The role of lower and middle K-theory in topology (Lecture I)

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Outline

- Introduce the projective class group $K_0(R)$.
- Discuss its algebraic and topological significance (e.g., finiteness obstruction).
- Introduce $K_1(R)$ and the Whitehead group $\mathrm{Wh}(G)$.
- Discuss its algebraic and topological significance (e.g., s-cobordism theorem).
- Introduce negative K-theory and the Bass-Heller-Swan decomposition.

The projective class group

Definition (Projective R-module)

An R-module P is called *projective* if it satisfies one of the following equivalent conditions:

- P is a direct summand in a free R-module;
- The following lifting problem has always a solution

$$
\begin{array}{ccccc}
M & \xrightarrow{p} & N & \longrightarrow & 0 \\
 & {\scriptstyle \overline{f}}\nwarrow & \uparrow{\scriptstyle f} & & \\
 & & P & &
\end{array}
$$

- If $0 \to M_0 \to M_1 \to M_2 \to 0$ is an exact sequence of R-modules, then $0 \to \hom_R(P, M_0) \to \hom_R(P, M_1) \to \hom_R(P, M_2) \to 0$ is exact.

- Over a field or, more generally, over a principal ideal domain every projective module is free.
- If R is a principal ideal domain, then a finitely generated R-module is projective (and hence free) if and only if it is torsionfree. For instance $\mathbb{Z}/n$ is for $n \geq 2$ never projective as $\mathbb{Z}$-module.
- Let R and S be rings and $R \times S$ be their product. Then $R \times \{0\}$ is a finitely generated projective $R \times S$-module which is not free.

Example (Representations of finite groups)

Let F be a field of characteristic p for p a prime number or 0. Let G be a finite group.
Then F with the trivial G-action is a projective FG-module if and only if $p = 0$ or p does not divide the order of G. It is a free FG-module only if G is trivial.

Definition (Projective class group $K_0(R)$)

Let R be an (associative) ring (with unit). Define its *projective class group*

$$K_0(R)$$

to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

- This is the same as the Grothendieck construction applied to the abelian monoid of isomorphism classes of finitely generated projective R-modules under direct sum.
- The *reduced projective class group* $\widetilde{K}_0(R)$ is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R-modules, or, equivalently, the cokernel of $K_0(\mathbb{Z}) \to K_0(R)$.

- Let P be a finitely generated projective R-module. It is stably free, i.e., $P \oplus R^m \cong R^n$ for appropriate $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$.
- $\widetilde{K}_0(R)$ measures the deviation of finitely generated projective R-modules from being stably finitely generated free.
- The assignment $P \mapsto [P] \in K_0(R)$ is the universal additive invariant or dimension function for finitely generated projective R-modules.
- Induction

  Let $f\colon R \to S$ be a ring homomorphism. Given an R-module M, let f_*M be the S-module $S \otimes_R M$. We obtain a homomorphism of abelian groups

$$f_*\colon K_0(R) \to K_0(S), \quad [P] \mapsto [f_*P].$$

- Compatibility with products

  The two projections from $R \times S$ to R and S induce isomorphisms

$$K_0(R \times S) \xrightarrow{\cong} K_0(R) \times K_0(S).$$

- Morita equivalence

  Let R be a ring and $M_n(R)$ be the ring of (n, n)-matrices over R.
  We can consider R^n as a $M_n(R)$-R-bimodule and as a R-$M_n(R)$-bimodule.
  Tensoring with these yields mutually inverse isomorphisms

$$\begin{array}{lcllcl} K_0(R) & \xrightarrow{\cong} & K_0(M_n(R)), & [P] & \mapsto & [{}_{M_n(R)}R^n{}_R \otimes_R P]; \\ K_0(M_n(R)) & \xrightarrow{\cong} & K_0(R), & [Q] & \mapsto & [{}_R R^n{}_{M_n(R)} \otimes_{M_n(R)} Q]. \end{array}$$

Example (Principal ideal domains)

If R is a principal ideal domain. Let F be its quotient field. Then we obtain mutually inverse isomorphisms

$$\begin{array}{lclcl} \mathbb{Z} & \xrightarrow{\cong} & K_0(R), & n & \mapsto & [R^n]; \\ K_0(R) & \xrightarrow{\cong} & \mathbb{Z}, & [P] & \mapsto & \dim_F(F \otimes_R P). \end{array}$$

Example (Representation ring)

Let G be a finite group and let F be a field of characteristic zero. Then the representation ring $R_F(G)$ is the same as $K_0(FG)$. Taking the character of a representation yields an isomorphism

$$R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{C} = K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \text{class}(G, \mathbb{C}),$$

where $\text{class}(G; \mathbb{C})$ is the complex vector space of class functions $G \to \mathbb{C}$, i.e., functions, which are constant on conjugacy classes.

Example (Dedekind domains)

- Let R be a Dedekind domain, for instance the ring of integers in an algebraic number field.
- Call two ideals I and J in R equivalent if there exists non-zero elements r and s in R with $rI = sJ$. The ideal class group C(R) is the abelian group of equivalence classes of ideals under multiplication of ideals.
- Then we obtain an isomorphism

$$C(R) \xrightarrow{\cong} \widetilde{K}_0(R), \quad [I] \mapsto [I].$$

- The structure of the finite abelian group

$$C(\mathbb{Z}[\exp(2\pi i/p)]) \cong \widetilde{K}_0(\mathbb{Z}[\exp(2\pi i/p)]) \cong \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$$

is only known for small prime numbers p.

Theorem (Swan (1960))

If G is finite, then $\widetilde{K}_0(\mathbb{Z}G)$ is finite.

- Topological K-theory

  Let X be a compact space. Let $K^0(X)$ be the Grothendieck group of isomorphism classes of finite-dimensional complex vector bundles over X.

  This is the zero-th term of a generalized cohomology theory $K^*(X)$ called topological K-theory. It is 2-periodic, i.e., $K^n(X) = K^{n+2}(X)$, and satisfies $K^0(\mathrm{pt}) = \mathbb{Z}$ and $K^1(\mathrm{pt}) = \{0\}$.
- Let $C(X)$ be the ring of continuous functions from X to $\mathbb{C}$.

Theorem (Swan (1962))

There is an isomorphism

$$K^0(X) \xrightarrow{\cong} K_0(C(X)).$$

Wall's finiteness obstruction

Definition (Finitely dominated)

A CW-complex X is called *finitely dominated* if there exists a finite (= compact) CW-complex Y together with maps $i\colon X \to Y$ and $r\colon Y \to X$ satisfying $r \circ i \simeq \mathrm{id}_X$.

- A finite CW-complex is finitely dominated.
- A closed manifold is a finite CW-complex.

Problem

Is a given finitely dominated CW-complex homotopy equivalent to a finite CW-complex?

Definition (Wall's finiteness obstruction)

A finitely dominated CW-complex X defines an element

$$o(X) \in K_0(\mathbb{Z}[\pi_1(X)])$$

called its *finiteness obstruction* as follows.

- Let $\widetilde{X}$ be the universal covering. The fundamental group $\pi = \pi_1(X)$ acts freely on $\widetilde{X}$.
- Let $C_*(\widetilde{X})$ be the cellular chain complex. It is a free $\mathbb{Z}\pi$-chain complex.
- Since X is finitely dominated, there exists a finite projective $\mathbb{Z}\pi$-chain complex P_* with $P_* \simeq_{\mathbb{Z}\pi} C_*(\widetilde{X})$.
- Define

$$o(X) := \sum_n (-1)^n \cdot [P_n] \in K_0(\mathbb{Z}\pi).$$

Theorem (Wall (1965))

A finitely dominated CW-complex X is homotopy equivalent to a finite CW-complex if and only if its reduced finiteness obstruction $\widetilde{o}(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ vanishes.

- A finitely dominated simply connected CW-complex is always homotopy equivalent to a finite CW-complex since $\widetilde{K}_0(\mathbb{Z}) = \{0\}$.
- Given a finitely presented group G and $\xi \in K_0(\mathbb{Z}G)$, there exists a finitely dominated CW-complex X with $\pi_1(X) \cong G$ and $o(X) = \xi$.

Theorem (Geometric characterization of $\widetilde{K}_0(\mathbb{Z}G) = \{0\}$)

The following statements are equivalent for a finitely presented group G:

- *Every finite dominated CW-complex with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex.*
- $\widetilde{K}_0(\mathbb{Z}G) = \{0\}$.

Conjecture (Vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ for torsionfree G)

If G is torsionfree, then

$$\widetilde{K}_0(\mathbb{Z}G) = \{0\}.$$

The Whitehead group

Definition (K_1-group $K_1(R)$)

Define the *K_1-group of a ring R*

$$K_1(R)$$

to be the abelian group whose generators are conjugacy classes $[f]$ of automorphisms $f \colon P \to P$ of finitely generated projective R-modules with the following relations:

- Given an exact sequence $0 \to (P_0, f_0) \to (P_1, f_1) \to (P_2, f_2) \to 0$ of automorphisms of finitely generated projective R-modules, we get $[f_0] + [f_2] = [f_1]$;
- $[g \circ f] = [f] + [g]$.

- This is the same as $GL(R)/[GL(R), GL(R)]$.
- An invertible matrix $A \in GL(R)$ can be reduced by elementary row and column operations and (de-)stabilization to the trivial empty matrix if and only if $[A] = 0$ holds in the *reduced K_1-group*

$$\widetilde{K}_1(R) := K_1(R)/\{\pm 1\} = \operatorname{cok}\left(K_1(\mathbb{Z}) \to K_1(R)\right).$$

- If R is commutative, the determinant induces an epimorphism

$$\det\colon K_1(R) \to R^{\times},$$

  which in general is not bijective.
- The assignment $A \mapsto [A] \in K_1(R)$ can be thought of the universal determinant for R.

Definition (Whitehead group)

The *Whitehead group* of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

Lemma

We have $\mathrm{Wh}(\{1\}) = \{0\}$.

Proof.

- The ring $\mathbb{Z}$ possesses an Euclidean algorithm.
- Hence every invertible matrix over $\mathbb{Z}$ can be reduced via elementary row and column operations and destabilization to a $(1,1)$-matrix (± 1).
- This implies that any element in $K_1(\mathbb{Z})$ is represented by ± 1.

□

Let G be a finite group. Then:

- Let F be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.
  Define $r_F(G)$ to be the number of irreducible F-representations of G.
  This is the same as the number of F-conjugacy classes of elements of G.
  Here $g_1 \sim_{\mathbb{C}} g_2$ if and only if $g_1 \sim g_2$, i.e., $gg_1g^{-1} = g_2$ for some $g \in G$. We have $g_1 \sim_{\mathbb{R}} g_2$ if and only if $g_1 \sim g_2$ or $g_1 \sim g_2^{-1}$ holds. We have $g_1 \sim_{\mathbb{Q}} g_2$ if and only if $\langle g_1 \rangle$ and $\langle g_1 \rangle$ are conjugated as subgroups of G.
- The Whitehead group $\mathrm{Wh}(G)$ is a finitely generated abelian group.
- Its rank is $r_{\mathbb{R}}(G) - r_{\mathbb{Q}}(G)$.
- The torsion subgroup of $\mathrm{Wh}(G)$ is the kernel of the map $K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G)$.
- In contrast to $\widetilde{K}_0(\mathbb{Z}G)$ the Whitehead group $\mathrm{Wh}(G)$ is computable.

Whitehead torsion

Definition (*h*-cobordism)

An *h-cobordism* over a closed manifold M_0 is a compact manifold W whose boundary is the disjoint union $M_0 \amalg M_1$ such that both inclusions $M_0 \to W$ and $M_1 \to W$ are homotopy equivalences.

Theorem (*s*-Cobordism Theorem, Barden, Mazur, Stallings, Kirby-Siebenmann)

Let M_0 be a closed (smooth) manifold of dimension ≥ 5. Let $(W; M_0, M_1)$ be an h-cobordism over M_0.
Then W is homeomorphic (diffeomorpic) to $M_0 \times [0,1]$ relative M_0 if and only if its Whitehead torsion

$$\tau(W, M_0) \in \mathrm{Wh}(\pi_1(M_0))$$

vanishes.

Conjecture (Poincaré Conjecture)

Let M be an n-dimensional topological manifold which is a homotopy sphere, i.e., homotopy equivalent to S^n.
Then M is homeomorphic to S^n.

Theorem

For $n \geq 5$ the Poincaré Conjecture is true.

Proof.

We sketch the proof for $n \geq 6$.

- Let M be a n-dimensional homotopy sphere.
- Let W be obtained from M by deleting the interior of two disjoint embedded disks D_1^n and D_2^n. Then W is a simply connected h-cobordism.
- Since $\mathrm{Wh}(\{1\})$ is trivial, we can find a homeomorphism $f\colon W \xrightarrow{\cong} \partial D_1^n \times [0,1]$ which is the identity on $\partial D_1^n = D_1^n \times \{0\}$.
- By the Alexander trick we can extend the homeomorphism $f|_{D_1^n \times \{1\}}\colon \partial D_2^n \xrightarrow{\cong} \partial D_1^n \times \{1\}$ to a homeomorphism $g\colon D_1^n \to D_2^n$.
- The three homeomorphisms $id_{D_1^n}$, f and g fit together to a homeomorphism $h\colon M \to D_1^n \cup_{\partial D_1^n \times \{0\}} \partial D_1^n \times [0,1] \cup_{\partial D_1^n \times \{1\}} D_1^n$. The target is obviously homeomorphic to S^n.

□

- The argument above does not imply that for a smooth manifold M we obtain a diffeomorphism $g\colon M \to S^n$.
  The Alexander trick does not work smoothly.
  Indeed, there exists so called exotic spheres, i.e., closed smooth manifolds which are homeomorphic but not diffeomorphic to S^n.
- The s-cobordism theorem is a key ingredient in the surgery program for the classification of closed manifolds due to Browder, Novikov, Sullivan and Wall.
- Given a finitely presented group G, an element $\xi \in \mathrm{Wh}(G)$ and a closed manifold M of dimension $n \geq 5$ with $G \cong \pi_1(M)$, there exists an h-cobordism W over M with $\tau(W, M) = \xi$.

Theorem (Geometric characterization of $\mathrm{Wh}(G) = \{0\}$)

The following statements are equivalent for a finitely presented group G and a fixed integer $n \geq 6$

- *Every compact n-dimensional h-cobordism W with $G \cong \pi_1(W)$ is trivial;*
- $\mathrm{Wh}(G) = \{0\}$.

