The Isomorphism Conjectures for arbitrary groups (Lecture V)

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Hangzhou, July 2007

Flashback

- We have introduced classifying spaces $E_{\mathcal{F}}(G)$ for a family $\mathcal{F}$ of subgroups.
- We have introduced the notion of an equivariant homology theory.
- We have formulated the Farrell-Jones Conjecture and the Baum-Connes Conjecture.
- We have already discussed application for torsionfree groups such as to the Kaplansky Conjecture and the Borel Conjecture.
- Cliffhanger

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What are the consequences of the Farrell-Jones Conjecture and the Baum-Connes Conjecture?

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Outline

- We give a review of the Farrell-Jones and the Baum-Connes Conjecture.
- We discuss the difference between the families $\mathcal{FIN}$ and $\mathcal{VCYC}$.
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- We discuss the difference between the families $\mathcal{FIN}$ and $\mathcal{VCYC}$.
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- G will always be a discrete group.

Definition (Classifying G-CW-complex for a family of subgroups)

Let $\mathcal{F}$ be a family of subgroups of G. A model for the *classifying G-CW-complex for the family $\mathcal{F}$* is a G-CW-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to X$.

We abbreviate $\underline{E}G := E_{\mathcal{FIN}}(G)$ and call it the *universal G-CW-complex for proper G-actions*.

We also write $EG = E_{\mathcal{TR}}(G)$.

Review of the Isomorphism Conjectures

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An *equivariant homology theory* $\mathcal{H}_*^?$ assigns to every group G a G-homology theory $\mathcal{H}_*^G$. These are linked together with the following so called *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a H-CW-pair (X, A), there are for all $n \in \mathbb{Z}$ natural homomorphisms

$$\operatorname{ind}_\alpha\colon \mathcal{H}_n^H(X, A) \quad\to\quad \mathcal{H}_n^G(\operatorname{ind}_\alpha(X, A))$$

satisfying

- Bijectivity
  If $\ker(\alpha)$ acts freely on X, then $\operatorname{ind}_\alpha$ is a bijection;
- Compatibility with the boundary homomorphisms;
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Conjecture (*K*-theoretic Farrell-Jones-Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (*L*-theoretic Farrell-Jones-Conjecture)

The L-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

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The Baum-Connes Conjecture predicts that the assembly map

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}^{\mathrm{top}}) \to H_n^G(pt, \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

- All assembly maps are the maps induced by the projection $E_{\mathcal{F}}(G) \to \mathrm{pt}$.
- These Conjecture can be thought of a kind of generalized induction theorem. They allow to compute the value of a functor such as $K_n(RG)$ on G in terms of its values for all virtually cyclic subgroups of G.

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Changing the family

Theorem (Transitivity Principle)

Let $\mathcal{F} \subseteq \mathcal{G}$ be two families of subgroups of G. Let $\mathcal{H}_*^?$ be an equivariant homology theory. Assume that for every element $H \in \mathcal{G}$ and $n \in \mathbb{Z}$ the assembly map

$$\mathcal{H}_n^H(E_{\mathcal{F}|_H}(H)) \to \mathcal{H}_n^H(pt)$$

is bijective, where $\mathcal{F}|_H = \{K \cap H \mid K \in \mathcal{F}\}$.
Then the relative assembly map induced by the up to G-homotopy unique G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$

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is bijective for all $n \in \mathbb{Z}$.

Changing the family

Theorem (Transitivity Principle)

Let $\mathcal{F} \subseteq \mathcal{G}$ be two families of subgroups of G. Let $\mathcal{H}^?_*$ be an equivariant homology theory. Assume that for every element $H \in \mathcal{G}$ and $n \in \mathbb{Z}$ the assembly map

$$\mathcal{H}^H_n(E_{\mathcal{F}|_H}(H)) \to \mathcal{H}^H_n(pt)$$

is bijective, where $\mathcal{F}|_H = \{K \cap H \mid K \in \mathcal{F}\}$.
Then the relative assembly map induced by the up to G-homotopy unique G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$

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Example (Passage from $\mathcal{FIN}$ to $\mathcal{VCYC}$ for the Baum-Connes Conjecture)

- The Baum-Connes Conjecture is known to be true for virtually cyclic groups. The Transitivity Principle implies that the relative assembly
$$K_n^G(\underline{E}G) \xrightarrow{\cong} K_n^G(E_{\mathcal{VCYC}}(G))$$
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- In general the relative assembly maps

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For instance the Bass-Heller Swan decomposition

$$K_{n-1}(R) \oplus K_n(R) \oplus \mathrm{NK}_n(R) \oplus \mathrm{NK}_n(R)) \xrightarrow{\cong} K_n(R[t,t^{-1}]) \cong K_n(R[\mathbb{Z}])$$

and the isomorphism

$$H_n^{\mathbb{Z}}(\underline{E}\mathbb{Z};\mathbf{K}_R) = H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_R) = H_n^{\{1\}}(S^1,\mathbf{K}_R) = K_{n-1}(R) \oplus K_n(R)$$

show that

$$H_n^{\mathbb{Z}}(\underline{E}\mathbb{Z};\mathbf{K}_R) \to H_n^{\mathbb{Z}}(\mathrm{pt};\mathbf{K}_R) = K_n(R\mathbb{Z})$$

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- An infinite virtually cyclic group G is called of type I if it admits an epimorphism onto $\mathbb{Z}$ and of type II otherwise. A virtually cyclic group is of type II if and only if admits an epimorphism onto D_∞.
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The following maps are bijective for all $n \in \mathbb{Z}$

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- The results above imply that the versions of the Farrell-Jones Conjecture for torsionfree groups which we have presented in the second lecture follow from the general versions.
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Consequence of the Isomorphism Conjectures

