Groups, Geometry and Actions: Isomorphism Conjectures

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Reminder about Classifying spaces for families

Definition (Family of subgroups)

A *family $\mathcal{F}$ of subgroups* of G is a set of subgroups of G which is closed under conjugation and taking subgroups.

Examples for $\mathcal{F}$ are:

$$\begin{aligned}
\mathcal{TR} &= \{\text{trivial subgroup}\}; \\
\mathcal{F}\text{in} &= \{\text{finite subgroups}\}; \\
\mathcal{VC}\text{yc} &= \{\text{virtually cyclic subgroups}\}; \\
\mathcal{ALL} &= \{\text{all subgroups}\}.
\end{aligned}$$

Definition (Classifying G-CW-complex for a family of subgroups, tom Dieck(1974))

Let $\mathcal{F}$ be a family of subgroups of G. A model for the *classifying G-CW-complex for the family $\mathcal{F}$* is a G-CW-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to X$.

We abbreviate $\underline{E}G := E_{\mathcal{F}\mathrm{in}}(G)$ and call it the *universal G-CW-complex for proper G-actions*.

We abbreviate $\underline{\underline{E}}G := E_{\mathcal{VC}\mathrm{yc}}(G)$.

We also write $EG = E_{\mathcal{TR}}(G)$.

Theorem (Homotopy characterization of $E_{\mathcal{F}}(G)$)

Let $\mathcal{F}$ be a family of subgroups.

- There exists a model for $E_{\mathcal{F}}(G)$ for any family $\mathcal{F}$;
- Two models for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent;
- A G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy groups belong to $\mathcal{F}$ and for each $H \in \mathcal{F}$ the H-fixed point set X^H is contractible.

Equivariant homology theories

Definition (*G*-homology theory)

A *G-homology theory* $\mathcal{H}_*$ is a covariant functor from the category of *G-CW*-pairs to the category of $\mathbb{Z}$-graded Λ-modules together with natural transformations

$$\partial_n(X, A) \colon \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- *G*-homotopy invariance;
- Long exact sequence of a pair;
- Excision;
- Disjoint union axiom.

Example (Bredon homology)

Consider any covariant functor

$$M \colon \mathrm{Or}\,G \to \mathbb{Z} - \mathrm{Modules}.$$

Then there is up to natural equivalence of G-homology theories precisely one G-homology theory $H_*^G(-, M)$, called Bredon homology, with the property that the covariant functor

$$H_n^G \colon \mathrm{Or}\,G \to \mathbb{Z} - \mathrm{Modules}, \quad G/H \mapsto H_n^G(G/H)$$

is trivial for $n \neq 0$ and naturally equivalent to M for $n = 0$.

- Let M be the constant functor with value the abelian group A. Then we get for every G-CW-complex X

$$H_n^G(X; M) \cong H_n(G \backslash X; A).$$

Definition (Equivariant homology theory)

An *equivariant homology theory* $\mathcal{H}_*^?$ assigns to every group G a
G-homology theory $\mathcal{H}_*^G$. These are linked together with the following so
called *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a
H-CW-pair (X, A), there are for all $n \in \mathbb{Z}$ natural homomorphisms

$$\operatorname{ind}_\alpha \colon \mathcal{H}_n^H(X, A) \;\; \to \;\; \mathcal{H}_n^G(\operatorname{ind}_\alpha(X, A))$$

satisfying

- Bijectivity
  If $\ker(\alpha)$ acts freely on X, then $\operatorname{ind}_\alpha$ is a bijection;
- Compatibility with the boundary homomorphisms;
- Functoriality in α;
- Compatibility with conjugation.

Example (Equivariant homology theories)

- Given a non-equivariant homology theory $\mathcal{K}_*$, put

$$\begin{aligned} \mathcal{H}_*^G(X) &:= \mathcal{K}_*(X/G); \\ \mathcal{H}_*^G(X) &:= \mathcal{K}_*(EG \times_G X) \quad \text{(Borel homology)}. \end{aligned}$$

- Equivariant bordism $\Omega_*^?(X)$;
- Equivariant topological K-homology $K_*^?(X)$ in the sense of Kasparov. Recall for $H \subseteq G$ finite

$$K_n^G(G/H) \cong K_n^H(\text{pt}) \cong \begin{cases} R_{\mathbb{C}}(H) & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

Example (Bredon homology)

- Consider a covariant functor

$$M \colon \mathrm{Groupoids} \to \mathbb{Z} - \mathrm{Modules}.$$

- Given a G-set S, let $\mathcal{G}^G(S)$ be the associated transport groupoid. The set of objects is S. The set of morphisms from s_1 to s_2 is $\{g \in G \mid gs_1 = s_2\}$.
- Composing M with the covariant functor $\mathcal{G}^G \colon \mathrm{Or}\,G \to \mathrm{Groupoids}$ yields a covariant functor $M^G \colon \mathrm{Or}\,G \to \mathbb{Z} - \mathrm{Modules}$.
- Let $H_*^G(X; M^G)$ be the G-homology theory given by the Bredon homology with coefficients in M^G.
- Then the collection H_*^G defines an equivariant homology theory $H_*^?(-; M)$.

Theorem (Equivalences of homology theories)

Let $\mathcal{H}_*^G$ and $\mathcal{K}_*^G$ be G-homology theories. Let $t_*^G \colon \mathcal{H}_*^G \to \mathcal{K}_*^G$ be a transformation of G-homology theories. Suppose that for any subgroup $H \subseteq G$ and $n \in \mathbb{Z}$, the map $t_n^G(G/H) \colon \mathcal{H}_n^G(G/H) \to \mathcal{K}_n^G(G/H)$ is bijective.

Then for every G-CW-complex X and $n \in \mathbb{Z}$ the map

$$t_n^G(X) \colon \mathcal{H}_n^G(X) \to \mathcal{K}_n^G(X)$$

is bijective.

Theorem (L.-Reich (2005))

Given a functor $\mathbf{E}\colon \mathrm{Groupoids} \to \mathrm{Spectra}$ *sending equivalences to weak equivalences, there exists an equivariant homology theory* $\mathcal{H}_*^?(-;\mathbf{E})$ *satisfying*

$$\mathcal{H}_n^H(pt) \cong \mathcal{H}_n^G(G/H) \cong \pi_n(\mathbf{E}(H)).$$

> **Theorem (Equivariant homology theories associated to K and L-theory, Davis-L. (1998))**
>
> *Let R be a ring (with involution). There exist covariant functors*
>
> $$\mathbf{K}_R \colon \text{Groupoids} \rightarrow \text{Spectra};$$
> $$\mathbf{L}_R^{\langle\infty\rangle} \colon \text{Groupoids} \rightarrow \text{Spectra};$$
> $$\mathbf{K}^{\text{top}} \colon \text{Groupoids}^{\text{inj}} \rightarrow \text{Spectra};$$
> $$\mathbf{K}^{l^1} \colon \text{Groupoids} \rightarrow \text{Spectra},$$
>
> *with the following properties:*

Theorem (Continued)

