notice that it is very convenient for the proof to allow arbitrary families of subgroups and to have the definition of $\mathcal{H}_*^G(X)$ at hand for arbitrary (not necessarily proper) G-CW-complexes X.

Lemma

Then the homology theories

$$H_*^?(-;\mathbf{K}_R)$$
$$H_*^?(-;\mathbf{L}_R^{\langle -\infty\rangle})$$
$$H_*^?(-;\mathbf{K}_{l^1}^{\mathrm{top}})$$

are strongly continuous.

- For instance one has to show that the canonical map induced by the various structure maps $G_i \longrightarrow G$ induce an isomorphism

$$\operatorname*{colim}_{i\in I} K_n\left(l^1\left(G_i\right)\right) \xrightarrow{\cong} K_n\left(l^1\left(\operatorname*{colim}_{i\in I} G_i\right)\right).$$

- This statement does not make sense for the reduced group C^*-algebra since it is not functorial under arbitrary group homomorphisms.
- For instance, $C_r^*(\mathbb{Z} * \mathbb{Z})$ is a simple C^*-algebra and hence no epimorphism $C_r^*(\mathbb{Z} * \mathbb{Z}) \to C_r^*(\{1\})$ exists.

(Co-)homology for extensions of $\mathbb{Z}^n$ by $\mathbb{Z}/p$ for an odd prime p

- Fix an extension
$$1 \to \mathbb{Z}^n \xrightarrow{i} \Gamma \xrightarrow{q} \mathbb{Z}/p \to 1$$
for an odd prime p such that the conjugation action of $\mathbb{Z}/p$ on $\mathbb{Z}^n$ is free outside the origin.

- Define natural numbers for $m, j, k \in \mathbb{Z}$, $m, k, j \geq 0$
$$\begin{aligned} r_m &:= \operatorname{rk}_{\mathbb{Z}} \left(\left(\Lambda^m(\mathbb{Z}[\zeta]^k) \right)^{\mathbb{Z}/p} \right); \\ a_j &:= \left| \{ (\ell_1, \ldots, \ell_k) \in \mathbb{Z}^k \mid \ell_1 + \cdots + \ell_k = j, 0 \leq \ell_i \leq p-1 \} \right|; \\ s_m &:= \sum_{j=0}^{m-1} a_j. \end{aligned}$$

- We have

$$\begin{aligned} r_0 &= 1; \\ r_m &= 0 \quad \text{for } m \geq p; \\ r_m &= \frac{1}{p} \cdot \left(\binom{p-1}{m} + (-1)^m \cdot (p-1) \right) \quad \text{for } 2 \leq m < p. \end{aligned}$$

- If $k = 1$, we get

$$\begin{aligned} a_j &= 1 && \text{for} \quad 0 \leq j \leq p-1; \\ a_j &= 0 && \text{for} \quad p \leq j; \\ s_m &= m && \text{for} \quad 0 \leq j \leq p; \\ s_m &= p && \text{for} \quad p \leq j. \end{aligned}$$

Theorem (Cohomology of $B\Gamma$ and $\underline{B}\Gamma$)

- *There is an integer k which is uniquely determined by $n = (p-1) \cdot k$;*
- *For $m \geq 0$ the map*

$$\iota^m \colon H^m(B\Gamma) \to H^m(\mathbb{Z}^n)^{\mathbb{Z}/p}$$

  is surjective and its kernel is isomorphic to $(\mathbb{Z}/p)^{s_m}$ if m is even and trivial, if m is odd;
- *For $m \geq 0$ we get*

$$H^m(B\mathbb{Z}^n)^{\mathbb{Z}/p} \cong \mathbb{Z}^{r_m};$$

Theorem (continued)

- *For $m \geq 0$ we get*

$$H^m(B\Gamma) \cong \begin{cases} \mathbb{Z}^{r_m} \oplus (\mathbb{Z}/p)^{s_m} & m \text{ even;} \\ \mathbb{Z}^{r_m} & m \text{ odd;} \end{cases}$$

- *For $m \geq 0$ we get for $\underline{B}\Gamma := \Gamma\backslash\underline{E}\Gamma$*

$$H^m(\underline{B}\Gamma) \cong \begin{cases} \mathbb{Z}^{r_m} & m \text{ even;} \\ \mathbb{Z}^{r_m} \oplus (\mathbb{Z}/p)^{p^k - s_m} & m \text{ odd}, m \geq 3; \\ \mathbb{Z}^{r_1} = 0 & m = 1. \end{cases}$$

- We want to give some information about its proof. This needs some preparation.
- Let
$$N = t^0 + t + \cdots + t^{p-1} \in \mathbb{Z}[\mathbb{Z}/p]$$
be the *norm element*.
Denote by $I(\mathbb{Z}/p)$ the *augmentation ideal*, i.e., the kernel of the augmentation homomorphism $\mathbb{Z}[\mathbb{Z}/p] \to \mathbb{Z}$.
Let $\zeta = e^{2\pi i/p} \in \mathbb{C}$ be a primitive p-th root of unity.
- We have isomorphism of $\mathbb{Z}[\mathbb{Z}/p]$-modules
$$\mathbb{Z}[\mathbb{Z}/p]/N \cong I(\mathbb{Z}/p) \cong \mathbb{Z}[\zeta].$$
- $\mathbb{Z}[\zeta]$ is a Dedekind domain

Example ($\mathbb{Z}^n_\rho = I(\mathbb{Z}/p)$)

If we take for ρ the one given by the $(p-1) \times (p-1)$ matrix

$$\rho = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -1 \\ 1 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 1 & \cdots & 0 & 0 & -1 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & \cdots & 1 & 0 & -1 \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}$$

then $\mathbb{Z}^{p-1}_\rho$ is $\mathbb{Z}[\mathbb{Z}/p]$-isomorphic to $\mathbb{Z}[\mathbb{Z}/p]/N \cong I(\mathbb{Z}/p) \cong \mathbb{Z}[\zeta]$.

Lemma

1. *We have*
$$\mathbb{Z}_\rho^n \cong I_1 \oplus \cdots \oplus I_k,$$
where the I_j are non-zero ideals of $\mathbb{Z}[\zeta]$ and $n = k(p-1)$;

2. *The map π splits as a group homomorphism. Conversely the semidirect product of $\mathbb{Z}/p$ acting on a finite direct sum of nonzero ideals of $\mathbb{Z}[\zeta]$ by multiplication by ζ satisfies the conditions placed on Γ above;*

3. *Each non-trivial finite subgroup P of Γ is isomorphic to $\mathbb{Z}/p$ and its Weyl group $W_\Gamma P := N_\Gamma P/P$ is trivial;*

4. There are bijections

$$\begin{array}{rcl} H^1(\mathbb{Z}/p;\mathbb{Z}^n_\rho) & \xrightarrow{\cong} & \operatorname{cok}(\rho - \operatorname{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n); \\ H^1(\mathbb{Z}/p;\mathbb{Z}^n_\rho) & \cong & (\mathbb{Z}/p)^k; \\ \operatorname{cok}\bigl(\rho - \operatorname{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n\bigr) & \xrightarrow{\cong} & \mathcal{P} := \{(P) \mid P \subset \Gamma, 1 < |P| < \infty\}. \end{array}$$

   If we fix a generator $s \in \mathbb{Z}/p$, the last bijection sends the element $\overline{u} \in \mathbb{Z}^n_\rho/(1-s)\mathbb{Z}^n_\rho$ the subgroup of order p generated by us;

5. We have $|\mathcal{P}| = p^k$;

6. $[\Gamma,\Gamma] = \operatorname{im}(\rho - \operatorname{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n)$;

7. $\Gamma/[\Gamma,\Gamma] \cong \operatorname{cok}(\rho - \operatorname{id}\colon \mathbb{Z}^n \to \mathbb{Z}^n) \oplus \mathbb{Z}/p = (\mathbb{Z}/p)^{k+1}$.

Proof.

We only give the proof of assertion (1)

- Let $u \in \mathbb{Z}^n_\rho$. Then $N \cdot u$ is fixed by conjugation with $t \in \mathbb{Z}/p$ and hence is zero by assumption.
- Thus $\mathbb{Z}^n_\rho$ is a finitely generated module over the Dedekind domain $\mathbb{Z}[\mathbb{Z}/p]/\Sigma = \mathbb{Z}[\zeta]$.
- Any finitely generated torsion-free module over a Dedekind domain is isomorphic to a direct sum of nonzero ideals.
- Since $I_j \otimes \mathbb{Q} \cong \mathbb{Q}[\zeta]$, we see $\mathrm{rk}_{\mathbb{Z}}(I_j) = p - 1$.

$\square$

- Next will analyze the Hochschild-Serre Spectral sequence associated to the extension
$$1 \to \mathbb{Z}^n \xrightarrow{i} \Gamma \xrightarrow{q} \mathbb{Z}/p \to 1;$$
- Recall that its E_2-term is
$$E_2^{i,j} = H^i(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho))$$
and it converges to $H^{i+j}(\Gamma)$.
- Recall that $\widehat{H}^*(G;M)$ denotes the Tate cohomology of a finite group G with coefficients in the $\mathbb{Z}G$-module M.
- We have
$$\begin{array}{lcll} \widehat{H}^i(G;M) & = & H^i(G;M) & \text{for } i \geq 1; \\ \widehat{H}^i(G;M) & = & H_{-i-1}(G;M) & \text{for } i \leq -2. \end{array}$$
- There is an exact sequence
$$0 \to \widehat{H}^{-1}(G;M) \to M \otimes_{\mathbb{Z}G} \mathbb{Z} \xrightarrow{N} M^G \to \widehat{H}^0(G;M) \to 0.$$

Lemma

1. $$\widehat{H}^i(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)) \cong \bigoplus_{\substack{\ell_1+\cdots+\ell_k=j \\ 0\le\ell_q\le p-1}} \widehat{H}^{i+j}(\mathbb{Z}/p) = \begin{cases} (\mathbb{Z}/p)^{a_j} & i+j \text{ even}; \\ 0 & i+j \text{ odd}; \end{cases}$$

2. *The Hochschild-Serre spectral sequence associated to the extension*

$$1 \to \mathbb{Z}^n \xrightarrow{i} \Gamma \xrightarrow{q} \mathbb{Z}/p \to 1$$

collapses.

Proof.

- We begin with assertion (1).
- We obtain a $\mathbb{Z}[\mathbb{Z}/p]$-isomorphism

$$H^j(\mathbb{Z}^n_\rho) \quad \cong \quad \Lambda^j \mathbb{Z}^n_{\rho^*}. \tag{0.5}$$

- Given an ideal $I \subseteq \mathbb{Z}[\zeta]$, the inclusion of I into $\mathbb{Z}[\zeta]$ induces an isomorphism of $\mathbb{Z}_{(p)}[\zeta]$-modules

$$I \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \xrightarrow{\cong} \mathbb{Z}[\zeta] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[\zeta].$$

- One deduces from this the existence of an isomorphism

$$\widehat{H}^i(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)) \cong \widehat{H}^i\left(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)_{(p)}\right) \cong \widehat{H}^i(\mathbb{Z}/p; \Lambda^j \mathbb{Z}[\zeta]^k).$$

$\square$

Proof continued.

