The Isomorphism Conjectures in the torsionfree case (Lecture II)

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Flashback

- We have introduced $K_n(R)$ for $n \in \mathbb{Z}, n \leq 1$.
- We have discussed the topological relevance of $K_0(RG)$ and the Whitehead group $\mathrm{Wh}(G)$, e.g., the finiteness obstruction and the s-cobordism theorem.
- We have stated the conjectures that $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ vanish for torsionfree G.
- We have presented the Bass-Heller-Swan decomposition and indicated some similarities between $K_n(RG)$ and group homology.
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Question (K-theory of group rings and group homology)

Is there a relation between $K_n(RG)$ and the group homology of G?

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Outline

- We introduce spectra and how they yield homology theories.
- We state the Farrell-Jones-Conjecture and the Baum-Connes Conjecture for torsionfree groups.
- We discuss applications of these conjectures such as the Kaplansky Conjecture and the Borel Conjecture.
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Homology theories and spectra

Definition (Spectrum)

A *spectrum*

$$\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$$

is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps*

$$\sigma(n) \colon E(n) \wedge S^1 \longrightarrow E(n+1).$$

A *map of spectra*

$$\mathbf{f} \colon \mathbf{E} \to \mathbf{E}'$$

is a sequence of maps $f(n) \colon E(n) \to E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e., $f(n+1) \circ \sigma(n) \; = \; \sigma'(n) \circ \left(f(n) \wedge \mathrm{id}_{S^1}\right)$ holds for all $n \in \mathbb{Z}$.

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- Given two pointed spaces $X = (X, x_0)$ and $Y = (Y, y_0)$, their one-point-union and their smash product are defined to be the pointed spaces

$$\begin{aligned} X \vee Y &:= \{(x, y_0) \mid x \in X\} \cup \{(x_0, y) \mid y \in Y\} \subseteq X \times Y; \\ X \wedge Y &:= (X \times Y)/(X \vee Y). \end{aligned}$$

We have $S^{n+1} \cong S^n \wedge S^1$.

- The sphere spectrum $\mathbf{S}$ has as n-th space S^n and as n-th structure map the homeomorphism $S^n \wedge S^1 \xrightarrow{\cong} S^{n+1}$.
- Let X be a pointed space. Its suspension spectrum $\Sigma^\infty X$ is given by the sequence of spaces $\{X \wedge S^n \mid n \geq 0\}$ with the homeomorphism $(X \wedge S^n) \wedge S^1 \cong X \wedge S^{n+1}$ as structure maps. We have $\mathbf{S} = \Sigma^\infty S^0$.

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Definition (Ω-spectrum)

Given a spectrum $\mathbf{E}$, we can consider instead of the structure map $\sigma(n) \colon E(n) \wedge S^1 \to E(n+1)$ its adjoint

$$\sigma'(n) \colon E(n) \to \Omega E(n+1) = \operatorname{map}(S^1, E(n+1)).$$

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Definition (Homotopy groups of a spectrum)

Given a spectrum $\mathbf{E}$, define for $n \in \mathbb{Z}$ its *n-th homotopy group*

$$\pi_n(\mathbf{E}) := \operatorname*{colim}_{k \to \infty} \pi_{k+n}(E(k))$$

to be the abelian group which is given by the colimit over the directed system indexed by $\mathbb{Z}$ with k-th structure map

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- Notice that a spectrum can have in contrast to a space non-trivial negative homotopy groups.
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- **Eilenberg-MacLane spectrum**

  Let A be an abelian group. The n-th Eilenberg-MacLane space $EM(A,n)$ associated to A for $n \geq 0$ is a CW-complex with $\pi_m(EM(A,n)) = A$ for $m = n$ and $\pi_m(EM(A,n)) = \{0\}$ for $m \neq n$. The associated Eilenberg-MacLane spectrum $\mathbf{H}(A)$ has as n-th space $EM(A,n)$ and as n-th structure map a homotopy equivalence $EM(A,n) \to \Omega EM(A,n+1)$.