- *They send equivalences of groupoids to weak equivalences of spectra;*
- *For every group G and all $n \in \mathbb{Z}$ we have*

$$\begin{aligned}
\pi_n(\mathbf{K}_R(G)) &\cong K_n(RG); \\
\pi_n(\mathbf{L}_R^{\langle -\infty \rangle}(G)) &\cong L_n^{\langle -\infty \rangle}(RG); \\
\pi_n(\mathbf{K}^{\text{top}}(G)) &\cong K_n(C_r^*(G)); \\
\pi_n(\mathbf{K}^{l^1}(G)) &\cong K_n(l^1(G)).
\end{aligned}$$

Example (Equivariant homology theories associated to K and L-theory)

We get equivariant homology theories

$$H_*^?(-; \mathbf{K}_R);$$
$$H_*^?(-; \mathbf{L}_R^{\langle -\infty \rangle});$$
$$H_*^?(-; \mathbf{K}^{\mathrm{top}});$$
$$H_*^?(-; \mathbf{K}^{l^1}),$$

satisfying for $H \subseteq G$

$$
\begin{aligned}
H_n^G(G/H; \mathbf{K}_R) &\cong H_n^H(\mathrm{pt}; \mathbf{K}_R) &&\cong K_n(RH); \\
H_n^G(G/H; \mathbf{L}_R^{\langle -\infty \rangle}) &\cong H_n^H(\mathrm{pt}; \mathbf{L}_R^{\langle -\infty \rangle}) &&\cong L_n^{\langle -\infty \rangle}(RH); \\
H_n^G(G/H; \mathbf{K}^{\mathrm{top}}) &\cong H_n^H(\mathrm{pt}; \mathbf{K}^{\mathrm{top}}) &&\cong K_n(C_r^*(H)): \\
H_n^G(G/H; \mathbf{K}^{l^1}) &\cong H_n^H(\mathrm{pt}; \mathbf{K}^{l^1}) &&\cong K_n(l^1(H)).
\end{aligned}
$$

Isomorphism Conjectures

Conjecture (Isomorphism Conjecture)

Let $\mathcal{H}_*^?$ be an equivariant homology theory. It satisfies the *Isomorphism Conjecture* for the group G and the family $\mathcal{F}$ if the projection $E_{\mathcal{F}}(G) \to pt$ induces for all $n \in \mathbb{Z}$ a bijection

$$\mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(pt).$$

- The point is to find a as small as possible family $\mathcal{F}$.
- The Isomorphism Conjecture is always true for $\mathcal{F} = \mathcal{ALL}$ since it becomes a trivial statement because of $E_{\mathcal{ALL}}(G) = pt$.
- The philosophy is to be able to compute the functor of interest for G by knowing it on the values of elements in $\mathcal{F}$.

Conjecture (*K*-theoretic Farrell-Jones-Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- The assembly map is the map induced by the projection $E_{\mathcal{VC}yc}(G) \to \text{pt}$.

Conjecture (*L*-theoretic Farrell-Jones-Conjecture)

The L-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(pt, \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (Baum-Connes Conjecture)

The *Baum-Connes Conjecture* predicts that the assembly map

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{F}in}(G), \mathbf{K}^{\text{top}}) \to H_n^G(pt, \mathbf{K}^{\text{top}}) = K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (Bost Conjecture)

The *Bost Conjecture* predicts that the assembly map

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{F}in}(G), \mathbf{K}^{l^1}) \to H_n^G(pt, \mathbf{K}^{\text{top}}) = K_n(l^1(G))$$

is bijective for all $n \in \mathbb{Z}$.

- The Baum-Connes assembly map factorizes over the Bost assembly map is the map induced by the inclusion $l^1(G) \to C_r^*(G)$.

$$K_n^G(\underline{E}G) \to K_n(l^1(G)) \to K_n(C_r^*(G)).$$

Changing the family

- Fix an equivariant homology theory $\mathcal{H}_*^?$.

Theorem (Transitivity Principle)

Suppose $\mathcal{F} \subseteq \mathcal{G}$ are two families of subgroups of G. Assume that for every element $H \in \mathcal{G}$ the group H satisfies the Isomorphism Conjecture for $\mathcal{F}|_H = \{K \subseteq H \mid K \in \mathcal{F}\}$.

Then the map

$$\mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{G}}(G))$$

is bijective for all $n \in \mathbb{Z}$.

Moreover, $(G, \mathcal{G})$ satisfies the Isomorphism Conjecture if and only if $(G, \mathcal{F})$ satisfies the Isomorphism Conjecture.

Sketch of proof.

- For a G-CW-complex X with isotropy group in $\mathcal{G}$ consider the natural map induced by the projection

$$s_*^G(X) \colon \mathcal{H}_*^G(X \times E_{\mathcal{F}}(G)) \to \mathcal{H}_*^G(X).$$

- This a natural transformation of G-homology theories defined for G-CW-complexes with isotropy groups in $\mathcal{G}$.
- In order to show that it is a natural equivalence it suffices to show that $s_n^G(G/H)$ is an isomorphism for all $H \in \mathcal{G}$ and $n \in \mathbb{Z}$.

$\square$

Sketch of proof (continued).

- The G-space $G/H \times E_{\mathcal{F}}(G)$ is G-homeomorphic to $G \times_H \operatorname{res}_G^H E_{\mathcal{F}}(G)$ and $\operatorname{res}_G^H E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F}|_H}(H)$.

- Hence by the induction structure $s_n^G(G/H)$ can be identified with the assembly map
$$\mathcal{H}_*^H(E_{\mathcal{F}|_H}(H)) \to \mathcal{H}_*^H(\mathrm{pt}),$$
which is bijective by assumption.

- Now apply this to $X = E_{\mathcal{G}}(G)$ and observe that $E_{\mathcal{G}}(G) \times E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F}}(G)$.

$\square$

Example (Passage from $\mathcal{F}$in to $\mathcal{VC}$yc for the Baum-Connes Conjecture)

- Consider the Baum-Connes setting, i.e., take $\mathcal{H}_*^? = K_*^?$.

- Consider the families $\mathcal{F}$in $\subseteq \mathcal{VC}$yc.

- For every virtually cyclic group V the Baum-Connes Conjecture is true, i.e.,

$$K_n^G(E_{\mathcal{F}in}(V)) \to K_n(C_r^*(V))$$

is bijective for $n \in \mathbb{Z}$.

- Hence by the Transitivity principle the following map is bijective for all groups G and all $n \in \mathbb{Z}$

$$K_n^G(\underline{E}G) = K_n^G(E_{\mathcal{F}in}(G)) \to K_n^G(E_{\mathcal{VC}yc}(G)).$$

- This explains why in the Baum-Connes setting it is enough to deal with $\mathcal{F}$in instead of $\mathcal{VC}$yc.
- This is not true in the Farrell-Jones setting and causes many extra difficulties there (NIL and UNIL-phenomena).
- This difference is illustrated by the following isomorphisms due to Pimsner-Voiculescu and Bass-Heller-Swan:

$$
\begin{aligned}
K_n(C_r^*(\mathbb{Z})) &\cong K_n(\mathbb{C}) \oplus K_{n-1}(\mathbb{C}); \\
K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R).
\end{aligned}
$$

- Due to Matthey-Mislin and Lück the map

$$
K_n^G(E_{\mathcal{FC}\mathrm{yc}}(G)) \xrightarrow{\cong} K_n^G(\underline{E}G)
$$

is bijective for all $n \in \mathbb{Z}$.