- Since

$$\Lambda^*\Big(\bigoplus_k \mathbb{Z}[\zeta]\Big) = \bigotimes_k \Lambda^*(\mathbb{Z}[\zeta])$$

  and $\Lambda^l(\mathbb{Z}[\zeta]) = 0$ for $l \geq p$, we get

$$\Lambda^j(\mathbb{Z}[\zeta]^k)) = \bigoplus_{\substack{\ell_1+\cdots+\ell_k=j\\ 0\leq\ell_q\leq p-1}} \Lambda^{\ell_1}\mathbb{Z}[\zeta] \otimes \cdots \otimes \Lambda^{\ell_k}\mathbb{Z}[\zeta].$$

- Therefore we obtain an isomorphism

$$\widehat{H}^i(\mathbb{Z}/p; H^j(\mathbb{Z}^n_\rho)) \cong \bigoplus_{\substack{\ell_1+\cdots+\ell_k=j\\ 0\leq\ell_q\leq p-1}} \widehat{H}^i\left(\mathbb{Z}/p; \Lambda^{\ell_1}\mathbb{Z}[\zeta] \otimes \cdots \otimes \Lambda^{\ell_k}\mathbb{Z}[\zeta]\right).$$

$\square$

Proof continued.

- Next we proof assertion (2).
- Next we want to show that the differentials $d_r^{i,j}$ are zero for all $r \geq 2$ and i, j.
- By the checkerboard pattern of the E_2-term it suffices to show for $r \geq 2$ and that the differentials $d_r^{0,j}$ are trivial for $r \geq 2$ and all odd $j \geq 1$.
- This is equivalent to show that for every odd $j \geq 1$ the edge homomorphism

$$\iota^j \colon H^j(\Gamma) \to H^j(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} = E_2^{0,j}$$

is surjective.

□

Proof continued.

- Let
$$\mathrm{trf}^j \colon H^j(\mathbb{Z}^n) \to H^j(\Gamma)$$
be the transfer map associated to ι.
- The composite $\iota^j \circ \mathrm{trf}^j$ agrees with the norm map $N \colon H^j(\mathbb{Z}^n_\rho) \to H^j(\mathbb{Z}^n_\rho)$ given by multiplication with the norm element N. Its image is contained in $H^j(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$.
- Hence it suffices to show that the cokernel of $N \colon H^j(\mathbb{Z}^n_\rho) \to H^j(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$ is trivial.
- Since this cokernel is isomorphic to $\widehat{H^0}(\mathbb{Z}/p, H^j(\mathbb{Z}^n_\rho))$, the claim follows from assertion (1).

$\square$

Proof continued.

- It remains to show that all extensions are trivial.
  Since the composite

$$H^{i+j}(\Gamma) \xrightarrow{\mathrm{trf}^{i+j}} H^{i+j}(\mathbb{Z}^n) \xrightarrow{\iota^{i+j}} H^{i+j}(\Gamma)$$

  is multiplication with p, the torsion in $H^{i+j}(\Gamma)$ has exponent p.
- Since $p \cdot E_\infty^{i,j} = p \cdot E_2^{i,j} = 0$ for $i > 0$, all extensions are trivial and

$$H^m\Gamma \cong \bigoplus_{i+j=m} E_\infty^{i,j} = \bigoplus_{i+j=m} E_2^{i,j}.$$

$\square$

Proof continued.

- Hence it suffices to show for $l_1, \ldots, l_k$ in $\{0, 1, \ldots, (p-1)\}$
$$\widehat{H}^i \left(\mathbb{Z}/p; \Lambda^{\ell_1}\mathbb{Z}[\zeta] \otimes \cdots \otimes \Lambda^{\ell_k}\mathbb{Z}[\zeta]\right) \cong \widehat{H}^{i+\sum_{a=1}^k \ell_a}(\mathbb{Z}/p).$$
- This is done by induction over over $j = \sum_{a=1}^k l_a$.
- The main ingredient is an exact sequence of $\mathbb{Z}[\mathbb{Z}/p]$-modules with a free $\mathbb{Z}[\mathbb{Z}/p]$-module in the middle
$$\begin{aligned} 1 \to \Lambda^{l_1-1}\mathbb{Z}[\zeta] \otimes \Lambda^{\ell_2}\mathbb{Z}[\zeta] \otimes \cdots \otimes \Lambda^{\ell_k}\mathbb{Z}[\zeta] \\ \to \Lambda^{l_1}\mathbb{Z}[\mathbb{Z}/p] \otimes \Lambda^{\ell_2}\mathbb{Z}[\zeta] \otimes \cdots \otimes \Lambda^{\ell_k}\mathbb{Z}[\zeta] \\ \to \Lambda^{l_1}\mathbb{Z}[\zeta] \otimes \Lambda^{\ell_2}\mathbb{Z}[\zeta] \otimes \cdots \otimes \Lambda^{\ell_k}\mathbb{Z}[\zeta] \to 1. \end{aligned}$$

□

- The first four assertion of Theorem about the cohomology of $B\Gamma$ and $\underline{B}\Gamma$ are direct consequences of the last two lemmas.
- Hence it remains to prove the last assertion.
- Before we do this, we look at an example about what we have achieved so far.

Example

For $k = 1$ and $p = 5$ for $E_2^{i,j} = E_\infty^{i,j}$

$$
\begin{array}{ccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & \cdots \\
\mathbb{Z} & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \cdots \\
0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & \cdots \\
\mathbb{Z}^2 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \cdots \\
0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & \cdots \\
\mathbb{Z} & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \mathbb{Z}_5 & 0 & \cdots
\end{array}
$$

Example (Example continued)

Moreover

$$H^n(\Gamma) = \begin{cases} \mathbb{Z} & n = 0; \\ 0 & n \geq 1, n \text{ odd}; \\ \mathbb{Z}^2 \oplus (\mathbb{Z}/5)^2 & n = 2; \\ \mathbb{Z} \oplus (\mathbb{Z}/5)^4 & n = 4; \\ (\mathbb{Z}/5)^5 & n \geq 6, n \text{ even}. \end{cases}$$

- We have earlier constructed the cellular pushout

$$\begin{array}{ccc} \coprod_{(P)\in\mathcal{P}} BP & \xrightarrow{j_0} & B\Gamma \\ \Big\downarrow{\scriptstyle \coprod_{(P)\in\mathcal{P}} \overline{\mathrm{pr}}_P} & & \Big\downarrow{\scriptstyle \overline{f}} \\ \coprod_{(P)\in\mathcal{P}} \mathrm{pt} & \xrightarrow{j_1} & \underline{B}\Gamma \end{array} \tag{0.6}$$

- It yields for $m \geq 0$ the long exact sequence of reduced integral cohomology groups

$$0 \to \widetilde{H}^{2m}(\underline{B}\Gamma) \xrightarrow{\overline{f}^*} \widetilde{H}^{2m}(\Gamma) \xrightarrow{\varphi^{2m}} \bigoplus_{(P)\in\mathcal{P}} \widetilde{H}^{2m}(P) \xrightarrow{\delta^{2m}} \widetilde{H}^{2m+1}(\underline{B}\Gamma) \xrightarrow{\overline{f}^*} \widetilde{H}^{2m+1}(\Gamma) \to 0 \tag{0.7}$$

where ϕ^m is the map induced by the various inclusions $P \subset G$ for $(P) \in \mathcal{P}$.

- In order to compute this sequence we need the following result whose proof is based on a spectral sequence argument.

Lemma

Let $K^{2m} \subseteq H^{2m}(\Gamma)$ *be the kernel of*

$$\varphi^{2m} \colon \widetilde{H}^{2m}(\Gamma) \to \bigoplus_{(P) \in \mathcal{P}} \widetilde{H}^{2m}(P).$$

Let $\beta \in H^2(\mathbb{Z}/p) \cong \mathbb{Z}/p$ *be a generator.*
Let L^{2m} *be the kernel of*

$$- \cup \pi^*(\beta)^n \colon H^{2m}(\Gamma) \to H^{2m+2n}(\Gamma).$$

Then for $m \geq 1$*:*

1. We have $K^{2m} = L^{2m}$;
2. The intersection L^{2m} with the kernel of the epimorphism
$$H^{2m}(\Gamma) \to H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}$$
is trivial;
3. The image of L^{2m} under the epimorphism
$$H^{2m}(\Gamma) \to H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p} = H^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho))$$
is the kernel of the projection
$$H^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho)) \to \widehat{H}^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho));$$
4. We have $K^{2m} \cong \mathbb{Z}^{r_m}$;
5. The quotient $H^{2m}(\Gamma)/K^{2m}$ is isomorphic to
$$\ker\left(H^{2m}(\Gamma) \to H^{2m}(\mathbb{Z}^n_\rho)^{\mathbb{Z}/p}\right) \oplus \widehat{H}^0(\mathbb{Z}/p; H^{2m}(\mathbb{Z}^n_\rho)) \cong (\mathbb{Z}/p)^{s_{2m+1}}.$$

Proof.

- We give only the proof of the first assertion.
- The following diagram computes

$$
\begin{tikzcd}
H^{2m}(\Gamma) \arrow[r, "\varphi^{2m}"] \arrow[d, "-\cup\pi^*(\beta)^n"] & \bigoplus_{(P)\in\mathcal{P}} H^{2m}(P) \arrow[d, "-\cup\phi^{2m+2n}(\pi^*(\beta)^n)"] \\
H^{2m+2n}(\Gamma) \arrow[r, "\varphi^{2m+2n}"] & \bigoplus_{(P)\in\mathcal{P}} H^{2m+2n}(P)
\end{tikzcd}
$$

- Since $\dim(\underline{B}\Gamma) \leq n$, we have $H^{i+2n}(\underline{B}\Gamma) = 0$ for $i \geq 1$.
- Hence the lower horizontal arrow is bijective by a long excat sequence from above. The right vertical arrow is bijective.

$\square$

- We conclude from the already proved fourth assertion of the Theorem about the cohomology of $B\Gamma$ and $\underline{B}\Gamma$. and the Lemma above:

Corollary

For $m \geq 1$ the long exact sequence

$$0 \to \widetilde{H}^{2m}(\underline{B}\Gamma) \xrightarrow{\overline{f}^*} \widetilde{H}^{2m}(\Gamma) \xrightarrow{\varphi^{2m}} \bigoplus_{(P)\in\mathcal{P}} \widetilde{H}^{2m}(P)$$
$$\xrightarrow{\delta^{2m}} \widetilde{H}^{2m+1}(\underline{B}\Gamma) \xrightarrow{\overline{f}^*} \widetilde{H}^{2m+1}(\Gamma) \to 0 \quad (0.8)$$

can be identified with

$$0 \to \mathbb{Z}^{r_{2m}} \to \mathbb{Z}^{r_{2m}} \oplus (\mathbb{Z}/p)^{s_{2m}} \to (\mathbb{Z}/p)^{p^k} \to \mathbb{Z}^{r_{2m+1}} \oplus (\mathbb{Z}/p)^{p^k - s_{2m+1}} \to \mathbb{Z}^{r_{2m+1}}$$

- The corollary above implies fifth assertion of the Theorem about the cohomology of $B\Gamma$ and $\underline{B}\Gamma$.

