Algebraic K-theory

Wolfgang Lück
Münster
Germany
email lueck@math.uni-muenster.de
http://www.math.uni-muenster.de/u/lueck/

summer term 2010

The projective class group

Definition (Projective R-module)

An R-module P is called *projective* if it satisfies one of the following equivalent conditions:

- P is a direct summand in a free R-module;
- The following lifting problem has always a solution

$$\begin{array}{ccccc} M & \xrightarrow{p} & N & \longrightarrow & 0 \\ & \nwarrow^{\overline{f}} & \uparrow f & & \\ & & P & & \end{array}$$

- If $0 \to M_0 \to M_1 \to M_2 \to 0$ is an exact sequence of R-modules, then $0 \to \hom_R(P, M_0) \to \hom_R(P, M_1) \to \hom_R(P, M_2) \to 0$ is exact.

- Over a field or, more generally, over a principal ideal domain every projective module is free.
- If R is a principal ideal domain, then a finitely generated R-module is projective (and hence free) if and only if it is torsionfree.
  For instance $\mathbb{Z}/n$ is for $n \geq 2$ never projective as $\mathbb{Z}$-module.
- Let R and S be rings and $R \times S$ be their product. Then $R \times \{0\}$ is a finitely generated projective $R \times S$-module which is not free.

Example (Representations of finite groups)

Let F be a field of characteristic p for p a prime number or 0. Let G be a finite group.
Then F with the trivial G-action is a projective FG-module if and only if $p = 0$ or p does not divide the order of G. It is a free FG-module only if G is trivial.

Definition (Projective class group $K_0(R)$)

Let R be an (associative) ring (with unit). Define its *projective class group*

$$K_0(R)$$

to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

- This is the same as the Grothendieck construction applied to the abelian monoid of isomorphism classes of finitely generated projective R-modules under direct sum.
- The *reduced projective class group* $\widetilde{K}_0(R)$ is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R-modules, or, equivalently, the cokernel of $K_0(\mathbb{Z}) \to K_0(R)$.

- Let P be a finitely generated projective R-module. It is stably free, i.e., $P \oplus R^m \cong R^n$ for appropriate $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$.
- $\widetilde{K}_0(R)$ measures the deviation of finitely generated projective R-modules from being stably finitely generated free.
- The assignment $P \mapsto [P] \in K_0(R)$ is the universal additive invariant or dimension function for finitely generated projective R-modules.
- Induction

  Let $f \colon R \to S$ be a ring homomorphism. Given an R-module M, let f_*M be the S-module $S \otimes_R M$. We obtain a homomorphism of abelian groups

$$f_* \colon K_0(R) \to K_0(S), \quad [P] \mapsto [f_*P].$$

- Compatibility with products

  The two projections from $R \times S$ to R and S induce isomorphisms

$$K_0(R \times S) \xrightarrow{\cong} K_0(R) \times K_0(S).$$

- Morita equivalence

  Let R be a ring and $M_n(R)$ be the ring of (n,n)-matrices over R. We can consider R^n as a $M_n(R)$-R-bimodule and as a R-$M_n(R)$-bimodule. Tensoring with these yields mutually inverse isomorphisms

$$\begin{array}{lclcl} K_0(R) & \xrightarrow{\cong} & K_0(M_n(R)), & [P] & \mapsto & [{}_{M_n(R)}R^n{}_R \otimes_R P]; \\ K_0(M_n(R)) & \xrightarrow{\cong} & K_0(R), & [Q] & \mapsto & [{}_R R^n{}_{M_n(R)} \otimes_{M_n(R)} Q]. \end{array}$$

Example (Principal ideal domains)

If R is a principal ideal domain. Let F be its quotient field. Then we obtain mutually inverse isomorphisms

$$\begin{array}{lclcll} \mathbb{Z} & \xrightarrow{\cong} & K_0(R), & n & \mapsto & [R^n]; \\ K_0(R) & \xrightarrow{\cong} & \mathbb{Z}, & [P] & \mapsto & \dim_F(F \otimes_R P). \end{array}$$

Example (Representation ring)

Let G be a finite group and let F be a field of characteristic zero. Then the representation ring $R_F(G)$ is the same as $K_0(FG)$. Taking the character of a representation yields an isomorphism

$$R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{C} = K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \operatorname{class}(G, \mathbb{C}),$$

where $\operatorname{class}(G; \mathbb{C})$ is the complex vector space of class functions $G \to \mathbb{C}$, i.e., functions, which are constant on conjugacy classes.

Example (Dedekind domains)

- Let R be a Dedekind domain, for instance the ring of integers in an algebraic number field.
- Call two ideals I and J in R equivalent if there exists non-zero elements r and s in R with $rI = sJ$. The ideal class group C(R) is the abelian group of equivalence classes of ideals under multiplication of ideals.
- Then we obtain an isomorphism

$$C(R) \xrightarrow{\cong} \widetilde{K}_0(R), \quad [I] \mapsto [I].$$

- The structure of the finite abelian group

$$C(\mathbb{Z}[\exp(2\pi i/p)]) \cong \widetilde{K}_0(\mathbb{Z}[\exp(2\pi i/p)]) \cong \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$$

is only known for small prime numbers p.

Theorem (Swan (1960))

If G is finite, then $\widetilde{K}_0(\mathbb{Z}G)$ is finite.

- Topological K-theory

  Let X be a compact space. Let $K^0(X)$ be the Grothendieck group of isomorphism classes of finite-dimensional complex vector bundles over X.

  This is the zero-th term of a generalized cohomology theory $K^*(X)$ called topological K-theory. It is 2-periodic, i.e., $K^n(X) = K^{n+2}(X)$, and satisfies $K^0(\mathrm{pt}) = \mathbb{Z}$ and $K^1(\mathrm{pt}) = \{0\}$.
- Let $C(X)$ be the ring of continuous functions from X to $\mathbb{C}$.

Theorem (Swan (1962))

There is an isomorphism

$$K^0(X) \xrightarrow{\cong} K_0(C(X)).$$

Wall's finiteness obstruction

Definition (Finitely dominated)

A CW-complex X is called *finitely dominated* if there exists a finite (= compact) CW-complex Y together with maps $i\colon X \to Y$ and $r\colon Y \to X$ satisfying $r \circ i \simeq \mathrm{id}_X$.

- A finite CW-complex is finitely dominated.
- A closed manifold is a finite CW-complex.

Problem

Is a given finitely dominated CW-complex homotopy equivalent to a finite CW-complex?

Definition (Wall's finiteness obstruction)

A finitely dominated CW-complex X defines an element

$$o(X) \in K_0(\mathbb{Z}[\pi_1(X)])$$

called its *finiteness obstruction* as follows.