Conjecture (Vanishing of $\mathrm{Wh}(G)$ for torsionfree G)

If G is torsionfree, then

$$\mathrm{Wh}(G) = \{0\}.$$

Negative K-theory

Definition (Bass-Nil-groups)

Define for $n = 0, 1$

$$\mathrm{NK}_n(R) := \operatorname{coker}\left(K_n(R) \to K_n(R[t])\right).$$

Theorem (Bass-Heller-Swan decomposition for K_1 (1964))

There is an isomorphism, natural in R,

$$K_0(R) \oplus K_1(R) \oplus \mathrm{NK}_1(R) \oplus \mathrm{NK}_1(R) \xrightarrow{\cong} K_1(R[t, t^{-1}]) = K_1(R[\mathbb{Z}]).$$

Definition (Negative K-theory)

Define inductively for $n = -1, -2, \ldots$

$$K_n(R) := \operatorname{coker}\left(K_{n+1}(R[t]) \oplus K_{n+1}(R[t^{-1}]) \to K_{n+1}(R[t, t^{-1}])\right).$$

Define for $n = -1, -2, \ldots$

$$\mathrm{NK}_n(R) := \operatorname{coker}\left(K_n(R) \to K_n(R[t])\right).$$

Theorem (Bass-Heller-Swan decomposition for negative K-theory)

For $n \leq 1$ there is an isomorphism, natural in R,

$$K_{n-1}(R) \oplus K_n(R) \oplus \mathrm{NK}_n(R) \oplus \mathrm{NK}_n(R) \xrightarrow{\cong} K_n(R[t, t^{-1}]) = K_n(R[\mathbb{Z}]).$$

Definition (Regular ring)

A ring R is called *regular* if it is Noetherian and every finitely generated R-module possesses a finite projective resolution.

- Principal ideal domains are regular. In particular $\mathbb{Z}$ and any field are regular.
- If R is regular, then $R[t]$ and $R[t, t^{-1}] = R[\mathbb{Z}]$ are regular.
- If R is regular, then RG in general is not Noetherian or regular.

Theorem (Bass-Heller-Swan decomposition for regular rings)

Suppose that R is regular. Then

$$\begin{aligned} K_n(R) &= 0 \quad \text{for } n \leq -1; \\ \mathrm{NK}_n(R) &= 0 \quad \text{for } n \leq 1, \end{aligned}$$

and the Bass-Heller-Swan decomposition reduces for $n \leq 1$ to the natural isomorphism

$$K_{n-1}(R) \oplus K_n(R) \xrightarrow{\cong} K_n(R[t, t^{-1}]) = K_n(R[\mathbb{Z}]).$$

- There are also higher algebraic K-groups $K_n(R)$ for $n \geq 2$ due to Quillen (1973).
- They are defined as homotopy groups of certain spaces or spectra. We refer to the lectures of Grayson.
- Most of the well known features of $K_0(R)$ and $K_1(R)$ extend to both negative and higher algebraic K-theory.
  For instance the Bass-Heller-Swan decomposition holds also for higher algebraic K-theory.

- Notice the following formulas for a regular ring R and a generalized homology theory $\mathcal{H}_*$, which look similar:

$$\begin{aligned} K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R); \\ \mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\mathrm{pt}) \oplus \mathcal{H}_{n-1}(\mathrm{pt}). \end{aligned}$$

- If G and K are groups, then we have the following formulas, which look similar:

$$\begin{aligned} \widetilde{K}_n(\mathbb{Z}[G * K]) &\cong \widetilde{K}_n(\mathbb{Z}G) \oplus \widetilde{K}_n(\mathbb{Z}K); \\ \widetilde{\mathcal{H}}_n(B(G * K)) &\cong \widetilde{\mathcal{H}}_n(BG) \oplus \widetilde{\mathcal{H}}_n(BK). \end{aligned}$$

Question (K-theory of group rings and group homology)

Is there a relation between $K_n(RG)$ and group homology of G?

To be continued

Stay tuned

The Isomorphism Conjectures in the torsionfree case (Lecture II)

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Flashback

- We have introduced $K_n(R)$ for $n \in \mathbb{Z}, n \leq 1$.
- We have discussed the topological relevance of $K_0(RG)$ and the Whitehead group $\mathrm{Wh}(G)$, e.g., the finiteness obstruction and the s-cobordism theorem.
- We have stated the conjectures that $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ vanish for torsionfree G.
- We have presented the Bass-Heller-Swan decomposition and indicated some similarities between $K_n(RG)$ and group homology.
- Cliffhanger

Question (K-theory of group rings and group homology)

Is there a relation between $K_n(RG)$ and the group homology of G?

Outline

- We introduce spectra and how they yield homology theories.
- We state the Farrell-Jones-Conjecture and the Baum-Connes Conjecture for torsionfree groups.
- We discuss applications of these conjectures such as the Kaplansky Conjecture and the Borel Conjecture.
- We explain that the formulations for torsionfree groups cannot extend to arbitrary groups.

Definition (Spectrum)

A *spectrum*

$$\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$$

is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps*

$$\sigma(n)\colon E(n) \wedge S^1 \longrightarrow E(n+1).$$

A *map of spectra*

$$\mathbf{f}\colon \mathbf{E} \to \mathbf{E}'$$

is a sequence of maps $f(n)\colon E(n) \to E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e., $f(n+1) \circ \sigma(n) \;=\; \sigma'(n) \circ (f(n) \wedge \mathrm{id}_{S^1})$ holds for all $n \in \mathbb{Z}$.

- Given two pointed spaces $X = (X, x_0)$ and $Y = (Y, y_0)$, their one-point-union and their smash product are defined to be the pointed spaces

$$
\begin{aligned}
X \vee Y &:= \{(x, y_0) \mid x \in X\} \cup \{(x_0, y) \mid y \in Y\} \subseteq X \times Y; \\
X \wedge Y &:= (X \times Y)/(X \vee Y).
\end{aligned}
$$

  We have $S^{n+1} \cong S^n \wedge S^1$.
- The sphere spectrum $\mathbf{S}$ has as n-th space S^n and as n-th structure map the homeomorphism $S^n \wedge S^1 \xrightarrow{\cong} S^{n+1}$.
- Let X be a pointed space. Its suspension spectrum $\Sigma^\infty X$ is given by the sequence of spaces $\{X \wedge S^n \mid n \geq 0\}$ with the homeomorphism $(X \wedge S^n) \wedge S^1 \cong X \wedge S^{n+1}$ as structure maps. We have $\mathbf{S} = \Sigma^\infty S^0$.

Definition (Ω-spectrum)

Given a spectrum **E**, we can consider instead of the structure map $\sigma(n)\colon E(n) \wedge S^1 \to E(n+1)$ its adjoint

$$\sigma'(n)\colon E(n) \to \Omega E(n+1) = \operatorname{map}(S^1, E(n+1)).$$

We call **E** an *Ω-spectrum* if each map $\sigma'(n)$ is a weak homotopy equivalence.

Definition (Homotopy groups of a spectrum)

Given a spectrum **E**, define for $n \in \mathbb{Z}$ its *n-th homotopy group*

$$\pi_n(\mathbf{E}) := \operatorname*{colim}_{k \to \infty} \pi_{k+n}(E(k))$$

to be the abelian group which is given by the colimit over the directed system indexed by $\mathbb{Z}$ with k-th structure map

$$\pi_{k+n}(E(k)) \xrightarrow{\sigma'(k)} \pi_{k+n}(\Omega E(k+1)) = \pi_{k+n+1}(E(k+1)).$$

- Notice that a spectrum can have in contrast to a space non-trivial negative homotopy groups.
- If **E** is an Ω-spectrum, then $\pi_n(\mathbf{E}) = \pi_n(E(0))$ for all $n \geq 0$.

- Eilenberg-MacLane spectrum

  Let A be an abelian group. The n-th Eilenberg-MacLane space $EM(A,n)$ associated to A for $n \geq 0$ is a CW-complex with $\pi_m(EM(A,n)) = A$ for $m = n$ and $\pi_m(EM(A,n)) = \{0\}$ for $m \neq n$. The associated Eilenberg-MacLane spectrum $\mathbf{H}(A)$ has as n-th space $EM(A,n)$ and as n-th structure map a homotopy equivalence $EM(A,n) \to \Omega EM(A,n+1)$.

- Algebraic K-theory spectrum

  For a ring R there is the algebraic K-theory spectrum $\mathbf{K}_R$ with the property

$$\pi_n(\mathbf{K}_R) = K_n(R) \quad \text{for } n \in \mathbb{Z}.$$

- Algebraic L-theory spectrum

  For a ring with involution R there is the algebraic L-theory spectrum $\mathbf{L}_R^{\langle -\infty \rangle}$ with the property

$$\pi_n(\mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(R) \quad \text{for } n \in \mathbb{Z}.$$

- Topological K-theory spectrum

  By Bott periodicity there is a homotopy equivalence

$$\beta \colon BU \times \mathbb{Z} \xrightarrow{\simeq} \Omega^2(BU \times \mathbb{Z}).$$

  The topological K-theory spectrum $\mathbf{K}^{\text{top}}$ has in even degrees $BU \times \mathbb{Z}$ and in odd degrees $\Omega(BU \times \mathbb{Z})$.
  The structure maps are given in even degrees by the map β and in odd degrees by the identity $\mathrm{id} \colon \Omega(BU \times \mathbb{Z}) \to \Omega(BU \times \mathbb{Z})$.

Definition (Homology theory)

Let Λ be a commutative ring, for instance $\mathbb{Z}$ or $\mathbb{Q}$.
A *homology theory* $\mathcal{H}_*$ with values in Λ-modules is a covariant functor from the category of CW-pairs to the category of $\mathbb{Z}$-graded Λ-modules together with natural transformations

$$\partial_n(X, A) \colon \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- Homotopy invariance
- Long exact sequence of a pair
- Excision

  If (X, A) is a CW-pair and $f \colon A \to B$ is a cellular map , then

$$\mathcal{H}_n(X, A) \xrightarrow{\cong} \mathcal{H}_n(X \cup_f B, B).$$

Definition (continued)

- Disjoint union axiom

$$\bigoplus_{i \in I} \mathcal{H}_n(X_i) \xrightarrow{\cong} \mathcal{H}_n\left(\coprod_{i \in I} X_i\right).$$

Definition (Smash product)

Let **E** be a spectrum and X be a pointed space. Define the smash product $X \wedge \mathbf{E}$ to be the spectrum whose n-th space is $X \wedge E(n)$ and whose n-th structure map is

$$X \wedge E(n) \wedge S^1 \xrightarrow{\mathrm{id}_X \wedge \sigma(n)} X \wedge E(n+1).$$

Theorem (Homology theories and spectra)

Let $\mathbf{E}$ *be a spectrum. Then we obtain a homology theory* $H_*(-;\mathbf{E})$ *by*

$$H_n(X,A;\mathbf{E}) := \pi_n\left((X \cup_A \operatorname{cone}(A)) \wedge \mathbf{E}\right).$$

It satisfies

$$H_n(pt;\mathbf{E}) = \pi_n(\mathbf{E}).$$

Example (Stable homotopy theory)

The homology theory associated to the sphere spectrum **S** is stable homotopy $\pi_*^s(X)$. The groups $\pi_n^s(\mathrm{pt})$ are finite abelian groups for $n \neq 0$ by a result of Serre (1953). Their structure is only known for small n.

Example (Singular homology theory with coefficients)

The homology theory associated to the Eilenberg-MacLane spectrum $\mathbf{H}(A)$ is singular homology with coefficients in A.

Example (Topological K-homology)

The homology theory associated to the topological K-theory spectrum $\mathbf{K}^{\mathrm{top}}$ is K-homology $K_*(X)$. We have

$$K_n(\mathrm{pt}) \cong \begin{cases} \mathbb{Z} & n \text{ even;} \\ \{0\} & n \text{ odd.} \end{cases}$$

The Isomorphism Conjectures for torsionfree groups

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(RG)$ is the algebraic K-theory of the group ring RG;
- $\mathbf{K}_R$ is the (non-connective) algebraic K-theory spectrum of R;
- $H_n(\mathrm{pt}; \mathbf{K}_R) \cong \pi_n(\mathbf{K}_R) \cong K_n(R)$ for $n \in \mathbb{Z}$.
- BG is the classifying space of the group G, i.e., the base space of the universal G-principal G-bundle $G \to EG \to BG$. Equivalently, $BG = EM(G, 1)$. The space BG is unique up to homotopy.

Conjecture (*L*-theoretic Farrell-Jones Conjecture for torsionfree groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$*.*

- $L_n^{\langle -\infty \rangle}(RG)$ is the algebraic L-theory of RG with decoration $\langle -\infty \rangle$;
- $\mathbf{L}_R^{\langle -\infty \rangle}$ is the algebraic L-theory spectrum of R with decoration $\langle -\infty \rangle$;
- $H_n(\mathrm{pt}; \mathbf{L}_R^{\langle -\infty \rangle}) \cong \pi_n(\mathbf{L}_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(R)$ for $n \in \mathbb{Z}$.

Conjecture (Baum-Connes Conjecture for torsionfree groups)

The Baum-Connes Conjecture for the torsionfree group predicts that the assembly map

$$K_n(BG) \to K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(BG)$ is the topological K-homology of BG, where $K_*(-) = H_*(-; \mathbf{K}^{\mathrm{top}})$ for $\mathbf{K}^{\mathrm{top}}$ the topological K-theory spectrum.
- $K_n(C_r^*(G))$ is the topological K-theory of the reduced complex group C^*-algebra $C_r^*(G)$ of G which is the closure in the norm topology of $\mathbb{C}G$ considered as subalgebra of $\mathcal{B}(l^2(G))$.
- There is also a real version of the Baum-Connes Conjecture

$$KO_n(BG) \to K_n(C_r^*(G; \mathbb{R})).$$

Consequences of the Isomorphism Conjectures for torsionfree groups

- In order to illustrate the depth of the Farrell-Jones Conjecture and the Baum-Connes Conjecture, we present some conclusions which are interesting in their own right.
- Let $\mathcal{FJ}_K(R)$ and $\mathcal{FJ}_L(R)$ respectively be the class of groups which satisfy the K-theoretic and L-theoretic respectively Farrell-Jones Conjecture for the coefficient ring R.
- Let $\mathcal{BC}$ be the class of groups which satisfy the Baum-Connes Conjecture.

Lemma

Let R be a regular ring. Suppose that G is torsionfree and $G \in \mathcal{FJ}_K(R)$. Then

- $K_n(RG) = 0$ *for* $n \leq -1$*;*
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective. In particular* $\widetilde{K}_0(RG)$ *is trivial if and only if* $\widetilde{K}_0(R)$ *is trivial.*

Lemma

Suppose that G is torsionfree and $G \in \mathcal{FJ}_K(\mathbb{Z})$. Then the Whitehead group $\operatorname{Wh}(G)$ *is trivial.*

Proof.

The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG; \mathbf{K}_R)$ whose E^2-term is given by

$$E^2_{p,q} = H_p(BG, K_q(R)).$$

Proof (continued).

- Since R is regular by assumption, we get $K_q(R) = 0$ for $q \leq -1$.
- Hence the edge homomorphism yields an isomorphism

$$K_0(R) = H_0(\mathrm{pt}, K_0(R)) \xrightarrow{\cong} H_0(BG; \mathbf{K}_R) \cong K_0(RG).$$

- We have $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = \{\pm 1\}$. We get an exact sequence

$$0 \to H_0(BG; K_1(\mathbb{Z})) = \{\pm 1\} \to H_1(BG; \mathbf{K}_{\mathbb{Z}}) \cong K_1(\mathbb{Z}G) \\ \to H_1(BG; K_0(\mathbb{Z})) = G/[G, G] \to 0.$$

- This implies $\mathrm{Wh}(G) := K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\} \cong 0$.

□

In particular we get for a torsionfree group $G \in \mathcal{FJ}_K(\mathbb{Z})$:

- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
- $\widetilde{K}_0(\mathbb{Z}G) = 0$;
- $\mathrm{Wh}(G) = 0$;
- Every finitely dominated CW-complex X with $G = \pi_1(X)$ is homotopy equivalent to a finite CW-complex;
- Every compact h-cobordism W of dimension ≥ 6 with $\pi_1(W) \cong G$ is trivial;
- If G belongs to $\mathcal{FJ}_K(\mathbb{Z})$, then it is of type FF if and only if it is of type FP (Serre's problem).