- In general the relative assembly maps

$$H_n^G(\underline{E}G; \mathbf{K}_R) \rightarrow H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{K}_R);$$
$$H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) \rightarrow H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{L}_R^{\langle -\infty \rangle}),$$

  are not bijective.
- Hence in the Farrell-Jones setting one has to pass to $\mathcal{VC}$yc and cannot use the easier to handle family $\mathcal{F}$in.

Example (Passage from $\mathcal{F}$in to $\mathcal{VC}$yc for the Farrell-Jones Conjecture)

For instance the Bass-Heller Swan decomposition

$$K_{n-1}(R) \oplus K_n(R) \oplus NK_n(R) \oplus NK_n(R)) \xrightarrow{\cong} K_n(R[t, t^{-1}]) \cong K_n(R[\mathbb{Z}])$$

and the isomorphism

$$H_n^{\mathbb{Z}}(\underline{E}\mathbb{Z}; \mathbf{K}_R) = H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{K}_R) = H_n^{\{1\}}(S^1, \mathbf{K}_R) = K_{n-1}(R) \oplus K_n(R)$$

show that

$$H_n^{\mathbb{Z}}(\underline{E}\mathbb{Z}; \mathbf{K}_R) \to H_n^{\mathbb{Z}}(\mathrm{pt}; \mathbf{K}_R) = K_n(R\mathbb{Z})$$

is bijective if and only if $NK_n(R) = 0$.

Conjecture (*K*-theoretic Farrell-Jones Conjecture for regular rings and torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (*K*-theoretic Farrell-Jones Conjecture for regular rings containing $\mathbb{Q}$)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R with $\mathbb{Q} \subseteq R$ predicts that the assembly map

$$H_n(\underline{E}G; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- By the Transitivity Principle the general version reduces to the version above if G is torsionfree and R is regular.
- Notice that the version above is close to the Baum-Connes Conjecture and that $\mathbb{C}$ is a regular ring.

- An infinite virtually cyclic group G is called of type I if it admits an epimorphism onto $\mathbb{Z}$ and of type II otherwise.
- A virtually cyclic group is of type II if and only if admits an epimorphism onto D_∞.
- Let $\mathcal{VC}yc_I$ or $\mathcal{VC}yc_{II}$ respectively be the family of subgroups which are either finite or which are virtually cyclic of type I or II respectively.

Theorem (Lück (2004), Quinn (2007), Reich (2007))

The following maps are bijective for all $n \in \mathbb{Z}$

$$
\begin{aligned}
H_n^G(E_{\mathcal{VC}yc_I}(G); \mathbf{K}_R) &\rightarrow H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{K}_R); \\
H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) &\rightarrow H_n^G(E_{\mathcal{VC}yc_I}(G); \mathbf{L}_R^{\langle -\infty \rangle}).
\end{aligned}
$$

Theorem (Cappell (1973), Grunewald (2005), Waldhausen (1978))

- The following maps are bijective for all $n \in \mathbb{Z}$.

$$H_n^G(\underline{E}G; \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \to \quad H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q};$$

$$H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) \left[\frac{1}{2}\right] \quad \to \quad H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \left[\frac{1}{2}\right];$$

- If R is regular and $\mathbb{Q} \subseteq R$, then for all $n \in \mathbb{Z}$ we get a bijection

$$H_n^G(\underline{E}G; \mathbf{K}_R) \to H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{K}_R).$$

Theorem (Bartels (2003))

For every $n \in \mathbb{Z}$ the two maps

$$
\begin{aligned}
H_n^G(\underline{E}G; \mathbf{K}_R) &\to H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{K}_R); \\
H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) &\to H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{L}_R^{\langle -\infty \rangle}),
\end{aligned}
$$

are split injective.

- Hence we get (natural) isomorphisms

$$H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{K}_R) \;\cong\; H_n^G(\underline{E}G; \mathbf{K}_R) \oplus H_n^G(E_{\mathcal{VC}yc}(G), \underline{E}G; \mathbf{K}_R);$$

$$H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{L}_R^{\langle -\infty \rangle})$$
$$\cong H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) \oplus H_n^G(E_{\mathcal{VC}yc}(G), \underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}).$$

- The analysis of the terms $H_n^G(E_{\mathcal{VC}yc}(G), \underline{E}G; \mathbf{K}_R)$ and $H_n^G(E_{\mathcal{VC}yc}(G), \underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle})$ boils down to investigating Nil-terms and UNil-terms in the sense of Waldhausen and Cappell.

Conjecture (*L*-theoretic Farrell-Jones Conjecture for torsionfree groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n\big(BG; \mathbf{L}_R^{\langle -\infty \rangle}\big) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

$$\begin{array}{ccc}
H_n^G(E_{\mathcal{F}\text{in}}(G); \mathbf{L}_\mathbb{Z}^p)[1/2] & \xrightarrow{\cong} & L_n^p(\mathbb{Z}G)[1/2] \\
\downarrow{\cong} & & \downarrow{\cong} \\
H_n^G(E_{\mathcal{F}\text{in}}(G); \mathbf{L}_\mathbb{R}^p)[1/2] & \xrightarrow{\cong} & L_n^p(\mathbb{R}G)[1/2] \\
\downarrow{\cong} & & \downarrow{\cong} \\
H_n^G(E_{\mathcal{F}\text{in}}(G); \mathbf{L}_{C_r^*(?;\mathbb{R})}^p)[1/2] & \xrightarrow{\cong} & L_n^p(C_r^*(G;\mathbb{R}))[1/2] \\
\downarrow{\cong} & & \downarrow{\cong} \\
H_n^G(E_{\mathcal{F}\text{in}}(G); \mathbf{K}_\mathbb{R}^{\text{top}})[1/2] & \xrightarrow{\cong} & K_n(C_r^*(G;\mathbb{R}))[1/2] \\
\downarrow & & \downarrow \\
H_n^G(E_{\mathcal{F}\text{in}}(G); \mathbf{K}_\mathbb{C}^{\text{top}})[1/2] & \xrightarrow{\cong} & K_n(C_r^*(G))[1/2]
\end{array}$$

Other versions of Isomorphism Conjectures

- There are functors $\mathcal{P}$ and A which assign to a space X the space of pseudo-isotopies and its A-theory.

- Composing it with the functor sending a groupoid to its classifying space yields functors $\mathbf{P}$ and $\mathbf{A}$ from $\mathrm{Groupoids}$ to $\mathrm{Spectra}$.

- Thus we obtain equivariant homology theories $H_*^?(-; \mathbf{P})$ and $H_*^?(-; \mathbf{A})$. They satisfy $H_n^G(G/H; \mathbf{P}) = \pi_n(\mathcal{P}(BH))$ and $H_n^G(G/H; \mathbf{A}) = \pi_n(A(BH))$.