- Let $\widetilde{X}$ be the universal covering. The fundamental group $\pi = \pi_1(X)$ acts freely on $\widetilde{X}$.
- Let $C_*(\widetilde{X})$ be the cellular chain complex. It is a free $\mathbb{Z}\pi$-chain complex.
- Since X is finitely dominated, there exists a finite projective $\mathbb{Z}\pi$-chain complex P_* with $P_* \simeq_{\mathbb{Z}\pi} C_*(\widetilde{X})$.
- Define

$$o(X) := \sum_n (-1)^n \cdot [P_n] \in K_0(\mathbb{Z}\pi).$$

Theorem (Wall (1965))

A finitely dominated CW-complex X is homotopy equivalent to a finite CW-complex if and only if its reduced finiteness obstruction $\widetilde{o}(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ vanishes.

- A finitely dominated simply connected CW-complex is always homotopy equivalent to a finite CW-complex since $\widetilde{K}_0(\mathbb{Z}) = \{0\}$.
- Given a finitely presented group G and $\xi \in K_0(\mathbb{Z}G)$, there exists a finitely dominated CW-complex X with $\pi_1(X) \cong G$ and $o(X) = \xi$.

Theorem (Geometric characterization of $\widetilde{K}_0(\mathbb{Z}G) = \{0\}$)

The following statements are equivalent for a finitely presented group G:

- *Every finite dominated CW-complex with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex.*
- $\widetilde{K}_0(\mathbb{Z}G) = \{0\}$.

Conjecture (Vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ for torsionfree G)

If G is torsionfree, then

$$\widetilde{K}_0(\mathbb{Z}G) = \{0\}.$$

The Whitehead group

Definition (K_1-group $K_1(R)$)

Define the *K_1-group of a ring R*

$$K_1(R)$$

to be the abelian group whose generators are conjugacy classes $[f]$ of automorphisms $f\colon P \to P$ of finitely generated projective R-modules with the following relations:

- Given an exact sequence $0 \to (P_0, f_0) \to (P_1, f_1) \to (P_2, f_2) \to 0$ of automorphisms of finitely generated projective R-modules, we get $[f_0] + [f_2] = [f_1]$;
- $[g \circ f] = [f] + [g]$.

- This is the same as $GL(R)/[GL(R), GL(R)]$.
- An invertible matrix $A \in GL(R)$ can be reduced by elementary row and column operations and (de-)stabilization to the trivial empty matrix if and only if $[A] = 0$ holds in the *reduced K_1-group*

$$\widetilde{K}_1(R) := K_1(R)/\{\pm 1\} = \operatorname{cok}(K_1(\mathbb{Z}) \to K_1(R)).$$

- If R is commutative, the determinant induces an epimorphism

$$\det\colon K_1(R) \to R^{\times},$$

  which in general is not bijective.
- The assignment $A \mapsto [A] \in K_1(R)$ can be thought of the universal determinant for R.

Definition (Whitehead group)

The *Whitehead group* of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

Lemma

We have $\mathrm{Wh}(\{1\}) = \{0\}$.

Proof.

- The ring $\mathbb{Z}$ possesses an Euclidean algorithm.
- Hence every invertible matrix over $\mathbb{Z}$ can be reduced via elementary row and column operations and destabilization to a $(1,1)$-matrix (± 1).
- This implies that any element in $K_1(\mathbb{Z})$ is represented by ± 1.

□

Let G be a finite group. Then:

- Let F be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.
  Define $r_F(G)$ to be the number of irreducible F-representations of G.
  This is the same as the number of F-conjugacy classes of elements of G.
  Here $g_1 \sim_{\mathbb{C}} g_2$ if and only if $g_1 \sim g_2$, i.e., $gg_1g^{-1} = g_2$ for some $g \in G$. We have $g_1 \sim_{\mathbb{R}} g_2$ if and only if $g_1 \sim g_2$ or $g_1 \sim g_2^{-1}$ holds.
  We have $g_1 \sim_{\mathbb{Q}} g_2$ if and only if $\langle g_1 \rangle$ and $\langle g_1 \rangle$ are conjugated as subgroups of G.
- The Whitehead group $\mathrm{Wh}(G)$ is a finitely generated abelian group.
- Its rank is $r_{\mathbb{R}}(G) - r_{\mathbb{Q}}(G)$.
- The torsion subgroup of $\mathrm{Wh}(G)$ is the kernel of the map $K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G)$.
- In contrast to $\widetilde{K}_0(\mathbb{Z}G)$ the Whitehead group $\mathrm{Wh}(G)$ is computable.

Whitehead torsion

Definition (*h*-cobordism)

An *h-cobordism* over a closed manifold M_0 is a compact manifold W whose boundary is the disjoint union $M_0 \amalg M_1$ such that both inclusions $M_0 \to W$ and $M_1 \to W$ are homotopy equivalences.

Theorem (*s*-Cobordism Theorem, Barden, Mazur, Stallings, Kirby-Siebenmann)

Let M_0 be a closed (smooth) manifold of dimension ≥ 5. Let $(W; M_0, M_1)$ be an h-cobordism over M_0.
Then W is homeomorphic (diffeomorphic) to $M_0 \times [0,1]$ relative M_0 if and only if its Whitehead torsion

$$\tau(W, M_0) \in \mathrm{Wh}(\pi_1(M_0))$$

vanishes.

Conjecture (Poincaré Conjecture)

Let M be an n-dimensional topological manifold which is a homotopy sphere, i.e., homotopy equivalent to S^n.
Then M is homeomorphic to S^n.

Theorem

For $n \geq 5$ the Poincaré Conjecture is true.

Proof.

We sketch the proof for $n \geq 6$.

- Let M be a n-dimensional homotopy sphere.
- Let W be obtained from M by deleting the interior of two disjoint embedded disks D_1^n and D_2^n. Then W is a simply connected h-cobordism.
- Since $\mathrm{Wh}(\{1\})$ is trivial, we can find a homeomorphism $f\colon W \xrightarrow{\cong} \partial D_1^n \times [0,1]$ which is the identity on $\partial D_1^n = D_1^n \times \{0\}$.
- By the Alexander trick we can extend the homeomorphism $f|_{D_1^n \times \{1\}}\colon \partial D_2^n \xrightarrow{\cong} \partial D_1^n \times \{1\}$ to a homeomorphism $g\colon D_1^n \to D_2^n$.
- The three homeomorphisms $id_{D_1^n}$, f and g fit together to a homeomorphism $h\colon M \to D_1^n \cup_{\partial D_1^n \times \{0\}} \partial D_1^n \times [0,1] \cup_{\partial D_1^n \times \{1\}} D_1^n$. The target is obviously homeomorphic to S^n.

□

- The argument above does not imply that for a smooth manifold M we obtain a diffeomorphism $g\colon M \to S^n$.
  The Alexander trick does not work smoothly.
  Indeed, there exists so called exotic spheres, i.e., closed smooth manifolds which are homeomorphic but not diffeomorphic to S^n.
- The s-cobordism theorem is a key ingredient in the surgery program for the classification of closed manifolds due to Browder, Novikov, Sullivan and Wall.
- Given a finitely presented group G, an element $\xi \in \mathrm{Wh}(G)$ and a closed manifold M of dimension $n \geq 5$ with $G \cong \pi_1(M)$, there exists an h-cobordism W over M with $\tau(W, M) = \xi$.

