

On hyperbolic groups with spheres as boundary

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Preview of the main result

Conjecture (Gromov (1994))

Let G be a hyperbolic group whose boundary is a sphere S^{n-1} . Then there is a closed aspherical manifold M with $\pi_1(M) \cong G$.

Theorem (Bartels-Lück-Weinberger (2009))

The Conjecture is true for $n \geq 6$.

Hyperbolic spaces and hyperbolic groups

Definition (Hyperbolic space)

A δ -hyperbolic space X is a geodesic space whose geodesic triangles are all δ -thin.

A geodesic space is called **hyperbolic** if it is δ -hyperbolic for some $\delta > 0$.

- A geodesic space with bounded diameter is hyperbolic.
- A tree is 0-hyperbolic.
- A simply connected complete Riemannian manifold M with $\sec(M) \leq \kappa$ for some $\kappa < 0$ is hyperbolic.
- \mathbb{R}^n is hyperbolic if and only if $n \leq 1$.

- Two geodesic rays $c_1, c_2: [0, \infty) \rightarrow X$ are called **equivalent** if there exists $C > 0$ satisfying $d_X(c_1(t), c_2(t)) \leq C$ for $t \in [0, \infty)$.

Definition (Boundary of a hyperbolic space)

Let X be a hyperbolic space. Define its **boundary** ∂X to be the set of equivalence classes of geodesic rays. Put

$$\bar{X} := X \amalg \partial X.$$

Lemma

There is a topology on \overline{X} with the properties:

- *\overline{X} is compact and metrizable;*
 - *The subspace topology $X \subseteq \overline{X}$ is the given one;*
 - *X is open and dense in \overline{X} .*
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- Let M be a simply connected complete Riemannian manifold M with $\sec(M) \leq \kappa$ for some $\kappa < 0$. Then M is hyperbolic and $\partial M = \mathcal{S}^{\dim(M)-1}$.

Definition (Quasi-isometry)

A map $f: X \rightarrow Y$ of metric spaces is called a **quasi-isometry** if there exist real numbers $\lambda, C > 0$ satisfying:

- The inequality

$$\lambda^{-1} \cdot d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq \lambda \cdot d_X(x_1, x_2) + C$$

holds for all $x_1, x_2 \in X$;

- For every y in Y there exists $x \in X$ with $d_Y(f(x), y) < C$.

Lemma (Švarc-Milnor Lemma)

Let X be a geodesic space. Suppose that G acts properly, cocompactly and isometrically on X . Choose a base point $x \in X$. Then the map

$$f: G \rightarrow X, \quad g \mapsto gx$$

is a quasiisometry.

Lemma (Quasi-isometry invariance of the Cayley graph)

The quasi-isometry type of the Cayley graph of a finitely generated group is independent of the choice of a finite set of generators.

Lemma (Quasi-isometry invariance of being hyperbolic)

The property “hyperbolic” is a quasi-isometry invariant of geodesic spaces.

Lemma (Quasi-isometry invariance of the boundary)

A quasi-isometry $f: X_1 \rightarrow X_2$ of hyperbolic spaces induces a homeomorphism

$$\partial X_1 \xrightarrow{\cong} \partial X_2.$$

Definition (Hyperbolic group)

A finitely generated group is called **hyperbolic** if its

- A group G is hyperbolic if and only if it acts properly, cocompactly and isometrically on a hyperbolic space. In this case $\partial G = \partial X$.
- Let M be a closed Riemannian manifold with $\sec(M) < 0$. Then $\pi_1(M)$ is hyperbolic with $S^{\dim(M)-1}$ as boundary.
- A subgroup of a hyperbolic group is either virtually cyclic or contains $\mathbb{Z} * \mathbb{Z}$ as subgroup.
- \mathbb{Z}^2 is not a subgroup of a hyperbolic group.
- If the boundary of a hyperbolic groups contains an open subset homeomorphic to \mathbb{R}^n , then the boundary is homeomorphic to S^n .
- A random finitely presented group is hyperbolic.

Gromov's Conjecture in low dimensions

Theorem (Casson-Jungreis (1994), Freden (1995), Gabai (1991))

A hyperbolic group has S^1 as boundary if and only if it is a Fuchsian group.