Conjecture (The Farrell-Jones Conjecture for pseudo-isotopies and A-theory)

The Farrell-Jones Conjecture for pseudo-isotopies and A-theory respectively is the Isomorphism Conjecture for $H_*^?(-; \mathbf{P})$ and $H_*^?(-; \mathbf{A})$ respectively for the family $\mathcal{VC}yc$.

Theorem (Relating pseudo-isotopy and K-theory)

The rational version of the K-theoretic Farrell-Jones Conjecture for coefficients in $\mathbb{Z}$ is equivalent Farrell-Jones Conjecture for Pseudoisotopies. In degree $n \leq 1$ this is even true integrally.

- Pseudo-isotopy and A-theory are important theories. In particular they are closely related to the space of selfhomeomorphisms and the space of selfdiffeomorphisms of closed manifolds.

- There are functors **THH** and **TC** which assign to a ring (or more generally to an S-algebra) a spectrum describing its topological Hochschild homology and its topological cyclic homology.

- These functors play an important role in K-theoretic computations.

- Composing them with the functor sending a groupoid to a kind of group ring yields functors $\mathbf{THH}_R$ and $\mathbf{TC}_R$ from $\mathrm{Groupoids}$ to $\mathrm{Spectra}$.

- Thus we obtain equivariant homology theories $H_*^?(-; \mathbf{THH}_R)$ and $H_*^?(-; \mathbf{TC}_R)$. They satisfy $H_n^G(G/H; \mathbf{THH}_R) = \mathbf{THH}_n(RH)$ and $H_n^G(G/H; \mathbf{TC}_R) = \mathbf{TC}_n(RH)$.

Conjecture (The Farrell-Jones Conjecture for topological Hochschild homology and cyclic homology)

The Farrell-Jones Conjecture for topological Hochschild homology and cyclic homology respectively is the Isomorphism Conjecture for $H_*^?(-; \mathbf{THH})$ and $H_*^?(-; \mathbf{TC})$ respectively for the family $\mathcal{C}yc$ of cyclic subgroups.

Theorem (Lück-Reich-Rognes-Varisco ($\geq$ 2010))

The Farrell-Jones Conjecture for topological Hochschild homology is true for all groups.

- There is a joint project by Lück-Rognes-Reich-Varisco aiming at this conjecture for TC and its application to the Farrell-Jones Conjecture generalizing the results of Bökstedt-Hsiang-Madsen.

Some prominent Conjectures

Conjecture (Kaplansky Conjecture)

The *Kaplansky Conjecture* says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Theorem (The Farrell-Jones Conjecture and the Kaplansky Conjecture, Bartels-L.-Reich(2007))

Let F be a skew-field and let G be a group satisfying the K-theoretic Farrell-Jones Conjecture for FG. Suppose that one of the following conditions is satisfied:

- F is commutative and has characteristic zero and G is torsionfree;
- G is torsionfree and sofic, e.g., residually amenable;
- The characteristic of F is p, all finite subgroups of G are p-groups and G is sofic.

Then 0 and 1 are the only idempotents in FG.

Proof.

- Let p be an idempotent in FG. We want to show $p \in \{0, 1\}$.
- Denote by $\epsilon \colon FG \to F$ the augmentation homomorphism sending $\sum_{g \in G} r_g \cdot g$ to $\sum_{g \in G} r_g$. Obviously $\epsilon(p) \in F$ is 0 or 1. Hence it suffices to show $p = 0$ under the assumption that $\epsilon(p) = 0$.
- Let $(p) \subseteq FG$ be the ideal generated by p which is a finitely generated projective FG-module.
- By assumption

$$i_* \colon K_0(F) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

  is surjective.
- Hence we can find a finitely generated projective F-module P and integers $k, m, n \geq 0$ satisfying

$$(p)^k \oplus FG^m \cong_{FG} i_*(P) \oplus FG^n.$$

Proof (continued).

- If we now apply $i_* \circ \epsilon_*$ and use $\epsilon \circ i = \text{id}$, $i_* \circ \epsilon_*(FG^l) \cong FG^l$ and $\epsilon(p) = 0$ we obtain
$$FG^m \cong i_*(P) \oplus FG^n.$$

- Inserting this in the first equation yields
$$(p)^k \oplus i_*(P) \oplus FG^n \cong i_*(P) \oplus FG^n.$$

- Our assumptions on F and G imply that FG is stably finite, i.e., if A and B are square matrices over FG with $AB = I$, then $BA = I$.
- This implies $(p)^k = 0$ and hence $p = 0$.

$\square$

Conjecture (Kadison Conjecture)

The Kaplansky Conjecture says for a torsionfree group G that 0 and 1 are the only idempotents in $C_r^*(G)$.

Theorem (The Baum-Connes Conjecture and the Kaplansky Conjecture)

Let G be a torsionfree group satisfying the Baum-Connes Conjecture. Then 0 and 1 are the only idempotents in $\mathbb{C}G$.

Proof.

- There is a trace map
$$\mathrm{tr}\colon C_r^*(G) \to \mathbb{C}$$
which sends $f \in C_r^*(G) \subseteq \mathcal{B}(l^2(G))$ to $\langle f(e), e\rangle_{l^2(G)}$.

- The L^2-index theorem due to Atiyah (1976) shows that the composite
$$K_0(BG) \to K_0(C_r^*(G)) \xrightarrow{\mathrm{tr}} \mathbb{C}$$
coincides with
$$K_0(BG) \xrightarrow{K_0(\mathrm{pr})} K_0(\mathrm{pt}) = \mathbb{Z} \xrightarrow{i} \mathbb{C}.$$

Proof (continued).

- Hence $\mathrm{tr}(p) \in \mathbb{Z}$.
- Since $\mathrm{tr}(1) = 1$, $\mathrm{tr}(0) = 0$, $0 \leq p \leq 1$ and $p^2 = p$, we get $\mathrm{tr}(p) \in \mathbb{R}$ and $0 \leq \mathrm{tr}(p) \leq 1$.
- We conclude $\mathrm{tr}(0) = \mathrm{tr}(p)$ or $\mathrm{tr}(1) = \mathrm{tr}(p)$.
- This implies already $p = 0$ or $p = 1$.

$\square$

Conjecture (Projective class groups)

Let R be a regular ring. Suppose that G is torsionfree. Then:

- $K_n(RG) = 0$ for $n \leq -1$;
- *The change of rings map $K_0(R) \to K_0(RG)$ is bijective;*
- *If R is a principal ideal domain, then $\widetilde{K}_0(RG) = 0$.*

Conjecture (Whitehead group)

If G is torsionfree, then the Whitehead group $\mathrm{Wh}(G)$ vanishes.