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Theorem (The Farrell-Jones Conjecture and the Kaplansky Conjecture, Bartels-L.-Reich(2007))

Let F be a skew-field and let G be a group with $G \in \mathcal{FJ}_K(F)$. Suppose that one of the following conditions is satisfied:

- *F is commutative and has characteristic zero and G is torsionfree;*
- *G is torsionfree and sofic, e.g., residually amenable;*
- *The characteristic of F is p, all finite subgroups of G are p-groups and G is sofic.*

Then 0 and 1 are the only idempotents in FG.

Proof.

- Let p be an idempotent in FG. We want to show $p \in \{0,1\}$.
- Denote by $\epsilon \colon FG \to F$ the augmentation homomorphism sending $\sum_{g \in G} r_g \cdot g$ to $\sum_{g \in G} r_g$. Obviously $\epsilon(p) \in F$ is 0 or 1. Hence it suffices to show $p = 0$ under the assumption that $\epsilon(p) = 0$.
- Let $(p) \subseteq FG$ be the ideal generated by p which is a finitely generated projective FG-module.
  Since $G \in \mathcal{FJ}_K(F)$, we can conclude that

$$i_* \colon K_0(F) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

  is surjective.
  Hence we can find a finitely generated projective F-module P and integers $k, m, n \geq 0$ satisfying

$$(p)^k \oplus FG^m \cong_{FG} i_*(P) \oplus FG^n.$$

Proof (continued).

- If we now apply $i_* \circ \epsilon_*$ and use $\epsilon \circ i = \mathrm{id}$, $i_* \circ \epsilon_*(FG^l) \cong FG^l$ and $\epsilon(p) = 0$ we obtain

$$FG^m \cong i_*(P) \oplus FG^n.$$

- Inserting this in the first equation yields

$$(p)^k \oplus i_*(P) \oplus FG^n \cong i_*(P) \oplus FG^n.$$

- Our assumptions on F and G imply that FG is stably finite, i.e., if A and B are square matrices over FG with $AB = I$, then $BA = I$. This implies $(p)^k = 0$ and hence $p = 0$.

□

Theorem (The Baum-Connes Conjecture and the Kaplansky Conjecture)

Let G be a torsionfree group with $G \in \mathcal{BC}$. Then 0 and 1 are the only idempotents in $\mathbb{C}G$.

Proof.

- There is a trace map
$$\mathrm{tr}\colon C_r^*(G) \to \mathbb{C}$$
which sends $f \in C_r^*(G) \subseteq \mathcal{B}(l^2(G))$ to $\langle f(e), e\rangle_{l^2(G)}$.
- The L^2-index theorem due to Atiyah (1976) shows that the composite
$$K_0(BG) \to K_0(C_r^*(G)) \xrightarrow{\mathrm{tr}} \mathbb{C}$$
coincides with
$$K_0(BG) \xrightarrow{K_0(\mathrm{pr})} K_0(\mathrm{pt}) = \mathbb{Z} \xrightarrow{i} \mathbb{C}.$$

Proof (continued).

- Hence $G \in \mathcal{BC}$ implies $\mathrm{tr}(p) \in \mathbb{Z}$.
- Since $\mathrm{tr}(1) = 1$, $\mathrm{tr}(0) = 0$, $0 \leq p \leq 1$ and $p^2 = p$, we get $\mathrm{tr}(p) \in \mathbb{R}$ and $0 \leq \mathrm{tr}(p) \leq 1$.
- We conclude $\mathrm{tr}(0) = \mathrm{tr}(p)$ or $\mathrm{tr}(1) = \mathrm{tr}(p)$.
- This implies already $p = 0$ or $p = 1$.

□

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that M and N are homeomorphic.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity. A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds of dimension ≥ 3 is homotopic to an isometric diffeomorphism.
- The Borel Conjecture is not true in the smooth category by results of Farrell-Jones(1989).
- There are also non-aspherical manifolds which are topologically rigid in the sense of the Borel Conjecture (see Kreck-L. (2005)).

Theorem (The Farrell-Jones Conjecture and the Borel Conjecture)

If the K- and L-theoretic Farrell-Jones Conjecture hold for G in the case $R = \mathbb{Z}$, then the Borel Conjecture is true in dimension ≥ 5 and in dimension 4 if G is good in the sense of Freedman.

- Thurston's Geometrization Conjecture implies the Borel Conjecture in dimension 3.
- The Borel Conjecture in dimension 1 and 2 is obviously true.

Definition (Structure set)

The *structure set* $S^{top}(M)$ of a manifold M consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold N as source.
Two such homotopy equivalences $f_0 \colon N_0 \to M$ and $f_1 \colon N_1 \to M$ are equivalent if there exists a homeomorphism $g \colon N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

Theorem

The Borel Conjecture holds for a closed manifold M if and only if $\mathcal{S}^{\text{top}}(M)$ consists of one element.

Theorem (Ranicki (1992))

There is an exact sequence of abelian groups, called algebraic surgery exact sequence, for an n-dimensional closed manifold M

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}} \mathcal{S}^{\mathrm{top}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

It can be identified with the classical geometric surgery sequence due to Sullivan and Wall in high dimensions.

- $\mathcal{S}^{\mathrm{top}}(M)$ consist of one element if and only if A_{n+1} is surjective and A_n is injective.
- $H_k(M;\mathbf{L}\langle 1\rangle) \to H_k(M;\mathbf{L})$ is bijective for $k \geq n+1$ and injective for $k = n$.

What happens for groups with torsion?

- The versions of the Farrell-Jones Conjecture and the Baum-Connes Conjecture above become false for finite groups unless the group is trivial.
- For instance the version of the Baum-Connes Conjecture above would predict for a finite group G

$$K_0(BG) \cong K_0(C_r^*(G)) \cong R_{\mathbb{C}}(G).$$

However, $K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} K_0(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} \mathbb{Q}$ and $R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} \mathbb{Q}$ holds if and only if G is trivial.

- If G is torsionfree, then the version of the K-theoretic Farrell-Jones Conjecture predicts

$$H_n(B\mathbb{Z};\mathbf{K}_R) = H_n(S^1;\mathbf{K}_R) = H_n(\mathrm{pt};\mathbf{K}_R) \oplus H_{n-1}(\mathrm{pt};\mathbf{K}_R)$$
$$= K_n(R) \oplus K_{n-1}(R) \cong K_n(R\mathbb{Z}).$$

  In view of the Bass-Heller-Swan decomposition this is only possible if $\mathrm{NK}_n(R)$ vanishes which is true for regular rings R but not for general rings R.
- We want to figure out what is needed for a general version which may be true for all groups.

- **Assembly**

  For a field F of characteristic zero and some groups G one knows that there is an isomorphism

$$\operatorname*{colim}_{\substack{H \subseteq G \\ |H| < \infty}} K_0(FH) \xrightarrow{\cong} K_0(FG).$$

  This indicates that one has at least to take into account the values for all finite subgroups to assemble $K_n(FG)$.

- **Degree Mixing**

  The Bass-Heller-Swan decomposition shows that the K-theory of finite subgroups in degree $m \leq n$ can affect the K-theory in degree n and that at least in the Farrell-Jones setting finite subgroups are not enough.

- In the Baum-Connes setting Nil-phenomena do not appear. Namely, a special case of a result due to Pimsner-Voiculescu (1982) says

$$K_n(C_r^*(G \times \mathbb{Z})) \cong K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)).$$

- Homological behaviour

  There is still a lot of homological behaviour known for $K_*(C_r^*(G))$. For instance there exists a long exact Mayer-Vietoris sequence associated to amalgamated products $G_1 *_{G_0} G_2$ and a Wang-sequence associated to semi-direct products $G \rtimes \mathbb{Z}$ by Pimsner-Voiculescu (1982).
  Similar versions under certain restrictions exist in K-and L-theory due to Cappell (1974) and Waldhausen (1978) if one makes certain assumptions on R or ignores certain Nil-phenomena.

Question (Classifying spaces for families)

Is there a version $E_{\mathcal{F}}(G)$ of the classifying space EG which takes the structure of the family of finite subgroups or other families $\mathcal{F}$ of subgroups into account and can be used for a general formulation of the Farrell-Jones Conjecture?

Question (Equivariant homology theories)

Can one define appropriate G-homology theories $\mathcal{H}_*^G$ that are in some sense computable and yield when applied to $E_{\mathcal{F}}(G)$ a term which potentially is isomorphic to the groups $K_n(RG)$, $L^{-\langle\infty\rangle}(RG)$ or $K_n(C_r^*(G))$?
In the torsionfree case they should reduce to $H_n(BG; \mathbf{K}_R)$, $H_n(BG; \mathbf{L}^{-\langle\infty\rangle})$ and $K_n(BG)$.

To be continued

Stay tuned

Classifying spaces for families (Lecture III)

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Flashback

- We have introduced the Farrell-Jones Conjecture and the Baum-Connes Conjecture for torsionfree groups:

$$\begin{array}{rcl} H_n(BG;\mathbf{K}_R) & \xrightarrow{\cong} & K_n(RG); \\ H_n(BG;\mathbf{L}_R^{\langle -\infty \rangle}) & \xrightarrow{\cong} & L_n^{\langle -\infty \rangle}(RG); \\ K_n(BG) & \xrightarrow{\cong} & K_n(C_r^*(G)). \end{array}$$

- We have discussed applications of these conjectures such as to the Kaplansky Conjecture and the Borel Conjecture.

- Cliffhanger

Question (Classifying spaces for families)

Is there a version $E_{\mathcal{F}}(G)$ of the classifying space EG which takes the structure of the family of finite subgroups or other families $\mathcal{F}$ of subgroups into account and can be used for a general formulation of the Farrell-Jones Conjecture?

Outline

- We introduce the notion of the classifying space of a family $\mathcal{F}$ of subgroups $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$.
- In the case, where $\mathcal{F}$ is the family $\mathcal{COM}$ of compact subgroups, we present some nice geometric models for $E_{\mathcal{F}}(G)$ and explain $E_{\mathcal{F}}(G) \simeq J_{\mathcal{F}}(G)$.
- We discuss finiteness properties of these classifying spaces.

Classifying spaces for families of subgroups

Definition (*G*-*CW*-complex)

A *G-CW-complex* X is a G-space together with a G-invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n \geq 0} X_n = X$$

such that X carries the colimit topology with respect to this filtration, and X_n is obtained from X_{n-1} for each $n \geq 0$ by attaching equivariant n-dimensional cells, i.e., there exists a G-pushout

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n \end{array}$$

- Group means locally compact Hausdorff topological group with a countable basis for its topology, unless explicitly stated differently.

Example (Simplicial actions)

Let X be a simplicial complex. Suppose that G acts simplicially on X. Then G acts simplicially also on the barycentric subdivision X', and all isotropy groups are open and closed. The G-space X' inherits the structure of a G-CW-complex.

Example (Smooth actions)

Let G be a Lie group acting properly and smoothly on a smooth manifold M.
Then M inherits the structure of G-CW-complex.

Definition (Proper G-action)

A G-space X is called *proper* if for each pair of points x and y in X there are open neighborhoods V_x of x and W_y of y in X such that the closure of the subset $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$ of G is compact.

Lemma

- *A proper G-space has always compact isotropy groups.*
- *A G-CW-complex X is proper if and only if all its isotropy groups are compact.*

Definition (Family of subgroups)

A *family $\mathcal{F}$ of subgroups* of G is a set of (closed) subgroups of G which is closed under conjugation and finite intersections.

Examples for $\mathcal{F}$ are:

$$
\begin{array}{lcl}
\mathcal{TR} & = & \{\text{trivial subgroup}\}; \\
\mathcal{FIN} & = & \{\text{finite subgroups}\}; \\
\mathcal{VCYC} & = & \{\text{virtually cyclic subgroups}\}; \\
\mathcal{COM} & = & \{\text{compact subgroups}\}; \\
\mathcal{COMOP} & = & \{\text{compact open subgroups}\}; \\
\mathcal{ALL} & = & \{\text{all subgroups}\}.
\end{array}
$$

Definition (Classifying G-CW-complex for a family of subgroups)

Let $\mathcal{F}$ be a family of subgroups of G. A model for the *classifying G-CW-complex for the family $\mathcal{F}$* is a G-CW-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to E_{\mathcal{F}}(G)$.

We abbreviate $\underline{E}G := E_{\mathcal{COM}}(G)$ and call it the *universal G-CW-complex for proper G-actions*.
We also write $EG = E_{\mathcal{TR}}(G)$.

Theorem (Homotopy characterization of $E_{\mathcal{F}}(G)$)

Let $\mathcal{F}$ be a family of subgroups.

- *There exists a model for $E_{\mathcal{F}}(G)$ for any family $\mathcal{F}$;*
- *Two model for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent;*
- *A G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy groups belong to $\mathcal{F}$ and for each $H \in \mathcal{F}$ the H-fixed point set X^H is weakly contractible.*

- A model for $E_{\mathcal{ALL}}(G)$ is G/G;
- $EG \to BG := G\backslash EG$ is the universal G-principal bundle for G-CW-complexes.

Example (Infinite dihedral group)

- Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 * \mathbb{Z}/2$ be the infinite dihedral group.
- A model for ED_∞ is the universal covering of $\mathbb{RP}^\infty \vee \mathbb{RP}^\infty$.
- A model for $\underline{E}D_\infty$ is $\mathbb{R}$ with the obvious D_∞-action.

Lemma

If G is totally disconnected, then $E_{\mathcal{COMOP}}(G) = \underline{E}G$.

Definition ($\mathcal{F}$-numerable G-space)

A *$\mathcal{F}$-numerable G-space* is a G-space, for which there exists an open covering $\{U_i \mid i \in I\}$ by G-subspaces satisfying:

- For each $i \in I$ there exists a G-map $U_i \to G/G_i$ for some $G_i \in \mathcal{F}$;
- There is a locally finite partition of unity $\{e_i \mid i \in I\}$ subordinate to $\{U_i \mid i \in I\}$ by G-invariant functions $e_i \colon X \to [0, 1]$.

- Notice that we do not demand that the isotropy groups of a $\mathcal{F}$-numerable G-space belong to $\mathcal{F}$.
- If $f \colon X \to Y$ is a G-map and Y is $\mathcal{F}$-numerable, then X is also $\mathcal{F}$-numerable.
- A G-CW-complex is $\mathcal{F}$-numerable if and only if each isotropy group appears as a subgroup of an element in $\mathcal{F}$.

- There is also a version $J_{\mathcal{F}}(G)$ of a classifying space for $\mathcal{F}$-numerable G-spaces.
- It is characterized by the property that $J_{\mathcal{F}}(G)$ is $\mathcal{F}$-numerable and for every $\mathcal{F}$-numerable G-space Y there is up to G-homotopy precisely one G-map $Y \to J_{\mathcal{F}}(G)$.
- We abbreviate $\underline{J}G = J_{\mathcal{COM}}(G)$ and $JG = J_{\mathcal{TR}}(G)$.
- $JG \to G\backslash JG$ is the universal G-principal bundle for numerable free proper G-spaces.