Theorem (Homology of $B\Gamma$ and $\underline{B}\Gamma$)

- *For $m \geq 0$ we get*

$$H_m(B\Gamma) \cong \begin{cases} \mathbb{Z}^{r_m} \oplus (\mathbb{Z}/p)^{s_{m+1}} & m \text{ odd;} \\ \mathbb{Z}^{r_m} & m \text{ even;} \end{cases}$$

- *For $m \geq 0$ we get*

$$H_m(\underline{B}\Gamma) \cong \begin{cases} \mathbb{Z}^{r_m} & m \text{ odd;} \\ \mathbb{Z}^{r_m} \oplus (\mathbb{Z}/p)^{p^k - s_{m+1}} & m \text{ even}, m \geq 2; \\ \mathbb{Z} & m = 0; \end{cases}$$

Proof.

Universal coefficient theorem □

The formulation of the Farrell-Jones Conjecture with additive categories

- It is convenient to generalize the Farrell-Jones Conjecture for group rings RG to additive categories with G-actions as explained next.
- We will give only some details for K-theory. The L-theory case is analogous but a little bit more complicated since one has to incorporate involutions.

Definition (Additive category)

An additive category $\mathcal{A}$ is a small category $\mathcal{A}$ such that for two objects A and B the morphism set $\operatorname{mor}_{\mathcal{A}}(A, B)$ has the structure of an abelian group and the direct sum $A \oplus B$ of two objects A and B exists and the obvious compatibility conditions hold.

A covariant functor of additive categories $F\colon \mathcal{A}_0 \to \mathcal{A}_1$ is a covariant functor such that for two objects A and B in $\mathcal{A}_0$ the map $\operatorname{mor}_{\mathcal{A}_0}(A, B) \to \operatorname{mor}_{\mathcal{A}_1}(F(A), F(B))$ sending f to $F(f)$ respects the abelian group structures and $F(A \oplus B)$ is a model for $F(A) \oplus F(B)$.

Example

Examples of additive categories are the category of R-modules, the category of R-chain complexes and the homotopy category of R-chain complexes.

- Notice that algebraic K-theory (and algebraic L-theory) can be defined for additive categories (with involution).
- If R is a ring (with involution) and $\mathcal{A}$ the additive category (with involution) of finitely generated projective R-modules, then the K-theory (and the L-theory) of $\mathcal{A}$ agrees with the K-theory (and the L-theory) of $\mathcal{A}$.

- Let $\mathcal{G}$ be a connected groupoid. Let $\mathcal{A}dd\text{-}\mathcal{C}at$ be the category of small additive categories.
- Given a contravariant functor $F\colon \mathcal{G} \to \mathcal{A}dd\text{-}\mathcal{C}at$, we define its homotopy colimit $\int_{\mathcal{G}} F$ as follows.
- An object is a pair (x, A) consisting of an object x in $\mathcal{G}$ and an object A in $F(x)$.
- A morphism in $\int_{\mathcal{G}} F$ from (x, A) to (y, B) is a formal sum

$$\sum_{f \in \mathrm{mor}_{\mathcal{G}}(x,y)} f \cdot \phi_f$$

  where $\phi_f\colon A \to F(f)(B)$ is a morphism in $F(x)$ and only finitely many coefficients $\phi_f \in \mathbb{Z}$ are different from zero.
- The composition law is determined by requiring

$$(g \cdot \psi) \circ (f \cdot \phi) = (g \circ f) \cdot (F(f)(\psi) \circ \phi).$$

- A model for the sum of two objects (x, A) and (x, B) is $(x, A \oplus B)$. Since $\mathcal{G}$ is connected, this yields the existence of a direct sum in general.

- Let $W\colon \mathcal{G}_1 \to \mathcal{G}_2$ be functor of connected groupoids. Let $F\colon \mathcal{G}_2 \to \mathrm{Add\text{-}Cat}$ be a contravariant functor. Then there is a functor of additive categories

$$W_*\colon \int_{\mathcal{G}_1} F \circ W \to \int_{\mathcal{G}_2} F.$$

- If W is an equivalence, then W_* is an equivalence.
- Let $\mathcal{G}$ be a connected groupoid. Let $S\colon F_1 \to F_2$ be a transformation of contravariant functors $\mathcal{G} \to \mathrm{Add\text{-}Cat}$ such that for every object x in $\mathcal{G}$ the functor $S(x)\colon F_0(x) \to F_1(x)$ is an equivalence of additive categories. Then there is an equivalence of additive categories

$$\int_{\mathcal{G}} S\colon \int_{\mathcal{G}} F_1 \to \int_{\mathcal{G}} F_2.$$

Definition (Transport groupoid)

Let G be a group and let ξ be a G-set. Define the transport groupoid $\mathcal{G}^G(\xi)$ to be the following groupoid. The set of objects is ξ itself. For $x_1, x_2 \in \xi$ the set of morphisms from x_1 to x_2 consists of those elements g in G for which $gx_1 = x_2$ holds. Composition of morphisms comes from the group multiplication in G.

- A G-map $\alpha\colon \xi \to \eta$ of G-sets induces a covariant functor $\mathcal{G}^G(\alpha)\colon \mathcal{G}^G(\xi) \to \mathcal{G}^G(\eta)$ by sending an object $x \in \xi$ to the object $\alpha(x) \in \eta$. A morphism $g\colon x_1 \to x_2$ is sent to the morphism $g\colon \alpha(x_1) \to \alpha(x_2)$.

- Fix a functor

$$\mathbf{E}\colon \mathrm{Add\text{-}Cat} \to \mathrm{Spectra}$$

  which sends weak equivalences of additive categories to weak homotopy equivalences of spectra.
- Fix a group G and an additive category $\mathcal{A}$ with right G-action.
- Let $\mathrm{pr}_G\colon \mathcal{G}^G(G/H) \to \mathcal{G}^G(G/G)$ be the obvious projection.
- We obtain a covariant functor

$$\mathrm{Or}(G) \to \mathrm{Spectra}, \quad G/H \mapsto \int_{\mathcal{G}^G(G/H)} \mathcal{A} \circ \mathrm{pr}_G .$$

- Composing it with $\mathbf{E}$ yields a covariant functor

$$\mathbf{E}_{\mathcal{A}}\colon \mathrm{Or}(G) \to \mathrm{Spectra}$$

- Associated to it is a G-homology theory $H_*^G(-;\mathbf{E}_{\mathcal{A}})$.

Definition (The K-theoretic Farrell-Jones Conjecture with additive categories as coefficients)

The K-theoretic Farrell-Jones Conjecture for G with additive categories as coefficients says that the projection $\underline{\underline{E}}G \to G/G$ induces for all $n \in \mathbb{Z}$ and all additive categories $\mathcal{A}$ with right G-action an isomorphism

$$H_n^G(\underline{\underline{E}}G;\mathbf{K}_{\mathcal{A}}) \xrightarrow{\cong} H_n^G(G/G;\mathbf{K}_{\mathcal{A}}) = K_n\left(\int_G \mathcal{A}\right).$$

- IF $\mathcal{A}$ is the additive category of finitely generated projective R-modules equipped with the trivial G-action, then the assembly map above can be identified with the classical one

$$H_n^G(E_{\mathcal{F}}(G);\mathbf{K}_R) \xrightarrow{\cong} H_n^G(G/G;\mathbf{K}_R) = K_n(RG)$$

which we have considered previously.
- The advantage of the approach via additive categories is that it includes the case of twisted group rings and more generally of crossed product rings and that it encompasses the fibered versions as well. In particular the inheritance to subgroups is built in.

- Let R be a ring and let G be a group. Let $e \in G$ be the unit in G and denote by 1 the multiplicative unit in R.
- Suppose that we are given maps of sets

$$\begin{aligned} c\colon G &\to \operatorname{aut}(R), \quad g \mapsto c_g \\ \tau\colon G \times G &\to R^{\times}. \end{aligned}$$

- We require

$$\begin{aligned} c_{\tau(g,g')} \circ c_{gg'} &= c_g \circ c_{g'}; && (0.9) \\ \tau(g,g') \cdot \tau(gg',g'') &= c_g(\tau(g',g'')) \cdot \tau(g,g'g''); && (0.10) \\ c_e &= \mathrm{id}_R; && (0.11) \\ \tau(e,g) &= 1; && (0.12) \\ \tau(g,e) &= 1, && (0.13) \end{aligned}$$

for $g, g', g'' \in G$, where $c_{\tau(g,g')}\colon R \to R$ is conjugation with $\tau(g,g')$, i.e., it sends r to $\tau(g,g')r\tau(g,g')^{-1}$.

- Let $R * G = R *_{c,\tau} G$ be the free R-module with the set G as basis.
- It becomes a ring with the following multiplication

$$\left(\sum_{g\in G} \lambda_g g\right) \cdot \left(\sum_{h\in G} \mu_h h\right) = \sum_{g\in G} \left(\sum_{\substack{g',g''\in G,\\ g'g''=g}} \lambda_{g'} c_{g'}(\mu_{g''})\tau(g',g'')\right) g.$$

- This multiplication is uniquely determined by the properties $g \cdot r = c_g(r) \cdot g$ and $g \cdot g' = \tau(g,g') \cdot (gg')$.
- The conditions (0.9) and (0.10) relating c and τ are equivalent to the condition that this multiplication is associative. The other conditions (0.11), (0.12) and (0.13) are equivalent to the condition that the element $1 \cdot e$ is a multiplicative unit in $R * G$.

Definition (Crossed product ring)

We call $R * G = R *_{c,\tau} G$ the crossed product of R and G with respect to c and τ.

Example (Group extensions)

- Let $1 \to H \xrightarrow{i} G \xrightarrow{p} Q \to 1$ be an extension of groups.
- Let $s\colon Q \to G$ be a map satisfying $p \circ s = \mathrm{id}$ and $s(e) = e$. We do not require s to be a group homomorphism.
- Define $c\colon Q \to \mathrm{aut}(RH)$ by $c_q(\sum_{h \in H} \lambda_h h) = \sum_{h \in H} \lambda_h s(q) h s(q)^{-1}$.
- Define $\tau\colon Q \times Q \to (RH)^{\times}$ by $\tau(q,q') = s(q)s(q')s(qq')^{-1}$.
- Notice that s is a group homomorphism if and only if τ is constant with value $1 \in R$.
- Then we obtain a ring isomorphism

$$RH * Q \xrightarrow{\cong} RG, \quad \sum_{q \in Q} \lambda_q q \mapsto \sum_{q \in Q} i(\lambda_q) s(q),$$

where $i\colon RH \to RG$ is the ring homomorphism induced by $i\colon H \to G$.

- Consider R, $c\colon G \to \operatorname{aut}(R)$ and $\tau\colon G \times G \to R^\times$ as above.
- Let R-FGF be the additive category of finitely generated free R-modules.
- One can associated an additive category with right G-action $R\text{-}\mathrm{FGF}_{c,\tau}$ to these data. Objects are pairs (M, g) for $g \in G$ and M an object in R-FGP.
- There is a natural equivalence between

$$\int_{\mathcal{G}^G(G/H)} R\text{-}\mathrm{FGP}_{c,\tau} \circ \mathrm{pr}_G$$

and

$$R *_{c|_H, \tau|_H} H\text{-}\mathrm{FGP}.$$

- Hence the Farrell-Jones assembly map above for the additive category $R\text{-FGF}_{c,\tau}$ with right G-action as coefficient becomes

$$H_n^G(\underline{\underline{E}}G;\mathbf{K}_{R\text{-FGF}_{c,\tau}}) \xrightarrow{\cong} H_n^G(G/G;\mathbf{K}_{R\text{-FGF}_{c,\tau}}) = K_n\left(\int_G R\text{-FGF}_{c,\tau}\right) = K_n(R *_{c,\tau} G).$$

  where

$$H_n^G(G/H;\mathbf{K}_{R\text{-FGF}_{c,\tau}}) = K_n(R *_{c|_H,\tau|_H} H).$$

- In the L-theory case we can also treat any choice of orientation homomorphisms into account using additive categories with involutions.