Lemma

Let R be a regular ring and let G be a torsionfree group such that K-theoretic Farrell-Jones Conjecture holds. Then

- *$K_n(RG) = 0$ for $n \leq -1$;*
- *The change of rings map $K_0(R) \to K_0(RG)$ is bijective. In particular $\widetilde{K}_0(RG)$ is trivial if and only if $\widetilde{K}_0(R)$ is trivial;*
- *The Whitehead group $\mathrm{Wh}(G)$ is trivial.*

Recall:

Conjecture (*K*-theoretic Farrell-Jones Conjecture for regular rings and torsionfree groups)

*The *K*-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map*

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG; \mathbf{K}_R)$ whose E^2-term is given by

$$E^2_{p,q} = H_p(BG, K_q(R)).$$

- Since R is regular by assumption, we get $K_q(R) = 0$ for $q \leq -1$.
- Hence the edge homomorphism yields an isomorphism

$$K_0(R) = H_0(\mathrm{pt}, K_0(R)) \xrightarrow{\cong} H_0(BG; \mathbf{K}_R) \cong K_0(RG).$$

- We have $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = \{\pm 1\}$. We get an exact sequence

$$0 \to H_0(BG; \mathbf{K}_{\mathbb{Z}}) = \{\pm 1\} \to H_1(BG; \mathbf{K}_{\mathbb{Z}}) \cong K_1(\mathbb{Z}G)$$
$$\to H_1(BG; K_0(\mathbb{Z})) = G/[G, G] \to 1.$$

This implies

$$\mathrm{Wh}(G) := K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\} \cong 0.$$

Conjecture (Moody's Induction Conjecture)

- Let R be a regular ring with $\mathbb{Q} \subseteq R$. Then the map given by induction from finite subgroups of G

$$\operatorname*{colim}_{\operatorname{Or}_{\mathcal{F}in}(G)} K_0(RH) \to K_0(RG)$$

  is bijective;

- Let F be a field of characteristic p for a prime number p. Then the map

$$\operatorname*{colim}_{\operatorname{Or}_{\mathcal{F}in}(G)} K_0(FH)[1/p] \to K_0(FG)[1/p]$$

  is bijective.

- If G is torsionfree, the Induction Conjecture says that everything comes from the trivial subgroup and we rediscover some of the previous conjectures.

Conjecture (Bass Conjecture)

Let R be a commutative integral domain and let G be a group. Let $g \neq 1$ be an element in G. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R.

Then the Bass Conjecture predicts that for every finitely generated projective RG-module P the value of its Hattori-Stallings rank $\mathrm{HS}_{RG}(P)$ at (g) is trivial.

- The Hattori-Stallings rank extends the notion of a character of a representation of a finite group to infinite groups.
- Roughly speaking, the Bass Conjecture extends basic facts of the representation theory of finite groups to infinite groups.
- If G is finite, the Bass Conjecture reduces to the Theorem of Swan.

Conjecture (L^2-torsion)

If X and Y are det-L^2-acyclic finite G-CW-complexes, which are G-homotopy equivalent, then their L^2-torsion agree:

$$\rho^{(2)}(X; \mathcal{N}(G)) = \rho^{(2)}(Y; \mathcal{N}(G)).$$

- The L^2-torsion of a closed Riemannian manifold M is defined in terms of the heat kernel on the universal covering.
- If M is hyperbolic and has odd dimension, its L^2-torsion is up to dimension constant its volume.
- The conjecture above allows to extend the notion of volume to word hyperbolic groups whose L^2-Betti numbers all vanish.

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f \colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\operatorname{sign}_x(M, f) := \langle \mathcal{L}(M) \cup f^* x, [M] \rangle$$

is an oriented homotopy invariant of (M, f), i.e., for every orientation preserving homotopy equivalence of closed oriented manifolds $g \colon M_0 \to M_1$ and homotopy equivalence $f_i \colon M_i \to BG$ with $f_1 \circ g \simeq f_2$ we have

$$\operatorname{sign}_x(M_0, f_0) = \operatorname{sign}_x(M_1, f_1).$$

- The Novikov Conjecture predicts for a homotopy equivalence $f : M \to N$ of closed aspherical manifolds

$$f_*(\mathcal{L}(M)) = \mathcal{L}(N).$$

- This is surprising since this is not true in general and in many case one could detect that two specific closed homotopy equivalent manifolds cannot be diffeomorphic by the failure of this equality to be true.

- A deep theorem of Novikov predicts that $f_*(\mathcal{L}(M)) = \mathcal{L}(N)$ holds for a homeomorphism. of closed manifolds.

- Hence an explanation why the Novikov Conjecture may be true for closed aspherical manifolds is due to the next conjecture.

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism.

In particular M and N are homeomorphic.

- This is the topological version of Mostow rigidity. One version of Mostow rigidity says that any homotopy equivalence between hyperbolic closed Riemannian manifolds is homotopic to an isometric diffeomorphism. In particular they are isometrically diffeomorphic if and only if their fundamental groups are isomorphic.

- The Borel Conjecture becomes in general false if one replaces homeomorphism by diffeomorphism. A counterexample is T^n for $n \geq 5$.

- In some sense the Borel Conjecture is opposed to the Poincaré Conjecture. Namely, in the Borel Conjecture the fundamental group can be complicated but there are no higher homotopy groups, whereas in the Poincaré Conjecture there is no fundamental group but complicated higher homotopy groups.

- A systematic study of topologically rigid manifolds is presented in a paper by Kreck-Lück (2006), where a kind of interpolation between the Poincaré Conjecture and the Borel Conjecture is studied.

- Thurston's Geometrization Conjecture implies the Borel Conjecture in dimension 3.

- The Borel Conjecture in dimension 1 and 2 is obviously true.

- All the conjectures above follow from the Farrell-Jones Conjecture provided some sometimes some dimension restrictions or conditions about R hold.
- We will also explain this for the Borel Conjecture.

Theorem

If the K-theoretic and the L-theoretic Farrell-Jones Conjecture holds for the group G, then the Borel Conjecture holds for any n-dimensional closed manifold with $\pi_1(M) \cong G$ provided that $n \geq 5$.

Recall:

Conjecture (*L*-theoretic Farrell-Jones Conjecture for torsionfree groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n\big(BG; \mathbf{L}_R^{\langle -\infty \rangle}\big) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Definition (Structure set)

The *structure set* $S^{top}(M)$ of a manifold M consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold N as source.

Two such homotopy equivalences $f_0 \colon N_0 \to M$ and $f_1 \colon N_1 \to M$ are equivalent if there exists a homeomorphism $g \colon N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

Theorem

The Borel Conjecture holds for a closed manifold M if and only if $S^{top}(M)$ consists of one element.

Theorem (Algebraic surgery sequence Ranicki (1992))

There is an exact sequence of abelian groups called *algebraic surgery exact sequence* for an n-dimensional closed manifold M

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M; \mathbf{L}\langle 1 \rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}}$$
$$\mathcal{S}^{\mathrm{top}}(M) \xrightarrow{\sigma_n} H_n(M; \mathbf{L}\langle 1 \rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

It can be identified with the classical geometric surgery sequence due to *Sullivan and Wall* in high dimensions.

- $\mathcal{S}^{\mathrm{top}}(M)$ consist of one element if and only if A_{n+1} is surjective and A_n is injective.
- $H_k(M; \mathbf{L}\langle 1 \rangle) \to H_k(M; \mathbf{L})$ is bijective for $k \geq n+1$ and injective for $k = n$ if both the K-theoretic and L-theoretic Farrell-Jones Conjecture hold for $G = \pi_1(M)$ and $R = \mathbb{Z}$.