Theorem (Geometric characterization of $\mathrm{Wh}(G) = \{0\}$)

The following statements are equivalent for a finitely presented group G and a fixed integer $n \geq 6$

- *Every compact n-dimensional h-cobordism W with $G \cong \pi_1(W)$ is trivial;*
- $\mathrm{Wh}(G) = \{0\}$.

Conjecture (Vanishing of $\mathrm{Wh}(G)$ for torsionfree G)

If G is torsionfree, then

$$\mathrm{Wh}(G) = \{0\}.$$

Negative K-theory

Definition (Bass-Nil-groups)

Define for $n = 0, 1$

$$NK_n(R) := \operatorname{coker}\left(K_n(R) \to K_n(R[t])\right).$$

Theorem (Bass-Heller-Swan decomposition for K_1 (1964))

There is an isomorphism, natural in R,

$$K_0(R) \oplus K_1(R) \oplus NK_1(R) \oplus NK_1(R) \xrightarrow{\cong} K_1(R[t, t^{-1}]) = K_1(R[\mathbb{Z}]).$$

Definition (Negative K-theory)

Define inductively for $n = -1, -2, \ldots$

$$K_n(R) := \operatorname{coker}\left(K_{n+1}(R[t]) \oplus K_{n+1}(R[t^{-1}]) \to K_{n+1}(R[t,t^{-1}])\right).$$

Define for $n = -1, -2, \ldots$

$$NK_n(R) := \operatorname{coker}\left(K_n(R) \to K_n(R[t])\right).$$

Theorem (Bass-Heller-Swan decomposition for negative K-theory)

For $n \leq 1$ there is an isomorphism, natural in R,

$$K_{n-1}(R) \oplus K_n(R) \oplus NK_n(R) \oplus NK_n(R) \xrightarrow{\cong} K_n(R[t,t^{-1}]) = K_n(R[\mathbb{Z}]).$$

Definition (Regular ring)

A ring R is called *regular* if it is Noetherian and every finitely generated R-module possesses a finite projective resolution.

- Principal ideal domains are regular. In particular $\mathbb{Z}$ and any field are regular.
- If R is regular, then $R[t]$ and $R[t, t^{-1}] = R[\mathbb{Z}]$ are regular.
- If R is regular, then RG in general is not Noetherian or regular.

Theorem (Bass-Heller-Swan decomposition for regular rings)

Suppose that R is regular. Then

$$\begin{aligned} K_n(R) &= 0 \quad \text{for } n \leq -1; \\ NK_n(R) &= 0 \quad \text{for } n \leq 1, \end{aligned}$$

and the Bass-Heller-Swan decomposition reduces for $n \leq 1$ to the natural isomorphism

$$K_{n-1}(R) \oplus K_n(R) \xrightarrow{\cong} K_n(R[t, t^{-1}]) = K_n(R[\mathbb{Z}]).$$

Construction of higher algebraic K-theory for rings

- There are also higher algebraic K-groups $K_n(R)$ for $n \geq 2$ due to Quillen (1973).
- Nowadays there several constructions: plus-construction, group completion, Q-construction, $S_\bullet$-construction.
- We give a quick review of the technically less demanding Q-construction.
- Most of the well known features of $K_0(R)$ and $K_1(R)$ extend to both negative and higher algebraic K-theory.
  For instance the Bass-Heller-Swan decomposition and Morita equivalence holds also for higher algebraic K-theory.

Definition (Acyclic space)

A space Z is called acyclic if it has the homology of a point, i.e., the singular homology with integer coefficients $H_n(Z)$ vanishes for $n \geq 1$ and is isomorphic to $\mathbb{Z}$ for $n = 0$.

- An acyclic space is path connected.
- The fundamental group π of an acyclic space is perfect, i.e., $\pi = [\pi, \pi]$, and satisfies $H_2(\pi; \mathbb{Z}) = 0$.

- In the sequel we will suppress choices of and questions about base points.
- The homotopy fiber $\operatorname{hofib}(f)$ of a map $f\colon X \to Y$ of path connected spaces has the property that it is the fiber of a fibration $p_f\colon E_f \to Y$ which comes with a homotopy equivalence $h\colon E_f \to X$ satisfying $p_f = f \circ h$.
- The long exact homotopy sequence associated to a map $f\colon X \to Y$ looks like

$$\cdots \xrightarrow{\partial_3} \pi_2(\operatorname{hofib}(f)) \xrightarrow{i_2} \pi_2(X) \xrightarrow{f_2} \pi_2(Y) \xrightarrow{\partial_2} \pi_1(\operatorname{hofib}(f)) \xrightarrow{i_1} \pi_1(X) \xrightarrow{f_1} \pi_1(Y) \xrightarrow{\partial_1} \pi_0(\operatorname{hofib}(f)) \xrightarrow{i_0} \pi_0(X) \xrightarrow{f_0} \pi_0(Y) \to \{0\}.$$

Definition (Acyclic map)

Let X and Y be path connected CW-complexes. A map $f\colon X \to Y$ is called acyclic if its homotopy fiber $\operatorname{hofib}(f)$ is acyclic.

- We conclude from the long exact homotopy sequence that $f_1\colon \pi_1(X) \to \pi_1(Y)$ is surjective and its kernel is a normal perfect subgroup P of $\pi_1(X)$ provided that f is acyclic.
- Namely, P is a quotient of the perfect group $\pi_2(\operatorname{hofib}(f))$ and $\pi_0(\operatorname{hofib})$ consists of one element.
- Obviously a space Z is acyclic if and only if the map $Z \to \mathrm{pt}$ is acyclic.

Definition (Plus-construction)

Let X be a connected CW-complex and $P \subseteq \pi_1(X)$ be a normal perfect subgroup. A map $f\colon X \to Y$ to a CW-complex is called a plus-construction of X relative to P if f is acyclic and the kernel of $f_1\colon \pi_1(X) \to \pi_1(Y)$ is P.