Theorem (Comparison of $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$, L. (2005))

- *There is up to G-homotopy precisely one G-map*

$$\phi\colon E_{\mathcal{F}}(G) \to J_{\mathcal{F}}(G);$$

- *It is a G-homotopy equivalence if one of the following conditions is satisfied:*
  - *Each element in $\mathcal{F}$ is open and closed;*
  - *G is discrete;*
  - *$\mathcal{F}$ is $\mathcal{COM}$;*
- *Let G be totally disconnected. Then $EG \to JG$ is a G-homotopy equivalence if and only if G is discrete.*

Special models for $\underline{E}G$

- We want to illustrate that the space $\underline{E}G = \underline{J}G$ often has very nice geometric models and appear naturally in many interesting situations.
- Let $C_0(G)$ be the Banach space of complex valued functions of G vanishing at infinity with the supremum-norm. The group G acts isometrically on $C_0(G)$ by $(g \cdot f)(x) := f(g^{-1}x)$ for $f \in C_0(G)$ and $g, x \in G$.
  Let $PC_0(G)$ be the subspace of $C_0(G)$ consisting of functions f such that f is not identically zero and has non-negative real numbers as values.

Theorem (Operator theoretic model, Abels (1978))

The G-space $PC_0(G)$ is a model for $\underline{J}G$.

Theorem

Let G be discrete. A model for $\underline{J}G$ is the space

$$X_G = \left\{ f \colon G \to [0,1] \;\middle|\; f \text{ has finite support}, \sum_{g \in G} f(g) = 1 \right\}$$

with the topology coming from the supremum norm.

Theorem (Simplicial Model)

Let G be discrete. Let $P_\infty(G)$ be the geometric realization of the simplicial set whose k-simplices consist of $(k+1)$-tupels $(g_0, g_1, \ldots, g_k)$ of elements g_i in G. This is a model for $\underline{E}G$.

- The spaces X_G and $P_\infty(G)$ have the same underlying sets but in general they have different topologies.
- The identity map induces a G-map $P_\infty(G) \to X_G$ which is a G-homotopy equivalence, but in general not a G-homeomorphism.

Theorem (Almost connected groups, Abels (1978).)

Suppose that G is almost connected, i.e., the group G/G^0 is compact for G^0 the component of the identity element.
Then G contains a maximal compact subgroup K which is unique up to conjugation, and the G-space G/K is a model for $\underline{J}G$.

- As a special case we get:

Theorem (Discrete subgroups of almost connected Lie groups)

Let L be a Lie group with finitely many path components.
Then L contains a maximal compact subgroup K which is unique up to conjugation, and the L-space L/K is a model for $\underline{E}L$.
If $G \subseteq L$ is a discrete subgroup of L, then L/K with the obvious left G-action is a finite dimensional G-CW-model for $\underline{E}G$.

Theorem (Actions on CAT(0)-spaces)

Let G be a (locally compact Hausdorff) topological group. Let X be a proper G-CW-complex. Suppose that X has the structure of a complete simply connected CAT(0)*-space for which G acts by isometries.*
Then X is a model for $\underline{E}G$.

- The result above contains as special case isometric G actions on simply-connected complete Riemannian manifolds with non-positive sectional curvature and G-actions on trees.

Theorem (Affine buildings)

Let G be a totally disconnected group. Suppose that G acts on the affine building Σ by simplicial automorphisms such that each isotropy group is compact.
Then Σ is a model for both $J_{\mathcal{COMOP}}(G)$ and $\underline{J}G$ and the barycentric subdivision Σ' is a model for both $E_{\mathcal{COMOP}}(G)$ and $\underline{E}G$.

- An important example is the case of a reductive p-adic algebraic group G and its associated affine Bruhat-Tits building $\beta(G)$. Then $\beta(G)$ is a model for $\underline{J}G$ and $\beta(G)'$ is a model for $\underline{E}G$ by the previous result.
- For more information about buildings we refer to the lectures of Abramenko.

- The Rips complex $P_d(G,S)$ of a group G with a symmetric finite set S of generators for a natural number d is the geometric realization of the simplicial set whose set of k-simplices consists of $(k+1)$-tuples $(g_0, g_1, \ldots g_k)$ of pairwise distinct elements $g_i \in G$ satisfying $d_S(g_i, g_j) \leq d$ for all $i, j \in \{0, 1, \ldots, k\}$.
- The obvious G-action by simplicial automorphisms on $P_d(G,S)$ induces a G-action by simplicial automorphisms on the barycentric subdivision $P_d(G,S)'$.

Theorem (Rips complex, Meintrup-Schick (2002))

Let G be a discrete group with a finite symmetric set of generators. Suppose that (G,S) is δ-hyperbolic for the real number $\delta \geq 0$. Let d be a natural number with $d \geq 16\delta + 8$.
Then the barycentric subdivision of the Rips complex $P_d(G,S)'$ is a finite G-CW-model for $\underline{E}G$.

- Arithmetic groups in a semisimple connected linear $\mathbb{Q}$-algebraic group possess finite models for $\underline{E}G$.
- Namely, let $G(\mathbb{R})$ be the $\mathbb{R}$-points of a semisimple $\mathbb{Q}$-group $G(\mathbb{Q})$ and let $K \subseteq G(\mathbb{R})$ be a maximal compact subgroup.
- If $A \subseteq G(\mathbb{Q})$ is an arithmetic group, then $G(\mathbb{R})/K$ with the left A-action is a model for $\underline{E}A$ as already explained above.
- The A-space $G(\mathbb{R})/K$ is not necessarily cocompact.

Theorem (Borel-Serre compactification)

The Borel-Serre compactification of $G(\mathbb{R})/K$ is a finite A-CW-model for $\underline{E}A$.

- For more information about arithmetic groups we refer to the lectures of Abramenko.

- Let $\Gamma_{g,r}^s$ be the mapping class group of an orientable compact surface F of genus g with s punctures and r boundary components.
  We will always assume that $2g + s + r > 2$, or, equivalently, that the Euler characteristic of the punctured surface F is negative.
- It is well-known that the associated Teichmüller space $\mathcal{T}_{g,r}^s$ is a contractible space on which $\Gamma_{g,r}^s$ acts properly.

Theorem (Teichmüller space)

The $\Gamma_{g,r}^s$-space $\mathcal{T}_{g,r}^s$ is a model for $\underline{E}\Gamma_{g,r}^s$.

- Let F_n be the free group of rank n.
- Denote by $\operatorname{Out}(F_n)$ the group of outer automorphisms of F_n, i.e., the quotient of the group of all automorphisms of F_n by the normal subgroup of inner automorphisms.
- Culler-Vogtmann (1996) have constructed a space X_n called outer space on which $\operatorname{Out}(F_n)$ acts with finite isotropy groups. It is analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface.
- The space X_n contains a spine K_n which is an $\operatorname{Out}(F_n)$-equivariant deformation retraction. This space K_n is a simplicial complex of dimension $(2n-3)$ on which the $\operatorname{Out}(F_n)$-action is by simplicial automorphisms and cocompact.

Theorem (Spine of outer space)

The barycentric subdivision K_n' is a finite $(2n-3)$-dimensional model of $\underline{E}\operatorname{Out}(F_n)$.

Example ($SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z})$)

- In order to illustrate some of the general statements above we consider the special example $SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z})$.
- Let $\mathbb{H}^2$ be the 2-dimensional hyperbolic space. The group $SL_2(\mathbb{R})$ acts by isometric diffeomorphisms on the upper half-plane by Moebius transformations. This action is proper and transitive. The isotropy group of $z = i$ is $SO(2)$. Since $\mathbb{H}^2$ is a simply-connected Riemannian manifold, whose sectional curvature is constant -1, the $SL_2(\mathbb{R})$-space $\mathbb{H}^2$ is a model for $\underline{E}SL_2(\mathbb{R})$.
- The group $SL_2(\mathbb{R})$ is a connected Lie group and $SO(2) \subseteq SL_2(\mathbb{R})$ is a maximal compact subgroup. Hence $SL_2(\mathbb{R})/SO(2)$ is a model for $\underline{E}SL_2(\mathbb{R})$
- Since the $SL_2(\mathbb{R})$-action on $\mathbb{H}^2$ is transitive and $SO(2)$ is the isotropy group at $i \in \mathbb{H}^2$, we see that the $SL_2(\mathbb{R})$-manifolds $SL_2(\mathbb{R})/SO(2)$ and $\mathbb{H}^2$ are $SL_2(\mathbb{R})$-diffeomorphic.

Example (continued)

- Since $SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\mathbb{R})$, the space $\mathbb{H}^2$ with the obvious $SL_2(\mathbb{Z})$-action is a model for $\underline{E}SL_2(\mathbb{Z})$.
- The group $SL_2(\mathbb{Z})$ is isomorphic to the amalgamated product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$. This implies that there is a tree on which $SL_2(\mathbb{Z})$ acts with finite stabilizers. The tree has alternately two and three edges emanating from each vertex. This is a 1-dimensional model for $\underline{E}SL_2(\mathbb{Z})$.
- The tree model and the other model given by $\mathbb{H}^2$ must be $SL_2(\mathbb{Z})$-homotopy equivalent. They can explicitly be related by the following construction.

Example (continued)

- Divide the Poincaré disk into fundamental domains for the $SL_2(\mathbb{Z})$-action. Each fundamental domain is a geodesic triangle with one vertex at infinity, i.e., a vertex on the boundary sphere, and two vertices in the interior. Then the union of the edges, whose end points lie in the interior of the Poincaré disk, is a tree T with $SL_2(\mathbb{Z})$-action which is the tree model above. The tree is a $SL_2(\mathbb{Z})$-equivariant deformation retraction of the Poincaré disk. A retraction is given by moving a point p in the Poincaré disk along a geodesic starting at the vertex at infinity, which belongs to the triangle containing p, through p to the first intersection point of this geodesic with T.

Example (continued)

- The tree T above can be identified with the Bruhat-Tits building of $SL_2(\widehat{\mathbb{Q}_p})$ and hence is a model for $\underline{E}SL_2(\widehat{\mathbb{Q}_p})$. Since $SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\widehat{\mathbb{Q}_p})$, we get another reason why this tree is a model for $SL_2(\mathbb{Z})$.

Finiteness properties

- Finiteness properties of the spaces EG and $\underline{E}G$ have been intensively studied in the literature. We mention a few examples and results. For more information we refer to the lectures of Brown.
- If EG has a finite-dimensional model, the group G must be torsionfree. There are often finite models for $\underline{E}G$ for groups G with torsion.
- Often geometry provides small model for $\underline{E}G$ in cases, where the models for EG are huge. These small models can be useful for computations concerning BG itself.

Theorem (Discrete subgroups of Lie groups)

Let L be a Lie group with finitely many path components. Let $K \subseteq L$ be a maximal compact subgroup K. Let $G \subseteq L$ be a discrete subgroup of L. Then L/K with the left G-action is a model for $\underline{E}G$.
Suppose additionally that G is virtually torsionfree, i.e., contains a torsionfree subgroup $\Delta \subseteq G$ of finite index.
Then we have for its virtual cohomological dimension

$$\operatorname{vcd}(G) \leq \dim(L/K).$$

Equality holds if and only if $G\backslash L$ is compact.

Theorem (A criterion for 1-dimensional models for BG, Stallings (1968), Swan (1969))

Let G be a discrete group. The following statements are equivalent:

- *There exists a 1-dimensional model for EG;*
- *There exists a 1-dimensional model for BG;*
- *The cohomological dimension of G is less or equal to one;*
- *G is a free group.*

Theorem (A criterion for 1-dimensional models for $\underline{E}G$, Dunwoody (1979))

Let G be a discrete group. Then there exists a 1-dimensional model for $\underline{E}G$ if and only if the cohomological dimension of G over the rationals $\mathbb{Q}$ is less or equal to one.

Theorem (Virtual cohomological dimension and $\dim(\underline{E}G)$, L. (2000))

Let G be a discrete group which is virtually torsionfree.

- *Then*

  $$\operatorname{vcd}(G) \leq \dim(\underline{E}G)$$

  for any model for $\underline{E}G$.

- *Let $l \geq 0$ be an integer such that for any chain of finite subgroups $H_0 \subsetneq H_1 \subsetneq \ldots \subsetneq H_r$ we have $r \leq l$.*
  Then there exists a model for $\underline{E}G$ of dimension $\max\{3, \operatorname{vcd}(G)\} + l$.

- The following problem has been stated by Brown (1979) and has created a lot of activities.

Problem

For which discrete groups G, which are virtually torsionfree, does there exist a G-CW-model for $\underline{E}G$ of dimension $\operatorname{vcd}(G)$?

- The results above do give some evidence for a positive answer.
- However, Leary-Nucinkis (2003) have constructed groups, where the answer is negative.

Theorem (Leary-Nucinkis (2001))

Let X be a CW-complex. Then there exists a group G with $X \simeq G\backslash\underline{E}G$.

Question (Homological Computations based on nice models for $\underline{E}G$)

Can nice geometric models for $\underline{E}G$ be used to compute the group homology and more general homology and cohomology theories of a group G?

Question (K-theory of group rings and group homology)

Is there a relation between $K_n(RG)$ and the group homology of G?

Question (Isomorphism Conjectures and classifying spaces of families)

Can classifying spaces of families be used to formulate a version of the Farrell-Jones Conjecture and the Baum-Connes Conjecture which may hold for all group G and all rings?

To be continued

Stay tuned

Equivariant homology theories (Lecture IV)

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Flashback

- We have introduced the Farrell-Jones Conjecture and the Baum-Connes Conjecture for torsionfree groups and discussed applications of these conjectures such as to the Kaplansky Conjecture and the Borel Conjecture.
- We have explained that the formulations for torsionfree groups cannot extend to arbitrary groups.
  Our goal is to find a formulation which makes sense for all groups and all rings.
- For this purpose we have introduced classifying spaces for families of subgroups of a group G which we will recall next.
- In the sequel group will mean discrete group.

Definition (Family of subgroups)

A *family $\mathcal{F}$ of subgroups* of G is a set of subgroups of G which is closed under conjugation and finite intersections.

Examples for $\mathcal{F}$ are:

$$\begin{array}{lcl} \mathcal{TR} & = & \{\text{trivial subgroup}\}; \\ \mathcal{FIN} & = & \{\text{finite subgroups}\}; \\ \mathcal{FCYC} & = & \{\text{finite cyclic subgroups}\}; \\ \mathcal{VCYC} & = & \{\text{virtually cyclic subgroups}\}; \\ \mathcal{ALL} & = & \{\text{all subgroups}\}. \end{array}$$

Definition (Classifying *G-CW*-complex for a family of subgroups)

Let $\mathcal{F}$ be a family of subgroups of G. A model for the *classifying G-CW-complex for the family $\mathcal{F}$* is a G-CW-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to X$.

We abbreviate $\underline{E}G := E_{\mathcal{FIN}}(G)$ and call it the *universal G-CW-complex for proper G-actions*.
We also write $EG = E_{\mathcal{TR}}(G)$.

- A model for $E_{\mathcal{F}}(G)$ exists and is unique up to G-homotopy.

- Cliffhanger

Question (Homological computations based on nice models for $\underline{E}G$)

Can nice geometric models for $\underline{E}G$ be used to compute the group homology and more general homology and cohomology theories of a group G?

Question (K-theory of group rings and group homology)

Is there a relation between $K_n(RG)$ and the group homology of G?

