# Classifying spaces for families (Lecture III)

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# Flashback

- We have introduced the Farrell-Jones Conjecture and the Baum-Connes Conjecture for torsionfree groups:

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\begin{array}{rcl}
H_n(BG;\mathbf{K}_R) & \xrightarrow{\cong} & K_n(RG); \\
H_n(BG;\mathbf{L}_R^{\langle -\infty\rangle}) & \xrightarrow{\cong} & L_n^{\langle -\infty\rangle}(RG); \\
K_n(BG) & \xrightarrow{\cong} & K_n(C_r^*(G)).
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**Question (Classifying spaces for families)**

*Is there a version $E_{\mathcal{F}}(G)$ of the classifying space $EG$ which takes the structure of the family of finite subgroups or other families $\mathcal{F}$ of subgroups into account and can be used for a general formulation of the Farrell-Jones Conjecture?*

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# Outline

- We introduce the notion of the classifying space of a family $\mathcal{F}$ of subgroups $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$.
- In the case, where $\mathcal{F}$ is the family $\mathcal{COM}$ of compact subgroups, we present some nice geometric models for $E_{\mathcal{F}}(G)$ and explain $E_{\mathcal{F}}(G) \simeq J_{\mathcal{F}}(G)$.
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# Classifying spaces for families of subgroups

**Definition ($G$-$CW$-complex)**

A *$G$-$CW$-complex* $X$ is a $G$-space together with a $G$-invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n \geq 0} X_n = X$$

such that $X$ carries the colimit topology with respect to this filtration, and $X_n$ is obtained from $X_{n-1}$ for each $n \geq 0$ by attaching equivariant $n$-dimensional cells, i.e., there exists a $G$-pushout

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n \end{array}$$

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- Group means locally compact Hausdorff topological group with a countable basis for its topology, unless explicitly stated differently.

### Example (Simplicial actions)

Let $X$ be a simplicial complex. Suppose that $G$ acts simplicially on $X$. Then $G$ acts simplicially also on the barycentric subdivision $X'$, and all isotropy groups are open and closed. The $G$-space $X'$ inherits the structure of a $G$-$CW$-complex.

### Example (Smooth actions)

Let $G$ be a Lie group acting properly and smoothly on a smooth manifold $M$.
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## Definition (Proper $G$-action)

A $G$-space $X$ is called *proper* if for each pair of points $x$ and $y$ in $X$ there are open neighborhoods $V_x$ of $x$ and $W_y$ of $y$ in $X$ such that the closure of the subset $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$ of $G$ is compact.

## Lemma

- *A proper G-space has always compact isotropy groups.*
- *A G-CW-complex X is proper if and only if all its isotropy groups are compact.*

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## Definition (Family of subgroups)

A *family $\mathcal{F}$ of subgroups* of $G$ is a set of (closed) subgroups of $G$ which is closed under conjugation and finite intersections.

Examples for $\mathcal{F}$ are:

$$\begin{array}{lcl} \mathcal{TR} & = & \{\text{trivial subgroup}\}; \\ \mathcal{FIN} & = & \{\text{finite subgroups}\}; \\ \mathcal{VCYC} & = & \{\text{virtually cyclic subgroups}\}; \\ \mathcal{COM} & = & \{\text{compact subgroups}\}; \\ \mathcal{COMOP} & = & \{\text{compact open subgroups}\}; \\ \mathcal{ALL} & = & \{\text{all subgroups}\}. \end{array}$$

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**Definition (Classifying $G$-$CW$-complex for a family of subgroups)**

Let $\mathcal{F}$ be a family of subgroups of $G$. A model for the *classifying $G$-$CW$-complex for the family $\mathcal{F}$* is a $G$-$CW$-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any $G$-$CW$-complex $Y$, whose isotropy groups belong to $\mathcal{F}$, there is up to $G$-homotopy precisely one $G$-map $Y \to E_{\mathcal{F}}(G)$.

We abbreviate $\underline{E}G := E_{\mathcal{COM}}(G)$ and call it the *universal $G$-$CW$-complex for proper $G$-actions*.

We also write $EG = E_{\mathcal{TR}}(G)$.