Definition (Poincaré duality group)

A Poincaré duality group G of dimension n is a finitely presented group satisfying:

- G is of type FP;
- $H^i(G; \mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

Lemma

Let X be a closed aspherical ANR-homology manifold of dimension n. Then its fundamental group is a Poincaré duality group of dimension n.

Theorem (Poincaré duality groups and ANR-homology manifolds Bartels-Lück-Weinberger (2009))

Let G be a torsion-free group. Suppose that its satisfies the K- and L-theoretic Farrell-Jones Conjecture. Consider $n \geq 6$.

Then the following statements are equivalent:

1. *G is a Poincaré duality group of dimension n;*
2. *There exists a closed aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$;*
3. *There exists a closed aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$ which has the DDP.*

If the first statements holds, then the homology ANR-manifold M appearing above is unique up to s-cobordism of ANR-homology manifolds.

Theorem (Resolution obstruction Quinn (1987))

There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ for homology ANR-manifolds with the following properties:

- *if $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;*
- *Let M be a homology ANR-manifold of dimension ≥ 5. Then M is a topological manifold if and only if M has the DDP and $\iota(M) = 1$.*

Question

Does the Quinn obstruction always vanishes for aspherical closed homology ANR-manifolds?

- If the answer is yes, we can replace "closed ANR-homology manifold" by "closed topological manifold" in the theorem above.
- In general the Quinn obstruction is not a homotopy invariant but it is a homotopy invariant for aspherical closed ANR-homology manifolds provided the integral Novikov Conjecture holds.
- However, most experts expect the answer no.

Definition (Bott manifold)

A *Bott manifold* is any simply connected closed Spin-manifold B of dimension 8 whose $\widehat{A}$-genus $\widehat{A}(B)$ is 8.

- We fix such a choice. (The particular choice does not matter.)
- Notice that the index defined in terms of the Dirac operator $\operatorname{ind}_{C_r^*(\{1\};\mathbb{R})}(B) \in KO_8(\mathbb{R}) \cong \mathbb{Z}$ is a generator and the product with this element induces the Bott periodicity isomorphisms $KO_n(C_r^*(G;\mathbb{R})) \xrightarrow{\cong} KO_{n+8}(C_r^*(G;\mathbb{R}))$.
- In particular

$$\operatorname{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) = \operatorname{ind}_{C_r^*(\pi_1(M\times B);\mathbb{R})}(M \times B),$$

if we identify $KO_n(C_r^*(\pi_1(M);\mathbb{R})) = KO_{n+8}(C_r^*(\pi_1(M);\mathbb{R}))$ via Bott periodicity.

- If M carries a Riemannian metric with positive scalar curvature, then the index

$$\mathrm{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) \in KO_n(C_r^*(\pi_1(M);\mathbb{R})),$$

which is defined in terms of the Dirac operator on the universal covering, must vanish by the Bochner-Lichnerowicz formula.

Conjecture ((Stable) Gromov-Lawson-Rosenberg Conjecture)

Let M be a closed connected Spin-manifold of dimension $n \geq 5$.
Then $M \times B^k$ carries for some integer $k \geq 0$ a Riemannian metric with positive scalar curvature if and only if

$$\mathrm{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) = 0 \quad \in KO_n(C_r^*(\pi_1(M);\mathbb{R})).$$

Theorem (Stolz (2002))

Suppose that the assembly map for the real version of the Baum-Connes Conjecture

$$H_n^G(\underline{E}G; \mathbf{K}O^{\mathrm{top}}) \to KO_n(C_r^*(G; \mathbb{R}))$$

is injective for the group G.

Then the Stable Gromov-Lawson-Rosenberg Conjecture true for all closed Spin-manifolds of dimension ≥ 5 with $\pi_1(M) \cong G$.

- The requirement $\dim(M) \geq 5$ is essential in the Stable Gromov-Lawson-Rosenberg Conjecture, since in dimension four new obstructions, the Seiberg-Witten invariants, occur.
- The unstable version of the Gromov-Lawson-Rosenberg Conjecture says that M carries a Riemannian metric with positive scalar curvature if and only if $\text{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) = 0$.
- Schick(1998) has constructed counterexamples to the unstable version using minimal hypersurface methods due to Schoen and Yau.
- It is not known whether the unstable version is true or false for finite fundamental groups.
- Since the Baum-Connes Conjecture is true for finite groups (for the trivial reason that $\underline{E}G = \text{pt}$ for finite groups G), the Stable Gromov-Lawson Conjecture holds for finite fundamental groups.

The status of the Farrell-Jones Conjecture

Theorem (Bartels-Lück (2009))

Let $\mathcal{FJ}$ be the class of groups for which both the K-theoretic and the L-theoretic Farrell-Jones Conjecture hold with coefficients in any additive G-category (with involution) is true has the following properties:

- *Hyperbolic group and virtually nilpotent groups belongs to $\mathcal{FJ}$;*
- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ belongs to $\mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}(R)$ for $i \in I$. Then $\mathrm{colim}_{i \in I} G_i$ belongs to $\mathcal{F}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *If we demand on the K-theory version only that the assembly map is 1-connected and keep the full L-theory version, then the properties above remain valid and the class $\mathcal{FJ}$ contains also all CAT(0)-groups.*

- Limit groups in the sense of Zela are CAT(0)-groups (Alibegovic-Bestvina (2005)).

- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.

- On examples is the construction of groups with expanders due to Gromov. These yield counterexamples to the Baum-Connes Conjecture with coefficients (see Higson-Lafforgue-Skandalis (2002)).

- However, our results show that these groups do satisfy the Farrell-Jones Conjecture in its most general form and hence also the other conjectures mentioned above.

- Bartels-Echterhoff-Lück(2008) show that the Bost Conjecture with coefficients in C^*-algebras is true for colimits of hyperbolic groups. Thus the failure of the Baum-Connes Conjecture with coefficients comes from the fact that the change of rings map

$$K_0\big(\mathcal{A} \rtimes_{l^1} G\big) \to K_0\big(\mathcal{A} \rtimes_{C_r^*} G\big)$$

is not bijective for all G-C^*-algebras $\mathcal{A}$.

- Mike Davis (1983) has constructed exotic closed aspherical manifolds using Gromov's hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.

- However, in all cases the universal coverings are CAT(0)-spaces and hence the fundamental groups are CAT(0)-groups.

- Hence by our main theorem they satisfy the Farrell-Jones Conjecture and hence the Borel Conjecture in dimension ≥ 5.

- There are still many interesting groups for which the Farrell-Jones Conjecture in its most general form is open. Examples are:
    - Amenable groups;
    - $Sl_n(\mathbb{Z})$ for $n \geq 3$;
    - Mapping class groups;
    - $Out(F_n)$;
- If one looks for a counterexample, there seems to be no good candidates which do not fall under our main theorems and have some exotic properties which may cause the failure of the Farrell-Jones Conjecture.
- One needs a property which can be used to detect a non-trivial element which is not in the image of the assembly map or is in its kernel.

Equivariant Chern character

Theorem (Dold (1962))

Let $\mathcal{H}_*$ be a generalized homology theory with values in Λ-modules for $\mathbb{Q} \subseteq \Lambda$.