Theorem (Induction for additive categories)

Let $\phi\colon K \to G$ be a group homomorphism. Let $\mathcal{A}$ be an additive category with right K-action.
Then one can define in a natural way an additive category with right G-action $\operatorname{ind}_\phi \mathcal{A}$ together with a natural equivalence of G-homology theories

$$\tau_*\colon H^K\big(\phi^*(-);\mathbf{E}_{\mathcal{A}}\big) \xrightarrow{\cong} H^G\big(-;\mathbf{E}_{(\operatorname{ind}_\phi \mathcal{A})}\big).$$

Definition (Fibered K-theoretic Farrell-Jones Conjecture)

A group G satisfies the fibered K-theoretic Farrell-Jones Conjecture with additive categories as coefficients if for any group homomorphism $\phi\colon K \to G$ and additive category with right G-action the assembly map

$$\operatorname{asmb}_n^{\phi,\mathcal{A}}\colon H_*^K\left(E_{\phi^*\mathcal{VC}\mathrm{yc}}(G);\mathbf{K}_{\phi^*\mathcal{A}}\right) \to K_n\left(\textstyle\int_K \phi^*\mathcal{A}\right).$$

is bijective for all $n \in \mathbb{Z}$, where the family $\phi^*\mathcal{VC}\mathrm{yc}$ of subgroups of K consists of subgroups $L \subseteq K$ with $\phi(L)$ virtually cyclic and $\phi^*\mathcal{A}$ is the additive K-category with involution obtained from $\mathcal{A}$ by restriction with ϕ.

- Obviously the fibered version for the group G implies the unfibered version for the group G , take $\phi = \mathrm{id}$.
- Thanks to our setting with additive categories the converse is also true:

Theorem (Additive categories versus fibered)

Let G be a group. Then G satisfies the fibered K-theoretic Farrell-Jones Conjecture for additive categories as coefficients if and only if G satisfies the K-theoretic Farrell-Jones Conjecture for additive categories as coefficients.

- The actual proofs do not become harder when one considers additive categories instead of rings as coefficients.
- Recall the slogan that the coefficients are dummy variables.

The formulation of the main result

- Next we formulate the main result whose proof we want to describe.
- For this purpose we need the following class of groups.

Definition (The class of groups $\mathcal{B}$)

Let $\mathcal{B}$ be the smallest class of groups satisfying the following conditions:

1. Hyperbolic groups belong to $\mathcal{B}$;
2. If G acts properly cocompactly and isometrically on a finite-dimensional CAT(0)-space, then $G \in \mathcal{B}$;
3. The class $\mathcal{B}$ is closed under taking subgroups;
4. Let $\pi\colon G \to H$ be a group homomorphism. If $H \in \mathcal{B}$ and $\pi^{-1}(V) \in \mathcal{B}$ for all virtually cyclic subgroups V of H, then $G \in \mathcal{B}$;
5. $\mathcal{B}$ is closed under finite direct products;
6. $\mathcal{B}$ is closed under finite free products;
7. The class $\mathcal{B}$ is closed under directed colimits, i.e., if $\{G_i \mid i \in I\}$ is a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{B}$ for $i \in I$, then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{B}$.

Theorem (Bartels-Lück(2009))

Let $G \in \mathcal{B}$. Then:

1. *The K-theoretic assembly map*

$$H_n^G(E_{\mathcal{F}}(G);\mathbf{K}_{\mathcal{A}}) \xrightarrow{\cong} H_n^G(G/G;\mathbf{K}_{\mathcal{A}}) = K_n\left(\int_G \mathcal{A}\right).$$

is bijective in degree $n \leq 0$ and surjective in degree $n = 1$ for any additive G-category $\mathcal{A}$;

2. *The L-theoretic Farrell-Jones assembly map*

$$H_n^G(E_{\mathcal{F}}(G);\mathbf{L}_{\mathcal{A}}^{\langle -\infty\rangle}) \xrightarrow{\cong} H_n^G(G/G;\mathbf{K}_{\mathcal{A}}) = L_n^{\langle -\infty\rangle}\left(\int_G \mathcal{A}\right)$$

with coefficients in any additive G-category $\mathcal{A}$ with involution is an isomorphism for all $n \in \mathbb{Z}$.

- If we drop in the definition of the class $\mathcal{B}$ the property about CAT(0)-groups, then the K-theoretic assembly map is bijective for all $n \in \mathbb{Z}$.

Outline of the proof

- The proof of the inheritance properties (subgroups, directed colimits, products) are based on general results on equivariant homology theory and have been explained already before
- The hard part is the proof of the Farrell-Jones Conjecture for hyperbolic groups or CAT(0)-groups.
- We will present an axiomatic approach which will ensure that under certain geometric conditions the Farrell-Jones Conjecture holds for K and L-theory.

- Very roughly, these conditions assert the existence of a compact space X with a homotopy G-action and the existence of a "long thin" G-equivariant cover of $G \times X$.
- Allowing homotopy G-actions instead of honest actions was one of the key ideas to be able to handle CAT(0)-groups.
- Here is an outline of the general strategy.
- Controlled algebra is used to set up an obstruction category whose K-respectively L-theory gives the homotopy fiber of the assembly map in question. So one needs to show the vanishing of the K-or L-theory of this obstruction category.

- We will mostly study K_1 and L_0 of these categories.
- In K-theory we represent elements by automorphisms or more generally by self-chain homotopy equivalences. In L-theory we represent elements by quadratic forms or more generally by 0-dimensional ultra-quadratic Poincaré complexes.
- For this outline it will be convenient to call these representatives cycles. In all cases these cycles come with a notion of size.
- More precisely, the obstruction category depends on a free G-space Z (in the simplest case this space is G, but it is important to keep this space variable) and associated to any cycle is a subset (its support) of $Z \times Z$.

- If Z is a metric space, then a cycle is said to be α-controlled over Z for some number $\alpha > 0$ if $d_Z(x,y) \le \alpha$ for all (x,y) in the support of the cycle.
- The Stability Theorem for the obstruction category asserts (for a class of metric spaces), that there is $\epsilon > 0$ such that the K-theory respectively L-theory class of every ϵ-controlled cycle is trivial.
- The strategy of the proof is then to prove that the K-theory respectively L-theory of the obstruction category is trivial by showing that every cycle is equivalent to an ϵ-controlled cycle.
- This is achieved in two steps.

- Firstly, a transfer replacing G by $G \times X$, where a suitable compact space X is used. The space X will gves us extra room to arrange things.
- Secondly, the "long thin" cover of $G \times X$ is used to construct a contracting map from $G \times X$ to a $\mathcal{VC}$yc-CW-complex.
- More precisely, this map is contracting with respect to the G-coordinate, but expanding with respect to the X-coordinate.
- The transfer will ensure that this bad expanding direction can be handled.
- It is crucial that the output of the transfer is a cycle that is ϵ-controlled over X for very small ϵ.
- To a significant extent, the argument in the L-theory case and the K-theory case are very similar. For example, the formalism of controlled algebra works for L-theory in the same way as for K-theory. This is because both functors have very similar properties.
- However, the L-theory transfer is quite different from the K-theory transfer and requires new ideas.

- The next two results indicates why one may be able to hope for a stability theorem as mentioned above.

Theorem (Controlled h-Cobordism Theorem, Ferry (1977))

Let M be a compact Riemannian manifold of dimension ≥ 5. Then there exists an $\epsilon = \epsilon_M > 0$, such that every ϵ-controlled h-cobordism over M is trivial.

Theorem (α-approximation theorem, Ferry (1979))

If M is a closed topological manifold of dimension ≥ 5 and α is an open cover of M, then there is an open cover β of M with the following property: If N is a topological manifold of the same dimension and $f \colon N \to M$ is a proper β-homotopy equivalence, then f is α-close to a homeomorphism.

- Another motivation for the appearance of controlled topology is the following.
- The Farrell-Jones Conjecture predicts that certain K and L-groups are given in terms of homology.
- The extra property of homology is excision, homotopy invariance exists already on the K-and L-theory itself.
- In general proofs of excisions are based on methods to make cycles small.
- The proof of excision for singular homology is essential based on the fact that one can achieve by barycentric subdivisions that a cycle representing a given class can be arranged to be very small.

Axiomatic formulation of the proof

- In order to formulate the axiomatic approach we need some definitions and notations.

Definition (Homotopy S-action)

Let S be a finite subset of a group G. Assume that S contains the trivial element $e \in G$. Let X be a space.
A homotopy S-action (φ, H) on X consists of continuous maps $\varphi_g \colon X \to X$ for $g \in S$ and homotopies $H_{g,h} \colon X \times [0,1] \to X$ for $g, h \in S$ with $gh \in S$ such that $H_{g,h}(-,0) = \varphi_g \circ \varphi_h$ and $H_{g,h}(-,1) = \varphi_{gh}$ holds for $g, h \in S$ with $gh \in S$. Moreover, we require that $H_{e,e}(-,t) = \varphi_e = \mathrm{id}_X$ for all $t \in [0,1]$

- No higher coherences of the homotopies are required.

- We need some notation associated to a homotopy S-action a (φ, H) on X.
- For $g \in S$ let $F_g(\varphi, H)$ be the set of all maps $X \to X$ of the form $x \mapsto H_{r,s}(x,t)$ where $t \in [0,1]$ and $r, s \in S$ with $rs = g$;
- For $(g,x) \in G \times X$ and $n \in \mathbb{N}$, let $S^n_{\varphi,H}(g,x)$ be the subset of $G \times X$ consisting of all (h,y) with the following property: There are $x_0, \dots, x_n \in X$, $a_1, b_1, \dots, a_n, b_n \in S$, $f_1, \widetilde{f}_1, \dots, f_n, \widetilde{f}_n \colon X \to X$, such that $x_0 = x$, $x_n = y$, $f_i \in F_{a_i}(\varphi, H)$, $\widetilde{f}_i \in F_{b_i}(\varphi, H)$, $f_i(x_{i-1}) = \widetilde{f}_i(x_i)$ and $h = g a_1^{-1} b_1 \dots a_n^{-1} b_n$.

$$
\begin{tikzcd}
x_0 = x \arrow[dr, "f_1", "a_1"'] & & x_1 \arrow[dl, "\widetilde{f_1}"', "b_1^{-1}"] \arrow[dr, "f_2", "a_2"'] & & x_2 \arrow[dl, "\widetilde{f_2}"', "b_2^{-1}"] \arrow[dr, "f_3", "a_3"'] & \\
& f_1(x_0) = \widetilde{f_1}(x_1) & & f_2(x_1) = \widetilde{f_2}(x_2) & & {}
\end{tikzcd}
$$

Definition (S-long covering)

Let (φ, H) be a homotopy S-action on X and $\mathcal{U}$ be an open cover of $G \times X$. We say that $\mathcal{U}$ is S-long with respect to (φ, H) if for every $(g, x) \in G \times X$ there is $U \in \mathcal{U}$ containing $S^{|S|}_{\varphi,H}(g, x)$ where $|S|$ is the cardinality of S.

