1.section
Survey over the Casson invariant

We start with describing the Casson invariant axiomatically.

1.1 The *Casson invariant* assigns to any oriented homology 3-sphere M an integer

$$\lambda(M) \in \mathbf{Z}$$

such that the following conditions are satisfied :

1.) If M and N are oriented homeomorphic, then $\lambda(M) = \lambda(N)$.

2.) Let $K \subset M$ be a knot. Suppose that $M(K, 1/n)$ is obtained from M by $\frac{1}{n}$-surgery on K. Then $M(K, 1/n)$ is again a homology 3-sphere . Let Δ_K be the symmetrized and normalized Alexander-Conway polynomial of $K \subset M$. Then :

$$\lambda(M(K, 1/(n+1)) - \lambda(M(K, 1/n)) = \frac{1}{2} \cdot \Delta_K''(1)$$

3.) $\lambda(S^3) = 0$ ■

We start with explaining the various terms appearing in the axioms above and then derive some conclusions and give applications. Finally we indicate the construction of the Casson invariant.

Definition 1.2 *Let R be a commutative associative ring with unit and n a positive integer. A R-homology n-sphere M is a n-dimensional manifold M satisfying $H_*(M; R) = H_*(S^n; R)$. We call M a homology sphere or integral homology sphere, if R is $\mathbf{Z}$, and a rational homology sphere, if R is $\mathbf{Q}$* ■

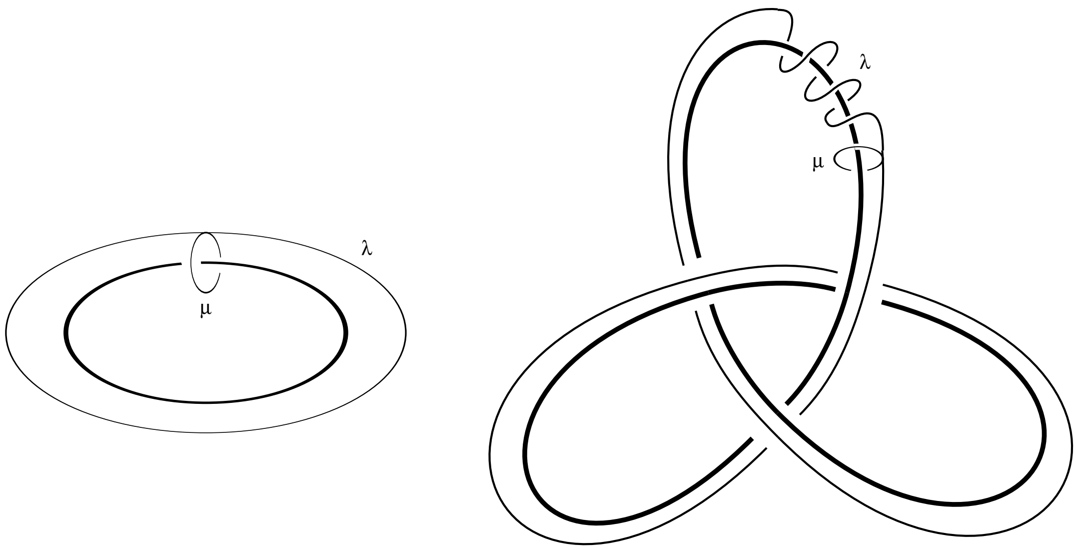
Any R-homology n-sphere is a closed orientable manifold. The following result is a direct consequence of Poincaré duality and the fact that any reducible SO(3)-representation is in fact a S^1-representation and factorizes over $\pi_1(M) \longrightarrow H_1(M)$.

Lemma 1.3 *Let M be an orientable closed 3-manifold. Then the following assertions are equivalent :*

1. *M is a homology sphere.*
2. *The fundamental group $\pi_1(M)$ is perfect.*
3. *$H_1(M) = H_1(M; \mathbf{Z})$ is zero.*
4. *There are no non-trivial reducible $SO(3)$-representations of $\pi_1(M)$.* ■

A *link* K in a 3-manifold M is an oriented closed submanifold of codimension 2. In other words, it is a disjoint union of embedded 1-dimensional spheres in M. A *knot* is a link which has only one component. Consider a knot K in the oriented homology 3-sphere M. Let $N(K)$ be a closed regular neighbourhood of K. Define $M(K) = M - int(N(K))$.By Alexander duality $H_1(M(K)) \cong H^2(M, K)$. We derive from the long cohomology sequence $H^2(M, K) \cong H^1(K)$, since M is a homology sphere. Hence $H_1(M(K))$ is isomorphic to $\mathbf{Z}$. We denote by μ resp. λ a generator of the kernel of $i_* : H_1(\partial N(K)) \longrightarrow H_1(N(K))$ resp. $j_* : H_1(\partial N(K)) \longrightarrow H_1(M(K))$, where i and j are the inclusions. These generators are only unique up to multiplication with ± 1. They can also be characterized by the property that they are represented by simple curves in $\partial N(K)$ such that the linking number of μ resp. λ with K is ± 1 resp. 0. The orientation of M induces an orientation on $N(K)$ and hence an orientation on ∂N using the outward normal and the decomposition $\nu(\partial N(K), N(K)) \oplus T\partial N(K) = TN(K) \mid_{\partial N(K)}$. We always assume that the intersection number of λ and μ in ∂N is $+1$. Then there are only two choices for the pair (λ, μ). Fix such a choice. We call λ the *longitudinal* and μ the *meridian* of $K \subset M$.

1.4 Longitudinal and meridian



Suppose we have fixed integers p and q satisfying $(p, q) = (1)$. Let $\sigma : S^1 \times \partial D^2 \longrightarrow \partial N(K)$ be a homeomorphism assigning $p\mu + q\lambda$ to the class of $\{1\} \times \partial D^2 \in H_1(S^1 \times \partial D^2)$. Then define $M(K, p/q)$ by the push out

1.5
$$
\begin{tikzcd}
S^1 \times \partial D^2 \arrow[r, "i \circ \sigma"] \arrow[d] & M(K) \arrow[d] \\
S^1 \times D^2 \arrow[r] & M(K, p/q)
\end{tikzcd}
$$

Equip $M(K,p/q)$ with the orientation induced from $M(K) \subset M$. We claim that the oriented homeomorphism type of $M(K,p/q)$ depends only on the underlying set of K and the element $p/q \in \mathbf{Q} \cup \{\infty\}$ and is independent of the choices of λ, μ,p, q and σ and the orientation of the knot. Suppose that we have made different choices λ', μ', p', q' and σ'. The composition $\sigma^{-1} \circ \sigma' : S^1 \times \partial D^2 \longrightarrow S^1 \times \partial D^2$ sends the class of $\{1\} \times \partial D^2$ to itself up to a possible sign. Isotopy classes of self maps of $S^1 \times S^1$ are classified by the induced map on homology. Hence a homeomorphism $f : S^1 \times S^1 \longrightarrow S^1 \times S^1$ extends to a homeomorphism $F : S^1 \times D^2 \longrightarrow S^1 \times D^2$ if and only if there is a homomorphism g making the following diagram commutative

$$
\begin{tikzcd}
H_1(S^1 \times S^1) \arrow[r, "H_1(f)"] \arrow[d, "i_*"] & H_1(S^1 \times S^1) \arrow[d, "i_*"'] \\
H_1(S^1 \times D^2) \arrow[r, "g"] & H_1(S^1 \times D^2)
\end{tikzcd}
$$

This implies the existence of an extension $\phi : S^1 \times D^2 \longrightarrow S^1 \times D^2$ of $\sigma^{-1} \circ \sigma'$. Now the desired homeomorphism is induced by ϕ, $\tau^{-1} \circ \sigma$ and the identity on $M(K)$. Given a knot K in a homology 3-sphere and an element $r \in \mathbf{Q} \cup \{\infty\}$, we say that $M(K,r)$ *is obtained from* M *by* r*-Dehn surgery on* K. Notice that $M(K,\infty) = M$ holds.

Lemma 1.6 *Let K be a knot in the homology 3-sphere M. Let p and q be integers satisfying $(p,q) = (1)$. Then*
$$H_1(M(K,p/q)) = \mathbf{Z}/p$$
In particular $M(K,1/n)$ is again a homology 3-sphere .

Proof : This follows from the Mayer-Vietoris-sequence of 1.5 which gives an exact sequence

$$\mathbf{Z} \oplus \mathbf{Z} \xrightarrow{(p,1)} \mathbf{Z} \oplus \mathbf{Z} \longrightarrow H_1(M(K,p/q)) \longrightarrow \{0\} \qquad \blacksquare$$

Let K be a knot in a homology 3-sphere M. Denote by X the knot complement $M - K$. We have already shown above using Alexander duality that $H_1(X)$ is isomorphic to $\mathbf{Z}$. Let $p : \widehat{X} \longrightarrow X$ be the corresponding cyclic covering. Then there is an exact sequence for some $r \geq 0$

$$0 \longrightarrow \oplus_r \mathbf{Z}[\mathbf{Z}] \xrightarrow{i} \oplus_r \mathbf{Z}[\mathbf{Z}] \longrightarrow H_1(\widehat{X}) \longrightarrow 0$$

Let $det(i) \in \mathbf{Z}[\mathbf{Z}]$ be the determinant of i. Then there is exactly one finite Laurent series with integral coefficients $p(t)$ such that $p(1) = 1$, $p(t) = p(t^{-1})$ and $p(t) = \pm t^m \cdot det(i)$ for appropiate $m \in \mathbf{Z}$ holds. This Laurent series denoted by Δ_K is an invariant of the knot K, called the *(symmetrized and normalized) Alexander polynomial.* Now we have defined all terms in the axiomatic characterization of the Casson invariant 1.1.