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\mathrm{sign}_x(M,f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M,f), i.e., for every orientation preserving homotopy equivalence of closed oriented manifolds $g\colon M_0 \to M_1$ and homotopy equivalence $f_i\colon M_0 \to M_1$ with $f_1 \circ g \simeq f_2$ we have

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Theorem (The Farrell-Jones, the Baum-Connes and the Novikov Conjecture)

Suppose that one of the following assembly maps

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Theorem ($K_0(RG)$ and induction from finite subgroups, Bartels-L.-Reich (2007))

- *Let R be a regular ring with $\mathbb{Q} \subseteq R$. Suppose $G \in \mathcal{FJ}_K(R)$. Then the map given by induction from finite subgroups of G*

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Let F be a field of characteristic zero. Suppose that $G \in \mathcal{FJ}_K(F)$. Then for every finitely generated projective FG-module P there exists a positive integer k and finitely many finite subgroups $H_1, H_2, \ldots, H_r$ such that

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Definition (Hattori-Stallings rank)

Define the Hattori-Stallings rank of P to be the class function

$$\mathrm{HS}_{RG}(P) := \mathrm{tr}_{RG}(A).$$

- This definition is independent of the choice of Q and ϕ.
- Let G be a finite group and let F be a field of characteristic zero. Then a finitely generated RG-module P is the same as a finite dimensional G-representation over F and the Hattori-Stallings rank can be identified with the character of the G-representation.

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Conjecture (Bass Conjecture)

Let R be a commutative integral domain and let G be a group. Let $g \neq 1$ be an element in G. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R.
Then the Bass Conjecture predicts that for every finitely generated projective RG-module P the value of its Hattori-Stallings rank $\mathrm{HS}_{RG}(P)$ at (g) is trivial.

- If G is finite, this is just the Theorem of Swan (1960).
- Another version of it would predict for the quotient field F of R that

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Let G be a group. Suppose that

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Conjecture (Vanishing of Bass-Nil-groups)

Let R be a regular ring with $\mathbb{Q} \subseteq R$. Then we get for all groups G and all $n \in \mathbb{Z}$ that

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Conjecture (Homotopy invariance of $L2$-torsion)

Let X and Y be det-L^2-acyclic finite G-CW-complexes, which are G-homotopy equivalent. Then their L^2-torsion agree:

$$\rho^{(2)}(X;\mathcal{N}(G)) = \rho^{(2)}(Y;\mathcal{N}(G)).$$

- The L^2-torsion of closed Riemannian manifold M is defined in terms of the heat kernel on the universal covering. If M is hyperbolic and has odd dimension, its L^2-torsion is up to a (non-vanishing) dimension constant its volume.
- The conjecture above allows to extend the notion of volume to hyperbolic groups whose L^2-Betti numbers all vanish.

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Suppose that $G \in \mathcal{FJ}_K(\mathbb{Z})$. Then G satisfies the Conjecture above.

- Deninger can define a *p*-adic Fuglede-Kadison determinant for a group G and relate it to *p*-adic entropy provided that $\mathrm{Wh}^{\mathbb{F}_p}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is trivial.
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plays a role (33 %) in a program to prove the Atiyah Conjecture. It says that for a closed Riemannian manifold with torsionfree fundamental group the L^2-Betti numbers of its universal covering are all integers. The Atiyah Conjecture is rather surprising in view of the analytic definition of the L^2-Betti numbers by

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Definition (Bott manifold)

A *Bott manifold* is any simply connected closed Spin-manifold B of dimension 8 whose $\widehat{A}$-genus $\widehat{A}(B)$ is 8.

- We fix such a choice. (The particular choice does not matter.)
- Notice that the index defined in terms of the Dirac operator $\mathrm{ind}_{C_r^*(\{1\};\mathbb{R})}(B) \in KO_8(\mathbb{R}) \cong \mathbb{Z}$ is a generator and the product with this element induces the Bott periodicity isomorphisms $KO_n(C_r^*(G;\mathbb{R})) \xrightarrow{\cong} KO_{n+8}(C_r^*(G;\mathbb{R}))$.
- In particular

$$\mathrm{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) = \mathrm{ind}_{C_r^*(\pi_1(M\times B);\mathbb{R})}(M \times B),$$

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- If M carries a Riemannian metric with positive scalar curvature, then the index

$$\operatorname{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) \in KO_n(C_r^*(\pi_1(M);\mathbb{R})),$$

which is defined in terms of the Dirac operator on the universal covering, must vanish by the Bochner-Lichnerowicz formula.

Conjecture ((Stable) Gromov-Lawson-Rosenberg Conjecture)

Let M be a closed connected Spin*-manifold of dimension $n \geq 5$. Then $M \times B^k$ carries for some integer $k \geq 0$ a Riemannian metric with positive scalar curvature if and only if*

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Theorem (Stolz (2002))

Suppose that the assembly map for the real version of the Baum-Connes Conjecture

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- The requirement $\dim(M) \geq 5$ is essential in the Stable Gromov-Lawson-Rosenberg Conjecture, since in dimension four new obstructions, the Seiberg-Witten invariants, occur.
- The unstable version of the Gromov-Lawson-Rosenberg Conjecture says that M carries a Riemannian metric with positive scalar curvature if and only if $\mathrm{ind}_{C_r^*(\pi_1(M);\mathbb{R})}(M) = 0$.
- Schick(1998) has constructed counterexamples to the unstable version using minimal hypersurface methods due to Schoen and Yau.
- It is not known whether the unstable version is true or false for finite fundamental groups.
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Question (Status)

For which groups are the Farrell-Jones Conjecture and the Baum-Connes Conjecture known to be true? What are open interesting cases?

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