Example (Honest group action)

If the homotopy action is the restriction of a G-action to S and S is symmetric with respect to $s \mapsto s^{-1}$, then $\varphi_g(x) = x$, $H_{g,h}(x, t) = ghx$ for all t and $S^n_{\varphi,H}(g, x) = \{(ga^{-1}, ax) \mid a = s_1 \dots s_{2|S|}, s_i \in S\}$.

Example (CAT(0)-space)

- Let Y be a CAT(0)-space with an isometric G action.
- Let $S \subseteq G$ be a finite subset and $R > 0$ and $x \in X$ be given.
- Let $X = B_R(x)$ be the ball of radius R around x. Let $r\colon Y \to B_R(x)$ the radial projection onto the ball which is the identity on $B_R(x)$.
- Define a homotopy S-action on X by putting

$$\varphi_g\colon B_R(X) \xrightarrow{i} Y \xrightarrow{l_g} Y \xrightarrow{r} B_R(x).$$

Define the homotopies by convex combination:

$$H_{g,h} B_R(x) \times [0,1] \to B_R(x), \quad (x,t) \mapsto t \cdot \varphi_g \circ \varphi_h(x) + (1-t) \cdot \varphi_{gh}.$$

- Notice that in the example above for large R the homotopies are not stationary in a neighborhood of the boundary of $\partial B_R(x)$ which is small in comparison with $B_R(x)$ itself. Here it is crucial that S is finite.
- If Y is locally compact, then $B_R(x)$ is compact.
- We will be able to restrict to a finite subset S of G, because our cycles for elements in the algebraic K-theory or L-theory of the obstruction category will involve only a finite number of group elements.
- The key to extend the result that the K-theoretic assembly map is 1-connected to being a weak homotopy equivalence is probably to consider also higher homotopies and use them to define the transfer map also for higher algebraic K-theory.
- In our proof we will consider individual elements in the K-theory and certain estimates will depend on the particular element. This is a problem when one wants to deal with higher K-theory or A-theory.

Definition (N-dominated space)

Let X be a metric space and $N \in \mathbb{N}$. We say that X is controlled N-dominated if for every $\epsilon > 0$ there is a finite CW-complex K of dimension at most N, maps $i\colon X \to K$, $p\colon K \to X$ and a homotopy $H\colon X \times [0,1] \to X$ between $p \circ i$ and id_X such that for every $x \in X$ the diameter of $\{H(x,t) \mid t \in [0,1]\}$ is at most ϵ.

Definition (Open $\mathcal{F}$-cover)

- Let Y be a G-space. Let $\mathcal{F}$ be a family of subgroups of G. A subset $U \subseteq Y$ is called an $\mathcal{F}$-subset if
  - For $g \in G$ and $U \in \mathcal{U}$ we have $g(U) = U$ or $U \cap g(U) = \emptyset$, where $g(U) := \{gx \mid x \in U\}$;
  - The subgroup $G_U := \{g \in G \mid g(U) = U\}$ lies in $\mathcal{F}$.
- An open $\mathcal{F}$-cover of Y is a collection $\mathcal{U}$ of open $\mathcal{F}$-subsets of Y such that the following conditions are satisfied:
  - $Y = \bigcup_{U \in \mathcal{U}} U$;
  - For $g \in G$, $U \in \mathcal{U}$ the set $g(U)$ belongs to $\mathcal{U}$.

- Let $\mathcal{U}$ be an open $\mathcal{F}$-cover of Y. Then its nerve $\mathcal{N}(\mathcal{U})$ comes with a simplicial G action whose isotropy groups belong to $\mathcal{F}$.

Assumption (Main Assumption)

There exists a number N such that for every finite subset S of G there are:

- *a contractible compact controlled N-dominated metric space X;*
- *a homotopy S-action (φ, H) on X;*
- *a cover $\mathcal{U}$ of $G \times X$ by open sets,*

such that the following holds for the G-action on $G \times X$ given by $g \cdot (h, x) = (gh, x)$:

1. $\dim \mathcal{U} \leq N$;
2. *$\mathcal{U}$ is S-long with respect to (φ, H);*
3. *$\mathcal{U}$ is an open $\mathcal{F}$-covering.*

Theorem (Axiomatic Formulation)

Let $\mathcal{F}$ be a family of subgroups of the group G. If G satisfies the main assumption above with respect to $\mathcal{F}$, then the following holds:

1. *Let $\mathcal{A}$ be an additive G-category, i.e., an additive category with right G-action by functors of additive categories. Then the assembly map*
$$H_m^G(E_{\mathcal{F}}(G);\mathbf{K}_{\mathcal{A}}) \to K_m\left(\int_G \mathcal{A}\right)$$
is an isomorphism for $m < 1$ and surjective for $m = 1$;
2. *Let $\mathcal{A}$ be an additive G-category with involution. Then the assembly map*
$$H_m^G(E_{\mathcal{F}_2}(G);\mathbf{L}_{\mathcal{A}}^{-\infty}) \to L_m^{\langle -\infty \rangle}\left(\int_G \mathcal{A}\right)$$
is an isomorphism for all $m \in \mathbb{Z}$. Here $\mathcal{F}_2$ is the family of all subgroups $V \subseteq G$ for which there is $F \subseteq V$ such that $F \in \mathcal{F}$ and $[V : F] \leq 2$.

Outline of the proof that hyperbolic groups satisfy the main assumptions

Theorem (Hyperbolic groups)

Every hyperbolic group satisfies the Main Assumption with respect to the family $\mathcal{VC}yc$ of virtually cyclic subgroups.

- We give an outline of the proof.

- Let d_G be a δ-hyperbolic left-invariant word-metric on the hyperbolic group G.
- Let $P_d(G)$ be the associated Rips complex for $d > 4\delta + 6$. It is a finite-dimensional contractible locally finite simplicial complex with proper cocompact simplicial G-action and is a model for $\underline{E}G$.
- This space can be compactified to a metrizable compact space $X := P_d(G) \cup \partial G$, where ∂G is the Gromov boundary of G. The boundary ∂X is far away from being a simplicial complex in general and is more of the type of a Cantor set.
- The proper cocompact simplicial action of G on $P_d(G)$ extends to a G-action on X. The induced action on ∂X can be very complicated and has infinite isotropy groups. There is no simplicial structure on X for which this action is simplicial.

- According to a result of Bestvina-Mess(1991) the subspace $\partial P_d(G) \subseteq X$ satisfies the Z-set condition.
- We only need to know that this implies the existence of a homotopy $H\colon X \times [0,1] \to X$, such that $H_0 = \mathrm{id}_X$ and $H_t(X) \subset P_d(G)$ for all $t > 0$.
- The compactness of X implies that for $t > 0$, $H_t(X)$ is contained in a finite subcomplex of $P_d(G)$.
- Therefore X is controlled N-dominated, where N is the dimension of $P_d(G)$.

- The main result of Bartels-Lück-Reich (2008) implies that there is a number N such that for every $\alpha > 0$ exists an open cover $\mathcal{V}_\alpha$ of $G \times X$ such that:
  - $\dim \mathcal{V}_\alpha \leq N$;
  - For every $(g, y) \in G \times X$ there is $V \in \mathcal{V}_\alpha$ such that

    $$\{(gh, h^{-1}y) \mid h \in e^\alpha\} \subseteq V.$$

    (We denote by e^α the open ball of radius α around the unit element e of G.)
  - $\mathcal{V}_\alpha$ is a $\mathcal{VC}$yc-cover with respect to the left G-action $g \cdot (h, x) = (gh, x)$.

- Consider a finite subset S of G containing e. Put $n = |S|$.
- Pick $\alpha > 0$ such that

$$\{l \in G \mid l = a_1^{-1} b_1 \dots a_n^{-1} b_n \text{ for } a_i, b_i \in S\} \subseteq e^{\alpha}.$$

- The G-action on X induces a homotopy S-action (φ, H) on X where φ_g is given by $l_g \colon X \to X,\ x \mapsto gx$ for $g \in S$, and $H_{g,h}(-,t) = l_{gh}$ for $g, h \in S$ with $gh \in S$ and $t \in [0,1]$.
- Notice that in this case

$$\begin{aligned} F_g(\varphi, H) &= \{l_g \colon X \to X\}; \\ S^n_{\varphi,H}(g,x) &= \{gl, l^{-1}x) \mid l = a_1^{-1} b_1 \dots a_n^{-1} b_n \text{ for } a_i, b_i \in S\}. \end{aligned}$$

- Hence $\mathcal{V}_\alpha$ is S-long with respect to (φ, H).

- Notice that in the sketch above the main open point is the construction of the open cover $\mathcal{V}_\alpha$. Here the flow space associated to a hyperbolic group will enter.
- We mention:

Theorem (Cat(0)-groups)

Every finite dimensional CAT(0)*-group satisfies the Main Assumption with respect to the family $\mathcal{VC}yc$ of virtually cyclic subgroups.*

Review of the transfer map in K-theory and L-theory

- We review the transfer map for the Whitehead group for a fibration $F \to E \xrightarrow{p} B$ of connected finite CW-complexes.
- The geometric version is defined as follows.
- Given an element $\eta \in \mathrm{Wh}(\pi(B))$, choose a homotopy equivalence $f\colon X \to B$ of finite CW-complexes whose Whitehead torsion $\tau(f)$ is η.
- Let $\overline{f}\colon Y \to E$ be the homotopy equivalence obtained from f by the pullback construction.
- The transfer $\mathrm{trf}\colon \mathrm{Wh}(\pi(B)) \to \mathrm{Wh}(\pi(E))$ sends η to $\tau(\overline{f})$.
- Next we want to describe its algebraic version.

- For simplicity we will assume that $\pi_1(p)\colon \pi_1(E) \to \pi_1(B)$ is bijective and we will identify in the sequel $G := \pi_1(E) = \pi_1(B)$.
- Recall that the fiber transport gives a homomorphism of monoids $G \to [F, F]$.
- Thus we obtain a finite free $\mathbb{Z}$-chain complex $C = C_*(F)$, namely, the cellular $\mathbb{Z}$-chain complex of F, together with an operation of G up to chain homotopy, i.e., a homomorphism of monoids $\rho\colon G \to [C, C]_{\mathbb{Z}}$ to the monoid of chain homotopy classes of $\mathbb{Z}$-chain maps $C \to C$.
- Given an element $a = \sum_{g \in G} \lambda_g g \in \mathbb{Z}G$, define a $\mathbb{Z}G$-chain map of finitely generated free $\mathbb{Z}G$-chain complexes, unique up to $\mathbb{Z}G$-chain homotopy, by

$$a \otimes_t C\colon \mathbb{Z}G \otimes_{\mathbb{Z}} C \to \mathbb{Z}G \otimes_{\mathbb{Z}} C, \quad g' \otimes x \mapsto \sum_{g \in G} \lambda_g \cdot g'g^{-1} \otimes r(g)(x),$$

where $r(g)\colon C \to C$ is some representative of $\rho(g)$.