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**Theorem (Homotopy characterization of $E_{\mathcal{F}}(G)$)**

*Let $\mathcal{F}$ be a family of subgroups.*

- *There exists a model for $E_{\mathcal{F}}(G)$ for any family $\mathcal{F}$;*
- *Two model for $E_{\mathcal{F}}(G)$ are $G$-homotopy equivalent;*
- *A $G$-$CW$-complex $X$ is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy groups belong to $\mathcal{F}$ and for each $H \in \mathcal{F}$ the $H$-fixed point set $X^H$ is weakly contractible.*

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- A model for $E_{\mathcal{ALL}}(G)$ is $G/G$;
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- Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 * \mathbb{Z}/2$ be the infinite dihedral group.
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## Definition ($\mathcal{F}$-numerable $G$-space)

A *$\mathcal{F}$-numerable $G$-space* is a $G$-space, for which there exists an open covering $\{U_i \mid i \in I\}$ by $G$-subspaces satisfying:

- For each $i \in I$ there exists a $G$-map $U_i \to G/G_i$ for some $G_i \in \mathcal{F}$;
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- Notice that we do not demand that the isotropy groups of a $\mathcal{F}$-numerable $G$-space belong to $\mathcal{F}$.
- If $f \colon X \to Y$ is a $G$-map and $Y$ is $\mathcal{F}$-numerable, then $X$ is also $\mathcal{F}$-numerable.
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- There is also a version $J_{\mathcal{F}}(G)$ of a classifying space for $\mathcal{F}$-numerable $G$-spaces.
- It is characterized by the property that $J_{\mathcal{F}}(G)$ is $\mathcal{F}$-numerable and for every $\mathcal{F}$-numerable $G$-space $Y$ there is up to $G$-homotopy precisely one $G$-map $Y \to J_{\mathcal{F}}(G)$.
- We abbreviate $\underline{J}G = J_{\mathcal{COM}}(G)$ and $JG = J_{\mathcal{TR}}(G)$.
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## Theorem (Comparison of $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$, L. (2005))

- *There is up to G-homotopy precisely one G-map*

$$\phi\colon E_{\mathcal{F}}(G) \to J_{\mathcal{F}}(G);$$

- *It is a G-homotopy equivalence if one of the following conditions is satisfied:*
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# Special models for $\underline{E}G$

- We want to illustrate that the space $\underline{E}G = \underline{J}G$ often has very nice geometric models and appear naturally in many interesting situations.
- Let $C_0(G)$ be the Banach space of complex valued functions of $G$ vanishing at infinity with the supremum-norm. The group $G$ acts isometrically on $C_0(G)$ by $(g \cdot f)(x) := f(g^{-1}x)$ for $f \in C_0(G)$ and $g, x \in G$.
  Let $PC_0(G)$ be the subspace of $C_0(G)$ consisting of functions $f$ such that $f$ is not identically zero and has non-negative real numbers as values.

**Theorem (Operator theoretic model, Abels (1978))**

*The $G$-space $PC_0(G)$ is a model for $\underline{J}G$.*

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## Theorem

*Let $G$ be discrete. A model for $\underline{J}G$ is the space*

$$X_G = \left\{ f \colon G \to [0,1] \;\middle|\; f \text{ has finite support}, \sum_{g \in G} f(g) = 1 \right\}$$

*with the topology coming from the supremum norm.*

## Theorem (Simplicial Model)

*Let $G$ be discrete. Let $P_\infty(G)$ be the geometric realization of the simplicial set whose $k$-simplices consist of $(k+1)$-tupels $(g_0, g_1, \ldots, g_k)$ of elements $g_i$ in $G$. This is a model for $\underline{E}G$.*

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*Let G be discrete. Let $P_\infty(G)$ be the geometric realization of the simplicial set whose k-simplices consist of $(k+1)$-tupels $(g_0, g_1, \ldots, g_k)$ of elements $g_i$ in G. This is a model for $\underline{E}G$.*

## Theorem

*Let G be discrete. A model for $\underline{J}G$ is the space*

$$X_G = \left\{ f\colon G \to [0,1] \;\middle|\; f \text{ has finite support}, \sum_{g \in G} f(g) = 1 \right\}$$

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*Suppose that G is almost connected, i.e., the group $G/G^0$ is compact for $G^0$ the component of the identity element.*
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**Theorem (Discrete subgroups of almost connected Lie groups)**

*Let L be a Lie group with finitely many path components.*
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**Theorem (Actions on CAT(0)-spaces)**

*Let $G$ be a (locally compact Hausdorff) topological group. Let $X$ be a proper $G$-$CW$-complex. Suppose that $X$ has the structure of a complete simply connected $\mathrm{CAT}(0)$-space for which $G$ acts by isometries.*
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**Theorem (Affine buildings)**