- **Algebraic K-theory spectrum**

  For a ring R there is the algebraic K-theory spectrum $\mathbf{K}_R$ with the property

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- Algebraic L-theory spectrum

  For a ring with involution R there is the algebraic L-theory spectrum $\mathbf{L}_R^{\langle -\infty \rangle}$ with the property

$$\pi_n(\mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(R) \quad \text{for } n \in \mathbb{Z}.$$

- Topological K-theory spectrum

  By Bott periodicity there is a homotopy equivalence

$$\beta \colon BU \times \mathbb{Z} \xrightarrow{\simeq} \Omega^2(BU \times \mathbb{Z}).$$

  The topological K-theory spectrum $\mathbf{K}^{\mathrm{top}}$ has in even degrees $BU \times \mathbb{Z}$ and in odd degrees $\Omega(BU \times \mathbb{Z})$.
  The structure maps are given in even degrees by the map β and in odd degrees by the identity $\mathrm{id} \colon \Omega(BU \times \mathbb{Z}) \to \Omega(BU \times \mathbb{Z})$.

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Let Λ be a commutative ring, for instance $\mathbb{Z}$ or $\mathbb{Q}$.
A *homology theory* $\mathcal{H}_*$ with values in Λ-modules is a covariant functor from the category of CW-pairs to the category of $\mathbb{Z}$-graded Λ-modules together with natural transformations

$$\partial_n(X, A)\colon \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- Homotopy invariance
- Long exact sequence of a pair
- Excision

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$$\bigoplus_{i \in I} \mathcal{H}_n(X_i) \xrightarrow{\cong} \mathcal{H}_n\left(\coprod_{i \in I} X_i\right).$$

Definition (Smash product)

Let $\mathbf{E}$ be a spectrum and X be a pointed space. Define the smash product $X \wedge \mathbf{E}$ to be the spectrum whose n-th space is $X \wedge E(n)$ and whose n-th structure map is

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Let $\mathbf{E}$ *be a spectrum. Then we obtain a homology theory* $H_*(-;\mathbf{E})$ *by*

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Example (Stable homotopy theory)

The homology theory associated to the sphere spectrum **S** is stable homotopy $\pi_*^s(X)$. The groups $\pi_n^s(\mathrm{pt})$ are finite abelian groups for $n \neq 0$ by a result of Serre (1953). Their structure is only known for small n.

Example (Singular homology theory with coefficients)

The homology theory associated to the Eilenberg-MacLane spectrum $\mathbf{H}(A)$ is singular homology with coefficients in A.

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The homology theory associated to the topological K-theory spectrum $\mathbf{K}^{\mathrm{top}}$ is K-homology $K_*(X)$. We have

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The Isomorphism Conjectures for torsionfree groups

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(RG)$ is the algebraic K-theory of the group ring RG;
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Conjecture (*L*-theoretic Farrell-Jones Conjecture for torsionfree groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

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Conjecture (Baum-Connes Conjecture for torsionfree groups)

The Baum-Connes Conjecture for the torsionfree group predicts that the assembly map

$$K_n(BG) \to K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(BG)$ is the topological K-homology of BG, where $K_*(-) = H_*(-; \mathbf{K}^{\mathrm{top}})$ for $\mathbf{K}^{\mathrm{top}}$ the topological K-theory spectrum.
- $K_n(C_r^*(G))$ is the topological K-theory of the reduced complex group C^*-algebra $C_r^*(G)$ of G which is the closure in the norm topology of $\mathbb{C}G$ considered as subalgebra of $\mathcal{B}(l^2(G))$.
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Consequences of the Isomorphism Conjectures for torsionfree groups

- In order to illustrate the depth of the Farrell-Jones Conjecture and the Baum-Connes Conjecture, we present some conclusions which are interesting in their own right.
- Let $\mathcal{FJ}_K(R)$ and $\mathcal{FJ}_L(R)$ respectively be the class of groups which satisfy the K-theoretic and L-theoretic respectively Farrell-Jones Conjecture for the coefficient ring R.
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Lemma

Let R be a regular ring. Suppose that G is torsionfree and $G \in \mathcal{FJ}_K(R)$. Then

- *$K_n(RG) = 0$ for $n \leq -1$;*
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Suppose that G is torsionfree and $G \in \mathcal{FJ}_K(\mathbb{Z})$. Then the Whitehead group $\operatorname{Wh}(G)$ is trivial.

Proof.