Theorem (Properties of the plus-construction)

Let Z be a connected CW-complex and let $P \subseteq \pi_1(X)$ be a normal perfect subgroup. Then:

- *There exists a plus-construction $f\colon X \to Y$ relative P;*
- *Let $f\colon X \to Y$ be a plus-construction relative P and let $g\colon X \to Z$ be a map such that the kernel of $g_1\colon \pi_1(X) \to \pi_1(Z)$ contains P. Then there is a map $\overline{g}\colon Y \to Z$ which is up to homotopy uniquely determined by the property that $\overline{g} \circ f$ is homotopic to g;*
- *If $f_1\colon X \to Y_1$ and $f_2\colon X \to Y_2$ are two plus-constructions for X relative P, then there exists a homotopy equivalence $g\colon Y_1 \to Y_2$ which is up to homotopy uniquely determined by the property $g \circ f_1 \simeq f_2$;*

Theorem (continued)

- *The map* $f_1 \colon \pi_1(X) \to \pi_1(X^+)$ *can be identified with the canonical projection* $\pi_1(X) \to \pi_1(X)/P$*;*
- *The map* $H_n(f;M) \colon H_n(X; f^*M) \to H_n(X^+;M)$ *is bijective for every* $n \geq 0$ *and every local coefficient systems* M *on* X^+*.*

- Every group G has a unique largest perfect subgroup $P \subseteq G$, called the perfect radical
- In the sequel we will always use the prefect radical of G for P unless explicitly stated differently.

Definition (Higher algebraic K-groups of a ring)

Let $BGL(R) \to BGL(R)^+$ be a plus-construction for the classifying space $BGL(R)$ of $GL(R)$ (with respect to the perfect radical of $GL(R)$ which is $E(R)$).

Define the K-theory space associated to R

$$K(R) := K_0(R) \times BGL(R)^+,$$

where we view $K_0(R)$ with the discrete topology.

Define the n-th algebraic K-group

$$K_n(R) := \pi_n(K(R)) \quad \text{for } n \geq 0.$$

- This definition makes sense because of the Theorem above.
- Notice that for $n \geq 1$ we have $K_n(R) = \pi_n(BGL(R)^+)$.
- For $n = 0, 1$ the last definition coincides with the ones given earlier in terms of generator and relations.
- A ring homomorphism $f\colon R \to S$ induces maps $GL(R) \to GL(S)$ and hence maps $BGL(R) \to BGL(S)$ and $BGL(R)^+ \to BGL(S)^+$. We have a map $K_0(R) \to K_0(S)$. Therefore f induces a maps

$$\begin{aligned} K(f)\colon K(R) &\to K(S); \\ K_n(f)\colon K_n(R) &\to K_n(S); \end{aligned}$$

Definition (Relative K-groups)

Define for a two-sided ideal $I \subseteq R$ and $n \geq 0$

$$K_n(R, I) := \pi_n\bigl(\mathrm{hofib}(K(\mathrm{pr})) \colon K(R) \to K(R/I)\bigr).$$

for $\mathrm{pr} \colon R \to R/I$ the projection.

Theorem (Long exact sequence of an ideal for algebraic K-theory)

Let $I \subseteq R$ be a two sided ideal. Then there is a long exact sequence, infinite to both sides

$$\cdots \xrightarrow{\partial_3} K_2(R, I) \xrightarrow{j_2} K_2(R) \xrightarrow{\mathrm{pr}_2} K_2(R/I) \xrightarrow{\partial_2} K_1(R, I) \xrightarrow{j_1} K_1(R)$$
$$\xrightarrow{\mathrm{pr}_1} K_1(R/I) \xrightarrow{\partial_1} K_0(R, I) \xrightarrow{j_1} K_0(R) \xrightarrow{\mathrm{pr}_0} K_0(R/I)$$
$$\xrightarrow{\partial_0} K_{-1}(R, I) \xrightarrow{j_1} K_{-1}(R) \xrightarrow{\mathrm{pr}_0} K_{-1}(R/I) \xrightarrow{\partial_{-1}} \cdots .$$

Definition (Spectrum)

- A spectrum

$$\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$$

  is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called structure maps

$$\sigma(n) \colon E(n) \wedge S^1 \longrightarrow E(n+1).$$

- A map of spectra

$$\mathbf{f} \colon \mathbf{E} \to \mathbf{E}'$$

  is a sequence of maps $f(n) \colon E(n) \to E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e., $f(n+1) \circ \sigma(n) \;=\; \sigma'(n) \circ (f(n) \wedge \mathrm{id}_{S^1})$ holds for all $n \in \mathbb{Z}$.
- The category of spectra is denoted by $\mathcal{S}\mathrm{pectra}$.

Definition (Homotopy groups of a spectrum)

The i-th homotopy group of a spectrum $\mathbf{E}$ is defined by

$$\pi_i(\mathbf{E}) \quad := \quad \operatorname*{colim}_{k \to \infty} \pi_{i+k}(E(k)), \qquad (0.1)$$

where the system $\pi_{i+k}(E(k))$ is given by the composite

$$\pi_{i+k}(E(k)) \xrightarrow{S} \pi_{i+k+1}(E(k) \wedge S^1) \xrightarrow{\sigma(k)_*} \pi_{i+k+1}(E(k+1))$$

of the suspension homomorphism S and the homomorphism induced by the structure map.

- The homotopy groups of a spectrum can be non-trivial also in negative degrees.

Definition (Weak equivalence)

A weak equivalence of spectra is a map $\mathbf{f} \colon \mathbf{E} \to \mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups.

- Given a spectrum $\mathbf{E}$, a classical construction in algebraic topology assigns to it a homology theory $H_*(-, \mathbf{E})$ with the property

$$H_n(\mathrm{pt}; \mathbf{E}) = \pi_n(\mathbf{E}).$$

  Put

$$H_n(X; \mathbf{E}) := \pi_n\left(X_+ \wedge \mathbf{E}\right).$$

- One also gets a cohomology theory $H^*(-, \mathbf{E})$ with the property

$$H^n(\mathrm{pt}; \mathbf{E}) = \pi_{-n}(\mathbf{E}).$$

- The basic example of a spectrum is the sphere spectrum $\mathbf{S}$. Its n-th space is S^n and its n-th structure map is the standard homeomorphism $S^n \wedge S^1 \xrightarrow{\cong} S^{n+1}$.
  Its associated homology theory is stable homotopy $\pi_*^s(-) = H_*(-; \mathbf{S})$.

Definition (Non-connective algebraic K-theory spectrum)

One can assign to ring R a spectrum $\mathbf{K}(R)$, the so called non-connective algebraic K-theory spectrum such that we get for all $n \in \mathbb{Z}$

$$\pi_n(\mathbf{K}(R)) \cong K_n(R).$$

Main properties of higher algebraic K-theory for rings

Theorem (Algebraic K-theory and finite products)

Let R_0 and R_1 be rings. Denote by $\mathrm{pr}_i \colon R_0 \times R_1 \to R_i$ for $i = 0, 1$ the projection. Then we obtain for $n \in \mathbb{Z}$ isomorphisms

$$(\mathrm{pr}_0)_n \times (\mathrm{pr}_1)_n \colon K_n(R_0 \times R_1) \xrightarrow{\cong} K_n(R_0) \times K_n(R_1)$$

Theorem (Morita equivalence for algebraic K-theory)

For every ring R and integer $k \geq 1$ there are for all $n \in \mathbb{Z}$ natural isomorphisms

$$\mu_n \colon K_n(R) \xrightarrow{\cong} K_n(M_k(R)).$$

Theorem (Algebraic K-theory and directed colimits)

Let $\{R_i \mid i \in I\}$ be a directed system of rings. Then the canonical map

$$\operatorname*{colim}_{i \in I} K_n(R_i) \xrightarrow{\cong} K_n\big(\operatorname*{colim}_{i \in I} R_i\big)$$

is bijective for $n \in \mathbb{Z}$.