Question (Isomorphism Conjectures and classifying spaces of families)

Can classifying spaces of families be used to formulate a version of the Farrell-Jones Conjecture and the Baum-Connes Conjecture which may hold for all groups and all rings?

Outline

- We intoduce the notion of an equivariant homology theory.
- We present the general formulation of the Farrell-Jones Conjecture and the Baum-Connes Conjecture.
- We discuss equivariant Chern characters.
- We present some explicit computations of equivariant topological K-groups and of homology groups associated to classifying spaces of groups.

Equivariant homology theories

Definition (*G*-homology theory)

A *G-homology theory* $\mathcal{H}_*$ is a covariant functor from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded Λ-modules together with natural transformations

$$\partial_n(X, A)\colon \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- G-homotopy invariance;
- Long exact sequence of a pair;
- Excision;
- Disjoint union axiom.

Definition (Equivariant homology theory)

An *equivariant homology theory* $\mathcal{H}_*^?$ assigns to every group G a G-homology theory $\mathcal{H}_*^G$. These are linked together with the following so called *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a H-CW-pair (X, A), there are for all $n \in \mathbb{Z}$ natural homomorphisms

$$\operatorname{ind}_\alpha\colon \mathcal{H}_n^H(X, A) \quad \to \quad \mathcal{H}_n^G(\operatorname{ind}_\alpha(X, A))$$

satisfying

- Bijectivity
  If $\ker(\alpha)$ acts freely on X, then $\operatorname{ind}_\alpha$ is a bijection;
- Compatibility with the boundary homomorphisms;
- Functoriality in α;
- Compatibility with conjugation.

Example (Equivariant homology theories)

- Given a non-equivariant homology theory $\mathcal{K}_*$, put

$$\begin{aligned} \mathcal{H}_*^G(X) &:= \mathcal{K}_*(X/G); \\ \mathcal{H}_*^G(X) &:= \mathcal{K}_*(EG \times_G X) \quad \text{(Borel homology)}. \end{aligned}$$

- Equivariant bordism $\Omega_*^?(X)$;
- Equivariant topological K-theory $K_*^?(X)$.

Theorem (L.-Reich (2005))

Given a functor $\mathbf{E}\colon \mathsf{Groupoids} \to \mathsf{Spectra}$ *sending equivalences to weak equivalences, there exists an equivariant homology theory* $\mathcal{H}_*^?(-;\mathbf{E})$ *satisfying*

$$\mathcal{H}_n^H(pt) \cong \mathcal{H}_n^G(G/H) \cong \pi_n(\mathbf{E}(H)).$$

Theorem (Equivariant homology theories associated to K and L-theory, Davis-L. (1998))

Let R be a ring (with involution). There exist covariant functors

$$\begin{aligned} \mathbf{K}_R \colon \mathrm{Groupoids} &\to \mathrm{Spectra}; \\ \mathbf{L}_R^{\langle\infty\rangle} \colon \mathrm{Groupoids} &\to \mathrm{Spectra}; \\ \mathbf{K}^{\mathrm{top}} \colon \mathrm{Groupoids}^{\mathrm{inj}} &\to \mathrm{Spectra} \end{aligned}$$

with the following properties:

- *They send equivalences of groupoids to weak equivalences of spectra;*
- *For every group G and all $n \in \mathbb{Z}$ we have*

$$\begin{aligned} \pi_n(\mathbf{K}_R(G)) &\cong K_n(RG); \\ \pi_n(\mathbf{L}_R^{\langle-\infty\rangle}(G)) &\cong L_n^{\langle-\infty\rangle}(RG); \\ \pi_n(\mathbf{K}^{\mathrm{top}}(G)) &\cong K_n(C_r^*(G)). \end{aligned}$$

Example (Equivariant homology theories associated to K and L-theory)

We get equivariant homology theories

$$H_*^?(-;\mathbf{K}_R);$$
$$H_*^?(-;\mathbf{L}_R^{\langle -\infty\rangle});$$
$$H_*^?(-;\mathbf{K}^{\mathrm{top}}),$$

satisfying for $H \subseteq G$

$$\begin{array}{lclcl} H_n^G(G/H;\mathbf{K}_R) & \cong & H_n^H(\mathrm{pt};\mathbf{K}_R) & \cong & K_n(RH); \\ H_n^G(G/H;\mathbf{L}_R^{\langle -\infty\rangle}) & \cong & H_n^H(\mathrm{pt};\mathbf{L}_R^{\langle -\infty\rangle}) & \cong & L_n^{\langle -\infty\rangle}(RH); \\ H_n^G(G/H;\mathbf{K}^{\mathrm{top}}) & \cong & H_n^H(\mathrm{pt};\mathbf{K}^{\mathrm{top}}) & \cong & K_n(C_r^*(H)). \end{array}$$

The general formulation of the Isomorphism Conjectures

Conjecture (*K*-theoretic Farrell-Jones-Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- The assembly map is the map induced by the projection $E_{\mathcal{VCYC}}(G) \to \mathrm{pt}$.

Conjecture (*L*-theoretic Farrell-Jones-Conjecture)

The L-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(pt, \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (Baum-Connes Conjecture)

The Baum-Connes Conjecture predicts that the assembly map

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}^{\mathrm{top}}) \to H_n^G(pt, \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

- We will discuss these conjectures and their applications in the next lecture.
- We will now continue with equivariant homology theories.

Equivariant Chern characters

- Let $\mathcal{H}_*$ be a (non-equivariant) homology theory. There is the Atiyah-Hirzebruch spectral sequence which converges to $\mathcal{H}_{p+q}(X)$ and has as E^2-term

$$E^2_{p,q} = H_p(X; \mathcal{H}_q(\mathrm{pt})).$$

- Rationally it collapses completely. Namely, one has the following result

Theorem (Non-equivariant Chern character, Dold (1962))

Let $\mathcal{H}_*$ be a homology theory with values in Λ-modules for $\mathbb{Q} \subseteq \Lambda$. Then there exists for every $n \in \mathbb{Z}$ and every CW-complex X a natural isomorphism

$$\bigoplus_{p+q=n} H_p(X; \Lambda) \otimes_\Lambda \mathcal{H}_q(pt) \xrightarrow{\cong} \mathcal{H}_n(X).$$

Dold's Chern character for a CW-complex X is given by the following composite:

$$\mathrm{ch}_n \colon \bigoplus_{p+q=n} H_p(X;\mathcal{H}_q(*)) \xrightarrow{\alpha^{-1}} \bigoplus_{p+q=n} H_p(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{H}_q(*)$$

$$\xrightarrow{\bigoplus_{p+q=n} (\mathrm{hur} \otimes \mathrm{id})^{-1}} \bigoplus_{p+q=n} \pi_p^s(X_+,*) \otimes_{\mathbb{Z}} \mathcal{H}_q(*) \xrightarrow{\bigoplus_{p+q=n} D_{p,q}} \mathcal{H}_n(X),$$

where $D_{p,q}$ sends $[f\colon (S^{p+k},\mathrm{pt}) \to (S^k \wedge X_+,\mathrm{pt})] \otimes \eta$ to the image of η under the composite

$$\mathcal{H}_q(*) \cong \mathcal{H}_{p+k+q}(S^{p+k},\mathrm{pt}) \xrightarrow{\mathcal{H}_{p+k+q}(f)} \mathcal{H}_{p+k+q}(S^k \wedge X_+,\mathrm{pt}) \cong \mathcal{H}_{p+q}(X).$$

- We want to extend this to the equivariant setting.
- This requires an extra structure on the coefficients of an equivariant homology theory $\mathcal{H}_*^?$.
- We define a covariant functor called induction

$$\mathrm{ind}\colon \mathcal{FGI} \to \Lambda\text{-}\mathrm{Mod}$$

from the category $\mathcal{FGI}$ of finite groups with injective group homomorphisms as morphisms to the category of Λ-modules as follows. It sends G to $\mathcal{H}_n^G(\mathrm{pt})$ and an injection of finite groups $\alpha\colon H \to G$ to the morphism given by the induction structure

$$\mathcal{H}_n^H(\mathrm{pt}) \xrightarrow{\mathrm{ind}_\alpha} \mathcal{H}_n^G(\mathrm{ind}_\alpha \mathrm{pt}) \xrightarrow{\mathcal{H}_n^G(\mathrm{pr})} \mathcal{H}_n^G(\mathrm{pt}).$$

Definition (Mackey extension)

We say that $\mathcal{H}_*^?$ has a Mackey extension if for every $n \in \mathbb{Z}$ there is a contravariant functor called restriction

$$\mathrm{res}\colon \mathcal{FGI} \to \Lambda\text{-}\mathrm{Mod}$$

such that these two functors ind and res agree on objects and satisfy the double coset formula ,i.e., we have for two subgroups $H, K \subset G$ of the finite group G

$$\mathrm{res}_G^K \circ \mathrm{ind}_H^G = \sum_{KgH \in K \backslash G/H} \mathrm{ind}_{c(g)\colon H \cap g^{-1}Kg \to K} \circ \mathrm{res}_H^{H \cap g^{-1}Kg},$$

where $c(g)$ is conjugation with g, i.e., $c(g)(h) = ghg^{-1}$.

- In every case we will consider such a Mackey extension does exist and is given by an actual restriction.
- For instance for $H_0^?(-; \mathbf{K}^{\text{top}})$ induction is the functor complex representation ring $R_{\mathbb{C}}$ with respect to induction of representations. The restriction part is given by the restriction of representations.

Theorem (Equivariant Chern character, L. (2002))

Let $\mathcal{H}_*^?$ be a equivariant homology theory with values in Λ-modules for $\mathbb{Q} \subseteq \Lambda$. Suppose that $\mathcal{H}_*^?$ has a Mackey extension. Let I be the set of conjugacy classes (H) of finite subgroups H of G.
Then there is for every group G, every proper G-CW-complex X and every $n \in \mathbb{Z}$ a natural isomorphism called equivariant Chern character

$$\mathrm{ch}_n^G : \bigoplus_{p+q=n} \bigoplus_{(H)\in I} H_p(C_GH\backslash X^H;\Lambda) \otimes_{\Lambda[W_GH]} S_H\left(\mathcal{H}_q^H(*)\right) \xrightarrow{\cong} \mathcal{H}_n^G(X).$$

- C_GH is the centralizer and N_GH the normalizer of $H \subseteq G$;
- $W_GH := N_GH/H \cdot C_GH$ (This is always a finite group);
- $S_H\left(\mathcal{H}_q^H(*)\right) := \operatorname{cok}\left(\bigoplus_{\substack{K\subset H\\ K\neq H}} \operatorname{ind}_K^H : \bigoplus_{\substack{K\subset H\\ K\neq H}} \mathcal{H}_q^K(*) \to \mathcal{H}_q^H(*)\right)$;
- $\mathrm{ch}_*^?$ is an equivalence of equivariant homology theories.

Theorem (Artin's Theorem)

Let G be finite. Then the map

$$\bigoplus_{C\subset G} \operatorname{ind}_C^G \colon \bigoplus_{C\subset G} R_{\mathbb{C}}(C) \to R_{\mathbb{C}}(G)$$

is surjective after inverting $|G|$, where $C \subset G$ runs through the cyclic subgroups of G.

Let C be a finite cyclic group. The Artin defect is the cokernel of the map

$$\bigoplus_{D\subset C, D\neq C} \operatorname{ind}_D^C \colon \bigoplus_{D\subset C, D\neq C} R_{\mathbb{C}}(D) \to R_{\mathbb{C}}(C).$$

For an appropriate idempotent $\theta_C \in R_{\mathbb{Q}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right]$ the Artin defect is after inverting the order of $|C|$ canonically isomorphic to

$$\theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right].$$

- Let $K_*^G = H_*^?(-;\mathbf{K}^{\text{top}})$ be equivariant topological K-theory.
- We get for a finite subgroup $H \subseteq G$

$$K_n^G(G/H) = K_n^H(\text{pt}) = \begin{cases} R_{\mathbb{C}}(H) & \text{if } n \text{ is even;} \\ \{0\} & \text{if } n \text{ is odd.} \end{cases}$$

- $S_H\left(K_q^H(*)\right) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ if H is not cyclic and q is even or if q is odd.
- $S_C\left(K_q^C(*)\right) \otimes_{\mathbb{Z}} \mathbb{Q} = \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Q}$ if C is finite cyclic and q is even.

- Recall

$$\mathrm{ch}_n^G\colon \bigoplus_{p+q=n} \bigoplus_{(H)\in I} H_p(C_G H\backslash X^H;\Lambda) \otimes_{\Lambda[W_G H]} S_H\left(\mathcal{H}_q^H(*)\right) \xrightarrow{\cong} \mathcal{H}_n^G(X).$$

Example (Improvement of Artin's Theorem)

Let G be finite, $X = \{*\}$ and $\mathcal{H}_*^? = K_*^?$. Then we get an improvement of Artin's theorem. Namely, the equivariant Chern character induces an isomorphism

$$\mathrm{ch}_0^G(\mathrm{pt})\colon \bigoplus_{(C)} \mathbb{Z} \otimes_{\mathbb{Z}[W_G C]} \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|G|}\right] \xrightarrow{\cong} R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|G|}\right]$$

where (C) runs over the conjugacy classes of finite cyclic subgroups.

Corollary (Rational computation of $K_*^G(\underline{E}G)$)

For every group G and every $n \in \mathbb{Z}$ we obtain an isomorphism

$$\bigoplus_{(C)} \bigoplus_{k} H_{p+2k}(BC_GC) \otimes_{\mathbb{Z}[W_GC]} \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n^G(\underline{E}G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

- If the Baum-Connes Conjecture holds for G, this gives a computation of $K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$.

- The last corollary follows from the equivariant Chern character

$$\mathrm{ch}_n^G \colon \bigoplus_{p+q=n} \bigoplus_{(H)\in I} H_p(C_G H\backslash X^H;\Lambda) \otimes_{\Lambda[W_G H]} S_H\left(\mathcal{H}_q^H(*)\right) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

  using the following facts.
- $\underline{E}G^C$ is a contractible proper $C_G C$- space. Hence the canonical map $BC_G C \to C_G C\backslash\underline{E}G^C$ induces an isomorphism

$$H_p(BC_G C) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} H_p(C_G C\backslash\underline{E}G^C) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

- $S_H\left(K_q^H(*)\right) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ if H is not cyclic and q is even or if q is odd.
- $S_C\left(K_q^C(*)\right) \otimes_{\mathbb{Z}} \mathbb{Q} = \theta_C \cdot R_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Q}$ if C is finite cyclic and q is even.

Topological K-theory of classifying spaces

- For a prime p denote by $r(p) = |\operatorname{con}_p(G)|$ the number of conjugacy classes (g) of elements $g \neq 1$ in G of p-power order.
- $\mathbb{I}_G$ is the augmentation ideal of $R_{\mathbb{C}}(G)$.
- Let $\mathbb{I}_p(G)$ be the image of the restriction homomorphism $\mathbb{I}(G) \to \mathbb{I}(G_p)$.