1.7 The Casson invariant has also the following properties :

a.) If M^- denotes the manifold M with orientation reversed, then :

$$\lambda(M) = -\lambda(M^-)$$

b) If $\lambda(M) \neq 0$, then there is a non-trivial representation of $\pi_1(M)$ in $SU(2)$. In particular the Casson invariant vanishes for homotopy 3-spheres, i. e. oriented 3-manifolds with the homotopy type of S^3.

c.) Let $\mu(M) \in \mathbf{Z}/2$ be the Rohlin invariant of the homology 3-sphere M. Then :

$$\lambda(M) \equiv \mu(M) \quad mod\, 2$$

In particular the Rohlin invariant of a homotopy sphere is zero.

d.) Let $M\sharp N$ be the connected sum of the homology 3-spheres M and N. Then $M\sharp N$ is again a homology 3-sphere satisfying

$$\lambda(M\sharp N) = \lambda(M) + \lambda(N)$$

e.) The Casson invariant is uniquely determined by its axioms 1.1 ■

We recall the definition of the *Rohlin invariant* $\mu(M)$ of a $\mathbf{Z}/2$-homology 3-sphere. For any such M there is a 4-dimensional PL-manifold W with $\partial W = M$ and vanishing first and second Stiefel Whitney classes $w_1(W)$ and $w_2(W)$. These conditions are equivalent to the existence of a *Spin*-structure on W. In particular W is orientable. The Rohlin invariant $\mu(M) \in \mathbf{Z}/16$ is the class represented by the signature $\sigma(W)$. This is well defined by Rohlin's Theorem that the signature of an orientable closed 4-dimensional PL-manifold with a *Spin*-structure is divisible by 16. Namely, if V is another such manifold, the closed 4-manifold $W \cup_M V^-$ has trivial first and second Stiefel Whitney class because Stiefel-Whitney classes are natural and the restriction maps $H^i(W \cup_M V^-; \mathbf{Z}/2) \longrightarrow H^i(W; \mathbf{Z}/2) \oplus H^i(V^-; \mathbf{Z}/2)$ are injective, and the signature is additive : $\sigma(W \cup_M V) = \sigma(W) + \sigma(V)$. If M is a $\mathbf{Z}$-homology 3-sphere, the signature of W is always divisible by 8. Then one defines the Rohlin invariant $\mu(M) \in \mathbf{Z}/2$ to be the class of $\sigma(W)/8$. We will always use the last definition of the Rohlin invariant for a homology 3-sphere .

Next we make some comments on the properties 1.7 of the Casson invariant listed above.

1.8 We will later prove that for any oriented homology 3-sphere M there is a sequence of oriented homology 3-spheres M_0, M_1, ... , M_r such that M_j is obtained from M_{j-1} by $1/n$-surgery on a knot $K_{j-1} \subset M_{j-1}$ and M_0 is S^3 and M_r is M. This implies e.), the uniqueness of the Casson invariant. ■

1.9 Any integer can occur as the value of the Casson invariant. Because the Casson invariant is addititive under connected sum, it suffices to realize the value 1. Consider the trefoil T in S^3. Its Alexander-Conway polynomial is $\Delta_T(t) = t - 1 + t^{-1}$. Notice that $S^3(T,1)$ is the so-called Poincaré sphere, which is defined as the quotient of $SU(2)$ by the binary dodecahedral group A_5^* of order 120. This is the universal central extension of A_5 by $Z/2$. Since $\lambda(S^3(T,0)) = \lambda(S^3) = 0$, we get

$$\lambda(S^3(T,1)) = 1 \qquad ■$$

1.10 The Casson invariant is not an invariant of the fundamental group. Let M be a oriented homology 3-sphere with non-trivial Casson invariant . Then the Casson invariant of $M\sharp M^-$ is zero because of

$$\lambda(M\sharp M^-) = \lambda(M) + \lambda(M^-) = \lambda(M) - \lambda(M) = 0$$

On the other hand $\lambda(M\sharp M)$ is $2 \cdot \lambda(M)$ and hence different from zero. But $M\sharp M$ and $M\sharp M^-$ have the same fundamental group by the Theorem of Seifert-von Kampen, namely the amalgam of $\pi_1(M)$ with itself. ■

1.11 The Casson invariant is not invariant under homology bordism. A homology bordism from M to N is a bordism W from M to N such that the inclusion of both M and N in W is a homology equivalence. Namely, there is a oriented homology 3-sphere M bounding a smooth contractible 4-manifold W with non-trivial Casson invariant. Notice that $W - int(D^4)$ is a homology bordism between M and S^3 for any imbedded $D^4 \subset W$. Recall that the Rohlin invariant is an invariant under homology bordism. ■

1.12 If the oriented homology 3-sphere M possesses an orientation reversing diffeomorphism, then its Casson invariant vanishes because of $\lambda(M) = \lambda(M^-) = -\lambda(M)$. In particular the Rohlin invariant of M is zero. ■

This conclusion is important because of the following result

Theorem 1.13 (Galewski-Stern) *The following assertions are equivalent :*

- *Each topological manifold of dimension ≥ 7 can be triangulated.*
- *There is a homology 3-sphere H such that $H\sharp H$ bounds a contractible 4-dimensional PL-manifold and $\mu(H) = 1$.* ■

A strategy to construct such an oriented homology 3-sphere is to construct an oriented homology 3-sphere H with $\mu(1) = 1$ carrying an orientation reversing involution. Then $H\sharp H$ is oriented diffeomorphic to $H\sharp H^-$. As $((H - int(D^3)) \times I) - int(D^4)$ is a homology bordism from $H\sharp H^-$ to ∂D^4, $H\sharp H$ is the boundary of an acyclic 4-dimensional manifold. But such a H does not exist by 1.12.

We give some explanations of the Theorem 1.13 of Galewski and Stern. A *polyhedron* P is a subset $P \subset \mathbf{R}^n$ such that any point $p \in P$ possesses a cone neighbourhood of the shape $N = \{p\} * K$ for a compact subset $K \subset P$ where $*$ denotes the join. We call N a *star* and K a *link* of p in P. A map $f : P \longrightarrow Q$ between polyhedra is *piecewise linear* or PL for short if each point $p \in P$ has a star $N = \{p\} * K$ such that $f(\lambda a + \mu x) = \lambda f(a) + \mu f(x)$ holds. As $\mathbf{R}^n$ has a canonical structure of a polyhedron, the notion of a PL-structure on a topological manifold is obvious. A *triangulation* (K, t) of a topological space X is a simplical complex K together with a homeomorphism $t :\mid K \mid \longrightarrow X$. A simplical complex which is PL-homeomorphic to a PL-manifold is called a *combinatorial manifold* . It is characterized by the fact that any link of each simplex is PL-homeomorphic to a PL-sphere or PL-ball. A PL-triangulation of a polyhedron P is a triangulation (K, t) with the property that t is a PL-homeomorphism. Any polyhedron possesses a PL-triangulation. If $f :\mid K \mid \longrightarrow \mid L \mid$ is a PL-homeomorphism of the underlying polyhedra of simplicial complexes K and L, then there are subdivisions K' and L' such that f is induced from a simplicial map from K' to L'. A topological manifold M has a PL-structure if and only if it has a triangulation by a combinatorial manifold. We will see the existence of a non-combinatorial triangulation of some PL-manifold and of topological manifolds possessing no triangulation. There are also topological manifolds possessing a triangulation but not a PL-structure.

There are classifying spaces BPL, $BTRI$ and $BTOP$ for PL-manifolds, topological manifolds with triangulation and topological manifolds and natural maps $BPL \longrightarrow BTRI$ and $BTRI \longrightarrow BTOP$. A topological manifold possesses a triangulation if and only if its classifying map into $BTOP$ has a lift to $BTRI$ and similiar for BPL, provided that the dimensions are large enough. Let Θ_3^h be the abelian group of homology bordism classes of oriented homology 3-spheres modulo oriented homology 3-spheres which are the boundary of acyclic 4-dimensional PL-manifolds. The structure of Θ_3^h is at the time of writing not known, at least one knows that it is not finitely generated. The Rohlin invariant defines a homomorphism $\mu : \Theta_3^h \longrightarrow \mathbf{Z}/2$. We get an exact sequence

1.14 $$0 \longrightarrow ker(\mu) \longrightarrow \Theta_3^h \xrightarrow{\mu} \mathbf{Z}/2 \longrightarrow 0$$

There are fibrations

1.15 $K(\mathbf{Z}/2,3) \longrightarrow BPL \longrightarrow BTOP$

$K(\Theta_3^h,3) \longrightarrow BPL \longrightarrow BTRI$

$K(ker(\mu),4) \longrightarrow BTRI \longrightarrow BTOP$

Let $\Delta(M) \in H^4(M;\mathbf{Z}/2)$ be the *Kirby-Siebenmann obstruction* for the existence of a PL-structure on a topological manifold M. The short exact sequence 1.14 above defines a Bockstein homomorphism

1.16 $\beta : H^4(M;\mathbf{Z}/2) \longrightarrow H^5(M;ker(\mu))$

Put $\nabla(M) := \beta(\Delta(M))$. Then $\nabla(M)$ is the obstruction for the existence of a triangulation of M. The existence of an oriented homology 3-sphere H with the properties that $\mu(H) = 1$ and $H\sharp H$ is the boundary of an acyclic 4-dimensional PL-manifold, is equivalent to the existence of a section for the sequence 1.14. If such a section exists, the Bockstein homomorphism and hence $\nabla(M)$ vanishes.

A PL-structure on a manifold is more than the existence of a triangulation. For $n \geq 5$ there is a triangulation on S^n which is not combinatorial. Namely, let H be a homology 3-sphere which not homotopic to S^3 such that there is a homeomorphism from$\Sigma^2 H$ to S^5. Such H exists by the Double Suspension Theorem of Edwards. Choose a triangulation on H. It induces a triangulation on $\Sigma^2 H$ and by the homeomorphism above on S^5. We have an embedding $S^1 \subset \Sigma^2 H$ coming from suspending $\emptyset \subset H$ twice. If the triangulation on H were combinatorial, then this embedding would be a PL-embedding. Hence it would be isotopic to the standard embedding of S^1 into S^5. This would imply $S^5 - S^1 \simeq S^3$, a contradiction to $\Sigma^2 H - S^1 \simeq H$.

1.17 Maybe the most important application of the Casson invariant is the conclusion that the Rohlin invariant of a homotopy 3-sphere is zero. A lot of strategies for disproving the 3-dimensional Poincaré conjecture that any homotopy 3-sphere is homeomorphic to S^3 were based on finding a homotopy 3-sphere with non-trivial Rohlin invariant (see Mandelbaum [29]).