Theorem (Bass-Heller-Swan decomposition for algebraic K-theory)

- *The following maps are isomorphisms of abelian groups for $n \in \mathbb{Z}$*

$$NK_n(R) \oplus K_n(R) \xrightarrow{\cong} K_n(R[t]);$$
$$K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R) \xrightarrow{\cong} K_n(R[t,t^{-1}]).$$

- *The following sequence is natural in R and split exact (with in R natural splitting) for $n \in \mathbb{Z}$*

$$0 \to K_n(R) \xrightarrow{(k_+)_* \oplus -(k_-)_*} K_n(R[t]) \oplus K_n(R[t^{-1}])$$
$$\xrightarrow{(\tau_+)_* \oplus (\tau_-)_*} K_n(R[t,t^{-1}]) \xrightarrow{C} K_{n-1}(R) \to 0.$$

- *If R is regular, then*

$$\begin{aligned} NK_n(R) &= \{0\} \quad \text{for } n \in \mathbb{Z}; \\ K_n(R) &= \{0\} \quad \text{for } n \leq -1. \end{aligned}$$

Theorem (Algebraic K-theory of finite fields Quillen(1973))

Let $\mathbb{F}_q$ be a finite field of order q. Then $K_n(\mathbb{F}_q)$ vanishes if $n = 2k$ for some integer $k \geq 1$, and is a finite cyclic group of order $q^k - 1$ if $n = 2k - 1$ for some integer $k \geq 1$.

Theorem (Rational Algebraic K-theory of ring of integers of number fields Borel(1972))

Let R be a ring of integers in an algebraic number field. Let r_1 be the number of distinct embeddings of F into $\mathbb{R}$ and let r_2 be the number of distinct conjugate pairs of embeddings of F into $\mathbb{C}$ with image not contained in $\mathbb{R}$. Then:

$$K_n(R) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \begin{cases} \{0\} & n \text{ even} \quad \text{or} \quad n \leq -1; \\ \mathbb{Q} & n = 0; \\ \mathbb{Q}^{r_1+r_2-1} & n = 1; \\ \mathbb{Q}^{r_1+r_2} & n \geq 2 \quad \text{and} \quad n = 1 \mod 4; \\ \mathbb{Q}^{r_2} & n \geq 2 \quad \text{and} \quad n = 3 \mod 4. \end{cases}$$

Theorem (Localization sequence in K-theory for Dedekind domains Quillen(1973))

Let R be a Dedekind domain with quotient field F. Then there is an exact sequence

$$\cdots \to K_{n+1}(F) \to \bigoplus_P K_n(R/P) \to K_n(R) \to K_n(F) \to \bigoplus_P K_{n-1}(R/P)$$
$$\to \cdots \to \bigoplus_P K_0(R/P) \to K_0(R) \to K_0(F) \to 0,$$

where P runs through the maximal ideals of R.

- We have $K_n(\mathbb{Z}) = \{0\}$ for $n \leq -1$ and the first values of $K_n(\mathbb{Z})$ for $n = 0, 1, 2, 3, 4, 5, 6, 7$ are given by $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, $\mathbb{Z}/48$, $\{0\}$, $\mathbb{Z}$, $\{0\}$, $\mathbb{Z}/240$.

Algebraic K-Theory with Coefficients

- By invoking the Moore space associated to $\mathbb{Z}/k$, one can introduce K-theory $K_n(R;\mathbb{Z}/k)$ for $n \in \mathbb{Z}$ with coefficients in $\mathbb{Z}/k$ for any integer $k \geq 2$.
- They fit into long exact sequences

$$\cdots \to K_{n+1}(R;\mathbb{Z}/k) \to K_n(R) \xrightarrow{k\cdot \mathrm{id}} K_n(R) \to K_n(R;\mathbb{Z}/k) \\ \to K_{n-1}(R) \xrightarrow{k\cdot \mathrm{id}} K_{n-1}(R) \to K_{n-1}(R;\mathbb{Z}/k) \to \cdots$$

Theorem (Algebraic K-theory mod k of algebraically closed fields Suslin(1983))

The inclusion of algebraically closed fields induces isomorphisms on $K_*(-;\mathbb{Z}/k)$.

- Let p be a prime number. Quillen(1973a) has computed the algebraic K-groups for any algebraic extension of the field $\mathbb{F}_p$ of p-elements for every prime p.
- One can determine $K_n(\overline{\mathbb{F}_p};\mathbb{Z}/k)$ for the algebraic closure of $\mathbb{F}_p$ from the long exact sequence above.
- Hence one obtains $K_n(F;\mathbb{Z}/k)$ for any algebraically closed field of prime characteristic p by Suslin's Theorem.

Theorem (Algebraic and topological K-theory mod k for $\mathbb{R}$ and $\mathbb{C}$ Suslin(1983))

The comparison map from algebraic to topological K-theory induces for all integers $k \geq 2$ and all $n \geq 0$ isomorphisms

$$\begin{array}{ccc} K_n(\mathbb{R};\mathbb{Z}/k) & \xrightarrow{\cong} & K_n^{\mathrm{top}}(\mathbb{R};\mathbb{Z}/k); \\ K_n(\mathbb{C};\mathbb{Z}/k) & \xrightarrow{\cong} & K_n^{\mathrm{top}}(\mathbb{C};\mathbb{Z}/k). \end{array}$$

- For every algebraically closed field F of characteristic 0 we have an injection $\overline{Q} \to F$ for the algebraically closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$,
- Hence the theorems above imply for every algebraically closed field F of characteristic zero:

$$K_n(F;\mathbb{Z}/k) \cong \begin{cases} \mathbb{Z}/k & n \geq 0, n \text{ even}; \\ \{0\} & n \geq 1, n \text{ odd}. \\ \{0\} & n \leq -1. \end{cases}$$

Algebraic K-Theory of spaces

- So far we have only considered algebraic K-theory of algebraic objects, e.g., of rings, modules or exact categories.
- Next we want to describe the most general version of algebraic K-theory which applies to spaces and is due to Waldhausen.
- This will allow to get information about spaces of topological or smooth automorphisms of topological or smooth manifolds.
- Other relevant theories are spaces of pseudoisotopies and spaces of h-cobordism over a manifold.
- We begin with the relevant generalization of an exact category.
- A category $\mathcal{C}$ is called pointed if it comes with a distinguished zero-object i.e., an object with is both initial and terminal.