*Let $G$ be a totally disconnected group. Suppose that $G$ acts on the affine building $\Sigma$ by simplicial automorphisms such that each isotropy group is compact.*
*Then $\Sigma$ is a model for both $J_{\mathcal{COMOP}}(G)$ and $\underline{J}G$ and the barycentric subdivision $\Sigma'$ is a model for both $E_{\mathcal{COMOP}}(G)$ and $\underline{E}G$.*

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**Theorem (Rips complex, Meintrup-Schick (2002))**

*Let $G$ be a discrete group with a finite symmetric set of generators. Suppose that $(G,S)$ is $\delta$-hyperbolic for the real number $\delta \geq 0$. Let $d$ be a natural number with $d \geq 16\delta + 8$.*
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- Arithmetic groups in a semisimple connected linear $\mathbb{Q}$-algebraic group possess finite models for $\underline{E}G$.
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  We will always assume that $2g + s + r > 2$, or, equivalently, that the Euler characteristic of the punctured surface $F$ is negative.
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- Denote by $\text{Out}(F_n)$ the group of outer automorphisms of $F_n$, i.e., the quotient of the group of all automorphisms of $F_n$ by the normal subgroup of inner automorphisms.
- Culler-Vogtmann (1996) have constructed a space $X_n$ called outer space on which $\text{Out}(F_n)$ acts with finite isotropy groups. It is analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface.
- The space $X_n$ contains a spine $K_n$ which is an $\text{Out}(F_n)$-equivariant deformation retraction. This space $K_n$ is a simplicial complex of dimension $(2n-3)$ on which the $\text{Out}(F_n)$-action is by simplicial automorphisms and cocompact.

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## Example ($SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z})$)

- In order to illustrate some of the general statements above we consider the special example $SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z})$.
- Let $\mathbb{H}^2$ be the 2-dimensional hyperbolic space. The group $SL_2(\mathbb{R})$ acts by isometric diffeomorphisms on the upper half-plane by Moebius transformations. This action is proper and transitive. The isotropy group of $z = i$ is $SO(2)$. Since $\mathbb{H}^2$ is a simply-connected Riemannian manifold, whose sectional curvature is constant $-1$, the $SL_2(\mathbb{R})$-space $\mathbb{H}^2$ is a model for $\underline{E}SL_2(\mathbb{R})$.
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## Example (continued)

- Since $SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\mathbb{R})$, the space $\mathbb{H}^2$ with the obvious $SL_2(\mathbb{Z})$-action is a model for $\underline{E}SL_2(\mathbb{Z})$.
- The group $SL_2(\mathbb{Z})$ is isomorphic to the amalgamated product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$. This implies that there is a tree on which $SL_2(\mathbb{Z})$ acts with finite stabilizers. The tree has alternately two and three edges emanating from each vertex. This is a 1-dimensional model for $\underline{E}SL_2(\mathbb{Z})$.
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- Divide the Poincaré disk into fundamental domains for the $SL_2(\mathbb{Z})$-action. Each fundamental domain is a geodesic triangle with one vertex at infinity, i.e., a vertex on the boundary sphere, and two vertices in the interior. Then the union of the edges, whose end points lie in the interior of the Poincaré disk, is a tree $T$ with $SL_2(\mathbb{Z})$-action which is the tree model above. The tree is a $SL_2(\mathbb{Z})$-equivariant deformation retraction of the Poincaré disk. A retraction is given by moving a point $p$ in the Poincaré disk along a geodesic starting at the vertex at infinity, which belongs to the triangle containing $p$, through $p$ to the first intersection point of this geodesic with $T$.

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- Finiteness properties of the spaces $EG$ and $\underline{E}G$ have been intensively studied in the literature. We mention a few examples and results. For more information we refer to the lectures of Brown.
- If $EG$ has a finite-dimensional model, the group $G$ must be torsionfree. There are often finite models for $\underline{E}G$ for groups $G$ with torsion.
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**Theorem (Discrete subgroups of Lie groups)**

*Let $L$ be a Lie group with finitely many path components. Let $K \subseteq L$ be a maximal compact subgroup $K$. Let $G \subseteq L$ be a discrete subgroup of $L$. Then $L/K$ with the left $G$-action is a model for $\underline{E}G$.*

*Suppose additionally that $G$ is virtually torsionfree, i.e., contains a torsionfree subgroup $\Delta \subseteq G$ of finite index.*

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*Let $G$ be a discrete group which is virtually torsionfree.*

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**Problem**

*For which discrete groups $G$, which are virtually torsionfree, does there exist a $G$-$CW$-model for $\underline{E}G$ of dimension $\operatorname{vcd}(G)$?*

- The results above do give some evidence for a positive answer.
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To be continued

Stay tuned

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