Another consequence is the existence of 4-dimensional topological manifolds having no triangulation. By the celebrated result of Freedman (see Freedman [12]), there is a closed, 1-connected, almost parallelizable, almost-smooth 4-dimensional topological manifold M with intersection matrix E_8. "Almost" means that the property holds for $M - \{point\}$. Suppose that M has a triangulation (K,t). Let S resp. L be the star resp. link of a vertex v. Then L is homotopy 3-sphere and a homology 3-manifold. This implies already that L is a 3-manifold bounding a smooth 4-manifold $\widehat{M}$ obtained from M by taking out the interior of S. As M and $\widehat{M}$ have the same intersection form, the signature of $\widehat{M}$ is 8. Since $\widehat{M}$ is parallelizable, the Rohlin invariant of L is 1. But it must be 0 as L is a homotopy 3-sphere, a contradiction. $\blacksquare$

Next we make some comments on the construction of the Casson invariant . We need some notation.

Notation 1.18

Let $W_g = W$ be the standard modell of the 3-dimensional handle body of genus g. Namely W is the g-fold connected sum of $S^1 \times D^2$.

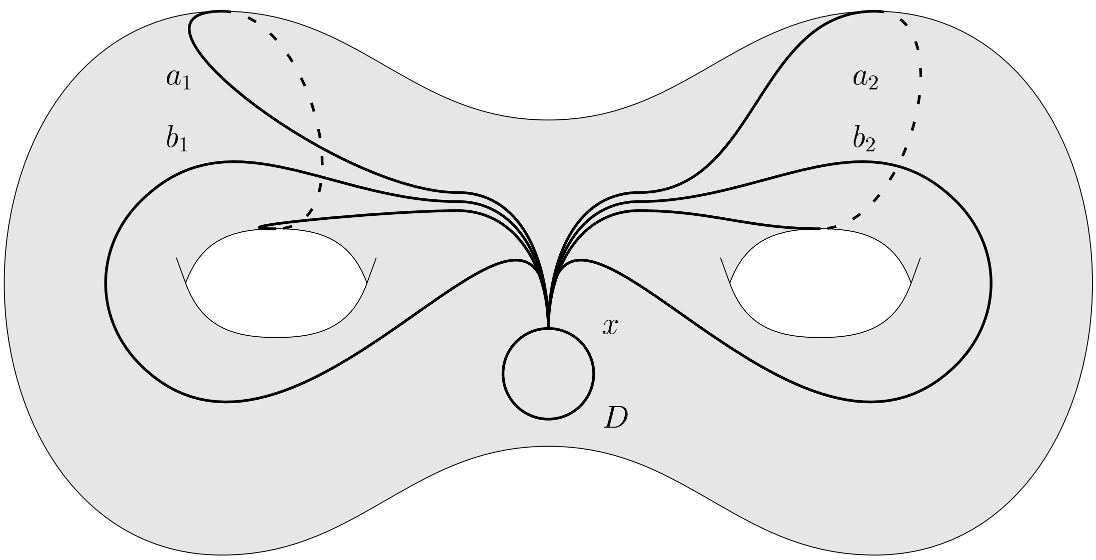
Denote by F the boundary of W. This is the surface of genus g, or in other words, the g-fold connected sum of $S^1 \times S^1$.

Let $D \subset F$ be a fixed embedded 2-disk.

Put $F^* := F - D$ and $S^1 := \partial D$.

Fix a base point $x \in D$ ■

1.19



The standard orientation of $\mathbf{R}^3$ induces an orientation on W. Then F, F^*, D and S^1 inherits orientations by the general agreement that an oriented manifold induces an orientation on its boundary using the decomposition $\nu(\partial M, M) \oplus T\partial M = TM|_{\partial M}$ and the outward normal field.

Definition 1.20 (Heegard modell) *Consider an orientation reversing homeomorphism $h : (F, D, x) \longrightarrow (F, D, x)$. Define the* Heegard modell of h *by*

$$| (W, H) | := W \cup_h W \qquad ■$$

Definition 1.21 (Heegard splitting) *Let M be an oriented closed* 3*-manifold. A* Heegard splitting *of M is a pair (W_1, W_2) consisting of submanifolds $W_1, W_2 \subset M$ of codimension* 0 *satisfying*

$$W_1 \cup W_2 = M \qquad \partial W_1 = W_1 \cap W_2 = \partial W_2 \qquad W_1 \cong W_2 \qquad \blacksquare$$

Any oriented closed 3-manifold has a Heegard-decomposition. For a handle body decomposition of M with exactly one 0- and one 3-handle put W_1 resp. W_2 to be the union of all 0 and 1-handles resp. all 2- and 3-handles. If (W_1, W_2) is a Heegard decomposition of M and $f_i : W_i \longrightarrow W$ is a homeomorphism to the standard handle body for $i = 1, 2$ such that the composition $f_2 \circ f_1^{-1}$ induces an orientation reversing homeomorphism $h : (F, D, x) \longrightarrow (F, D, x)$, then M and $\mid (W, h) \mid$ are oriented homeomorphic. Two Heegard decomposition of the same manifold are equivalent in the sense that after stabilization they become isotopic. The stabilization process consists of taking out a so called unknotted handle in W_1 and increases the genus by 1. It may happen that two Heegard splittings of the same genus are not isotopic, although appropiate stabilizations of them are.

Given a Heegard splitting (W_1, W_2) of M, we obtain a diagram of inclusions of spaces

1.22

$$\begin{tikzcd}
 & & & W_1 \arrow[dr] & \\
\partial F^* \arrow[r] & F^* \arrow[r] & F \arrow[ur] \arrow[dr] & & M \\
 & & & W_2 \arrow[ur] &
\end{tikzcd}$$

Applying the fundamental group with respect to the base point x gives a diagram of homomorphisms of groups

1.23

$$\begin{tikzcd}
 & & & \pi_1(W_1) \arrow[dr] & \\
\pi_1(\partial F^*) \arrow[r] & \pi_1(F^*) \arrow[r] & \pi_1(F) \arrow[ur] \arrow[dr] & & \pi_1(M) \\
 & & & \pi_1(W_2) \arrow[ur] &
\end{tikzcd}$$

That all the maps in the diagram 1.23 are epimorphisms except for the first one, follows from the following presentations of the fundamental groups if M is the Heegard modell $\mid (W,h) \mid$. The paths a_i and b_i on F^* are indicated in diagram 1.19 and $i : F \longrightarrow W$ is the inclusion.

1.24
$$\begin{aligned}
&\pi_1(F^*, x) =< a_1, a_2, ..., a_g, b_1, b_2, ...b_g > \\
&\pi_1(F, x) =< a_1, a_2, ..., a_g, b_1, b_2, ...b_g \mid \textstyle\prod_{i=1}^{g}[a_i, b_i] = 1 > \\
&\pi_1(W_1, x) =< a_1, a_2, ..., a_g, b_1, b_2, ...b_g \mid b_1 = b_2 = ... = b_g = 1 > \\
&\pi_1(W_2, x) =< a_1, a_2, ..., a_g, b_1, b_2, ...b_g \mid (i \circ h^{-1})_*(b_j) = 1 \quad 1 \leq j \leq g > \\
&\pi_1(M, x) =< a_1, a_2, ..., a_g, b_1, b_2, ..., b_g \mid b_j = 1, (i \circ h)_*^{-1}(b_j) = 1 \quad 1 \leq j \leq g >
\end{aligned}$$

Let G be a discrete group. The *space of representations of G in $SU(2)$* is defined by

1.25 $R(G) := Hom(G, SU(2)) \subset map(G, SU(2))$

The conjugation operation of $SU(2)$ on itself induces an operation of $SO(3) = SU(2)/\mathbf{Z}/2 = SU(2)/center(SU(2))$ on $SU(2)$ and hence an $SO(3)$-operation on $R(G)$. If we apply this functor $R(?)$ to the diagram 1.23 and define :

Notation 1.26
$$\begin{aligned}
&R_\partial := R(\pi_1(\partial F^*)) \\
&R^* := R(\pi_1(F^*)) \\
&R := R(\pi_1(F)) \\
&Q_i := R(\pi_1(W_i)) \textit{ for } i = 1,2 \qquad \blacksquare
\end{aligned}$$

we obtain a diagram where all maps except ∂ are inclusions:

1.27

$$\begin{tikzcd}
 & & & Q_1 \arrow[dl] & \\
R_\partial & R^* \arrow[l, "\partial"'] & R \arrow[l] & & R(\pi_1(M)) \arrow[ul] \arrow[dl] \\
 & & & Q_2 \arrow[ul] &
\end{tikzcd}$$

We derive from the presentation of the fundamental groups 1.24 since $\pi_1(F^*)$ and $\pi_1(W_i)$ are free:

1.28 $R^* = \prod_{j=1}^{2g} S^3$
$Q_i = \prod_{j=1}^{g} S^3$
$R_\partial = S^3$

Hence the intersection number $< Q_1, Q_2 >_{R^*}$ of Q_1 and Q_2 in R^* is defined. One key result is the following :

Proposition 1.29 *Let M be an oriented 3-manifold. Then :*

1. *$< Q_1, Q_2 >_{R^*}$ is different from 0 if and only if M is a rational homology sphere.*
2. *If M is a rational homology sphere, then*

$$|< Q_1, Q_2 >_{R^*}| = | H_1(M; \mathbf{Z}) |$$

3. *$< Q_1, Q_2 >_{R^*}$ is ± 1 if and only if M is an integral homology sphere.*
4. *Q_1 and Q_2 intersect at 1 transversely if and only if M is a rational homology sphere.*

Next we examine what happens after dividing out the $SO(3)$-action on the representation spaces. Recall that a representation is *reducible* if it contains a proper invariant linear subspace and *irreducible* otherwise. Let the map

1.30 $\partial : R^* \longrightarrow R_\partial$

be induced from the inclusion $i : \partial F^* \longrightarrow F^*$. Denote for a (discrete group) G

1.31 $S(G) = \{\rho \in R(G) \mid \rho \text{ is irreducible } \}$

The key result for the construction of the Casson invariant is :

Proposition 1.32

1. *The map ∂ is surjective.*
2. *The set of critical points is the set S of reducible representations.*
3. *$S(\pi_1(F^*, x)) = S(\pi_1(F, x))$*
4. *$R = \partial^{-1}(1)$*

5. *$R - S$ is an open smooth manifold of dimension $6g-3$ and carries a free proper $SO(3)$-action.*

We will deal with its proof in a later lecture. As $SO(3)$ is compact, we get smooth, free and proper $SO(3)$-actions on R, Q_1 and Q_2.