Conjecture (Cannon's Conjecture)

A hyperbolic group G has S^2 as boundary if and only if it acts properly, cocompactly and isometrically on \mathbb{H}^3 .

Theorem (Bestvina-Mess (1991))

Let G be an infinite hyperbolic group which is the fundamental group of a closed irreducible 3-manifold M .

- In dimension four the only hyperbolic groups which are known to be good in the sense of Freedman are virtually cyclic.
- Possibly our results hold also in dimension 5.

ANR-homology manifolds

Definition (Absolute neighborhood retract (ANR))

A topological space X is called **absolute neighborhood retract (ANR)** if it is normal and for every normal space Z , closed subset $Y \subseteq Z$ and map $f: Y \rightarrow X$ there is an open neighborhood $U \subseteq Z$ of Y and a map $F: U \rightarrow X$ extending f .

Definition (Homology ANR-manifold)

A **homology ANR-manifold** X is an ANR satisfying:

- X has a countable basis for its topology;
- The topological dimension of X is finite;
- X is locally compact;
- for every $x \in X$ we have for the singular homology

$$H_i(X, X - \{x\}; \mathbb{Z}) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$$

If X is additionally compact, it is called a **closed ANR-homology manifold**.

There is also the notion of a **compact ANR-homology manifold with boundary**.

- Every closed topological manifold is a closed ANR-homology manifold.
- Let M be homology sphere with non-trivial fundamental group. Then its suspension ΣM is a closed ANR-homology manifold but not a topological manifold.

Definition (Disjoint Disk Property (DDP))

A homology ANR-manifold M has the **disjoint disk property (DDP)**, if for any $\epsilon > 0$ and maps

$f, g: D^2 \rightarrow M$, there are maps $f', g': D^2 \rightarrow M$ so that f' is ϵ -close to f , g' is ϵ -close to g and $f'(D^2) \cap g'(D^2) = \emptyset$

- A topological manifold of dimension ≥ 5 is a closed ANR-homology manifold, which has the DDP by transversality.

Poincaré duality groups

Definition (Poincaré duality group)

A **Poincaré duality group** G of dimension n is a finitely presented group satisfying:

- G is of type FP;
- $H^i(G; \mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

Lemma

Let X be a closed aspherical ANR-homology manifold of dimension n . Then its fundamental group is a Poincaré duality group of dimension n .

Theorem (Poincaré duality groups and ANR-homology manifolds)

Bartels-Lück-Weinberger (2009)

Let G be a torsion-free group. Suppose that it satisfies the K - and L -theoretic Farrell-Jones Conjecture.

Consider $n \geq 6$.

Then the following statements are equivalent:

- 1 G is a Poincaré duality group of dimension n ;*
- 2 There exists a closed aspherical n -dimensional ANR-homology manifold M with $\pi_1(M) \cong G$;*
- 3 There exists a closed aspherical n -dimensional ANR-homology manifold M with $\pi_1(M) \cong G$ which has the DDP.*

The proof of the result above relies on:

- Surgery theory as developed by [Browder, Novikov, Sullivan, Wall](#) for smooth manifolds and its extension to topological manifolds using the work of [Kirby-Siebenmann](#).
- The algebraic surgery theory of [Ranicki](#)
- The surgery theory for ANR-manifolds due to [Bryant-Ferry-Mio-Weinberger](#) and basic ideas of [Quinn](#).
- The Farrell-Jones Conjecture.

The Farrell-Jones Conjecture

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups)

The *K*-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the *assembly map*

$$H_n(BG; \mathbf{K}_R) \rightarrow K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- There is also a version for L -theory.
- The most general version makes sense for all groups and all possible coefficient rings and twistings.