Then there exists for every $n \in \mathbb{Z}$ and every CW-complex X a natural isomorphism

$$\bigoplus_{p+q=n} H_p(X; \Lambda) \otimes_\Lambda \mathcal{H}_q(pt) \xrightarrow{\cong} \mathcal{H}_n(X).$$

- This means that the Atiyah-Hirzebruch spectral sequence collapses in the strongest sense.
- The assumption $\mathbb{Q} \subseteq \Lambda$ is necessary.

Dold's Chern character for a *CW*-complex X is given by the following composite

$$
\mathrm{ch}_n \colon \bigoplus_{p+q=n} H_p(X; \mathcal{H}_q(*)) \xleftarrow{\alpha} \bigoplus_{p+q=n} H_p(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{H}_q(*)
$$

$$
\xleftarrow[\cong]{\bigoplus_{p+q=n} \mathrm{hur} \otimes \mathrm{id}} \bigoplus_{p+q=n} \pi_p^s(X_+, *) \otimes_{\mathbb{Z}} \mathcal{H}_q(*)
$$

$$
\xrightarrow{\bigoplus_{p+q=n} D_{p,q}} \mathcal{H}_n(X).
$$

- We want to extend this to the equivariant setting.
- This requires an extra structure on the coefficients of an equivariant homology theory $\mathcal{H}^?_*$.
- We define a covariant functor called induction

$$\text{ind}\colon \text{FGINJ} \to \Lambda\text{-}\text{Mod}$$

from the category FGINJ of finite groups with injective group homomorphisms as morphisms to the category of Λ-modules as follows.

- It sends G to $\mathcal{H}^G_n(\text{pt})$ and an injection of finite groups $\alpha\colon H \to G$ to the morphism given by the induction structure

$$\mathcal{H}^H_n(\text{pt}) \xrightarrow{\text{ind}_\alpha} \mathcal{H}^G_n(\text{ind}_\alpha\,\text{pt}) \xrightarrow{\mathcal{H}^G_n(\text{pr})} \mathcal{H}^G_n(\text{pt}).$$

Definition (Mackey extension)

We say that $\mathcal{H}_*^?$ has a Mackey extension if for every $n \in \mathbb{Z}$ there is a contravariant functor called restriction

$$\mathrm{res} \colon \mathrm{FGI} \to \Lambda\text{-}\mathrm{Mod}$$

such that these two functors ind and res agree on objects and satisfy the double coset formula, i.e., we have for two subgroups $H, K \subset G$ of the finite group G

$$\mathrm{res}_G^K \circ \mathrm{ind}_H^G \;=\; \sum_{KgH \in K \backslash G / H} \mathrm{ind}_{c(g):H \cap g^{-1}Kg \to K} \circ \mathrm{res}_H^{H \cap g^{-1}Kg},$$

where $c(g)$ is conjugation with g, i.e., $c(g)(h) = ghg^{-1}$.

- In every case we will consider such a Mackey extension does exist and is given by an actual restriction.
- For instance for $H_0^?(-; \mathbf{K}^{\text{top}})$ induction is the functor complex representation ring $R_{\mathbb{C}}$ with respect to induction of representations. The restriction part is given by the restriction of representations.

Theorem (Lück (2002))

Let $\mathcal{H}_*^?$ be a proper equivariant homology theory with values in Λ-modules for $\mathbb{Q} \subseteq \Lambda$. Suppose that $\mathcal{H}_*^?$ has a Mackey extension. Let I be the set of conjugacy classes (H) of finite subgroups H of G.

Then there is for every group G, every proper G-CW-complex X and every $n \in \mathbb{Z}$ a natural isomorphism called equivariant Chern character

$$\mathrm{ch}_n^G \colon \bigoplus_{p+q=n} \bigoplus_{(H)\in I} H_p(C_G H \backslash X^H; \Lambda) \otimes_{\Lambda[W_G H]} S_H\left(\mathcal{H}_q^H(*)\right) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

- $C_G H$ is the centralizer and $N_G H$ the normalizer of $H \subseteq G$;
- $W_G H := N_G H / H \cdot C_G H$ (This is always a finite group);
- $S_H\left(\mathcal{H}_q^H(*)\right) := \mathrm{cok}\left(\bigoplus_{\substack{K \subset H \\ K \neq H}} \mathrm{ind}_K^H \colon \bigoplus_{\substack{K \subset H \\ K \neq H}} \mathcal{H}_q^K(*) \to \mathcal{H}_q^H(*)\right).$
- $\mathrm{ch}_*^?$ is an equivalence of equivariant homology theories.

Theorem (Lück (2002))

Let G be a group. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. There is a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{p+q=n} \bigoplus_{(g) \in T} H_p(BC_G\langle g \rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\
\downarrow & & \downarrow \\
\bigoplus_{p+q=n} \bigoplus_{(g) \in T} H_p(BC_G\langle g \rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \longrightarrow & K_n^{\mathrm{top}}(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C}
\end{array}$$

- The vertical arrows come from the obvious change of rings and of K-theory maps.

- The horizontal arrows can be identified with the assembly maps occurring in the Farrell-Jones Conjecture and the Baum-Connes Conjecture by the equivariant Chern character.

- Splitting principle.

Groups with special maximal finite subgroups

- Let G be a discrete group. Let $\mathcal{MF}\text{in}$ be the subset of $\mathcal{F}\text{in}$ consisting of elements in $\mathcal{F}\text{in}$ which are maximal in $\mathcal{F}\text{in}$.
- Assume that G satisfies the following assertions:

  (M) Every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup;

  (NM) $M \in \mathcal{MF}\text{in}, M \neq \{1\} \Rightarrow N_G M = M$.

- Here are some examples of groups G which satisfy conditions (M) and (NM):
  - Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$;
  - Fuchsian groups;
  - One-relator groups G.

- For such a group there is a nice model for $\underline{E}G$ with as few non-free cells as possible. Let $\{(M_i) \mid i \in I\}$ be the set of conjugacy classes of maximal finite subgroups of $M_i \subseteq G$. By attaching free G-cells we get an inclusion of G-CW-complexes $j_1 \colon \coprod_{i \in I} G \times_{M_i} EM_i \to EG$.

- Define $\underline{E}G$ as the G-pushout

$$
\begin{array}{ccc}
\coprod_{i \in I} G \times_{M_i} EM_i & \xrightarrow{\ j_1\ } & EG \\
\downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle f_1} \\
\coprod_{i \in I} G/M_i & \xrightarrow{\ k_1\ } & \underline{E}G
\end{array}
$$

where u_1 is the obvious G-map obtained by collapsing each EM_i to a point.

- Next we explain why $\underline{E}G$ is a model for the classifying space for proper actions of G.
- Its isotropy groups are all finite. We have to show for $H \subseteq G$ finite that $\underline{E}G^H$ contractible.
- We begin with the case $H \neq \{1\}$. Because of conditions (M) and (NM) there is precisely one index $i_0 \in I$ such that H is subconjugated to M_{i_0} and is not subconjugated to M_i for $i \neq i_0$. We get

$$\left(\coprod_{i \in I} G/M_i \right)^H = (G/M_{i_0})^H = \text{pt.}$$

Hence $\underline{E}G^H = \text{pt.}$
- It remains to treat $H = \{1\}$. Since u_1 is a non-equivariant homotopy equivalence and j_1 is a cofibration, f_1 is a non-equivariant homotopy equivalence. Hence $\underline{E}G$ is contractible.