The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG; \mathbf{K}_R)$ whose E^2-term is given by

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$$0 \to H_0(BG; K_1(\mathbb{Z})) = \{\pm 1\} \to H_1(BG; \mathbf{K}_{\mathbb{Z}}) \cong K_1(\mathbb{Z}G) \\ \to H_1(BG; K_0(\mathbb{Z})) = G/[G, G] \to 0.$$

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In particular we get for a torsionfree group $G \in \mathcal{FJ}_K(\mathbb{Z})$:

- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
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- $\mathrm{Wh}(G) = 0$;
- Every finitely dominated CW-complex X with $G = \pi_1(X)$ is homotopy equivalent to a finite CW-complex;
- Every compact h-cobordism W of dimension ≥ 6 with $\pi_1(W) \cong G$ is trivial;
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Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Theorem (The Farrell-Jones Conjecture and the Kaplansky Conjecture, Bartels-L.-Reich(2007))

Let F be a skew-field and let G be a group with $G \in \mathcal{FJ}_K(F)$. Suppose that one of the following conditions is satisfied:

- *F is commutative and has characteristic zero and G is torsionfree;*
- *G is torsionfree and sofic, e.g., residually amenable;*
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- *The characteristic of F is p, all finite subgroups of G are p-groups and G is sofic.*

Then 0 and 1 are the only idempotents in FG.

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 *and* 1 *are the only idempotents in RG.*

Theorem (The Farrell-Jones Conjecture and the Kaplansky Conjecture, Bartels-L.-Reich(2007))

Let F be a skew-field and let G be a group with $G \in \mathcal{FJ}_K(F)$*. Suppose that one of the following conditions is satisfied:*

- *F is commutative and has characteristic zero and G is torsionfree;*
- *G is torsionfree and sofic, e.g., residually amenable;*
- *The characteristic of F is p, all finite subgroups of G are p-groups and G is sofic.*

Then 0 *and* 1 *are the only idempotents in FG.*

Proof.

- Let p be an idempotent in FG. We want to show $p \in \{0, 1\}$.
- Denote by $\epsilon\colon FG \to F$ the augmentation homomorphism sending $\sum_{g \in G} r_g \cdot g$ to $\sum_{g \in G} r_g$. Obviously $\epsilon(p) \in F$ is 0 or 1. Hence it suffices to show $p = 0$ under the assumption that $\epsilon(p) = 0$.
- Let $(p) \subseteq FG$ be the ideal generated by p which is a finitely generated projective FG-module.
  Since $G \in \mathcal{FJ}_K(F)$, we can conclude that
  $$i_*\colon K_0(F) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$
  is surjective.
  Hence we can find a finitely generated projective F-module P and integers $k, m, n \geq 0$ satisfying
  $$(p)^k \oplus FG^m \cong_{FG} i_*(P) \oplus FG^n.$$

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- If we now apply $i_* \circ \epsilon_*$ and use $\epsilon \circ i = \mathrm{id}$, $i_* \circ \epsilon_*(FG^l) \cong FG^l$ and $\epsilon(p) = 0$ we obtain

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- Inserting this in the first equation yields

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- Our assumptions on F and G imply that FG is stably finite, i.e., if A and B are square matrices over FG with $AB = I$, then $BA = I$. This implies $(p)^k = 0$ and hence $p = 0$.

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Theorem (The Baum-Connes Conjecture and the Kaplansky Conjecture)

Let G be a torsionfree group with $G \in \mathcal{BC}$. Then 0 and 1 are the only idempotents in $\mathbb{C}G$.

Proof.

- There is a trace map
$$\mathrm{tr}\colon C_r^*(G) \to \mathbb{C}$$
which sends $f \in C_r^*(G) \subseteq \mathcal{B}(l^2(G))$ to $\langle f(e), e \rangle_{l^2(G)}$.
- The L^2-index theorem due to Atiyah (1976) shows that the composite
$$K_0(BG) \to K_0(C_r^*(G)) \xrightarrow{\mathrm{tr}} \mathbb{C}$$
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- Hence $G \in \mathcal{BC}$ implies $\text{tr}(p) \in \mathbb{Z}$.
- Since $\text{tr}(1) = 1$, $\text{tr}(0) = 0$, $0 \leq p \leq 1$ and $p^2 = p$, we get $\text{tr}(p) \in \mathbb{R}$ and $0 \leq \text{tr}(p) \leq 1$.
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Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that M and N are homeomorphic.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity. A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds of dimension ≥ 3 is homotopic to an isometric diffeomorphism.
- The Borel Conjecture is not true in the smooth category by results of Farrell-Jones(1989).
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If the K- and L-theoretic Farrell-Jones Conjecture hold for G in the case $R = \mathbb{Z}$, then the Borel Conjecture is true in dimension ≥ 5 and in dimension 4 if G is good in the sense of Freedman.