Theorem (Completion Theorem, Atiyah-Segal (1969))

Let G be a finite group.
Then there are isomorphisms of abelian groups

$$\begin{aligned} K^0(BG) &\cong R_{\mathbb{C}}(G)\widehat{_{\mathbb{I}_G}} \\ &\cong \mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(G) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}_p} \cong \mathbb{Z} \times \prod_{p \text{ prime}} (\widehat{\mathbb{Z}_p})^{r(p)}; \\ K^1(BG) &\cong 0. \end{aligned}$$

Theorem (L. (2005))

Let G be a discrete group. Denote by $K^*(BG)$ the topological (complex) K-theory of its classifying space BG. Suppose that there is a cocompact G-CW-model for the classifying space $\underline{E}G$ for proper G-actions.
Then there is a $\mathbb{Q}$-isomorphism

$$\overline{\mathrm{ch}}_G^n \colon K^n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \left(\prod_{i \in \mathbb{Z}} H^{2i+n}(BG;\mathbb{Q})\right) \times \prod_{p \text{ prime}} \prod_{(g) \in \mathrm{con}_p(G)} \left(\prod_{i \in \mathbb{Z}} H^{2i+n}(BC_G\langle g\rangle;\widehat{\mathbb{Q}_p})\right).$$

- The multiplicative structure can also be determined.
- There are many groups for which a cocompact G-CW-model for $\underline{E}G$ exists, e.g., hyperbolic groups.

Example ($SL_3(\mathbb{Z})$)

- It is well-known that its rational cohomology satisfies $\widetilde{H}^n(BSL_3(\mathbb{Z});\mathbb{Q}) = 0$ for all $n \in \mathbb{Z}$.
- Actually, by a result of Soule (1978) the quotient space $SL_3(\mathbb{Z})\backslash \underline{E}SL_3(\mathbb{Z})$ is contractible and compact.
- From the classification of finite subgroups of $SL_3(\mathbb{Z})$ we see that $SL_3(\mathbb{Z})$ contains up to conjugacy two elements of order 2, two elements of order 4 and two elements of order 3 and no further conjugacy classes of non-trivial elements of prime power order.
- The rational homology of each of the centralizers of elements in $\operatorname{con}_2(G)$ and $\operatorname{con}_3(G)$ agrees with the one of the trivial group.
- Hence we get

$$\begin{aligned} K^0(BSL_3(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong \mathbb{Q} \times (\mathbb{Q}\widehat{_2})^4 \times (\mathbb{Q}\widehat{_3})^2; \\ K^1(BSL_3(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} &\cong 0. \end{aligned}$$

Example (Continued)

- The identification of $K^0(BSL_3(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q}$ above is compatible with the multiplicative structures.
- Actually the computation using Brown-Petersen cohomology and the Conner-Floyd relation by Tezuka-Yagita (1992) gives the integral computation

$$\begin{aligned} K^0(BSL_3(\mathbb{Z})) &\cong \mathbb{Z} \times (\widehat{\mathbb{Z}_2})^4 \times (\widehat{\mathbb{Z}_3})^2; \\ K^1(BSL_3(\mathbb{Z})) &\cong 0. \end{aligned}$$

- Soule (1978) has computed the integral cohomology of $SL_3(\mathbb{Z})$.

A computation

- Let G be a discrete group. Let $\mathcal{MFIN}$ be the subset of $\mathcal{FIN}$ consisting of elements in $\mathcal{FIN}$ which are maximal in $\mathcal{FIN}$.
- Assume that G satisfies the following assertions:
  - (M) Every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup;
  - (NM) $M \in \mathcal{MFIN}, M \neq \{1\} \Rightarrow N_G M = M$.
- Here are some examples of groups G which satisfy conditions (M) and (NM):
  - Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$;
  - Fuchsian groups;
  - One-relator groups G.

- For such a group there is a nice model for $\underline{E}G$ with as few non-free cells as possible. Let $\{(M_i) \mid i \in I\}$ be the set of conjugacy classes of maximal finite subgroups of $M_i \subseteq G$. By attaching free G-cells we get an inclusion of G-CW-complexes $j_1 \colon \coprod_{i \in I} G \times_{M_i} EM_i \to EG$.
- Define $\underline{E}G$ as the G-pushout

$$\begin{array}{ccc} \coprod_{i \in I} G \times_{M_i} EM_i & \xrightarrow{j_1} & EG \\ \downarrow u_1 & & \downarrow f_1 \\ \coprod_{i \in I} G/M_i & \xrightarrow{k_1} & \underline{E}G \end{array}$$

where u_1 is the obvious G-map obtained by collapsing each EM_i to a point.

- Next we explain why $\underline{E}G$ is a model for the classifying space for proper actions of G.
- Its isotropy groups are all finite. We have to show for $H \subseteq G$ finite that $\underline{E}G^H$ contractible.
- We begin with the case $H \neq \{1\}$. Because of conditions (M) and (NM) there is precisely one index $i_0 \in I$ such that H is subconjugated to M_{i_0} and is not subconjugated to M_i for $i \neq i_0$. We get

$$\left(\coprod_{i \in I} G/M_i\right)^H = \left(G/M_{i_0}\right)^H = \text{pt}.$$

  Hence $\underline{E}G^H = \text{pt}$.
- It remains to treat $H = \{1\}$. Since u_1 is a non-equivariant homotopy equivalence and j_1 is a cofibration, f_1 is a non-equivariant homotopy equivalence. Hence $\underline{E}G$ is contractible.

- Consider the pushout obtained from the G-pushout above by dividing the G-action

$$\begin{array}{ccc} \coprod_{i\in I} BM_i & \longrightarrow & BG \\ \downarrow & & \downarrow \\ \coprod_{i\in I} \mathrm{pt} & \longrightarrow & G\backslash \underline{E}G \end{array}$$

- The associated Mayer-Vietoris sequence yields

$$\ldots \to \widetilde{H}_{p+1}(G\backslash \underline{E}G) \to \bigoplus_{i\in I} \widetilde{H}_p(BM_i) \to \widetilde{H}_p(BG) \to \widetilde{H}_p(G\backslash \underline{E}G) \to \ldots$$

- In particular we obtain an isomorphism for $p \geq \dim(\underline{E}G) + 2$

$$\bigoplus_{i\in I} H_p(BM_i) \xrightarrow{\cong} H_p(BG).$$

Theorem

Let G be a discrete group which satisfies the conditions (M) and (NM) above.
Then there is an isomorphism

$$K_1^G(\underline{E}G) \xrightarrow{\cong} K_1(G \backslash \underline{E}G),$$

and a short exact sequence

$$0 \to \bigoplus_{i \in I} \widetilde{R}_{\mathbb{C}}(M_i) \to K_0(\underline{E}G) \to K_0(G \backslash \underline{E}G) \to 0.$$

It splits if we invert the orders of all finite subgroups of G.

- If the Baum-Connes Conjecture is true for G, then

$$K_n(C_r^*(G)) \cong K_n^G(\underline{E}G).$$

- We see that for computations of group homology or of K- and L-groups of group rings and group C^*-algebras it is important to understand the spaces $G\backslash \underline{E}G$.
- Often geometric input ensures that $G\backslash \underline{E}G$ is a fairly small CW-complex.
- On the other hand recall the result due to Leary-Nucinkis (2001) that for any CW-complex X there exists a group G with $X \simeq G\backslash \underline{E}G$.

Question (Consequences)

What are the consequences of the Farrell-Jones Conjecture and the Baum-Connes Conjecture?

To be continued
Stay tuned

The Isomorphism Conjectures for arbitrary groups (Lecture V)

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Flashback

- We have introduced classifying spaces $E_{\mathcal{F}}(G)$ for a family $\mathcal{F}$ of subgroups.
- We have introduced the notion of an equivariant homology theory.
- We have formulated the Farrell-Jones Conjecture and the Baum-Connes Conjecture.
- We have already discussed application for torsionfree groups such as to the Kaplansky Conjecture and the Borel Conjecture.
- Cliffhanger

Question (Consequences)

What are the consequences of the Farrell-Jones Conjecture and the Baum-Connes Conjecture?

Outline

- We give a review of the Farrell-Jones and the Baum-Connes Conjecture.
- We discuss the difference between the families $\mathcal{FIN}$ and $\mathcal{VCYC}$.
- We discuss consequences of the Farrell-Jones and the Baum-Connes Conjecture.

Review of the Isomorphism Conjectures

- G will always be a discrete group.

Definition (Classifying G-CW-complex for a family of subgroups)

Let $\mathcal{F}$ be a family of subgroups of G. A model for the *classifying G-CW-complex for the family $\mathcal{F}$* is a G-CW-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to X$.

We abbreviate $\underline{E}G := E_{\mathcal{FIN}}(G)$ and call it the *universal G-CW-complex for proper G-actions*.

We also write $EG = E_{\mathcal{TR}}(G)$.

Definition (Equivariant homology theory)

An *equivariant homology theory* $\mathcal{H}_*^?$ assigns to every group G a G-homology theory $\mathcal{H}_*^G$. These are linked together with the following so called *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a H-CW-pair (X, A), there are for all $n \in \mathbb{Z}$ natural homomorphisms

$$\mathrm{ind}_\alpha\colon \mathcal{H}_n^H(X,A) \quad \to \quad \mathcal{H}_n^G(\mathrm{ind}_\alpha(X,A))$$

satisfying

- Bijectivity
  If $\ker(\alpha)$ acts freely on X, then ind_α is a bijection;
- Compatibility with the boundary homomorphisms;
- Functoriality in α;
- Compatibility with conjugation.

Conjecture (*K*-theoretic Farrell-Jones-Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (*L*-theoretic Farrell-Jones-Conjecture)

The L-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(pt, \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (Baum-Connes Conjecture)

The Baum-Connes Conjecture predicts that the assembly map

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}^{\text{top}}) \to H_n^G(pt, \mathbf{K}^{\text{top}}) = K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

- All assembly maps are the maps induced by the projection $E_{\mathcal{F}}(G) \to \text{pt}$.
- These Conjecture can be thought of a kind of generalized induction theorem. They allow to compute the value of a functor such as $K_n(RG)$ on G in terms of its values for all virtually cyclic subgroups of G.

Changing the family

Theorem (Transitivity Principle)

Let $\mathcal{F} \subseteq \mathcal{G}$ be two families of subgroups of G. Let $\mathcal{H}_*^?$ be an equivariant homology theory. Assume that for every element $H \in \mathcal{G}$ and $n \in \mathbb{Z}$ the assembly map

$$\mathcal{H}_n^H(E_{\mathcal{F}|_H}(H)) \to \mathcal{H}_n^H(pt)$$

is bijective, where $\mathcal{F}|_H = \{K \cap H \mid K \in \mathcal{F}\}$.
Then the relative assembly map induced by the up to G-homotopy unique G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$

$$\mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{G}}(G))$$

is bijective for all $n \in \mathbb{Z}$.

Example (Passage from $\mathcal{FIN}$ to $\mathcal{VCYC}$ for the Baum-Connes Conjecture)

- The Baum-Connes Conjecture is known to be true for virtually cyclic groups. The Transitivity Principle implies that the relative assembly

$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n^G(E_{\mathcal{VCYC}}(G))$$

  is bijective for all $n \in \mathbb{Z}$.
- Hence it does not matter in the context of the Baum-Connes Conjecture whether we consider the family $\mathcal{FIN}$ or $\mathcal{VCYC}$.

- In general the relative assembly maps

$$\begin{aligned} H_n^G(\underline{E}G; \mathbf{K}_R) &\to H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_R); \\ H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) &\to H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{L}_R^{\langle -\infty \rangle}), \end{aligned}$$

  are not bijective.
- Hence in the Farrell-Jones setting one has to pass to $\mathcal{VCYC}$ and cannot use the easier to handle family $\mathcal{FIN}$.

Example (Passage from $\mathcal{FIN}$ to $\mathcal{VCYC}$ for the Farrell-Jones Conjecture)

For instance the Bass-Heller Swan decomposition

$$K_{n-1}(R) \oplus K_n(R) \oplus \mathrm{NK}_n(R) \oplus \mathrm{NK}_n(R)) \xrightarrow{\cong} K_n(R[t,t^{-1}]) \cong K_n(R[\mathbb{Z}])$$

and the isomorphism

$$H_n^{\mathbb{Z}}(\underline{E}\mathbb{Z};\mathbf{K}_R) = H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_R) = H_n^{\{1\}}(S^1,\mathbf{K}_R) = K_{n-1}(R) \oplus K_n(R)$$

show that

$$H_n^{\mathbb{Z}}(\underline{E}\mathbb{Z};\mathbf{K}_R) \to H_n^{\mathbb{Z}}(\mathrm{pt};\mathbf{K}_R) = K_n(R\mathbb{Z})$$

is bijective if and only if $\mathrm{NK}_n(R) = 0$.

- An infinite virtually cyclic group G is called of type I if it admits an epimorphism onto $\mathbb{Z}$ and of type II otherwise. A virtually cyclic group is of type II if and only if admits an epimorphism onto D_∞.
- Let $\mathcal{VCYC}_I$ or $\mathcal{VCYC}_{II}$ respectively be the family of subgroups which are either finite or which are virtually cyclic of type I or II respectively.

Theorem (L. (2004), Quinn (2007), Reich (2007))

The following maps are bijective for all $n \in \mathbb{Z}$

$$\begin{aligned} H_n^G(E_{\mathcal{VCYC}_I}(G); \mathbf{K}_R) &\to H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_R); \\ H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) &\to H_n^G(E_{\mathcal{VCYC}_I}(G); \mathbf{L}_R^{\langle -\infty \rangle}). \end{aligned}$$

Theorem (Cappell (1973), Grunewald (2005), Waldhausen (1978))

- *The following maps are bijective for all $n \in \mathbb{Z}$.*

$$H_n^G(\underline{E}G; \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \to \quad H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q};$$

$$H_n^G(\underline{E}G; \mathbf{L}_R^{\langle -\infty \rangle}) \left[\frac{1}{2}\right] \quad \to \quad H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \left[\frac{1}{2}\right];$$

- *If R is regular and $\mathbb{Q} \subseteq R$, then for all $n \in \mathbb{Z}$ we get a bijection*

$$H_n^G(\underline{E}G; \mathbf{K}_R) \to H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_R).$$

Theorem (Bartels (2003))

For every $n \in \mathbb{Z}$ the two maps

$$\begin{aligned} H_n^G(\underline{E}G;\mathbf{K}_R) &\to H_n^G(E_{\mathcal{VCYC}}(G);\mathbf{K}_R); \\ H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty \rangle}) &\to H_n^G(E_{\mathcal{VCYC}}(G);\mathbf{L}_R^{\langle -\infty \rangle}), \end{aligned}$$

are split injective.

- Hence we get (natural) isomorphisms

$$H_n^G(E_{\mathcal{VCYC}}(G);\mathbf{K}_R) \cong H_n^G(\underline{E}G;\mathbf{K}_R) \oplus H_n^G(E_{\mathcal{VCYC}}(G),\underline{E}G;\mathbf{K}_R);$$

$$H_n^G(E_{\mathcal{VCYC}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \cong H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}) \oplus H_n^G(E_{\mathcal{VCYC}}(G),\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}).$$

- The analysis of the terms $H_n^G(E_{\mathcal{VCYC}}(G),\underline{E}G;\mathbf{K}_R)$ and $H_n^G(E_{\mathcal{VCYC}}(G),\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle})$ boils down to investigating Nil-terms and UNil-terms in the sense of Waldhausen and Cappell.