Notation 1.33 $\hat{R} := (R-S)/SO(3) \qquad \hat{Q}_i := (Q_i - S)/SO(3)$ ■

This implies

Proposition 1.34

1. *$\hat{R}$ is a smooth open manifold of dimension $6g-6$.*
2. *$\hat{Q}_i$ is a properly embedded open submanifold of dimension $3g-3$ in $\hat{R}$.*
3. *$\hat{Q}_1 \cap \hat{Q}_2$ is compact.*

If one has fixed orientations on $\hat{R}$ and $\hat{Q}_i$, then the intersection number $< \hat{Q}_1, \hat{Q}_2 >_{\hat{R}}$ is defined.

The orientation on M induces an orientation on W_1 and W_2 by restriction. Then F from W_1, F^* from F and ∂F^* from F^* inherit orientations by the general conventions for boundaries of oriented manifolds resp. by restriction. The orientation on ∂F^* determines a generator in $\pi_1(\partial F^*)$ and thus an orientation on R_∂. Fix any orientation on R^*. As $R-S$ sits in the preimage of 1 of the map $\partial : R^* \longrightarrow R_\partial$, the orientations of R^* and R_∂ induce an orientation on $R-S$. This determines also on orientation on $\hat{R}$. All in all we have explained, how an orientation of M induces an orientation on $\hat{R}$. Fix any orientations on Q_1 and Q_2. This induces orientations on $\hat{Q}_1$ and $\hat{Q}_2$. Now we define

Definition 1.35 (Casson invariant)
Let M be a oriented homology 3-sphere . Define :

$$\lambda(M) := \frac{(-1)^g \cdot < \hat{Q}_1, \hat{Q}_2 >_{\hat{R}}}{2 \cdot < Q_1, Q_2 >_{R^*}}$$

Obviously this is independent of the choice of orientation of R^*, Q_1 and Q_2. If we reverse the orientation of M, then the orientation of R_∂ and hence of $\hat{R}$ is reversed so that $\lambda(M^-) = -\lambda(M)$ holds. Evidently $\lambda(M)$ vanishes if there are no non-trivial representations of $SO(3)$ (cf. 1.7). The condition that M is a rational homology 3-sphere guarantees that $< Q_1, Q_2 >_{R^*}$ is not zero (see 1.29). We have to divide out this term to ensure that the choice of orientation on Q_1 and Q_2 do not matter. If we neglect this choice, the Casson invariant would reduce to a number $mod\, 2$ and hence just to the Rohlin invariant. But we even need that M is an integral homology sphere because then the only reducible $SO(3)$-representation of $\pi_1(M)$ is the trivial one (see Lemma 1.3). This is crucial for the proof that the intersection of $\hat{Q}_1$ and $\hat{Q}_2$ in $\hat{R}$ is defined (see Proposition 1.34)

3.section
The Alexander polynomial

The Alexander polynomial was introduced by Alexander in 1928 [3] and is still one of the most important invariants in knot theory. We will define it using Seifert surfaces and then give other tools for its computation.

Let S be an oriented Seifert surface for the ordered oriented link L in the oriented homology 3-sphere M. Choose a trivilization of the normal bundle $\nu(S, M)$ compatible with the orientation of M and S and a Riemannian metric on M. We obtain an embedding $i : F \times S^0 \longrightarrow M$. Let the embeddings :

$i^+ : S \longrightarrow M - S$
$i^- : S \longrightarrow M - S$

be the restrictions to $F \times \{+1\}$ and $F \times \{-1\}$. Notice that the isotopy classes of i^+ and i^- are independent of the choice of Riemannian metric on M. The *Seifert pairing*

3.1 $s : H_1(S) \times H_1(S) \longrightarrow \mathbf{Z}$

sends (u, v) to the linking number $link(u, i^+(v))$ of u and $i^+_*(v)$ in M. Choose an integral bases $b_1, b_2, \ldots, b_k$ of $H_1(S)$. Define the *Seifert matrix* $A := (s(b_i, b_j))_{(i,j=1,k)}$. Notice that $A^t = (link(b_j, i^-_*(b_i)))_{(i,j=1,k)}$ and $A - A^t$ is the intersection matrix of the Seifert surface. The polynomial $det(t \cdot A - t^{-1} \cdot A^t)$ is independent of the choice of bases where A^t is the transposed matrix.

Definition 3.2 *Let L be an oriented link in a oriented homology* 3*-sphere M with Seifert surface S. If S is not a disk, we define the Alexander polynomial by :*

$$\Delta_L(t) = det(t \cdot A - t^{-1} \cdot A^t).$$

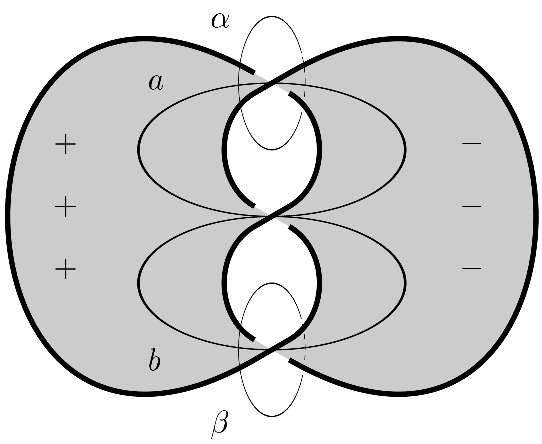
If S is a disk, define the Alexander polynomial by

$$\Delta_L(t) = 1 \qquad \blacksquare$$

For a proof that this definition is independent of the choice of Seifert surface we refer to [25] , page $192 - 200$.

Example 3.3 We illustrate the result above by computing the Alexander polynomial of the trefoil and the Hopf link again but now using Seifert surfaces. The following picture shows the Seifert surface of the trefoil together with a standard base a, b of its first homology and a fixed base α, β of the first homology group of the complement of the Seifert surface.

3.4

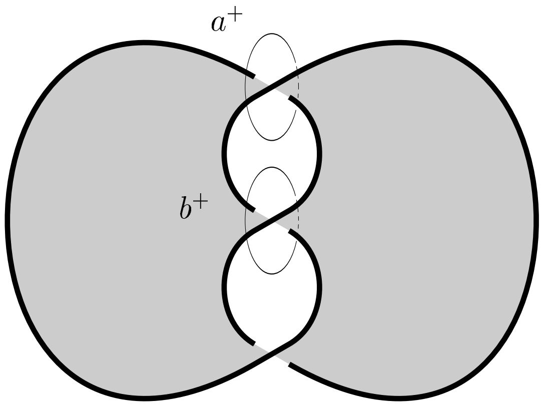


One easily checks :

$$link(a, \alpha) = link(b, \beta) = 1 \qquad\qquad link(b, \alpha) = link(a, \beta) = 0$$

The positive push-offs a^+ and b^+ look as indicated below

3.5



Hence we obtain :

$$a^+ = -\alpha \qquad\qquad b^+ = \alpha - \beta$$

Then the Seifert matrix looks like :

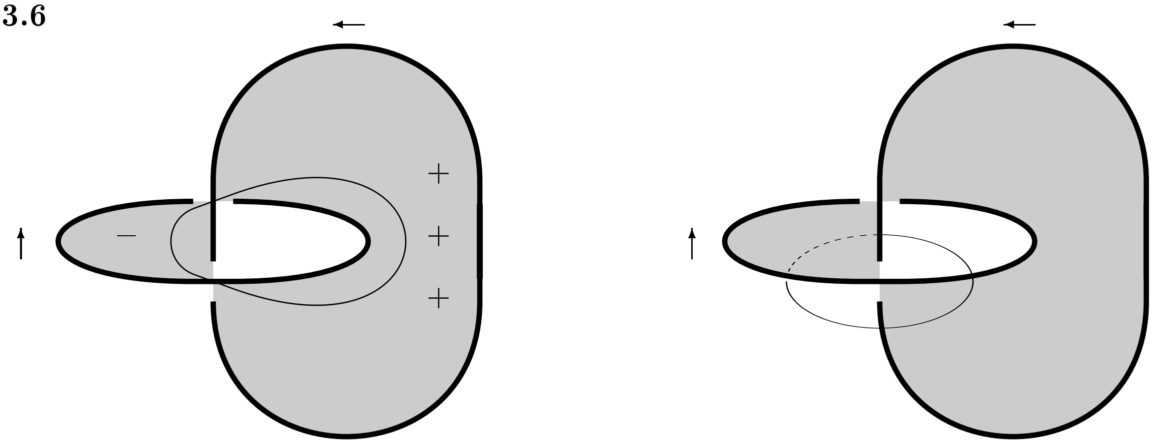
$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

We obtain for the Alexander polynomial :

$$\Delta_T(t) = t^2 - 1 + t^{-2}$$

For the Hopf link we obtain the following picture

3.6



The Seifert matrix is (1). Hence we obtain :

$$\Delta_H(t) = t - t^{-1} \qquad \blacksquare$$

The next lemma collects the main properties of this invariant:

Lemma 3.7 *1. If K is an oriented knot in an oriented homology 3-sphere M, we get :*

$$\Delta_K(1) = 1$$

2. *If L is an oriented link with r-components in a oriented homology 3-sphere , its Alexander polynomial is $(-1)^r$-symmetric:*

$$\Delta_L(t) = (-1)^r \cdot \Delta_L(t^{-1})$$

3. *Let L be an oriented link with two components L_1 and L_2 in an oriented homology 3-sphere M. If the Alexander polynomial Δ_L is zero, then the linking number $link(L_1, L_2)$is zero. If Δ_L is different from zero, we obtain :*

$$\frac{1}{2} \cdot \frac{d}{dt}\Delta_L\bigg|_{t=1} = link(L_1, L_2)$$

4. $\Delta_{K\sharp L} = \Delta_K \cdot \Delta_L$

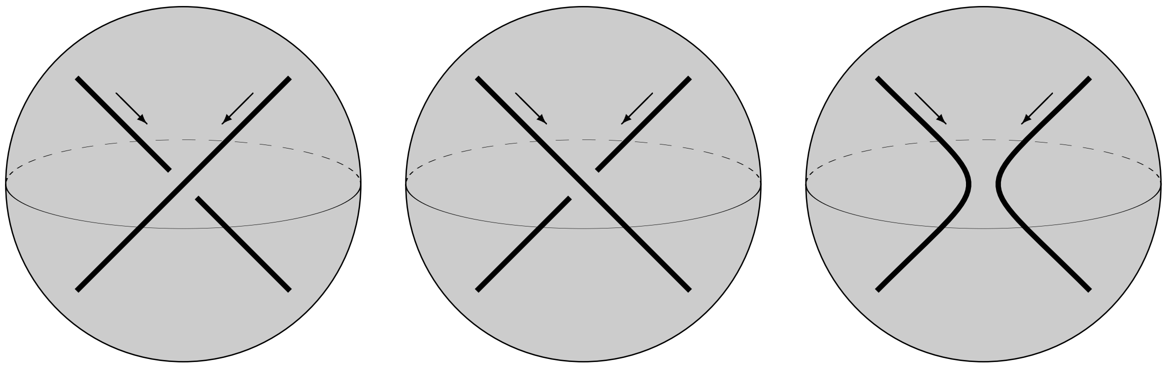
5. $\Delta_{K \coprod L} = 0$

6. *Let L be an oriented link and K be a knot in the oriented homology 3-sphere M. Suppose that there are Seifert surfaces S_L and S_K such that $S_K \cap S_L = \emptyset$ holds. Let q be an integer. Then :*

$$\Delta(L \subset M) = \Delta(L \subset M(K, 1/q))$$

Let L_+, L_- and L_0 be links in an oriented 3-manifold M. We call (L_+, L_-, L_0) a *skein triple* if there is an embedded ball $D^3 \subset M$ such that L_+, L_- and L_0 are equal in $M - int(D^3)$ and within D^3 look as follows

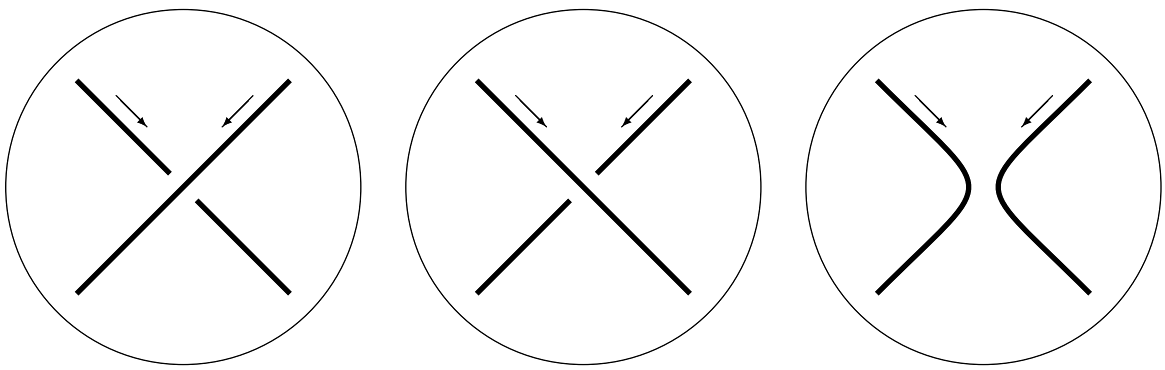
3.8 Skein triple of links



L_+ L_- L_0

We say that L_+, L_- and L_0 are skein related *if (L_+, L_-, L_0) is a skein triple. We call link diagramms L_+, L_- and L_0* skein related *if there is a ball $D^2 \subset \mathbf{R}^2$ such that the diagrams are identical outside D^2 and are given inside D^2 be the pictures below.*

3.9 Skein triple of link diagrams



L_+ L_- L_0

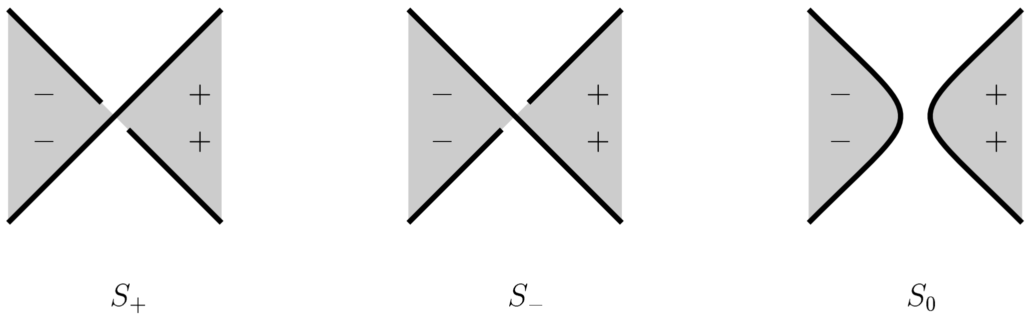
Of course three link diagrams L_+, L_- and L_0 are skein related if and only if the links, they discribe in S^3, are skein related.

Lemma 3.10 *Let L_+, L_- and L_0 be skein related links in an oriented homology 3-sphere M. Then :*

$$\Delta_{L_+} - \Delta_{L_-} - (t - t^{-1}) \cdot \Delta_{L_0} = 0$$

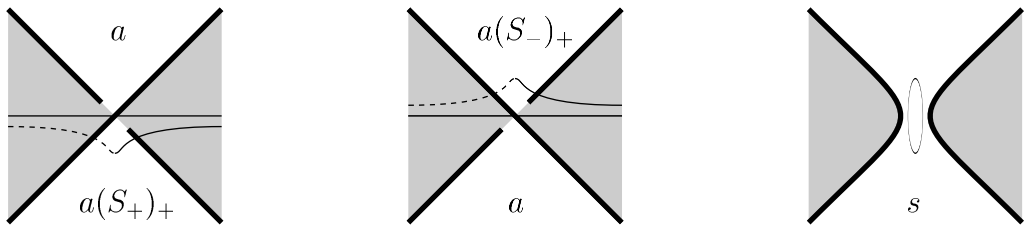
Proof : *We can find Seifert surfaces S_+, S_- and S_0 for L_+, L_- and L_0 such that they agree outside an embedded ball D^3 and look inside the ball as indicated below :*

3.11 Seifert surfaces of the skein triple



We obtain S_+ and S_- from S_0 by attaching a 1-handle $D^1 \times D^1$ to the boundary. Hence there is a curve a in $S_+ \cap S_-$ such that $H_1(S_\pm) =< a > \oplus H_1(S_0)$ holds. Let $a(S_+)_+$ and $a(S_-)_+$ be the positive push-offs of a for S_+ and S_-. These curves are indicated below :

3.12



The curve $s \subset D^3 \subset M$ satisfies $link(s, a) = 1$ and $a(S_+)_+ - a(S_-)_+ = s$. Hence we can *find Seifert matrices V_+,V_- and V_0 such that the following holds :*

$$V_+ = V_- + \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad\qquad V_+ = \begin{pmatrix} * & * \\ 0 & V_0 \end{pmatrix}$$

Now one calculates :

$\Delta_{L_+} - \Delta_{L_-} = det(t \cdot V_+ - t^{-1} \cdot V_+^t) - det(t \cdot V_- - t^{-1} \cdot V_-^t) =$

$(t - t^{-1}) \cdot det(t \cdot V_0 - t^{-1} \cdot V_0^t) = (t - t^{-1}) \cdot \Delta_{L_0}$ ■

Let L be a link diagram in S^3. Denote by $c(L)$ the number of crossings, by $r(L)$ the number of components and by $n(L)$ the minimal number of crossings which must be changed in order to get a link diagram describing the trivial link of $r(L)$ components. The last number is well defined by the following argument. Choose an ordering and orientation of the components of L. For $i = 1, 2, \ldots, r(L) - 1$ do the following : Fix a point x on L_i and move along L_i in the positive direction from x to x and, if necessary, change the crossing with the components L_j for $j = i, i+1, \ldots, r(L)$ such that the arc, one is just moving on, is the overcrossing arc. The components of the resulting link are stacked one below the other and are hence unlinked. Moreover, each component bounds an embedded disk and is hence trivial. We call the pair $(c(L), N(L))$ the complexity *of a link diagram.*

Lemma 3.13 *Suppose that the function*

Δ *: { isotopy classes of oriented links in oriented homology 3-spheres }* $\longrightarrow \mathbf{Z}[t, t^{-1}]$

has the following properties :

a.) Let L be a link and K be a knot in the oriented homology 3-sphere M. Suppose that there are Seifert surfaces S_L and S_K such that $S_K \cap S_L = \emptyset$ holds. Let q be an integer. Then :

$$\Delta(L \subset M) = \Delta(L \subset M(K, 1/q))$$

b.) If (L_+, L_-, L_0) is a skein triple of links in an oriented homology 3-sphere M, then :

$$\Delta(L_+ \subset M) - \Delta(L_- \subset M) - (t - t^{-1}) \cdot \Delta(L_0 \subset M) = 0$$

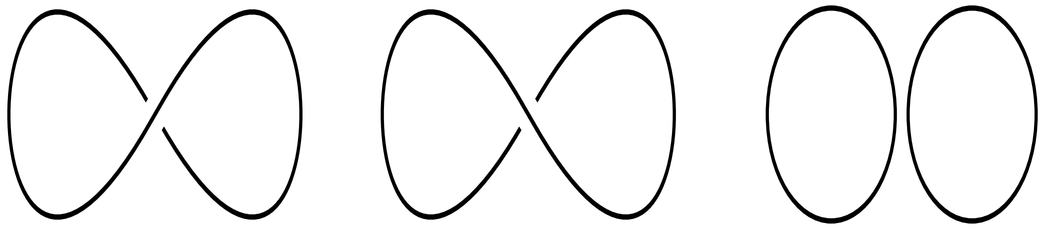
c.) $\Delta(unknot \subset S^3) = 1$

Then Δ is the Alexander polynomial. The Alexander polynomial has these properties

Proof : *The Alexander polynomial has the property a.) by lemma 3.7 and b.) by lemma 3.10 and c.) is easily verified. It remains to prove for the difference ∇ of Δ as above and the Alexander polynomial that ∇ is identically zero.*