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Definition (Structure set)

The *structure set* $\mathcal{S}^{top}(M)$ of a manifold M consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold N as source.

Two such homotopy equivalences $f_0 \colon N_0 \to M$ and $f_1 \colon N_1 \to M$ are equivalent if there exists a homeomorphism $g \colon N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

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Theorem (Ranicki (1992))

There is an exact sequence of abelian groups, called algebraic surgery exact sequence, for an n-dimensional closed manifold M

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}} \mathcal{S}^{\mathrm{top}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

It can be identified with the classical geometric surgery sequence due to Sullivan and Wall in high dimensions.

- $\mathcal{S}^{\mathrm{top}}(M)$ consist of one element if and only if A_{n+1} is surjective and A_n is injective.
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What happens for groups with torsion?

- The versions of the Farrell-Jones Conjecture and the Baum-Connes Conjecture above become false for finite groups unless the group is trivial.
- For instance the version of the Baum-Connes Conjecture above would predict for a finite group G

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However, $K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} K_0(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} \mathbb{Q}$ and $R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong_{\mathbb{Q}} \mathbb{Q}$ holds if and only if G is trivial.

- If G is torsionfree, then the version of the K-theoretic Farrell-Jones Conjecture predicts

$$\begin{aligned} H_n(B\mathbb{Z};\mathbf{K}_R) = H_n(S^1;\mathbf{K}_R) &= H_n(\mathrm{pt};\mathbf{K}_R) \oplus H_{n-1}(\mathrm{pt};\mathbf{K}_R) \\ &= K_n(R) \oplus K_{n-1}(R) \cong K_n(R\mathbb{Z}). \end{aligned}$$

  In view of the Bass-Heller-Swan decomposition this is only possible if $\mathrm{NK}_n(R)$ vanishes which is true for regular rings R but not for general rings R.

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- **Assembly**

  For a field F of characteristic zero and some groups G one knows that there is an isomorphism

$$\operatorname*{colim}_{\substack{H \subseteq G \\ |H| < \infty}} K_0(FH) \xrightarrow{\cong} K_0(FG).$$

  This indicates that one has at least to take into account the values for all finite subgroups to assemble $K_n(FG)$.

- **Degree Mixing**

  The Bass-Heller-Swan decomposition shows that the K-theory of finite subgroups in degree $m \leq n$ can affect the K-theory in degree n and that at least in the Farrell-Jones setting finite subgroups are not enough.

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- In the Baum-Connes setting Nil-phenomena do not appear. Namely, a special case of a result due to Pimsner-Voiculescu (1982) says

$$K_n(C_r^*(G \times \mathbb{Z})) \cong K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)).$$

- Homological behaviour

  There is still a lot of homological behaviour known for $K_*(C_r^*(G))$. For instance there exists a long exact Mayer-Vietoris sequence associated to amalgamated products $G_1 *_{G_0} G_2$ and a Wang-sequence associated to semi-direct products $G \rtimes \mathbb{Z}$ by Pimsner-Voiculescu (1982).

  Similar versions under certain restrictions exist in K-and L-theory due to Cappell (1974) and Waldhausen (1978) if one makes certain assumptions on R or ignores certain Nil-phenomena.

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Question (Classifying spaces for families)

Is there a version $E_{\mathcal{F}}(G)$ of the classifying space EG which takes the structure of the family of finite subgroups or other families $\mathcal{F}$ of subgroups into account and can be used for a general formulation of the Farrell-Jones Conjecture?

Question (Equivariant homology theories)

Can one define appropriate G-homology theories $\mathcal{H}_*^G$ that are in some sense computable and yield when applied to $E_{\mathcal{F}}(G)$ a term which potentially is isomorphic to the groups $K_n(RG)$, $L^{-\langle\infty\rangle}(RG)$ or $K_n(C_r^*(G))$?
In the torsionfree case they should reduce to $H_n(BG;\mathbf{K}_R)$, $H_n(BG;\mathbf{L}^{-\langle\infty\rangle})$ and $K_n(BG)$.

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