- Thus we obtain a ring homomorphism $\mathbb{Z}G \to [\mathbb{Z}G \otimes_{\mathbb{Z}} C, \mathbb{Z}G \otimes_{\mathbb{Z}} C]_{\mathbb{Z}G}$ to the ring of $\mathbb{Z}G$-chain homotopy classes of $\mathbb{Z}G$-chain maps $\mathbb{Z}G \otimes_{\mathbb{Z}} C \to \mathbb{Z}G \otimes_{\mathbb{Z}} C$.
- It extends in the obvious way to matrices over $\mathbb{Z}G$, namely, for a matrix $A \in M_{m,n}(\mathbb{Z}G)$ we obtain a $\mathbb{Z}G$-chain map, unique up to G-homotopy,

$$A \otimes_t C \colon \mathbb{Z}G^m \otimes_{\mathbb{Z}} C \to \mathbb{Z}G^n \otimes_{\mathbb{Z}} C.$$

- The algebraic transfer

$$p^* \colon \operatorname{Wh}(G) \to \operatorname{Wh}(G)$$

sends the class of an invertible matrix $A \in GL_n(\mathbb{Z}G)$ to the Whitehead torsion of the $\mathbb{Z}G$-self-chain homotopy equivalence $A \otimes_t C \colon \mathbb{Z}G^n \otimes_{\mathbb{Z}} C \to \mathbb{Z}G^n \otimes_{\mathbb{Z}} C$.

- The proof of the identification of the geometric and algebraic homomorphisms uses the following facts.
- Any element $\eta \in \mathrm{Wh}(\pi(B))$ can be realized by $\tau(r)$ for $r\colon X \to B$, where X its obtained from B by attaching cells in two consecutive dimensions and r is a deformation retraction. The matrix of the only non-trivial differential in $C_*(\widetilde{X}, \widetilde{B})$ is a representative of η.
- Then the simple homotopy type of the $\mathbb{Z}G$-chain complex $C_*(\widetilde{Y}; \widetilde{E})$ is given by the mapping cone of

$$A \otimes_t C\colon \mathbb{Z}G^m \otimes_{\mathbb{Z}} C \to \mathbb{Z}G^n \otimes_{\mathbb{Z}} C.$$

- An important tool is the down-up-formula which determines the composite

$$\mathrm{Wh}(\pi(B)) \xrightarrow{\mathrm{trf}} \mathrm{Wh}(\pi(E)) \xrightarrow{p_*} \mathrm{Wh}(\pi(E))$$

and which we review next.

- Let Sw(G) be the Swan group of G, i.e. the Grothendieck group $\mathbb{Z}G$-modules which are finitely generated as abelian groups.
- Let $\mathrm{Sw}^f(G)$ be the Grothendieck group $\mathbb{Z}G$-modules which are finitely generated free as abelian groups.
- The forgetful map $\mathrm{Sw}^f(G) \xrightarrow{\cong} \mathrm{Sw}(G)$ is an isomorphism.
- We obtain a pairing

$$\begin{aligned}\otimes\colon \mathrm{Wh}(G) \otimes \mathrm{Sw}^f(G) &\to \mathrm{Wh}(G),\\ [A] \otimes [M] &\mapsto [A \otimes_t \mathrm{id}_M \colon \mathbb{Z}G^n \otimes_{\mathbb{Z}} M \to \mathbb{Z}G^n \otimes_{\mathbb{Z}} M],\end{aligned}$$

  where $[A \otimes_t \mathrm{id}_M$ is defined analogously to $\otimes_t$ for chain complexes above.
- Hence we obtain a pairing

$$\otimes\colon \mathrm{Wh}(G) \otimes \mathrm{Sw}(G) \to \mathrm{Wh}(G)$$

- Since the fiber is a finite CW-complex and comes with a homotopy G-action, we obtain an element

$$h(F) := \sum_{n \geq 0} (-1)^n \cdot [H_n(F)] \quad \in \mathrm{Sw}(G).$$

- The down up formula says

$$p_* \circ \mathrm{trf} = - \otimes h(F).$$

- Suppose that G acts trivially on the homology of F and the Euler characteristic of F is one, e.g., F is contractible.
- Then $h(F)$ is given by the class of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ and hence $- \otimes h(F)$ is the identity.
- Since $\pi_1(p)$ is an isomorphism by assumption, p_* is an isomorphism.
- Hence trf is an isomorphism if F is contractible.

- To obtain an *L-theory transfer* $\operatorname{trf}\colon L_n(\pi(B)) \to L_n(\pi(E))$ we have additionally to assume that F is a finite n-dimensional Poincaré complex.
- For simplicity we assume that $n \equiv 0 \mod (4)$ and that F is an oriented n-dimensional Poincaré complex and the fiber transport $G \to [F, F]$ takes values in homotopy classes of orientation preserving self-homotopy equivalences and – as before – that $\pi_1(p)$ is bijective.
- The geometric version is defined by pulling back a surgery problem.
- The algebraic version is defined as follows.

- Because F is a Poincaré complex, there is a symmetric form $\varphi\colon C^{-*} \to C$, where C^{-*} denotes the dual of the cellular chain complex of F,i.e., $(C^{-*})_n = (C_{-n})^*$.
- If $\psi\colon M^* \to M$ is a quadratic form over $\mathbb{Z}[G]$, then the composition

$$\psi \otimes_t (C,\varphi)\colon (M \otimes C)^{-*} \cong M^* \otimes C^{-*} \xrightarrow{\operatorname{id} \otimes \varphi} M^* \otimes C \xrightarrow{\psi \otimes_t C} M \otimes C$$

  defines an ultra-quadratic form on $M \otimes C$.
- The L-theory transfer sends the class of $(M,\psi) \in L_0(\mathbb{Z}G)$ to the class of $(M \otimes C, \psi \otimes_t (C,\varphi)$.

- There is a version of the down-up-formula

$$\otimes \colon L_n(\mathbb{Z}G) \otimes W(G) \to L_n(\mathbb{Z}G)$$

where $W(G)$ is a kind of Witt group of symmetric bilinear $\mathbb{Z}$-forms over finitely generated free $\mathbb{Z}$-modules which come with a G-action.

- The intersection pairing in the middle homology modulo torsion of the fiber defines an element

$$s(F) \in W(G).$$

- The up-down formula says that the composite

$$L_n(\pi(B)) \xrightarrow{\mathrm{trf}} L_n(\pi(E)) \xrightarrow{p_*} L_n(\pi(E))$$

is given by $- \otimes s(F)$.

- Suppose that G acts trivially on the homology of F and the signature of F is one.
- Then $s(F)$ is given by the standard symmetric bilinear form with $\mathbb{Z}$ as underlying group and trivial G-action and $- \otimes s(F)$ is the identity.
- Since $\pi_1(p)$ is an isomorphism by assumption, p_* is an isomorphism.
- Hence trf is an isomorphism.

- A problem will be that we can construct appropriate space which are contractible so that the K-theory transfer is an isomorphism. However, the spaces at hand like spheres and so on do have signature zero and hence the transfer associated to them is zero.
- Hence one needs a construction which makes out of a space which represents the unit in $\mathrm{Sw}(G)$ a new space which presents the unit in $W(G)$.
- On a space level Farrell-Jones used the observation that $S^2 \times S^2/\mathbb{Z}/2$ for the flip action is $\mathbb{CP}^2$ which has signature one. The problem is that $S^n \times S^n/\mathbb{Z}/2$ for even $n \geq 4$ is a rational homology manifold of signature one but not a manifold. But this means what has to do equivariant or stratified surgery what can be very complicated.
- The following construction yields an easy algebraic solution to the problem.

- Given a finitely projective R-module P over the commutative ring R, define a symmetric bilinear R-form $H_{\otimes}(P)$ called multiplicative hyperbolic form by

$$\big(P \otimes P^*\big) \times \big(P \otimes P^*\big) \to R, \quad (p \otimes \alpha, q \otimes \beta) \mapsto \alpha(q) \cdot \beta(p).$$

  If one replaces $\otimes$ by $\oplus$ and $\cdot$ by $+$, this becomes the standard hyperbolic form.
- The multiplicative hyperbolic form induces a ring homomorphism

$$H_{\otimes} \colon K_0(R) \to L_p^0(R), \quad [P] \mapsto [H_{\otimes}(P)].$$

- We have to show that this is well-defined, i.e., we must prove

$$[H_{\otimes}(P \oplus Q)] = [H_{\otimes}(P)] + [H_{\otimes}(P)] \in L_p^0(\Lambda)$$

  for two finitely generated projective Λ-modules P and Q.

- This follows from the fact that we have an isomorphism of Λ-modules

$$\begin{aligned}
&(P \oplus Q)^* \otimes_\Lambda P \oplus Q \\
&\qquad \cong P^* \otimes_\Lambda P \oplus Q^* \otimes_\Lambda Q \oplus Q^* \otimes_\Lambda P \oplus P^* \otimes_\Lambda Q \\
&\qquad\quad \cong P^* \otimes_\Lambda P \oplus Q^* \otimes_\Lambda Q \oplus (Q^* \otimes_\Lambda P \oplus (Q^* \otimes_\Lambda P)^*)
\end{aligned}$$

which induces an isomorphism of non-singular symmetric Λ-forms

$$H_\otimes(P \oplus Q) \cong H_\otimes(P) \oplus H_\otimes(Q) \oplus H(Q^* \otimes_\Lambda P).$$

- For $R = \mathbb{Z}$ we obtain an isomorphism

$$H_\otimes \colon K_0(\mathbb{Z}) \xrightarrow{\cong} L^0(\mathbb{Z}).$$

Contracting maps coming from S-long covers

- Throughout this section we fix the following data. Let
  - G be a group;
  - (X, d_X) be a compact metric space. We equip $G \times X$ with the G-action $g(h, x) = (gh, x)$;
  - S be a finite subset of G (containing e);
  - (φ, H) be a homotopy S-action on X.
- For every number $\Lambda > 0$ we define a G-invariant (quasi-)metric $d_{S,\Lambda}$ on $G \times X$ as follows.

- For $(g,x),(h,y) \in G \times X$ consider $n \in \mathbb{Z}$, $n \geq 0$, elements $x_0, \ldots, x_n \in X$, $z_0, \ldots, z_n$ in X, elements $a_1, b_1, \ldots, a_n, b_n$ in S and maps $f_1, \widetilde{f}_1, \ldots, f_n, \widetilde{f}_n \colon X \to X$ such that

$$\begin{aligned}
&x = x_0, z_n = y, \\
&f_i \in F_{a_i}(\varphi, H), \widetilde{f}_i \in F_{b_i}(\varphi, H), f_i(z_{i-1}) = \widetilde{f}_i(x_i) \text{ for } i = 1, 2, \ldots n; \\
&h = g a_1^{-1} b_1 \ldots a_n^{-1} b_n.
\end{aligned}$$

- If $n = 0$, we just demand $x_0 = x$, $z_0 = y$, $g = h$ and no elements a_i, b_i, f_i and $\widetilde{f}_i$ occur.
- To this data we associate the number

$$n + \sum_{i=0}^{n} \Lambda \cdot d_X(x_i, z_i).$$

Definition (G-invariant quasi-metric on $G \times X$)

For $(g, x), (h, y) \in G \times X$ define

$$d_{S,\Lambda}((g, x), (h, y)) \in [0, \infty]$$

as the infimum of the numbers above over all possible choices of n, x_i, z_i, a_i, b_i, f_i and $\widetilde{f}_i$. If the set of possible choices is empty, then we put $d_{S,\Lambda}((g, x), (h, y)) := \infty$.