- The analysis of the terms $H_n^G(\underline{E}G;\mathbf{K}_R)$ and $H_n^G(\underline{E}G;\mathbf{L}_R^{\langle -\infty \rangle})$ is using the methods of the previous lecture (e.g., Chern characters).
- The results above imply that the versions of the Farrell-Jones Conjecture for torsionfree groups which we have presented in the second lecture follow from the general versions.
- The latter is obvious for the Baum-Connes Conjecture since for torsionfree G we have $EG = \underline{E}G$.

Consequence of the Isomorphism Conjectures

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\operatorname{sign}_x(M, f) := \langle \mathcal{L}(M) \cup f^* x, [M] \rangle$$

is an oriented homotopy invariant of (M, f), i.e., for every orientation preserving homotopy equivalence of closed oriented manifolds $g\colon M_0 \to M_1$ and homotopy equivalence $f_i\colon M_0 \to M_1$ with $f_1 \circ g \simeq f_2$ we have

$$\operatorname{sign}_x(M_0, f_0) = \operatorname{sign}_x(M_1, f_1).$$

Theorem (The Farrell-Jones, the Baum-Connes and the Novikov Conjecture)

Suppose that one of the following assembly maps

$$\begin{array}{rcl} H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{L}_R^{\langle -\infty \rangle}) & \to & H_n^G(pt, \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG); \\ K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}^{\text{top}}) & \to & H_n^G(pt, \mathbf{K}^{\text{top}}) = K_n(C_r^*(G)), \end{array}$$

is rationally injective.
Then the Novikov Conjecture holds for the group G.

Theorem ($K_0(RG)$ and induction from finite subgroups, Bartels-L.-Reich (2007))

- *Let R be a regular ring with $\mathbb{Q} \subseteq R$. Suppose $G \in \mathcal{FJ}_K(R)$. Then the map given by induction from finite subgroups of G*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{FIN}}(G)} K_0(RH) \to K_0(RG)$$

  is bijective;

- *Let F be a field of characteristic p for a prime number p. Suppose that $G \in \mathcal{FJ}_K(F)$. Then the map*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{FIN}}(G)} K_0(FH)[1/p] \to K_0(FG)[1/p]$$

  is bijective.

Theorem (Permutation Modules, Bartels-L.-Reich (2007))

Let F be a field of characteristic zero. Suppose that $G \in \mathcal{FJ}_K(F)$. Then for every finitely generated projective FG-module P there exists a positive integer k and finitely many finite subgroups H_1, H_2, ..., H_r such that

$$P^k \cong_{FG} F[G/H_1] \oplus F[G/H_2] \oplus \ldots \oplus F[G/H_r].$$

- Let R be commutative ring and let G be a group.
- Let $\text{class}(G, R)$ be the R-module of class functions $G \to R$, i.e., functions $G \to R$ which are constant on conjugacy classes.
- Let $\text{tr}_{RG}\colon RG \to \text{class}(G, R)$ be the obvious R-homomorphism. It extends to a map

$$\text{tr}_{RG}\colon M_n(RG) \to \text{class}(G, R)$$

  by taking the sums of the values of the diagonal entries.
- Let P be a finitely generated RG-module. Choose a finitely generated projective RG-module Q and an isomorphism $\phi\colon RG^n \xrightarrow{\cong} P \oplus Q$. Let $A \in M_n(RG)$ be a matrix such that $\phi^{-1} \circ (f \oplus \text{id}_q) \circ \phi\colon RG^n \to RG^n$ is given by A.

Definition (Hattori-Stallings rank)

Define the Hattori-Stallings rank of P to be the class function

$$\mathrm{HS}_{RG}(P) := \mathrm{tr}_{RG}(A).$$

- This definition is independent of the choice of Q and ϕ.
- Let G be a finite group and let F be a field of characteristic zero. Then a finitely generated RG-module P is the same as a finite dimensional G-representation over F and the Hattori-Stallings rank can be identified with the character of the G-representation.

Conjecture (Bass Conjecture)

Let R be a commutative integral domain and let G be a group. Let $g \neq 1$ be an element in G. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R.
Then the Bass Conjecture predicts that for every finitely generated projective RG-module P the value of its Hattori-Stallings rank $\mathrm{HS}_{RG}(P)$ at (g) is trivial.

- If G is finite, this is just the Theorem of Swan (1960).
- Another version of it would predict for the quotient field F of R that

$$K_0(RG) \to K_0(FG)$$

factorizes as

$$K_0(RG) \to K_0(R) \to K_0(F) \to K_0(FG).$$

Theorem (Linnell-Farrell (2003))

Let G be a group. Suppose that

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{FIN}}(G)} K_0(FH) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective for all fields F of prime characteristic. (This is true if $G \in \mathcal{FJ}_K(F)$ for every field F of prime characteristic).
Then the Bass Conjecture is satisfied for every integral domain R.

Conjecture (Vanishing of Bass-Nil-groups)

Let R be a regular ring with $\mathbb{Q} \subseteq R$. Then we get for all groups G and all $n \in \mathbb{Z}$ that

$$\mathrm{NK}_n(RG) = 0.$$

Theorem (Bartels-L.-Reich (2007))

Let R be a regular ring with $\mathbb{Q} \subseteq R$. If $G \in \mathcal{FJ}_K(R)$, then the conjecture above is true.

Conjecture (Homotopy invariance of $L2$-torsion)

Let X and Y be det-L^2-acyclic finite G-CW-complexes, which are G-homotopy equivalent. Then their L^2-torsion agree:

$$\rho^{(2)}(X;\mathcal{N}(G)) = \rho^{(2)}(Y;\mathcal{N}(G)).$$

- The L^2-torsion of closed Riemannian manifold M is defined in terms of the heat kernel on the universal covering. If M is hyperbolic and has odd dimension, its L^2-torsion is up to a (non-vanishing) dimension constant its volume.
- The conjecture above allows to extend the notion of volume to hyperbolic groups whose L^2-Betti numbers all vanish.

Theorem (L. (2002))

Suppose that $G \in \mathcal{FJ}_K(\mathbb{Z})$. Then G satisfies the Conjecture above.

- Deninger can define a p-adic Fuglede-Kadison determinant for a group G and relate it to p-adic entropy provided that $\mathrm{Wh}^{\mathbb{F}_p}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is trivial.
- The surjectivity of the map

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{FIN}}(G)} K_0(\mathbb{C}H) \to K_0(\mathbb{C}G)$$

plays a role (33 %) in a program to prove the Atiyah Conjecture. It says that for a closed Riemannian manifold with torsionfree fundamental group the L^2-Betti numbers of its universal covering are all integers. The Atiyah Conjecture is rather surprising in view of the analytic definition of the L^2-Betti numbers by

$$b_p^{(2)}(M) := \lim_{t\to\infty} \int_F e^{-t\widetilde{\Delta}_p}(\widetilde{x},\widetilde{x})\, dvol_{\widetilde{M}},$$

where F is a fundamental domain for the $\pi_1(M)$-action on $\widetilde{M}$.

Definition (Bott manifold)

A *Bott manifold* is any simply connected closed Spin-manifold B of dimension 8 whose $\widehat{A}$-genus $\widehat{A}(B)$ is 8.

- We fix such a choice. (The particular choice does not matter.)
- Notice that the index defined in terms of the Dirac operator $\operatorname{ind}_{C_r^*(\{1\};\mathbb{R})}(B) \in KO_8(\mathbb{R}) \cong \mathbb{Z}$ is a generator and the product with this element induces the Bott periodicity isomorphisms $KO_n(C_r^*(G;\mathbb{R})) \xrightarrow{\cong} KO_{n+8}(C_r^*(G;\mathbb{R}))$.
- In particular

$$\operatorname{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) = \operatorname{ind}_{C_r^*(\pi_1(M\times B);\mathbb{R})}(M \times B),$$

if we identify $KO_n(C_r^*(\pi_1(M);\mathbb{R})) = KO_{n+8}(C_r^*(\pi_1(M);\mathbb{R}))$ via Bott periodicity.

- If M carries a Riemannian metric with positive scalar curvature, then the index

$$\operatorname{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) \in KO_n(C_r^*(\pi_1(M);\mathbb{R})),$$

which is defined in terms of the Dirac operator on the universal covering, must vanish by the Bochner-Lichnerowicz formula.

Conjecture ((Stable) Gromov-Lawson-Rosenberg Conjecture)

Let M be a closed connected Spin*-manifold of dimension $n \geq 5$. Then $M \times B^k$ carries for some integer $k \geq 0$ a Riemannian metric with positive scalar curvature if and only if*

$$\operatorname{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) \; = \; 0 \quad \in KO_n(C_r^*(\pi_1(M);\mathbb{R})).$$

Theorem (Stolz (2002))

Suppose that the assembly map for the real version of the Baum-Connes Conjecture

$$H_n^G(\underline{E}G; \mathbf{KO}^{\mathrm{top}}) \to KO_n(C_r^*(G;\mathbb{R}))$$

is injective for the group G.
Then the Stable Gromov-Lawson-Rosenberg Conjecture true for all closed Spin*-manifolds of dimension* ≥ 5 *with* $\pi_1(M) \cong G$*.*

- The requirement $\dim(M) \geq 5$ is essential in the Stable Gromov-Lawson-Rosenberg Conjecture, since in dimension four new obstructions, the Seiberg-Witten invariants, occur.
- The unstable version of the Gromov-Lawson-Rosenberg Conjecture says that M carries a Riemannian metric with positive scalar curvature if and only if $\text{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) = 0$.
- Schick(1998) has constructed counterexamples to the unstable version using minimal hypersurface methods due to Schoen and Yau.
- It is not known whether the unstable version is true or false for finite fundamental groups.
- Since the Baum-Connes Conjecture is true for finite groups (for the trivial reason that $\underline{E}G = \text{pt}$ for finite groups G), the Stable Gromov-Lawson Conjecture holds for finite fundamental groups.

Question (Status)

For which groups are the Farrell-Jones Conjecture and the Baum-Connes Conjecture known to be true? What are open interesting cases?

Question (Methods of proof)

What are the methods of proof?

Question (Relations)

What are the relations between the Farrell-Jones Conjecture and the Baum-Connes Conjecture?

To be continued
Stay tuned

Summary, status and outlook (Lecture VI)

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Flashback

- We have formulated the Farrell-Jones Conjecture and the Baum-Connes Conjecture.
- We have already discussed applications.
- Cliffhanger

Question (Status)

For which groups are the Farrell-Jones Conjecture and the Baum-Connes Conjecture known to be true? What are open interesting cases?

Question (Relations)

What are the relations between the Farrell-Jones Conjecture and the Baum-Connes Conjecture?

Outline

- We briefly review the Farrell-Jones and the Baum-Connes Conjecture.
- We review applications of the Farrell-Jones and the Baum-Connes Conjecture.
- We mention other versions of the Isomorphism Conjectures.
- We explain relations between the Farrell-Jones and the Baum-Connes Conjecture.
- We give a status report about the Farrell-Jones and the Baum-Connes Conjecture.
- Miscellaneous.

Review of the Isomorphism Conjectures

Conjecture (*K*-theoretic Farrell-Jones-Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (*L*-theoretic Farrell-Jones-Conjecture)

The L-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(pt, \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (Baum-Connes Conjecture)

The Baum-Connes Conjecture predicts that the assembly map

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}^{\mathrm{top}}) \to H_n^G(pt, \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

Review of Applications

- The following results or conjectures are consequences of the Farrell-Jones or Baum-Connes Conjecture.
- $\mathcal{FJ}_K(R)$, $\mathcal{FJ}_L(R)$ or $\mathcal{BC}$ respectively are the classes of groups which satisfy the Farrell-Jones Conjecture for K- or L-theory with coefficients in R or the Baum-Connes Conjecture respectively.

Theorem ($K_n(\mathbb{Z}G)$ for $n \leq 1$ and torsionfree G)

We get for a torsionfree group $G \in \mathcal{FJ}(\mathbb{Z})$:

- $K_n(\mathbb{Z}G) = 0$ *for* $n \leq -1$;
- $\widetilde{K}_0(\mathbb{Z}G) = 0$;
- $\mathrm{Wh}(G) = 0$;
- *Every finitely dominated CW-complex X with $G = \pi_1(X)$ is homotopy equivalent to a finite CW-complex;*
- *Every compact h-cobordism $W = (W; M_0, M_1)$ of dimension ≥ 6 with $\pi_1(W) \cong G$ is trivial.*

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Theorem (The Farrell-Jones Conjecture and the Kaplansky Conjecture)

If F is a field of characteristic zero and the torsionfree group G belongs to $\mathcal{FJ}_K(F)$, then G and F satisfy the Kaplansky Conjecture.

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that M and N are homeomorphic.

Theorem (The Farrell-Jones Conjecture and the Borel Conjecture)

If G belongs to both $\mathcal{FJ}_K(\mathbb{Z})$ and $\mathcal{FJ}_L(\mathbb{Z})$, then the Borel Conjecture is true in dimension ≥ 5 and in dimension 4 if G is good in the sense of Freedman.

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f \colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\mathrm{sign}_x(M, f) := \langle \mathcal{L}(M) \cup f^* x, [M] \rangle$$

is an oriented homotopy invariant of (M, f)

Theorem (The Farrell-Jones, the Baum-Connes and the Novikov Conjecture)

If G belongs to $\mathcal{FJ}_L(\mathbb{Z})$ or to $\mathcal{BC}$, then the Novikov Conjecture holds for the group G.

Theorem ($K_0(RG)$ and induction from finite subgroups)

Let R be a regular ring with $\mathbb{Q} \subseteq R$. Suppose $G \in \mathcal{FJ}(R)$. Then the map given by induction from finite subgroups of G

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{FIN}}(G)} K_0(RH) \to K_0(RG)$$

is bijective.

Conjecture (Bass Conjecture)

Let R be a commutative integral domain and let G be a group. Let $g \neq 1$ be an element in G. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R.
Then the Bass Conjecture predicts that for every finitely generated projective RG-module P the value of its Hattori-Stallings rank $\mathrm{HS}_{RG}(P)$ at (g) is trivial.

Theorem (The Farrell-Jones Conjecture and the Bass Conjecture)

Let G be a group. Suppose that $G \in \mathcal{FJ}(F)$ for every field F of prime characteristic.
Then the Bass Conjecture is satisfied for every integral domain R.

Conjecture (Homotopy invariance of $L2$-torsion)

If X and Y are det-L^2-acyclic finite G-CW-complexes, which are G-homotopy equivalent, then their L^2-torsion agree:

$$\rho^{(2)}(X;\mathcal{N}(G)) = \rho^{(2)}(Y;\mathcal{N}(G)).$$

Theorem

Suppose that $G \in \mathcal{FJ}(\mathbb{Z})$. Then G satisfies the Conjecture above.