Definition (Category with cofibrations and weak equivalences)

A category with cofibrations and weak equivalences is a small pointed category $\mathcal{C}$ with a subcategory $co\mathcal{C}$, called category of cofibrations in $\mathcal{C}$ and a subcategory $w\mathcal{C}$, called category of weak equivalences in $\mathcal{C}$ such that the following axioms are satisfied:

- The isomorphisms in $\mathcal{C}$ are cofibrations, i.e., belong to $co\mathcal{C}$;
- For every object C the map $* \to C$ is a cofibration, where $*$ is the distinguished zero-object;
- If in the diagram $A \xleftarrow{i} B \xrightarrow{f} C$ the left arrow is a cofibration, the pushout

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow f & & \downarrow \overline{f} \\ C & \xrightarrow{\overline{i}} & D \end{array}$$

exists and $\overline{i}$ is a cofibration;

Definition (Continued)

- The isomorphisms in $\mathcal{C}$ are contained in $w\mathcal{C}$;
- If in the commutative diagram

$$\begin{array}{ccccc} B & \leftarrowtail & A & \longrightarrow & C \\ \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\ B' & \leftarrowtail & A' & \longrightarrow & C' \end{array}$$

  the horizontals arrow on the left are cofibrations, and all vertical arrows are weak equivalences, then the induced map on the pushout of the upper row to the pushout of the lower row is a weak isomorphism.

Example (Topological spaces)

- Let $\mathcal{S}paces$ be the category of pointed topological spaces.
- Ignoring the condition small, we obtain a category with cofibrations and weak equivalences is follows.
- The one-point space is the zero object.
- We declare cofibration of topological spaces to be the cofibrations.
- We declare the weak equivalences to consists of one of the following classes:
  - homeomorphisms
  - homotopy equivalences
  - weak homotopy equivalences
  - homology equivalences with respect to a given homology theory

Example (Exact categories are categories with cofibrations and weak equivalences)

- Let $\mathcal{P} \subseteq \mathcal{A}$ be an exact category.
- It becomes a category with cofibrations and weak equivalences as follows.
- The zero-object is just a zero-object in the abelian category $\mathcal{A}$.
- An admissible monomorphism in $\mathcal{P}$ is a morphism $i \colon A \to B$ which occurs in an exact sequence $0 \to A \to B \to C \to 0$ of $\mathcal{P}$. They form the cofibrations.
- The weak equivalences are given by the isomorphisms.

Example (The category $\mathcal{R}(X)$ of retractive spaces)

- Let X be a space.
- A retractive space over X is a triple (Y, r, s) consisting of a space Y and maps $s\colon X \to Y$ and $r\colon Y \to X$ satisfying $r \circ s = \mathrm{id}_X$.
- A morphism from (Y, r, s) to (Y', r', s') is a map $f\colon X \to X'$ satisfying $r' \circ f = r$ and $f \circ s = s'$.
- The zero-object is $(X, \mathrm{id}_X, \mathrm{id}_X)$.
- A morphism $f\colon (Y, r, s) \to (Y', r', s')$ is declared to be a cofibration if the underlying map of spaces $f\colon Y \to Y'$ is a cofibration.
- Now there are several possibilities to define weak equivalences. One may require that $f\colon Y \to Y'$ is a homeomorphism, a homotopy equivalence, weak homotopy equivalence or a homology equivalence with respect to some fixed homology theory.
- Then one obtains a category $\mathcal{R}(X)$ with cofibrations and weak equivalences except that $\mathcal{R}(X)$ is not small.

Example (The category $\mathcal{R}^f(X)$ of relatively finite retractive CW-complexes)

- The following subcategory $\mathcal{R}^f(X)$ of $\mathcal{R}(X)$ will be relevant for us and is indeed a (small) category with cofibrations and weak equivalences.
- We require that (Y, X) is a relative CW-complex which is relatively finite, and $s\colon X \to Y$ is the inclusion and morphisms to be cellular maps.
- We choose all weak homotopy equivalences as weak equivalences and inclusion of relative CW-complexes as cofibrations.

- We obtain a covariant functor from $\mathcal{S}paces$ to the category $\mathcal{C}atcofwe$ of categories with cofibrations and weak equivalences.

- Next we explain how to associate to a category with cofibrations and weak equivalences an infinite loop space.
- This is a generalization of the Q-construction.
- For an integer $n \geq 0$ let $[n]$ be the ordered set $\{0, 1, 2, \ldots, n\}$.
- Let Δ be the category whose set of objects is $\{[n] \mid n = 0, 1, 2 \ldots\}$ and whose set of morphisms from $[m]$ to $[n]$ consists of the order preserving maps.
- A simplicial category is a contravariant functor from Δ to the category $\mathcal{C}at$ of categories.
- Analogously, a simplicial category with cofibrations and weak equivalences is a contravariant functor from Δ to $\mathcal{C}atcof we$.

- Let $\mathcal{C}$ be a category with cofibrations and weak equivalences.
- We want to assign to it a a simplicial category with weak cofibrations and weak equivalences $S_\bullet\mathcal{C}$ as follows.
- Define $S_n\mathcal{C}$ to be the category for which an object is a sequence of cofibrations $A_{0,1} \xrightarrow{k_{0,1}} A_{0,2} \xrightarrow{k_{0,2}} \cdots \xrightarrow{k_{0,n-1}} A_{0,n}$ together with explicit choices of quotient objects $\mathrm{pr}_{i,j} \colon A_{0,j} \to A_{i,j} = A_{0,j}/A_{i,0}$ for $i, j \in \{1, 2, \ldots, n\}, i < j$, i.e., we fix pushouts

$$\begin{array}{ccc} A_{i,0} & \xrightarrow{k_{0,j-1}\circ\cdots\circ k_{0,i}} & A_{j,0} \\ \downarrow & & \downarrow \mathrm{pr}_{i,j} \\ 0 & \longrightarrow & A_{i,j} \end{array}$$

- Morphisms are given by collection of morphisms $\{f_{i,j}\}$ which make the obvious diagram commute.
- These explicit choices of quotient objects are needed to define the relevant face and degeneracy maps.
- For instance the face map $d_i \colon S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ is given for $i \geq 1$ by dropping $A_{0,i}$ and for $i = 0$ by passing to $A_{0,2}/A_{0,1} \to A_{0,3}/A_{0,1} \to \cdots \to A_{0,n}/A_{0,1}$.
- An arrow in $S_n\mathcal{C}$ is declared to be a cofibration if each arrow $A_{i,j} \to A_{i',j'}$ is a cofibration and analogously for weak equivalences.

- We obtain a simplicial category $wS_\bullet\mathcal{C}$ by considering the category of weak equivalences of $S_\bullet\mathcal{C}$.
- Let $|wS_\bullet\mathcal{C}|$ be the geometric realization of the simplicial category $wS_\bullet\mathcal{C}$ which is the geometric realization of the bisimplicial set obtained by the composite of the functor nerve of a category with $wS_\bullet\mathcal{C}$.