We first treat the case where M is S^3. Obviously ∇ applied to the unknot is zero. Because of the following skein relation

3.14



∇ *of the unlink of two components is zero. Inductively over the number of components one verifies that* ∇ *of the unlink is always zero. Given any link* L *with* $c(L) > 0$ *and* $n(L) > 0$, *there is a skein triple* (L_+, L_-, L_0) *containing* L *such that for the other members* $c(L)$ *or* $n(L)$ *is smaller. Hence we can prove by induction over the complexity of* L *that* $\nabla(L)$ *vanishes.*

Now we come to the general case of a link L *in an oriented homology* 3*-sphere* M. *We will later show that there is a sequence of oriented homology* 3*-spheres* M_0,M_1, ... ,M_r *such that* M_{i+1} *is obtained from* M_i *by* ± 1 *surgery on a knot* $K_i \subset M_i$ *and* M_0 *is* M *and* M_r *is* S^3. *We use induction over* r. *The induction begin* $M = S^3$ *is done above. Choose Seifert surfaces* S_L *for* L *and* S_0 *for* K_0 *in* M. *Since these are surfaces with boundary, there exists one-dimensional spines* F_L *and* F_0 *for* S_L *and* S_0 *such that* S_L *and* S_0 *are ambient isotopic to arbitrary small regular neighbourhoods of the spines. We can find an ambient isotopy of* K_0 *in* M *such that* F_L *and* F_0 *are disjoint because the sum of the dimensions of the spines is smaller than the dimension of* M. *By a second ambient isotopy of* K_0 *we can achieve that* S_0 *is disjoint from* F_L. *By an ambient isotopy of* L *we obtain that the Seifert surfaces are disjoint. Notice that these processes may require crossings of* K *and* L *but no self crossings of* L *and* K. *But now we derive from property a.) that* $\nabla(L \subset M) = \nabla(L \subset M_1)$. *Now apply the induction hypothesis to* $L \subset M_1$. ■

In the surgery formula for the Casson invariant a term involving the second derivative of the Alexander polynomial appears. We can characterize this term as follows.

Lemma 3.15 *Suppose that the function*

γ : { *isotopy classes of oriented links in oriented homology* 3*-spheres* } $\longrightarrow \mathbf{R}$

has the following properties :

a.) Let L *be a link and* K *be a knot in the oriented homology* 3*-sphere* M. *Suppose that there are Seifert surfaces* S_L *and* S_K *such that* $S_K \cap S_L = \emptyset$ *holds. Let* q *be an integer. Then :*

$$\gamma(L \subset M) = \gamma(L \subset M(K, 1/q))$$

b.) If (L_+, L_-, L_0) *is a skein triple of links in an oriented homology* 3*-sphere* M *such that* L_+ *is a knot. Then* L_- *is a knot and* L_0 *is a link of two components* L_0' *and* L_0'' *and we have:*

$$\gamma(L_+ \subset M) - \gamma(L_- \subset M) = link(L_0', L_0'')$$

c.) $\gamma(unknot \subset S^3) = 0$

Then $\gamma(L)$ *is* $\frac{1}{4} \cdot \frac{d^2}{dt^2} \Delta_L \Big|_{t=1}$.

Proof : *We only show that* $\frac{1}{4} \cdot \frac{d^2}{dt^2}\Delta_L\Big|_{t=1}$ *has the required properties, the verification of uniqueness is analogous to the proof in lemma 3.13. We get properties a.) and c.) directly from lemma 3.13. Since we have* $\Delta_{L_0}(t) = -\Delta_{L_0}(t^{-1})$, *we get :*

$$\Delta_{L_0}|_{t=1} = 0$$

We derive property b.) from from lemma 3.13, lemma 3.7 and the following calculation :

$\frac{d^2}{dt^2}\Delta_{L_+}\Big|_{t=1} - \frac{d^2}{dt^2}\Delta_{L_-}\Big|_{t=1} = \frac{d^2}{dt^2}\left((t-t^{-1})\cdot\Delta_{L_0}\right)\Big|_{t=1} =$

$0 \cdot \frac{d^2}{dt^2}\Delta_{L_0}\Big|_{t=1} + (+2)\cdot\frac{d}{dt}\Delta_{L_0}\Big|_{t=1} + (-2)\cdot\Delta_{L_0}|_{t=1} =$

$4 \cdot link(L_0', L_0'')$ ∎

Next we explain how one can compute the Alexander polynomial from the fundamental group. We start with introducing the differential calculus *due to Fox. Let* G *be a group. Let* $\epsilon : \mathbf{Z}[G] \longrightarrow \mathbf{Z}$ *be the* augmentation homomorphism *sending* $\sum_{g\in G}\lambda_g \cdot g$ *to* $\sum_{g\in G}\lambda_g$. *A* derivation *is a homomorphism* $\delta : \mathbf{Z}[G] \longrightarrow M$ *into a* $\mathbf{Z}[G]$*-module* M *satisfying :*

$\delta(u+v) = \delta(u) + \delta(v)$ *(linearity)*

$\delta(u\cdot v) = \delta(u)\cdot\epsilon(v) + u\cdot\delta(v)$ *(Leibniz rule)*

If $f : M \longrightarrow N$ *is a homomorphism of* $\mathbf{Z}[G]$*-modules and* δ *a derivation on* M, *then* $f\circ\delta$ *is a derivation on* N. *Hence the set of* $\mathbf{Z}[G]$*-derivations into a* $\mathbf{Z}[G]-\mathbf{Z}[H]$*-bimodule* M *inherits a right* $\mathbf{Z}[H]$*-module structure. The following rules are important for calculations* ($g \in G$)*:*

3.16 $\delta(m) = 0$ for $m \in \mathbf{Z}$
$\delta(g^{-1}) = -g^{-1}\cdot\delta(g)$
$\delta(g^n) = (1+g+g^2+\ \dots\ +g^{n-1})\cdot\delta(g)$
$\delta(g^{-n}) = -(g^{-1}+g^{-2}+\ \dots\ +g^{-n})\cdot\delta(g)$ for $n \geq 1$ ∎

If F *is the free group in generators* $s_1, s_2, \dots, s_n$, *then for any elements* $x_1, x_2, \dots, x_n$ *in a* $\mathbf{Z}[F]$*-bimodule* M *there is precisely one derivation sending* s_i *to* x_i. *Its construction and the verification of uniqueness is done by induction over the word length. Let* $\phi : F \longrightarrow G$ *be a group homomorphism. Then* $\mathbf{Z}[G]$ *becomes a* $\mathbf{Z}[F]-\mathbf{Z}[G]$*-bimodule. The derivations* $\frac{\partial}{\partial s_i} : \mathbf{Z}[F] \longrightarrow \mathbf{Z}[G]$ *sending* s_j *to* 1, *if* $i=j$, *and to* 0, *if* $i \neq j$, *are called the* partial derivations with respect to ϕ. *They form a basis for the right* $\mathbf{Z}[G]$*-module of* $\mathbf{Z}[F]$*-derivations into* $\mathbf{Z}[G]$. *The Fox derivatives are useful for computing cellular chain complexes of universal coverings.*

Lemma 3.17 *Let* X *be a finite* 2*-dimensional CW-complex with fundamental group* π *and universal covering* $\widetilde{X}$. *Suppose that* X *has only one* 0*-cell. Let*

$$< s_1, s_2, \ \dots\ , s_n \mid R_1, R_2, \ \dots\ , R_m >= \pi$$

be a cellular representation of the fundamental group, i.e. the generators s_i *correspond to the* 1*-cells and the relations* R_i *are defining relations for the* 2*-cells. Let* $\phi : F \longrightarrow \pi$ *be the* *canonical projection, if F is the free group in generators s_1, s_2, ..., s_n. Then the cellular $\mathbf{Z}[\pi]$-chain complex of the universal covering with respect to a cellular bases looks like*

$$\mathbf{Z}[\pi]^m \xrightarrow{A} \mathbf{Z}[\pi]^n \xrightarrow{B} \mathbf{Z}[\pi]$$

where the matrices A and B are given as follows :

$$A = \begin{pmatrix} \frac{\partial R_1}{\partial s_1} & \frac{\partial R_2}{\partial s_1} & \cdots & \frac{\partial R_m}{\partial s_1} \\ \frac{\partial R_1}{\partial s_2} & \frac{\partial R_2}{\partial s_2} & \cdots & \frac{\partial R_m}{\partial s_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial R_1}{\partial s_n} & \frac{\partial R_2}{\partial s_n} & \cdots & \frac{\partial R_m}{\partial s_n} \end{pmatrix}$$

$$B = \begin{pmatrix} s_1 - 1 & s_2 - 1 & \cdots & s_m - 1 \end{pmatrix}$$

<u>Proof</u> : *This is obvious for the matrix B, since the 1-cell corresponding to S_i lifts in the covering to a path from $\tilde{x}$ to $s_i \cdot \tilde{x}$ for a fixed lift $\tilde{x}$ of the only 0-cell x. Let w be a loop in X with base point x. Denote by $\tilde{w}$ a lift in $\widetilde{X}$ with starting point $\tilde{x}$. This defines an element in $C_1(\widetilde{X})$, also denoted by $\tilde{w}$. It depends only on the class of w in π. Hence we can define a map $\delta : \mathbf{Z}[\pi] \longrightarrow C_1(\widetilde{X})$ sending $\sum_{w\in\pi} \lambda_w \cdot w$ to $\sum_{w\in\pi} \lambda_w \cdot \tilde{w}$. One easily checks that δ is linear and satisfies $\delta(w \cdot v) = \delta(w) + w \cdot \delta(v)$. Let δ_i be the i^{th}-component of δ, if we identify $C_1(\widetilde{X})$ with $\mathbf{Z}[\mathbf{Z}^m]$ using the cellular bases $\tilde{s_1}$, $\tilde{s_2}$,..., $\tilde{s_m}$. Then δ_i is a derivation $\mathbf{Z}[F] \longrightarrow \mathbf{Z}[G]$ mapping $\phi(s_j)$ to 1, if $i = j$, and to 0 otherwise. Therefore it has to be $\frac{\partial}{\partial s_i}$. But $\delta(R_i)$ is just the image of the cellular base element corresponding to the 2-cell with defining relation R_i under the second differential in the cellular chain complex of $\widetilde{X}$.* ■

The proof of the next lemma is omitted. It is a consequence of lemma 3.17 and the description of the Alexander polynomial as a torsion invariant by Milnor [32].