Conjecture ((Stable) Gromov-Lawson-Rosenberg Conjecture)

Let M be a closed connected Spin*-manifold of dimension $n \geq 5$. Then $M \times B^k$ carries for some integer $k \geq 0$ a Riemannian metric with positive scalar curvature if and only if*

$$\mathrm{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) \; = \; 0 \qquad \in KO_n(C_r^*(\pi_1(M);\mathbb{R})).$$

Theorem (The Baum-Connes Conjecture and the stable Gromov-Lawson-Rosenberg Conjecture)

If $G \in \mathcal{BC}$, then the Stable Gromov-Lawson-Rosenberg Conjecture true for all closed Spin*-manifolds of dimension ≥ 5 with $\pi_1(M) \cong G$.*

Other versions of Isomorphism Conjectures

Conjecture (Isomorphism Conjecture)

Let $\mathcal{H}^?_*$ be an equivariant homology theory. It satisfies the Isomorphism Conjecture for the group G and the family $\mathcal{F}$ if the projection $E_{\mathcal{F}}(G) \to pt$ induces for all $n \in \mathbb{Z}$ a bijection

$$\mathcal{H}^G_n(E_{\mathcal{F}}(G)) \to \mathcal{H}^G_n(pt).$$

Example

- The Farrell-Jones Conjecture for K-theory or L-theory respectively with coefficients in R is the Isomorphism Conjecture for $\mathcal{H}^?_* = H_*(-;\mathbf{K}_R)$ or $\mathcal{H}^?_* = H_*(-;\mathbf{L}_R^{\langle -\infty \rangle})$ respectively and $\mathcal{F} = \mathcal{VCYC}$.
- The Baum-Connes Conjecture is the Isomorphism Conjecture for $\mathcal{H}^?_* = K^?_* = H^?_*(-;\mathbf{K}^{\mathrm{top}})$ and $\mathcal{F} = \mathcal{FIN}$.

- There are functors $\mathcal{P}$ and A which assign to a space X the space of pseudo-isotopies and its A-theory.
- Composing it with the functor sending a groupoid to its classifying space yields functors **P** and **A** from $\mathcal{G}roupoids$ to $\mathcal{S}pectra$.
- Thus we obtain equivariant homology theories $H_*^?(-;\mathbf{P})$ and $H_*^?(-;\mathbf{A})$. They satisfy $H_n^G(G/H;\mathbf{P}) = \pi_n(\mathcal{P}(BH))$ and $H_n^G(G/H;\mathbf{A}) = \pi_n(A(BH))$.

Conjecture (The Farrell-Jones Conjecture for pseudo-isotopies and A-theory)

The Farrell-Jones Conjecture for pseudo-isotopies and A-theory respectively is the Isomorphism Conjecture for $H_*^?(-;\mathbf{P})$ and $H_*^?(-;\mathbf{A})$ respectively for the family $\mathcal{VCYC}$.

Theorem (Relating pseudo-isotopy and K-theory)

The rational version of the K-theoretic Farrell-Jones Conjecture for coefficients in $\mathbb{Z}$ is equivalent Farrell-Jones Conjecture for Pseudoisotopies.
In degree $n \leq 1$ this is even true integrally.

- Pseudo-isotopy and A-theory are important theories. In particular they are closely related to the space of selfhomeomorphisms and the space of selfdiffeomorphisms of closed manifolds.

- There are functors **THH** and **TC** which assign to a ring (or more generally to an **S**-algebra) a spectrum describing its topological Hochschild homology and its topological cyclic homology.
- These functors play an important role in K-theoretic computations.
- Composing it with the functor sending a groupoid to a kind of group ring yields functors $\mathbf{THH}_R$ and $\mathbf{TC}_R$ from $\mathcal{G}roupoids$ to $\mathcal{S}pectra$.
- Thus we obtain equivariant homology theories $H_*^?(-;\mathbf{THH}_R)$ and $H_*^?(-;\mathbf{TC}_R)$. They satisfy $H_n^G(G/H;\mathbf{THH}_R) = \mathrm{THH}_n(RH)$ and $H_n^G(G/H;\mathbf{TC}_R) = \mathrm{TC}_n(RH)$.

Conjecture (The Farrell-Jones Conjecture for topological Hochschild homology and cyclic homology)

The Farrell-Jones Conjecture for topological Hochschild homology and cyclic homology respectively is the Isomorphism Conjecture for $H_*^?(-;\mathbf{THH})$ and $H_*^?(-;\mathbf{TC})$ respectively for the family $\mathcal{CYC}$ of cyclic subgroups.

- We can apply the functor topological K-theory also to Banach algebras such that $l^1(G)$.
- Let $\mathbf{K}_{l^1}^{\text{top}}$ be the functor from $\mathcal{G}roupoids$ to $\mathcal{S}pectra$ which assign to a groupoid the topological K-theory spectrum of its l^1-algebra.
- We obtain an equivariant homology theory $H_*^?(-;\mathbf{K}_{l^1}^{\text{top}})$. It satisfies $H_n^G(G/H,\mathbf{K}_{l^1}^{\text{top}}) = K_n(l^1(H))$.

Conjecture (Bost Conjecture)

The Bost Conjecture is the Isomorphism Conjecture for $H_*^?(-;\mathbf{K}_{l^1}^{\text{top}})$ and the family $\mathcal{FIN}$.

- The assembly map appearing in the Bost Conjecture

$$H_n^G(\underline{E}G;\mathbf{K}_{l^1}^{\text{top}}) \to H_n^G(\text{pt};\mathbf{K}_{l^1}^{\text{top}}) = K_n(l^1(G))$$

composed with the change of algebras homomorphism

$$K_n(l^1(G)) \to K_n(C_r^*(G))$$

is precisely the assembly map appearing in the Baum-Connes Conjecture

$$H_n^G(\underline{E}G;\mathbf{K}^{\text{top}}) = H_n^G(\underline{E}G;\mathbf{K}_{l^1}^{\text{top}}) \to H_n^G(\text{pt};\mathbf{K}^{\text{top}}) = K_n(C_r^*(G)).$$

Relations between the Farrell-Jones and the Baum-Connes Conjecture

- We discuss some relations between the Farrell-Jones Conjecture and the Baum-Connes Conjecture.
- Mainly these come from the sequence of inclusions of rings

$$\mathbb{Z}G \to \mathbb{R}G \to C_r^*(G;\mathbb{R}) \to C_r^*(G)$$

  and the change of theories from algebraic to topological K-theory and from algebraic L-theory to topological K-theory for C^*-algebras.

$$\begin{array}{ccc}
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_{\mathbb{Z}}^p)[1/2] & \xrightarrow{\cong} & L_n^p(\mathbb{Z}G)[1/2] \\
\downarrow\cong & & \downarrow\cong \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_{\mathbb{R}}^p)[1/2] & \xrightarrow{\cong} & L_n^p(\mathbb{R}G)[1/2] \\
\downarrow\cong & & \downarrow\cong \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_{C_r^*(?;\mathbb{R})}^p)[1/2] & \xrightarrow{\cong} & L_n^p(C_r^*(G;\mathbb{R}))[1/2] \\
\downarrow\cong & & \downarrow\cong \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathbb{R}}^{\mathrm{top}})[1/2] & \xrightarrow{\cong} & K_n(C_r^*(G;\mathbb{R}))[1/2] \\
\downarrow & & \downarrow \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathrm{top}})[1/2] & \xrightarrow{\cong} & K_n(C_r^*(G))[1/2]
\end{array}$$

Theorem (Rational computations of K-groups, L. (2002))

Let G be a group. Let T be the set of conjugacy classes (g) *of elements* $g \in G$ *of finite order.*
Then there is a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\
\downarrow & & \downarrow \\
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \longrightarrow & K_n^{\mathrm{top}}(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C}
\end{array}$$

- The horizontal arrows can be identified with the assembly maps occurring in the Farrell-Jones Conjecture and the Baum-Connes Conjecture by the equivariant Chern character.
- In particular they are isomorphisms if these conjecture hold for G.

- Splitting principle.

  The calculation of the relevant K-and L-groups often split into a universal group homology part which is independent of the theory, and a second part which essentially depends on the theory in question and the coefficients.

Status of the Farrell-Jones and the Baum-Connes Conjecture

Theorem (Bartels-L.-Reich (2007), Bartels-Echterhoff-Reich (2007))

Let R be a ring. Then:

- *Every hyperbolic group and every virtually nilpotent group belongs to $\mathcal{FJ}(R)$;*
- *If G_1 and G_2 belong to $\mathcal{FJ}(R)$, then $G_1 \times G_2$ belongs to $\mathcal{FJ}(R)$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}(R)$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}(R)$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}(R)$, then $H \in \mathcal{FJ}(R)$.*

- We emphasize that this result holds for all rings R. Actually we can even treat crossed product rings $R * G$. For more information about the last result and its proof we refer to the talks of Bartels.
- The groups above are certainly wild in Bridson's universe of groups.
- Many recent constructions of groups with exotic properties are given by colimits of directed systems of hyperbolic groups. Examples are.
  - groups with expanders;
  - Lacunary hyperbolic groups in the sense of Olshanskii-Osin-Sapir;
  - Tarski monsters, i.e., infinite groups whose proper subgroups are all finite cyclic of p-power order for a given prime p;
- Gromov's groups with expanders, for which the Baum-Connes Conjecture with coefficients fails by Higson-Lafforgue-Skandalis (2002), belong to $\mathcal{FJ}_K(R)$ for all R.

- If G is a torsionfree hyperbolic group and R any ring, then we get an isomorphism

$$H_n(BG;\mathbf{K}_R) \oplus \left(\bigoplus_{\substack{(C), C \subseteq G, C \neq 1 \\ C \text{ maximal cyclic}}} \mathrm{NK}_n(R) \right) \xrightarrow{\cong} K_n(RG).$$

- Bartels and L. have a program to prove $G \in \mathcal{FJ}_K(R)$ if G acts properly and cocompactly on a simply connected CAT(0)-space.
- This would imply $G \in \mathcal{FJ}_K(R)$ for all subgroups G of cocompact lattices in almost connected Lie groups and for all limit groups G.

Theorem (Farrell-Jones (1993))

Let G be a subgroup of a cocompact lattice in an almost connected Lie group.
Then the Farrell-Jones Conjecture for pseudo-isotopy is true for G.

Theorem (L.-Reich-Rognes-Varisco (2007))

The Farrell-Jones Conjecture for topological Hochschild homology is true for all groups.

- For more information about the theorems above and further results we refer to the talks by Bartels, Rosenthal and Varisco.

Theorem (Farrell-Jones (1991 - 1993))

The Borel Conjecture and the L-theoretic Farrell-Jones Conjecture with coefficients in $\mathbb{Z}$ are true for a group G if one of the following conditions are satisfied:

- *G is the fundamental group of a closed Riemannian manifold with non-positive curvature;*
- *G is the fundamental group of a complete Riemannian manifold with pinched negative curvature;*
- *G is a torsionfree subgroup of $GL(n, \mathbb{R})$.*

- Bartels and L. have a program to prove the L-theoretic Farrell-Jones Conjecture for all coefficient rings and the same class of groups for which the K-theoretic versions have been proved.
- Bartels and L. have a program to prove $G \in \mathcal{FJ}_L(R)$ if G acts properly and cocompactly on a simply connected CAT(0)-space. This would yield the same result for all subgroups of cocompact lattices in almost connected Lie groups.
- Recall that a group G which belongs to both $\mathcal{FJ}_K(\mathbb{Z})$ and $\mathcal{FJ}_L(\mathbb{Z})$ satisfies the Borel Conjecture.

Definition (a-T-menable group)

A group G is *a-T-menable*, or, equivalently, has the *Haagerup property* if G admits a metrically proper isometric action on some affine Hilbert space.

- The class of a-T-menable groups is closed under taking subgroups, under extensions with finite quotients and under finite products.
  It is not closed under semi-direct products.
- Examples of a-T-menable groups are:
  - countable amenable groups;
  - countable free groups;
  - discrete subgroups of $SO(n, 1)$ and $SU(n, 1)$;
  - Coxeter groups;
  - countable groups acting properly on trees, products of trees, or simply connected CAT(0) cubical complexes.

- A group G has *Kazhdan's property (T)* if, whenever it acts isometrically on some affine Hilbert space, it has a fixed point. An infinite a-T-menable group does not have property (T). Since $SL(n, \mathbb{Z})$ for $n \geq 3$ has property (T), it cannot be a-T-menable.

Theorem (Higson-Kasparov(2001))

A group G which is a-T-menable satisfies the Baum Connes Conjecture (with coefficients).

Theorem (Lafforgue (1998))

The Baum-Connes Conjecture is true for a certain class of groups which does contain some groups with property (T).

Theorem (Mineyev-Yu (2002))

The Baum-Connes Conjecture is true for subgroups of hyperbolic groups.

Theorem (Bartels-Echterhoff-L. (2007))

The Bost Conjecture is true for a colimit of a directed system of hyperbolic groups.

Miscellaneous

- The Baum-Connes Conjecture and the Farrell-Jones Conjecture are not known for $SL_n(\mathbb{Z})$ for $n \geq 3$, mapping class groups and $\mathrm{Out}(F_n)$;
- Certain groups with expanders yield counterexamples to the Baum-Connes Conjecture with coefficients by a construction due to Higson-Lafforgue-Skandalis (2002).
- The K-theoretic Farrell-Jones conjecture and the Bost Conjecture are true for these groups by recent results of Bartels-L.-Reich (2007) and Bartels-Echterhoff-L. (2007).
- It is not known whether there are counterexamples to the Farrell-Jones Conjecture or the Baum-Connes Conjecture.

- There seems to be no promising candidate of a group G which is a potential counterexample to the K- or L-theoretic Farrell-Jones Conjecture or the Bost Conjecture.
- The Baum-Connes Conjecture is the one for which it is most likely that there may exist a counterexample.
  One reason is the existence of counterexamples to the version with coefficients.
  Another reason is that $K_n(C_r^*(G))$ has certain failures concerning functoriality which do not exists for $K_n^G(\underline{E}G)$.
  For instance it is not functorial for arbitrary group homomorphisms since the reduced group C^*-algebra is not functorial for arbitrary group homomorphisms.
  These failures are not present for $K_n(RG)$, $L^{\langle -\infty \rangle}(RG)$ and $K_n(l^1(G))$.

- Most of the proofs of the Farrell-Jones Conjecture use methods from controlled topology.
- Roughly speaking, controlled topology means that one considers free modules with a basis and thinks of these basis elements as sitting in a metric space.
  Then a map between such modules can be visualized by arrows between these basis elements.
  Control means that these arrows are small.
- Our homological approach to the assembly map is good for structural investigations but not for proofs.
  For proofs of these Conjectures it is often helpful to get some geometric input.
- In the Farrell-Jones setting the door to geometry is opened by interpreting the assembly map as a forget control map.

- The task to show for instance surjectivity is to manipulate a representative of the K-or L-theory class such that its class is unchanged but one has gained control.
- This is done by geometric constructions which yield contracting maps.
- These constructions are possible if some geometry connected to the group is around, such as negative curvature.
- We refer to the lectures of Bartels for such controlled methods.

- The approach using topological cyclic homology goes back to Böckstedt-Hsiang-Madsen.
  It is of homotopy theoretic nature.
  We refer to the lecture of Varisco for more information about that approach.
- The methods of proof for the Baum-Connes Conjecture are of analytic nature.
  The most prominent one is the Dirac-Dual-Dirac method based on KK-theory due to Kasparov.
  KK-theory is a bivariant theory together with a product.
  The assembly map is given by multiplying with a certain element in a certain KK-group.
  The essential idea is to construct another element in a dual KK-group which implements the inverse of the assembly map.

- The analytic methods for the proof of the Baum-Connes Conjecture do not seem to be applicable to the Farrell-Jones setting.
- One would hope for a transfer of methods from the Farrell-Jones setting to the Baum-Connes Conjecture.
  So far not much has happened in this direction.

The end

Thank you for listening