Definition (Algebraic K-theory space of a category with cofibrations and weak equivalences)

Let $\mathcal{C}$ be a category with cofibrations and weak equivalences. Its algebraic K-theory space is defined by

$$K(\mathcal{C}) := \Omega|wS_\bullet\mathcal{C}|.$$

- There is a canonical map $|w\mathcal{C}| \to \Omega|wS_\bullet\mathcal{C}|$ which is the adjoint of the obvious identification of the 1-skeleton in the $S_\bullet$ direction of $|wS_\bullet\mathcal{C}|$ with the reduced suspension $|w\mathcal{C}| \wedge S^1$.
- Since one can iterate this construction, one obtains a sequence of maps

$$|w\mathcal{C}| \to \Omega|wS_\bullet\mathcal{C}| \to \Omega\Omega|wS_\bullet S_\bullet\mathcal{C}| \to \cdots$$

  It turns out that all these maps except the first one are weak homotopy equivalences.
- So $K(\mathcal{C})$ is an infinite loop space.

Definition ((Connective) A-theory)

Let X be a topological space. Let $\mathcal{R}^f(X)$ be the category with cofibrations and weak equivalences defined above. Define the A-theory space $A(X)$ associated to X to be the algebraic K-theory space $K(\mathcal{R}^f(X))$ defined above.

- Waldhausen's A-construction encompasses the Q-construction of Quillen.
- There are many other instances where linear constructions were generalized to constructions for spaces and thus yield significant improvements, e.g., topological Hochschild homology and topological cyclic homology.

- As in the case of algebraic K-theory of rings it will be crucial for us to consider a non-connective version.
- Vogell has defined a delooping of $A(X)$ yielding a non-connective Ω-spectrum $\mathbf{A}(X)$ for a topological space X.
- This construction actually yields a covariant functor

$$\mathbf{A} \colon \mathcal{S}paces \quad \to \quad \Omega\text{-}\mathcal{S}pectra$$

Definition (Non-connective A-theory)

We call $\mathbf{A}(X)$ the (non-connective) A-theory spectrum associated to the topological space X. We write for $n \in \mathbb{Z}$

$$A_n(X) := \pi_n(X)$$

- Let X be a connected space with fundamental group $\pi = \pi_1(X)$ which admits a universal covering $p_X \colon \widetilde{X} \to X$.
- Consider an object in $\mathcal{R}^f(X)$. Recall that it is given by a relatively finite relative CW-complex (Y, X) together with a map $r \colon X \to Y$ satisfying $r|_X = \mathrm{id}_X$.
- Let $\widetilde{Y} \to Y$ be the π-covering obtained from $p_X \colon \widetilde{X} \to X$ by the pullback construction applied to $r \colon X \to Y$.
- The cellular $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{X}, \widetilde{Y})$ of the relative free π-CW-complex $(\widetilde{X}, \widetilde{Y})$ is a finite free $\mathbb{Z}\pi$-chain complex.
- This gives essentially a functor of categories with cofibrations and weak equivalences from $\mathcal{R}^f(X)$ to the category of finite free $\mathbb{Z}\pi$-chain complexes.
- The algebraic K-theory of the category of finite free $\mathbb{Z}\pi$-chain complex agrees with the one of the finitely generated free $\mathbb{Z}\pi$-modules.
- Hence we get a natural map of spectra called linearization map

$$\mathbf{L} \colon \mathbf{A}(X) \quad \to \quad \mathbf{K}(\mathbb{Z}\pi)$$

Theorem (Connectivity of the linearization map)

Let X be an aspherical CW-complex. Then:

- *The linearization map $\mathbf{L}$ is 2-connected, i.e., the map*

$$L_n := \pi_n(\mathbf{L})\colon A_n(X) \to K_n(\mathbb{Z}\pi_1(X))$$

  is bijective for $n \leq 1$ and surjective for $n = 2$;
- *Rationally the map L_n is bijective for all $n \in \mathbb{Z}$.*

- This implies that the map of spectra $A(X) \to \mathbf{A}(X)$ is a weak homotopy equivalence if $X = \mathrm{pt}$, but not in general.

Pseudoisotopy and Whitehead spaces

- A topological pseudoisotopy of a compact manifold M is a homeomorphism $h\colon M \times I \to M \times I$, which restricted to $M \times \{0\} \cup \partial M \times I$ is the obvious inclusion.
- This is a weaker notion than the one of isotopy.
- The space $P(M)$ of pseudoisotopies is the group of all such homeomorphisms, where the group structure comes from composition. If we allow M to be non-compact, we will demand that h has compact support, i.e., there is a compact subset $C \subseteq M$ such that $h(x,t) = (x,t)$ for all $x \in M - C$ and $t \in [0,1]$.
- There is a stabilization map $P(M) \to P(M \times I)$ given by crossing a pseudoisotopy with the identity on the interval I.

- The stable pseudoisotopy space is defined as

$$\mathcal{P}(M) = \operatorname*{colim}_j P(M \times I^j).$$

- There exists also a smooth versions $P^{\mathrm{Diff}}(M)$ and $\mathcal{P}^{\mathrm{Diff}}(M)$.
- The natural inclusions $P(M) \to \mathcal{P}(M)$ and $P^{\mathrm{Diff}}(M) \to \mathcal{P}^{\mathrm{Diff}}(M)$ induces an isomorphism on the i-th homotopy group if the dimension n of M is large compared to i, roughly for $i \leq n/3$

- Next we want to define a delooping of $P(M)$.
- Let $p\colon M \times \mathbb{R}^k \times I \to \mathbb{R}^k$ denote the natural projection.
- For a manifold M the space $P_b(M;\mathbb{R}^k)$ of bounded pseudoisotopies is the space of all self-homeomorphisms $h\colon M \times \mathbb{R}^k \times I \to M \times \mathbb{R}^k \times I$ satisfying:
  - The restriction of h to $M \times \mathbb{R}^k \times \{0\} \cup \partial M \times \mathbb{R} \times [0,1]$ is the inclusion.
  - the map h is bounded in the $\mathbb{R}^i$-direction, i.e., the set $\{p \circ h(y) - p(y) \mid y \in M \times \mathbb{R}^k \times I\}$ is a bounded subset of $\mathbb{R}^k$.
  - the map h has compact support in the M-direction.
- There is an obvious stabilization map $P_b(M;\mathbb{R}^k) \to P_b(M \times I;\mathbb{R}^k)$.
- The stable bounded pseudoisotopy space is defined by

$$\mathcal{P}_b(M;\mathbb{R}^k) = \operatorname*{colim}_j P_b(M \times I^j;\mathbb{R}^k).$$

- There is a homotopy equivalence $\mathcal{P}_b(M;\mathbb{R}^k) \to \Omega\mathcal{P}_b(M;\mathbb{R}^{k+1})$.
- Hence the sequences of spaces $\mathcal{P}_b(M;\mathbb{R}^k)$ for $k = 0, 1, 2, \ldots$ and $\Omega^{-i}\mathcal{P}_b(M)$ for $i = 0, -1, -2, \ldots$ define an Ω-spectrum $\mathbf{P}(M)$.
- Analogously one defines the differentiable versions $\mathcal{P}_b^{\mathrm{Diff}}(M;\mathbb{R}^k)$ and $\mathbf{P}^{\mathrm{Diff}}(M)$.