Lemma 3.18 *Let L be an oriented link in an oriented homology 3-sphere. Let*

$$< s_1, s_2, \ \dots \ , s_n \mid R_1, R_2, \ \dots \ , R_{n-1} > = \ \pi$$

be a representation of the fundamental group π of the link complement with n generators and $n-1$ relations. Let $\phi : F \longrightarrow \pi$ be the canonical projection, if F is the free group in generators s_1, s_2, ..., s_n, and $\psi : \pi \longrightarrow \mathbf{Z}$ be the canonical epimorphism. Denote by A^ϕ the *matrix over* $\mathbf{Z}[\pi]$

$$A^{\phi} = \begin{pmatrix} \frac{\partial R_1}{\partial s_1} & \frac{\partial R_2}{\partial s_1} & \cdots & \frac{\partial R_{n-1}}{\partial s_1} \\ \frac{\partial R_1}{\partial s_2} & \frac{\partial R_2}{\partial s_2} & \cdots & \frac{\partial R_{n-1}}{\partial s_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial R_1}{\partial s_n} & \frac{\partial R_2}{\partial s_n} & \cdots & \frac{\partial R_{n-1}}{\partial s_n} \end{pmatrix}$$

Let i be any integer $1 \leq i \leq n$ such that $\phi(s_i) - 1 \neq 0$. Such i always exists. Denote by $A_i^{\psi\phi}$ the matrix obtained from A^{ϕ} by deleting the i^{th}-row and then applying the change of rings map $\mathbf{Z}[\pi] \longrightarrow \mathbf{Z}[\mathbf{Z}]$ induced by the Hopf map $\psi : \pi \longrightarrow \mathbf{Z}$ which sends a loop w in the complement of the link to the sum of the linking numbers of w with the components of the link. Then $\Delta_L(t)$ is different from zero if and only if det(A_i^{ϕ}) is non-zero, and we obtain in this case for some s and some sign $\pm$:

$$\Delta_L(t) = \pm t^s \cdot \det(A_i^{\phi})(t^2)$$

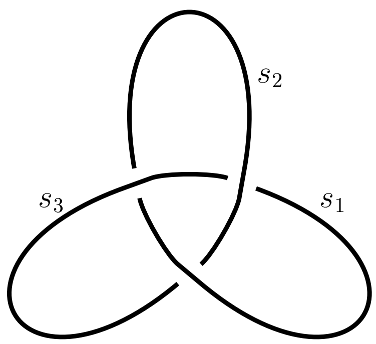
Remark 3.19 The computation above determines Δ_L if K is a knot and only up to a sign ± 1 if K is not a knot. Namely, let p be a polynomial satisfying $\Delta_L(t) = \pm t^s \cdot p(t)$ for some sign $\pm$ and some s. Since $\Delta_L(t^{-1}) = -(-1)^r \cdot \Delta_L(t)$ holds, we conclude

$$p(t) = (-1)^r \cdot t^{2s} \cdot p(t^{-1})$$

This determines s. If K is a knot, we can derive the sign in the equation above using the fact $\Delta_K(1) = 1$. Notice that both polynomials are independent of the orientations of M and L, provided that L is knot. ■

Example 3.20 Notice that the lemma **??** and lemma 3.18 give an algorithm to compute the Alexander polynomial of a link from a link diagram. We carry this out in the case of the trefoil using the following link diagram

3.21



The Wirtinger presentation looks like :

$\pi_1(S^3(T)) =< s_1, s_2, s_3 \mid s_2s_1 = s_1s_3, s_1s_3 = s_3s_2, s_3s_2 = s_2s_1 >$

We may omitt the third relation. We obtain the following matrix A

$$\begin{pmatrix} s_2 - s_2s_1s_3^{-1}s_1^{-1} & 1 \\ 1 & -s_1s_3s_2^{-1} \\ -s_2s_1s_3^{-1} & s_1 - s_1s_3s_2^{-1}s_3^{-1} \end{pmatrix}$$

We have to put $t = s_1 = s_2 = s_3$ since the Hopf map sends s_i to t, and obtain

$$\begin{pmatrix} t-1 & 1 \\ 1 & -t \\ -t & t-1 \end{pmatrix}$$

The minors in this matrix are $-t^2+t-1$, t^2-t+1 and $-t^2+t-1$. Hence the Alexander polynomial of the trefoil satisfies

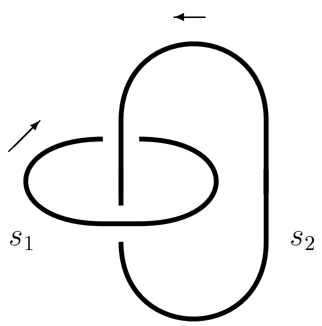
$$\Delta_T = \pm t^s \cdot (t^4 - t^2 + 1)$$

Hence we get:

$$\Delta_T = t^2 - 1 + t^{-2}$$

Example 3.22 We do the same for the Hopf link

3.23



We obtain the Wirtinger presentation :

$\pi_1(S^3(H)) =< s_1, s_2 \mid s_1s_2s_1^{-1}s_2^{-1} >$

Then the matrix A looks like :

$$\begin{pmatrix} 1 - s_1s_2s_1^{-1} \\ s_1 - s_1s_2s_1^{-1}s_2^{-1} \end{pmatrix}$$

Hence the Alexander polynomial of H satsifies

$$\Delta_H = \pm t^s \cdot (t^2 - 1)$$

This implies:

$$\delta_H(t) = \pm(t - t^{-1})$$

These computations are compatible with the previous calculation made using Seifert surfaces.

Finally we mention the curiosity that the term $\frac{1}{4}\cdot \frac{d^2}{dt}\Delta_L\Big|_{t=1}$ *is related to a homomorphism* κ. *Denote by* $\mathbf{Z}[\mathbf{Z}]_{(0)}$ *the quotient field of* $\mathbf{Z}[\mathbf{Z}]$. *Let* $\mathbf{Z}[\mathbf{Z}]^*_{(0)}$ *be the multiplicative group of units. We define an homomorphism of abelian groups*

3.24 $\kappa : \mathbf{Z}[\mathbf{Z}]^*_{(0)}/\{\pm t^n\} \longrightarrow \mathbf{R}$

with respect to the multiplicative structure on the source and the additive structure on the domain as follows. An element in $\mathbf{Z}[\mathbf{Z}]^*_{(0)}/\{\pm t^n\}$ *is represented by a quotient* $\frac{p}{q}$ *with* $p, q \in zz[\mathbf{Z}] - \{0\}$. *As* $\mathbf{Z}[\mathbf{Z}]$ *is factorial, there are unique non-negative integers* μ_p *and* μ_q *satisfying* $p = (t-1)^{\mu_p}\cdot p_0$ *and* $q = (t-1)^{\mu_q}\cdot q_0$ *such that* $p_0(1) \neq 0$ *and* $q_0(1) \neq 0$ *holds. We define :*

$$\kappa(\frac{p}{q}) := \frac{d^2}{dt^2}\,\frac{p_0(t)\cdot p_0(t^{-1})}{p_0(1)\cdot p_0(1)}\Bigg|_{t=1} - \frac{d^2}{dt^2}\,\frac{q_0(t)\cdot q_0(t^{-1})}{q_0(1)\cdot q_0(1)}\Bigg|_{t=1}$$

We have to show that this is independent of the various choices and that this is indeed an homomorphism. Any other representative of the class of $\frac{p}{q}$ *in* $\Lambda^*_{(0)}/\Lambda^*$ *looks like* $\frac{p\cdot r\cdot \epsilon\cdot t^n}{q\cdot r}$ *for some* $r \in \Lambda - \{0\}$, $\epsilon \in \{\pm 1\}$ *and* $n \in \mathbf{Z}$. *Choose* $r_0 \in \Lambda$ *and a non-negative integer* μ_r *such that* $r = r_0\cdot(t-1)^{\mu_r}$ *and* $r_0(1) \neq 0$ *holds. Let* $x_i(t)$ *for* $i = 1, 2$ *be elements in* Λ *satisfying* $x_i(t) = x_i(t^{-1})$ *and* $x_i(1) = 1$. *Then we get :*

$$\frac{d}{dt}x_i(t) = \frac{d}{dt}x_i(t^{-1}) = -t^{-2}\cdot\frac{d}{dt}x_i(t^{-1})$$

This implies :

$$\frac{d}{dt}x_i\Big|_{t=1} = 0$$

Hence we get :

$$\frac{d^2}{dt^2}(x_1\cdot x_2)\Big|_{t=1} = \frac{d^2}{dt^2}(x_1)\Big|_{t=1} + \frac{d^2}{dt^2}(x_2)\Big|_{t=1}$$

Applying this to $\frac{p(t)\cdot p_0(t^{-1})}{p_0(1)\cdot p_0(1)}$ *and the corresponding expression for* r_0 *shows that the map is well defined. The verification that it is a homomorphism is similar. We have for a knot* K *in an oriented homology 3-sphere* M

$$\kappa(\Delta_K) = 2\cdot\frac{d^2}{dt^2}\Delta_K\Big|_{t=1}$$

and will later see:

$$\lambda(M(K, 1/(n+1))) - \lambda(M(K), 1/n))) = \frac{1}{8}\cdot\kappa(\Delta_K)$$

Remark 3.25 The Alexander polynomial Δ_L of definition 3.2 and the Alexander-Conway polynomial Δ_L^{Con} is used in Akbulut-McCarthy [1] are related by $\Delta_L(t) = \Delta_L^{con}(t^2)$.

The Alexander polynomial is extensively treated in the textbooks Burde-Zieschang [6] and Rolfson [40]. For its connection to torsion invariants we refer to Turaev [43] . The skein invariance is treated in Conway [9] and Kauffman [24].

4.section
The Jones polynomial

Although we do not need the Jones polynomial for the Casson invariant, we spend some time on it, as it is a natural extension of the Alexander polynomial and interesting in its own right. In this sections all links are understood to be oriented links in S^3.