Theorem (Bartels-Lück-Reich(2008), Bartels-Lück(2009), Bartels-Farrell-Lück-Reich (2010))

Let \mathcal{FJ} be the class of groups for which both the K -theoretic and the L -theoretic Farrell-Jones Conjectures hold with coefficients in any additive G -category (with involution). It has the following properties:

- *Hyperbolic group and virtually nilpotent groups belongs to \mathcal{FJ} ;*
- *If G_1 and G_2 belong to \mathcal{FJ} , then $G_1 \times G_2$ and $G_1 * G_2$ belongs to \mathcal{FJ} ;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\text{colim}_i G_i$ belongs to \mathcal{FJ} .*

Theorem (continued)

- *If we demand for the K -theory version only that the assembly map is 1-connected and keep the full L -theory version, then the properties above remain valid and the class \mathcal{FJ} contains also all $\text{CAT}(0)$ -groups;*
 - *The last statement is also true all cocompact lattices in almost connected Lie groups.*
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- For all applications presented in these talks the version, where we demand for the K -theory version only that the assembly map is 1-connected and keep the full L -theory version, is sufficient.

Theorem (Bestvina-Mess (1991))

A hyperbolic G is a Poincaré duality group of dimension n if and only if its boundary and S^{n-1} have the same Čech cohomology.

Corollary

Let G be a torsion-free word-hyperbolic group. Let $n \geq 6$.

Then the following statements are equivalent:

- 1 The boundary ∂G has the integral Čech cohomology of S^{n-1} ;*
- 2 G is a Poincaré duality group of dimension n ;*
- 3 There exists a closed aspherical n -dimensional ANR-homology manifold M with $\pi_1(M) \cong G$;*

Quinn's resolution obstruction

Theorem (Quinn (1987))

There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ for homology ANR-manifolds with the following properties:

- *if $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;*
- *Let M be a homology ANR-manifold of dimension ≥ 5 . Then M is a topological manifold if and only if M has the DDP and $\iota(M) = 1$.*

Question

Does the Quinn obstruction always vanishes for aspherical closed homology ANR-manifolds?

- If the answer is yes, we can replace “closed ANR-homology manifold” by “closed topological manifold” in the theorem above.
- In general the Quinn obstruction is not a homotopy invariant but it is a homotopy invariant for aspherical closed ANR-homology manifolds provided the integral Novikov Conjecture holds.
- However, most experts expect the answer no.

Theorem (Quasi-isometry invariance of Quinn's resolution obstruction)

Bartels-Lück-Weinberger(2009)

Let G_1 and G_2 be torsionfree hyperbolic groups.

- Let G_1 and G_2 be quasi-isometric. Then G_1 is a Poincaré duality group of dimension n if and only if G_2 is;
- Let M_i be an aspherical closed ANR-homology manifold with $\pi_1(M_i) \cong G_i$. If ∂G_1 and ∂G_2 are homeomorphic, then the Quinn obstructions of M_1 and M_2 agree;
- Let G_1 and G_2 be quasi-isometric. Then there exists an aspherical closed topological manifold M_1 with $\pi_1(M_1) = G_1$ if and only if there exists an aspherical

Hyperbolic groups with spheres as boundary

Theorem (Hyperbolic groups with spheres as boundary Bartels-Lück-Weinberger(2009))

Let G be a torsion-free hyperbolic group and let n be an integer ≥ 6 .

Then the following statements are equivalent:

- 1 The boundary ∂G is homeomorphic to S^{n-1} ;*
- 2 There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering \tilde{M} is homeomorphic to \mathbb{R}^n and the compactification of \tilde{M} by ∂G is homeomorphic to D^n .*

Exotic Examples

By hyperbolization techniques due to Charney, Davis, Januskiewicz one can find the following examples:

Examples (Exotic universal coverings)

There are aspherical closed topological manifolds M with hyperbolic fundamental group $G = \pi_1(M)$ satisfying:

- For $n \geq 1$ the universal covering \tilde{M} is not homeomorphic to \mathbb{R}^n and ∂G is not homeomorphic to S^{n-1} .
- \tilde{M} is homeomorphic to \mathbb{R}^n but ∂G is not S^{n-1} .

Example (No smooth structures)

For every $k \geq 2$ there exists a torsionfree hyperbolic group G with $\partial G \cong S^{4k-1}$ such that there is no aspherical closed smooth manifold M with $\pi_1(M) \cong G$. In particular G is not the fundamental group of a closed smooth Riemannian manifold with $\sec(M) < 0$.