Definition ((Non-connective) pseudo-isotopy spectrum)

We call the Ω-spectra $\mathbf{P}(X)$ and $\mathbf{P}^{\mathrm{Diff}}(X)$ associated to a topological space X the (non-connective) pseudoisotopy spectrum and the smooth (non-connective) pseudoisotopy spectrum of X.

- There is a simplicial construction which allows to extend these definitions for manifolds to all topological spaces.

- Waldhausen defines the functor $\mathrm{Wh}^{\mathrm{PL}}$ from spaces to infinite loop spaces which can be viewed as connective Ω-spectra, and establishes a fibration sequence

$$X_+ \wedge A(\mathrm{pt}) \to A(X) \to \mathrm{Wh}^{\mathrm{PL}}(X).$$

- After taking homotopy groups, it can be compared with the algebraic K-theory assembly map via the commutative diagram

$$\begin{array}{ccc} \pi_n(X_+ \wedge A(\mathrm{pt})) & \longrightarrow & \pi_n(A(X)) \\ \downarrow \cong & & \downarrow \\ \pi_n(X_+ \wedge \mathbf{A}(\mathrm{pt})) = H_n(X; \mathbf{A}(\mathrm{pt})) & \longrightarrow & \pi_n(\mathbf{A}(X)) \\ \downarrow & & \downarrow \\ H_n(B\pi_1(X); \mathbf{K}(\mathbb{Z})) & \longrightarrow & K_n(\mathbb{Z}\pi_1(X)). \end{array}$$

- The left upper vertical arrow is bijective for $n \in \mathbb{Z}$.
- The right upper vertical arrow is bijective for $n \geq 1$.
- The vertical arrows from the second row to the third row come from the linearization map.
- The left lower vertical arrow is bijective for $n \leq 1$ and rationally bijective for $n \in \mathbb{Z}$.
- In the case where X is aspherical, the lower right vertical map is bijective for $n \leq 1$ and rationally bijective for all $n \in \mathbb{Z}$.

Theorem (Relating the Whitehead space and pseudisotopy)

$$\Omega^2 \operatorname{Wh}^{\mathrm{PL}}(X) \simeq \mathcal{P}(X).$$

Corollary

Suppose that M is a closed aspherical manifold. Suppose that the K-theoretic Farrell-Jones Conjecture holds for $R = \mathbb{Z}$ and $G = \pi_1(M)$, i.e., the lowest horizontal arrow in the diagram above is bijective. Then we get for all $n \geq 0$

$$\begin{aligned} \pi_n(\operatorname{Wh}^{\mathrm{PL}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} &= 0; \\ \pi_n(\mathcal{P}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} &= 0. \end{aligned}$$

- There is also a smooth version of the Whitehead space $\mathrm{Wh}^{\mathrm{Diff}}(X)$.
- We have $\Omega^2 \mathrm{Wh}^{\mathrm{Diff}}(M) \simeq \mathcal{P}^{\mathrm{Diff}}(M)$.
- A result of Waldhausen says that there is a natural splitting of connective spectra

$$A(X) \quad \simeq \quad \Sigma^{\infty}(X_{+}) \vee \mathrm{Wh}^{\mathrm{Diff}}(X).$$

Corollary

Let M be a closed aspherical manifold. Suppose that the K-theoretic Farrell-Jones Conjecture holds for $R = \mathbb{Z}$ and $G = \pi_1(M)$. Then we get for all $n \geq 0$

$$\begin{aligned} \pi_n(\mathrm{Wh}^{\mathrm{Diff}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong \bigoplus_{k=1}^{\infty} H_{n-4k-1}(M;\mathbb{Q}); \\ \pi_n(\mathcal{P}^{\mathrm{Diff}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong \bigoplus_{k=1}^{\infty} H_{n-4k+1}(M;\mathbb{Q}). \end{aligned}$$

- If one additionally also assumes the Farrell-Jones Conjectures for L-theory, one gets

Theorem (Homotopy Groups of $\operatorname{Top}(M)$)

Let M be an orientable closed aspherical manifold of dimension > 10 with fundamental group G. Suppose the L-theory assembly map

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \to L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$$

is an isomorphism for all n and suppose the K-theory assembly map

$$H_n(BG; \mathbf{K}(\mathbb{Z})) \to K_n(\mathbb{Z}G)$$

is an isomorphism for $n \leq 1$ and a rational isomorphism for $n \geq 2$. Then for $1 \leq i \leq (\dim M - 7)/3$ one has

$$\pi_i(\operatorname{Top}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \operatorname{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1, \\ 0 & \text{if } i > 1 \end{cases}$$

- In the differentiable case one additionally needs to study involutions on the higher K-theory groups. The corresponding result reads:

Theorem (Homotopy Groups of Diff(M))

Let M be an orientable closed aspherical differentiable manifold of dimension > 10 with fundamental group G. Suppose that the same assumptions as in the last theorem hold.
Then we have for $1 \le i \le (\dim M - 7)/3$

$$\pi_i(\mathrm{Diff}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathrm{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ \bigoplus_{j=1}^{\infty} H_{(i+1)-4j}(M;\mathbb{Q}) & \text{if } i > 1 \text{ and } \dim M \text{ odd}; \\ 0 & \text{if } i > 1 \text{ and } \dim M \text{ even}. \end{cases}$$

- There is also a space of parametrized h-cobordisms $H(M)$ for a compact topological manifold M.
- Roughly speaking, the space is designed such that a map $N \to H(M)$ is the same as a bundle over N whose fibers are h-cobordisms over M.
- The set of path components $\pi_0(H(M))$ agrees with the isomorphism classes of h-cobordisms over M.
- In particular the s-Cobordism Theorem is equivalent to the statement that for $\dim(M) \geq 5$ we obtain a bijection $\pi_0(H(M)) \xrightarrow{\cong} \operatorname{Wh}(\pi_1(M))$ coming from taking the Whitehead torsion, or, equivalently that we obtain a bijection

$$\pi_0(H(M)) \xrightarrow{\cong} \pi_0(\Omega \operatorname{Wh}(M)).$$

- There is also a stable version, the space of stable parametrized h-cobordisms

$$\mathcal{K}(M) := \operatorname{colim}_j H(M \times I^j).$$

Theorem (The stable parametrized *h*-cobordism Theorem)

If M is a compact topological manifold, then there is a homotopy equivalence

$$\mathcal{K}(M) \xrightarrow{\simeq} \Omega \operatorname{Wh}(M).$$