- Consider the pushout obtained from the G-pushout above by dividing the G-action

$$
\begin{array}{ccc}
\coprod_{i \in I} BM_i & \longrightarrow & BG \\
\downarrow & & \downarrow \\
\coprod_{i \in I} \mathrm{pt} & \longrightarrow & G \backslash \underline{E} G
\end{array}
$$

- The associated Mayer-Vietoris sequence yields

$$
\ldots \to \widetilde{H}_{p+1}(G \backslash \underline{E} G) \to \bigoplus_{i \in I} \widetilde{H}_p(BM_i) \to \widetilde{H}_p(BG)
$$
$$
\to \widetilde{H}_p(G \backslash \underline{E} G) \to \ldots
$$

- In particular we obtain an isomorphism for $p \geq \dim(\underline{E} G) + 1$

$$
\bigoplus_{i \in I} H_p(BM_i) \xrightarrow{\cong} H_p(BG).
$$

Theorem

Let G be a discrete group which satisfies the conditions (M) and (NM) above.
Then there is an isomorphism

$$K_1^G(\underline{E}G) \xrightarrow{\cong} K_1(G \backslash \underline{E}G),$$

and a short exact sequence

$$0 \to \bigoplus_{i \in I} \widetilde{R}_{\mathbb{C}}(M_i) \to K_0(\underline{E}G) \to K_0(G \backslash \underline{E}G) \to 0.$$

which splits of we invert the orders of the finite subgroups of G.

- If the Baum-Connes Conjecture is true for G, then

$$K_n(C_r^*(G)) \cong K_n^G(\underline{E}G).$$

Example (One-relator groups)

- Let $G = \langle s_1, s_2, \ldots s_g \mid r \rangle$ be a finitely generated one-relator-group.
- The Baum-Connes Conjecture is known to be true for G.
- If G is torsionfree, the presentation complex associated to the presentation above is a model for BG and we get

$$K_n(C_r^*(G)) \cong K_n(BG) \cong \begin{cases} H_0(BG) \oplus H_2(BG) & n \text{ even}; \\ H_1(BG) & n \text{ odd}. \end{cases}$$

- Now suppose that G is not torsionfree.

Example (continued)

- Let F be the free group with basis $\{q_1, q_2, \ldots, q_g\}$. Then r is an element in F. There exists an element $s \in F$ and an integer $m \geq 2$ such that $r = s^m$, the cyclic subgroup C generated by the class $\overline{s} \in Q$ represented by s has order m, any finite subgroup of G is subconjugated to C and for any $g \in G$ the implication $g^{-1}Cg \cap C \neq 1 \Rightarrow g \in C$ holds.

- Hence G satisfies (M) and (NM).

- There is an explicit two-dimensional model for $\underline{E}G$ with one 0-cell $G/C \times D^0$, g 1-cells $G \times D^1$ and one free 2-cell $G \times D^2$.

Example (continued)

- We conclude for $n \geq 3$

$$H_n(BC) \cong H_n(BG).$$

- We obtain for odd n

$$K_n(C_r^*(G)) \cong K_1(G \backslash \underline{E}G) \cong H_1(G \backslash \underline{E}G).$$

- We obtain for even n

$$K_n(C_r^*(G)) \cong \widetilde{R}_{\mathbb{C}}(C) \oplus H_0(G \backslash \underline{E}G) \oplus H_2(G \backslash \underline{E}G).$$

Torsionfree hyperbolic groups

Theorem (Torsionfree hyperbolic groups)

Let G be a torsionfree hyperbolic group. Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal infinite cyclic subgroups of G.

1. *We get for $n \in \mathbb{Z}$ an isomorphism*

$$H_n\big(BG; \mathbf{K}(R)\big) \oplus \bigoplus_{V \in \mathcal{M}} NK_n(R) \oplus NK_n(R) \;\xrightarrow{\cong}\; K_n(RG);$$

2. *We get for $n \in \mathbb{Z}$ an isomorphism*

$$H_n\big(BG; \mathbf{L}^{\langle -\infty \rangle}(R)\big) \;\xrightarrow{\cong}\; L_n^{\langle -\infty \rangle}(RG).$$

> **Example (Finitely generated free groups)**
>
> Let F_r be the finitely generated free group $*_{i=1}^r \mathbb{Z}$ of rank r. Since it acts freely on a tree, it is hyperbolic. We obtain for $n \in \mathbb{Z}$
>
> $$K_n(RF_r) \cong K_n(R) \oplus K_{n-1}(R)^r \oplus \bigoplus_{V \in \mathcal{M}} NK_n(R)^2;$$
>
> $$L_n^{\langle -\infty \rangle}(RF_r) \cong L_n^{\langle -\infty \rangle}(R) \oplus L_{n-1}^{\langle -\infty \rangle}(R)^r,$$
>
> where $\mathcal{M}$ is a complete system of representatives of the conjugacy classes of maximal infinite cyclic subgroups of F_r.

Example (Surface groups)

- Let Γ_g be the fundamental group of the orientable closed surface of genus $g \geq 2$.

- Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal infinite cyclic subgroups of Γ_g.

- Then Γ_g is hyperbolic and we conclude using the Atiyah-spectral sequence that

$$K_n(R\Gamma_g) \cong K_n(R) \oplus H_n\big(B\Gamma_g, \mathrm{pt}; \mathbf{K}_R\big) \oplus \bigoplus_{V \in \mathcal{M}} NK_n(R)^2;$$

$$L_n^{\langle -\infty \rangle}(R\Gamma_g) \cong L_n^{\langle -\infty \rangle}(R) \oplus H_n\big(B\Gamma_g, \mathrm{pt}; \mathbf{L}\langle -\infty \rangle\big),$$

and there are short exact sequences

$$0 \to K_{n-1}(R)^{2g} \to H_n\big(B\Gamma_g, \mathrm{pt}; \mathbf{K}_R\big) \to K_{n-2}(R) \to 0;$$

$$0 \to L_{n-1}^{\langle -\infty \rangle}(R)^{2g} \to H_n\big(B\Gamma_g, \mathrm{pt}; \mathbf{L}_R^{\langle -\infty \rangle}\big) \to L_{n-2}^{\langle -\infty \rangle}(R) \to 0.$$

Example (continued)

- Suppose $R = \mathbb{Z}$.
- Then we obtain for every $i \in \{1, 0, -1, \ldots\} \amalg \{-\infty\}$ that $L_n^{\langle i \rangle}(\mathbb{Z})$ is $\mathbb{Z}$ if $n \equiv 0 \mod 4$, $\mathbb{Z}/2$ if $n \equiv 2 \mod 4$, and is trivial otherwise, and hence

$$L_n^{\langle i \rangle}(\mathbb{Z}\Gamma_g) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & n \equiv 0, 2 \mod 4; \\ \mathbb{Z}^g & n \equiv 1 \mod 4; \\ (\mathbb{Z}/2)^g & n \equiv 3 \mod 4. \end{cases}$$