Lemma

- *For every* $\Lambda > 0$, $d_{S,\Lambda}$ *is a well-defined* G*-invariant quasi-metric on* $G \times X$. *The set* S *generates* G *if and only if* $d_{S,\Lambda}$ *is a metric;*
- *Let* $(g,x),(h,y) \in G \times X$ *and let* $m \in \mathbb{Z}$, $m \geq 1$. *If* $d_{S,\Lambda}((g,x),(h,y)) \leq m$ *for all* Λ, *then* $(h,y) \in S^m_{\varphi,H}(g,x)$;
- *For* $x,y \in X$ *and* $g \in G$ *we have* $d_{S,\Lambda}((g,x),(h,y)) < 1$ *if and only if* $g = h$ *and* $\Lambda \cdot d_X(x,y) < 1$ *hold. In this case we get*

$$d_{S,\Lambda}((g,x),(h,y)) = \Lambda \cdot d_X(x,y).$$

- *The topology on* $G \times X$ *induced by* $d_{S,\Lambda}$ *is the product topology on* $G \times X$ *for the discrete topology on* G *and the given one on* X.

Lemma

Let $\mathcal{U}$ be an S-long finite-dimensional G-equivariant cover of $G \times X$. Let m be any number with $m \leq |S|$.
Then there is $\Lambda > 0$ such that the Lebesgue number of $\mathcal{U}$ with respect to $d_{S,\Lambda}$ is at least $m/2$, i.e., for every (g,x) there is $U \in \mathcal{U}$ containing the open $m/2$-ball $B_{m/2,\Lambda}(g,x)$ around (g,x) with respect to the metric $d_{S,\Lambda}$.

- In the following proposition d^1 denotes the l^1-metric on simplicial complexes.

Theorem (Contracting map)

Let G be a finitely generated group that fulfills the Main Assumption for $\mathcal{F}$. Let N be the number appearing in the Main Assumption. Let $S \subseteq G$ be a finite subset which generates G. Let $\epsilon > 0$, $\beta > 0$.
Then there are:

- *a compact contractible controlled N-dominated metric space (X, d);*
- *a homotopy S-action (φ, H) on X;*
- *a positive real number Λ;*
- *a simplicial complex Σ of dimension $\leq N$ with a simplicial cell preserving G-action;*
- *a G-equivariant map $f \colon G \times X \to \Sigma$,*

satisfying:

1. *The isotropy groups of Σ are members of $\mathcal{F}$;*
2. *If $(g, x), (h, y) \in G \times X$ and $d_{S,\Lambda}((g, x), (h, y)) \leq \beta$, then*

$$d^1(f(g, x), f(h, y)) \leq \epsilon.$$

Sketch of proof.

- There exists by the Main Assumption a contractible compact controlled N-dominated space X, a homotopy S-action (φ, H) on X and an S-long cover $\mathcal{U}$ of $G \times X$ such that $\mathcal{U}$ is an N-dimensional open $\mathcal{F}$-covering.
- Using the result above we find $\Lambda > 0$ such that the Lebesgue number of $\mathcal{U}$ with respect to $d_{S,\Lambda}$ is at least $|S|/2$.
- Let $\Sigma := |\mathcal{U}|$ be the realization of the nerve of $\mathcal{U}$. Since $\mathcal{U}$ is an open $\mathcal{F}$-cover, Σ inherits a simplicial cell preserving G-action whose isotropy groups are members of $\mathcal{F}$.
- Let now $f \colon G \times X \to \Sigma$ be the map induced by $\mathcal{U}$, i.e.,

$$f(x) := \sum_{U \in \mathcal{U}} \frac{d_{S,\Lambda}(x, G \times X - U)}{\sum_{V \in \mathcal{U}} d_{S,\Lambda}(x, G \times X - V)} U.$$

- It has all the desired properties.

□

Some categories with control

- Let G be a group, Y a space and $\mathcal{A}$ be a additive category.
- Let $\mathcal{E} \subseteq \{E \mid E \subseteq Y \times Y\}$ and $\mathcal{F} \subseteq \{F \mid F \subseteq Y\}$ be collections satisfying certain conditions. (These conditions are designed to ensure that we indeed obtain an additive category and are satisfied in all cases of interest.)
- The category $\mathcal{C}(Y;\mathcal{E},\mathcal{F};\mathcal{A})$ is defined as follows.
- Objects are given by sequences $(M_y)_{y \in Y}$ of objects in $\mathcal{A}$ such that
  - M is $\mathcal{F}$-controlled: there is F in $\mathcal{F}$ such that the support $\operatorname{supp} M := \{y \mid M_y \neq 0\}$ is contained in F;
  - M has locally finite support: for every $y \in Y$ there is an open neighborhood U of y such that $U \cap \operatorname{supp} M$ is finite.

- A morphism ψ from $M = (M_y)_{y \in Y}$ to $M' = (M'_y)_{y \in Y}$ is given by a collection $(\psi_{y',y} \colon M_y \to M'_{y'})_{(y',y) \in Y \times Y}$ of morphisms in $\mathcal{A}$ such that
  - ψ is $\mathcal{E}$-controlled: there is $E \in \mathcal{E}$ such that the support $\operatorname{supp}(\psi) := \{(y',y) \mid \psi_{y',y} \neq 0\}$ is contained in E;
  - ψ is row and column finite: for every $y \in Y$ the sets $\{y' \in Y \mid (y,y') \in \operatorname{supp}\psi\}$ and $\{y' \in Y \mid (y',y) \in \operatorname{supp}\psi\}$ are finite.
- Composition of morphisms is given by matrix multiplication, i.e., $(\psi' \circ \psi)_{y'',y} = \sum_{y' \in Y} \psi_{y'',y'} \circ \psi_{y',y}$.

- Let now Y be a (left) G-space and assume that $\mathcal{A}$ is equipped with a (strict) right G-action, i.e., $\mathcal{A}$ is an additive G-category. Assume that the G-action on Y preserves both $\mathcal{F}$ and $\mathcal{E}$.
- Then $\mathcal{C}(Y, \mathcal{E}, \mathcal{F}; \mathcal{A})$ inherits a (right) G-action making it an additive G-category.

Definition ($\mathcal{C}^G(Y; \mathcal{E}, \mathcal{F}; \mathcal{A})$)

We will denote by $\mathcal{C}^G(Y; \mathcal{E}, \mathcal{F}; \mathcal{A})$ the subcategory of $\mathcal{C}(Y; \mathcal{E}, \mathcal{F}; \mathcal{A})$ that is fixed by G.

- Let (Z, d) be a metric space.
- Let $\mathcal{E}(Z, d) := \{E_\alpha \mid \alpha > 0\}$ where $E_\alpha := \{(z, z') \mid d(z, z') \leq \alpha\}$.
- For an additive category $\mathcal{A}$ we define

$$\mathcal{C}(Z, d; \mathcal{A}) := \mathcal{C}(Z; \mathcal{E}(Z, d), \{Z\}; \mathcal{A}).$$

- Let $\epsilon > 0$. A morphism ψ in $\mathcal{C}(Z, d; \mathcal{A})$ is said to be ϵ-controlled if $\operatorname{supp}(\psi) \subseteq E_\epsilon$.

- The idempotent completion $\operatorname{Idem}(\mathcal{A})$ of an additive category $\mathcal{A}$ is the following additive category.
- Objects are morphisms $p\colon M \to M$ in $\mathcal{A}$ satisfying $p^2 = p$.
- A morphism $f\colon (M,p) \to (N,q)$ in $\operatorname{Idem}(\mathcal{A})$ is a morphism $f\colon M \to N$ satisfying $q \circ f \circ p = f$.
- Composition and the additive structure are inherited from $\mathcal{A}$ in the obvious way.
- If $\mathcal{A}$ is an additive category which is equivalent to the category of finitely generated free R-modules, then $\operatorname{Idem}(\mathcal{A})$ is equivalent to the category of finitely generated projective R-modules.

- Let Y be a G-space and let (Z,d) be a metric space with isometric G-action. Let $\mathcal{A}$ be an additive G-category.
- There is the definition of equivariant continuous control condition $\mathcal{E}^Y_{Gcc} \subseteq \{E \subseteq (Y \times [1,\infty))^{\times 2}\}$
- Define $\mathcal{E}(Y,Z,d)$ as the collection of all $E \subseteq (G \times Z \times Y \times [1,\infty))^{\times 2}$ that satisfy the following conditions:
  - E is $\mathcal{E}^Y_{Gcc}$-controlled: there exists an element $E' \in \mathcal{E}^Y_{Gcc}$ with the property that $((g,z,y,t),(g',z',y',t')) \in E$ implies $((y,t),(y',t')) \in E'$;
  - E is bounded over G: there is a finite subset S of G with the property that $((g,z,y,t),(g',z',y',t')) \in E$ implies $g^{-1}g' \in S$;
  - E is bounded over Z: there is $\alpha > 0$ such that $((g,z,y,t),(g',z',y',t')) \in E$ implies $d(z,z') \leq \alpha$.

- We define $\mathcal{F}(Y, Z, d)$ to be the collection of all $F \subseteq G \times Z \times Y \times [1, \infty)$ for which there is a compact subset K of $G \times Z \times Y$ such that for $(g, z, y, t) \in F$ there is $h \in G$ satisfying $(hg, hz, hy) \in K$.

Definition ($\mathcal{O}^G(Y, Z, d; \mathcal{A})$)

We define obstruction category

$$\mathcal{O}^G(Y, Z, d; \mathcal{A}) := \mathcal{C}^G(G \times Z \times Y \times [1, \infty); \mathcal{E}(Y, Z, d), \mathcal{F}(Y, Z, d); \mathcal{A}),$$

where we use the G-action on $G \times Z \times Y \times [0, \infty)$ given by $g(h, z, y, t) := (gh, gz, gy, t)$.

- We will also use the case where Z is trivial, i.e., a point, in this case we write $\mathcal{O}^G(Y;\mathcal{A})$ and drop the point from the notation.
- We remark that all our constructions on this category will happen in the $G \times Z$ factor of $G \times Z \times Y \times [1,\infty)$; in particular, it will not be important to know the precise definition of the equivariant continuous control condition $\mathcal{E}^Y_{Gcc}$.

Definition (Flasque)

An additive category is called flasque if there is a functor of such categories $\Sigma^\infty\colon \mathcal{A} \to \mathcal{A}$ together with a natural equivalence of functors of such categories $\mathrm{id}_{\mathcal{A}} \oplus \Sigma^\infty \xrightarrow{\cong} \Sigma^\infty$.

- The property flasque allows to perform an Eilenberg swindle.

Theorem (Flasque categories, Karoubi filtrations and colimits)

- *If $\mathcal{A}$ is a flasque additive category, then $\mathbf{K}(\mathcal{A})$ is weakly contractible;*
- *If $\mathcal{A} \subseteq \mathcal{U}$ is a Karoubi filtration of additive categories, then*

  $$\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{U}) \to \mathbf{K}(\mathcal{U}/\mathcal{A})$$

  is a homotopy fibration sequence of spectra;
- *If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is an equivalence of additive categories, then $\mathbf{K}(\varphi)$ is a weak equivalence of spectra;*
- *If $\mathcal{A} = \operatorname{colim}_i \mathcal{A}_i$ is a colimit of additive categories over a directed system, then the natural map $\operatorname{colim}_i \mathbf{K}(\mathcal{A}_i) \to \mathbf{K}(\mathcal{A})$ is a weak equivalence.*

Theorem (Reduction of the proof to the obstruction category)

Let G be a group. Let $\mathcal{F}$ be a family of subgroups of G.

- *Suppose that there is $m_0 \in \mathbb{Z}$ such that*

$$\pi_{m_0}\big(\mathbf{K}(\mathcal{O}^G(E_{\mathcal{F}}G;\mathcal{A}))\big) = 0$$

  holds for all additive G-categories $\mathcal{A}$.
  Then the K-theoretic assembly map is an isomorphism for $m < m_0$ and surjective for $m = m_0$ for all such $\mathcal{A}$.

- *Suppose that there is $m_0 \in \mathbb{Z}$ such that*

$$\pi_{m_0}\big(\mathbf{L}^{\langle -\infty \rangle}(\mathcal{O}^G(E_{\mathcal{F}_2}(G);\mathcal{A}))\big) = 0$$

  holds for all additive G-categories $\mathcal{A}$ with involution.
  Then the L-theoretic assembly map is an isomorphism for all $m \in \mathbb{Z}$ and such $\mathcal{A}$.

Sketch of proof.

- Construct controlled categories which fit to a commutative diagram of the shape

$$\begin{array}{ccccc} \mathbb{K}^{-\infty}\mathcal{T}^G(E_{\mathcal{F}}G) & \longrightarrow & \mathbb{K}^{-\infty}\mathcal{O}^G(E_{\mathcal{F}}G) & \longrightarrow & \mathbb{K}^{-\infty}\mathcal{D}^G(E_{\mathcal{F}}G) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{K}^{-\infty}\mathcal{T}^G(\mathrm{pt}) & \longrightarrow & \mathbb{K}^{-\infty}\mathcal{O}^G(\mathrm{pt}) & \longrightarrow & \mathbb{K}^{-\infty}\mathcal{D}^G(\mathrm{pt}). \end{array}$$

- The right vertical can be identified with the assembly map appearing in the Farrell-Jones Conjecture.

$\square$

Sketch of proof (continued).

- The rows are homotopy fibrations thank to a Karoubi filtration argument.
- Check that that the left vertical map is induced by an equivalence of categories and is therefore an equivalence of spectra.
- Prove that the homotopy groups of the lower middle spectrum vanish.
- Now the claim follows by considering the long exact ladder of homotopy groups associated to the diagram above.

□

Stability

Theorem (Stability Theorem)

Let $N \in \mathbb{N}$ and let $\mathcal{F}$ be a family of subgroups of a group G. Let S be a finite subset of G. Let $\mathcal{A}$ be an additive G-category.
Then there exists a positive real number $\epsilon = \epsilon(N, \mathcal{A}, G, \mathcal{F}, S)$ with the following property:
If Σ is a simplicial complex of dimension $\leq N$ equipped with a simplicial action of G all whose isotropy groups are members of $\mathcal{F}$ and $\alpha\colon C \to C$ is an (ϵ, S)-chain homotopy equivalence over $\mathcal{O}^G(E_{\mathcal{F}}G, \Sigma, d^1; \mathcal{A})$ where C is concentrated in degrees $0, \ldots, N$, then

$$[(C, \alpha)] = 0 \in K_1(\mathcal{O}^G(E_{\mathcal{F}}G, \Sigma, d^1; \mathcal{A})).$$

Transfer

- Consider the following data:
  - G be a group;
  - $N \in \mathbb{N}$;
  - $(X, d) = (X, d_X)$ be a compact contractible controlled N-dominated metric space;
  - Y be a G-space;
  - $\mathcal{A}$ be an additive G-category.
- There is an obvious notion of a chain homotopy S-action.
- Just replace in the previous definition of a homotopy S-action on a space the space by a chain complex.

Theorem (Existence of a controlled chain complex for the transfer)

Let S be a finite subset of G (containing e) and (φ, H) be a homotopy S-action on X.

Then for every $\epsilon > 0$ there exists a homotopy S-chain complex $\mathbf{P} = (P, \varphi^P, H^P)$ over $\operatorname{Idem}(\mathcal{C}(X, d; \mathbb{Z}))$ satisfying:

- *P is concentrated in degrees $0, \ldots, N$;*
- *P is ϵ-controlled;*
- *There is a homotopy S-chain equivalence*

  $$f \colon \mathbf{P} \to \mathbf{T}_{x_0}$$

  to the trivial homotopy S-chain complex at $x_0 \in X$ for some (and hence all) $x_0 \in X$;
- *If $g \in S$ and $(x, y) \in \operatorname{supp} \varphi^P_g$, then $d(x, \varphi_g(y)) \leq \epsilon$;*
- *if $g, h \in S$ with $gh \in S$ and $(x, y) \in \operatorname{supp} H^P_{g,h}$, then there is $t \in [0, 1]$ such that $d(x, H_{g,h}(y, t)) \leq \epsilon$.*

Sketch of proof.

- Consider the subcomplex $C^{\mathrm{sing},\epsilon}(X)$ of the singular chain complex of X spanned by singular simplices of diameter bounded by an appropriate small constant ϵ
- This chain complex is in an ϵ-controlled way finitely dominated, because X is controlled N-dominated, and can therefore up to controlled homotopy be replaced by finite projective chain complex P.
- The homotopy S-action on X induces through this homotopy equivalence the chain homotopy S-action on P.

$\square$

Theorem (Improving cycles in $K_1(\mathrm{Idem}(\mathcal{O}^G(Y;\mathcal{A})))$)

Let $T \subseteq S$ be finite subsets of G (both containing e) such that for $g, h \in T$, we have $gh \in S$.
Let $\alpha \colon M \to M$ be a T-automorphism in $\mathcal{O}^G(Y;\mathcal{A})$. Let $\Lambda > 0$.
Then there is an $(S,2)$-chain homotopy equivalence $\hat{\alpha} \colon C \to C$ over $\mathrm{Idem}(\mathcal{O}^G(Y, G \times X, d_{S,\Lambda};\mathcal{A}))$ where C is concentrated in degrees $0, \ldots, N$, such that

$$[p(C,\hat{\alpha})] = [(M,\alpha)] \in K_1(\mathrm{Idem}(\mathcal{O}^G(Y;\mathcal{A})))$$

where $p \colon \mathrm{Idem}(\mathcal{O}^G(Y, G \times X, d_{S,\Lambda};\mathcal{A})) \to \mathrm{Idem}(\mathcal{O}^G(Y;\mathcal{A}))$ is the functor induced by the projection $G \times X \to pt$.

Sketch of proof.

- Put $\epsilon := 1/\Lambda$.
- Let $\mathbf{P} = (P, \varphi^P, H^P)$ be a homotopy S-chain complex over $\mathrm{Idem}(\mathcal{C}(X, d; \mathbb{Z}))$ satisfying the assertions of the last theorem. Then a transfer construction yields $\mathrm{tr}^{\mathbf{P}}(\alpha) \colon M \otimes P \to M \otimes P$ which is an S-chain homotopy equivalence over $\mathcal{O}^G(Y, X, d; \mathcal{A})$.
- the existence of the homotopy S-chain equivalence f in the theorem above yields

$$\begin{aligned} q[(M \otimes P, \mathrm{tr}^{\mathbf{P}} \alpha)] &= q[(M \otimes T, \mathrm{tr}^{\mathbf{T}_{x_0}} \alpha)] \\ &= [(M, \alpha)] \in K_1(\mathrm{Idem}(\mathcal{O}^G(Y; \mathcal{A}))). \end{aligned}$$

- Let $F \colon \mathcal{O}^G(Y, X, d; \mathcal{A}) \to \mathcal{O}^G(Y, G \times X, d_{S,\Lambda}; \mathcal{A})$ be the functor induced by the map $(g, x, y, t) \mapsto (g, g, x, y, t)$.

$\square$

Sketch of proof (continued).

- Let $q\colon \mathcal{O}^G(Y,X,d;\mathcal{A}) \to \mathcal{O}^G(Y;\mathcal{A})$ be the functor induced by $X \to \mathrm{pt}$.
- Put $(C,\hat{\alpha}) := F(M \otimes P, \mathrm{tr}^{\mathbf{P}}\, \alpha)$.
- Since $p \circ F = q$ we have $[p(C,\hat{\alpha})] = [\alpha]$.
- Then $\hat{\alpha}$ is a $(S,2)$-chain homotopy equivalence by construction.
- The key observation is that for $t \in T$ and $(x,y) \in \mathrm{supp}(\varphi_t^P)$ we have

$$d_{S,\Lambda}((e,x),(t,y)) \leq 1 + \Lambda \cdot d_X(x,\phi_g(x)) \leq 1 + \Lambda \cdot \epsilon = 2.$$

□

Sketch of proof that the Main Assumption implies the Main Theorem

- Let N be the number appearing in the Main Assumption.
- It suffices to show $K_1(\mathcal{O}^G(E_{\mathcal{F}}G;\mathcal{A})) = 0$ for every additive G-category $\mathcal{A}$.
- Consider $a \in K_1(\mathcal{O}^G(E_{\mathcal{F}}G;\mathcal{A}))$.
- Pick an automorphism $\alpha\colon M \to M$ in $\mathcal{O}^G(E_{\mathcal{F}}G;\mathcal{A}))$ such that $[(M,\alpha)] = a$.
- By definition α is an T-automorphism for some finite subset T of G (containing e).
- We can arrange that T generates G.

- Set $S := \{ab \mid a, b \in T\}$.
- Let $\epsilon = \epsilon(N, \mathcal{A}, G, \mathcal{F}, S)$ be the number appearing in the Stability Theorem.
- Set $\beta := 2$.
- Let (X, d),(φ, H), Λ, Σ and f be as in the Theorem about contracting maps
- Consider the following commuting diagram of functors

$$\begin{tikzcd}
\mathcal{O}^G(E_{\mathcal{F}}G, G \times X, d_{S,\Lambda}; \mathcal{A}) \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{O}^G(E_{\mathcal{F}}G, \Sigma, d^1; \mathcal{A}) \arrow[dl, "q"] \\
& \mathcal{O}^G(E_{\mathcal{F}}G; \mathcal{A}) &
\end{tikzcd}$$

where p resp. q are induced by projecting $G \times X$ resp. Σ to a point and F is induced by f.

- By the Theorem about improving cycles there is a (β, S)-chain homotopy equivalence $[(C, \hat{\alpha})]$ over $\mathrm{Idem}(\mathcal{O}^G(E_{\mathcal{F}}, G \times X, d_{S,\Lambda}; \mathcal{A}))$ such that $[p(C, \hat{\alpha})] = a$.
- The Theorem about contracting maps implies that $F(\hat{\alpha})$ is an (ϵ, S)-chain homotopy equivalence over $\mathrm{Idem}(\mathcal{O}^G(E_{\mathcal{F}}G, \Sigma, d^1; \mathcal{A}))$.
- By the Stability Theorem $[F(C, \hat{\alpha})] = 0$.
- Therefore $a = [q \circ F(C, \hat{\alpha})] = 0$.