Definition 4.1 *A* skein invariant *is a function*

$\gamma : \{$ *isotopy classes of oriented links in* $S^3\} \longrightarrow R$

into an associative commutative ring R with unit 1 with the following properties :

a.) There exist units $a_+, a_-, a_0 \in R^*$ such that for any skein triple (L_+, L_-, L_0) the following relation holds :

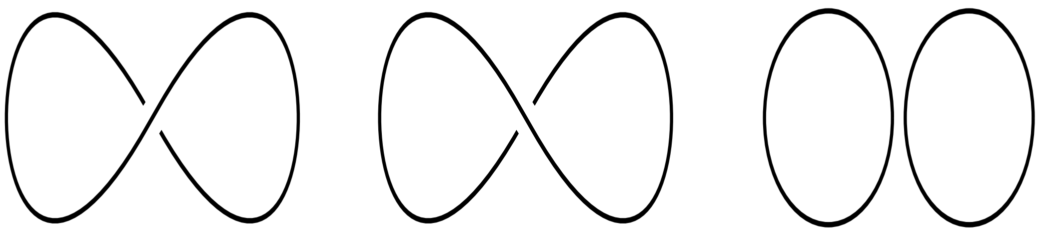
$$a_+ \cdot \gamma(L_+) + a_- \cdot \gamma(L_-) + a_0 \cdot \gamma(L_0) = 0$$

b.) $\gamma(unknot) = 1$ ■

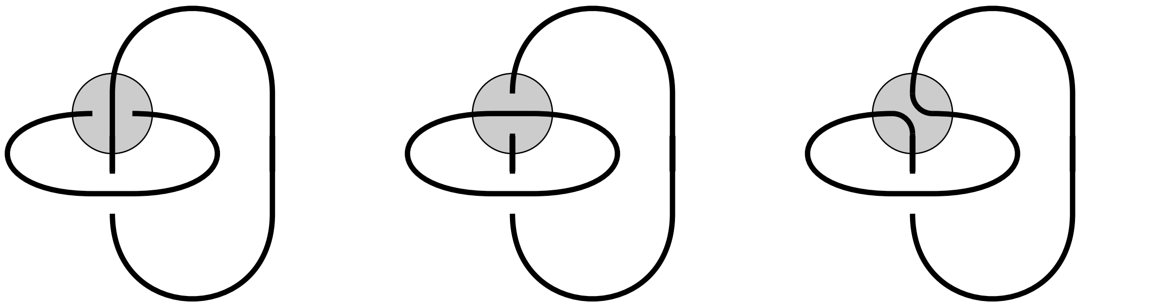
Example 4.2 The Alexander polynomial is a skein invariant for $R = \mathbf{Z}[t, t^{-1}]$ and skein coefficients $1, -1, -(t - t^{-1})$ by lemma 3.10. ■

The skein relation is effective for computations. Consider a skein invariant γ with skein coefficients a_+, a_-, a_0. Let $unlink^r$ be the unlink of r components, H be the Hopf link (with linking number $+1$) and T the trefoil (with positive crossings). Given a link L, denote by $L^\times$ its mirrow image, i.e. the image of L under an orientation reversing homeomorphism $S^3 \longrightarrow S^3$. If L is given by a link diagram, a link diagram for $L^\times$ is obtained by changing all crossings. One easily computes from the following skein relations

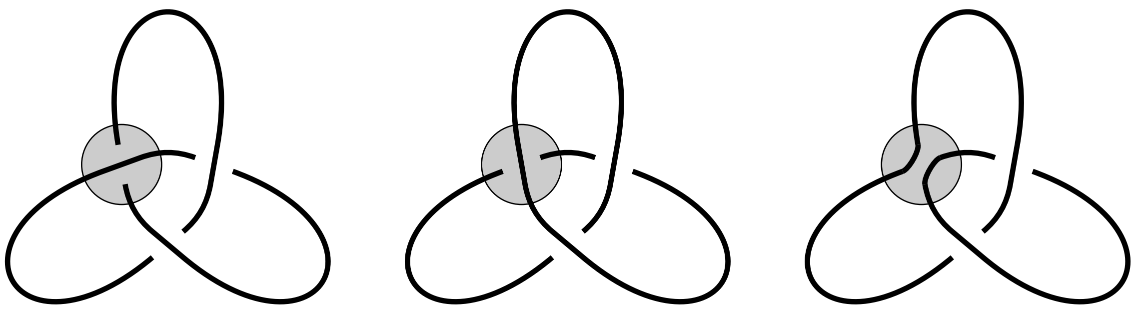
4.3



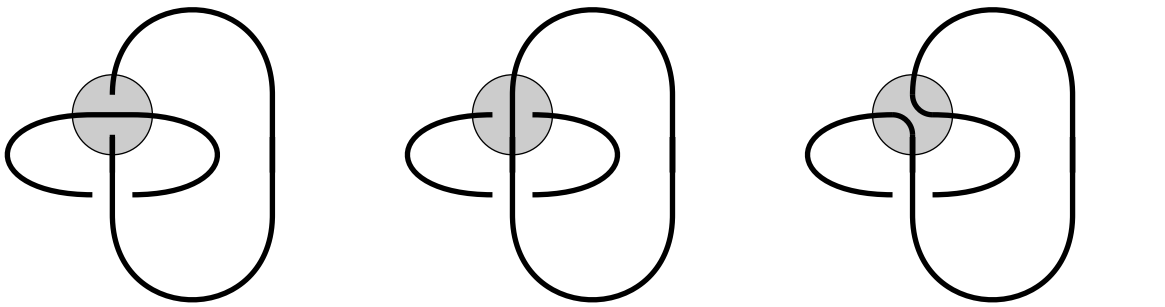
4.4



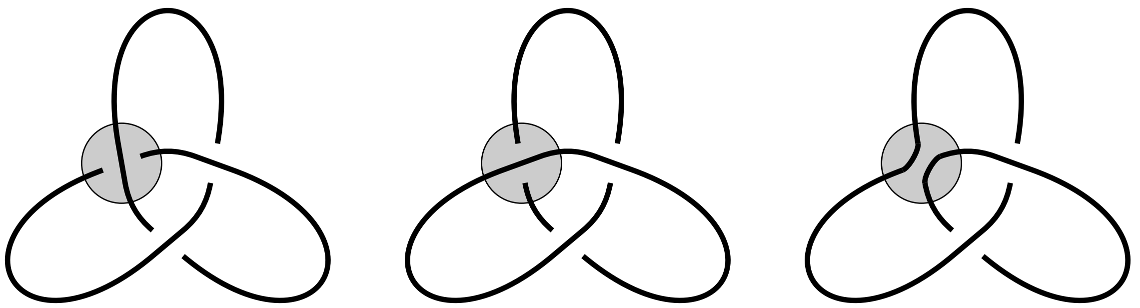
4.5



4.6



4.7



4.8 $\gamma(unlink^2) = -a_+a_0^{-1} - a_-a_0^{-1}$
$\gamma(H) = a_-a_0^{-1} + a_+^{-1}a_-^2a_0^{-1} - a_+^{-1}a_0$
$\gamma(T) = -2a_+^{-1}a_- - a_+^{-2}a_-^2 + a_+^{-2}a_0^2$
$\gamma(H^\times) = a_+a_0^{-1} + a_+^2a_-^{-1}a_0^{-1} - a_-^{-1}a_0$
$\gamma(T^\times) = -2a_+a_-^{-1} - a_+^2a_-^{-2} + a_-^{-2}a_0^2$ ∎

In particular we get for the Alexander polynomials the same values as we got before in example 3.20 and example 3.3.

Lemma 4.9 *Let* $\gamma : \{$ *isotopy classes of oriented links in* $S^3\} \longrightarrow R$ *be a skein invariant.*

1.) γ *is determined by the skein coefficients.*

2.) The following assertions are equivalent :

a.) There are skein coefficients a_+, a_-, a_0 *for* γ *satisfying* $a_+ + a_- + a_0 = 0$
b.) $\gamma(unlink^2) = 1$
c.) $\gamma(L) = 1$ *for all links* L.

3.) The following assertions are equivalent :

a.) There are skein coefficients a_+, a_-, a_0 *for* γ *satisfying* $a_+ + a_- - a_0 = 0$
b.) $\gamma(unlink^2) = -1$
c.) $\gamma(L) = (-1)^{r(L)+1}$ *for all links* L *where* $r(L)$ *is the number of components.*

4.) The following assertions are equivalent :

a.) $\gamma(unlink^2) \notin \{\pm 1\}$
b.) The skein coefficients of γ *satisfy :*
$a_+ \cdot (\gamma(H) - \gamma(unlink^2)) = a_0 \cdot (\gamma(unlink^2)^2 - 1)$
$a_- = -a_+ - a_0 \cdot \gamma(unlink^2) = 0$

5.) Suppose that R *has no zero divisors. Then the values of* $\gamma(unlink^2)$ *and* $\gamma(H)$ *determine* γ. *They also determine the skein coefficients up to multiplication with a common unit, provided that* $\gamma(unlink^2) \notin \{\pm 1\}$ *holds.*

Proof : *We use by induction over the complexity of the link. We get 2.) from the skein triple 4.3. We derive 3.) from the observation for a skein triple* L_+, L_-, L_0 *that* $r(L_+) = r(L_-) = r(L_0) \pm 1$ *holds. We conclude 4.) from the skein triples 4.3 and 4.4. Now 5.) is a direct consequence.* ∎

Definition 4.10 *A skein invariant*

$\Gamma : \{$ *isotopy classes of oriented links in* $S^3\} \longrightarrow R$

is a universal skein invariant *, if for any skein invariant*

$\gamma : \{$ *isotopy classes of oriented links in* $S^3\} \longrightarrow S$

there is a ring homomorphism $\Phi : R \longrightarrow S$ *satisfying* $\gamma = \Phi \circ \Gamma$ ∎

It will turn out that there is a universal skein invariant, the two-variable Jones polynomial. Before we construct the universal skein invariant, we derive its main properties from the universal property. Notice that we do not require that the homomorphism Φ appearing in the definition of a universal skein invariant is unique.

Lemma 4.11 *Suppose there is a skein invariant*

$\Gamma : \{$ *isotopy classes of oriented links in* $S^3\} \longrightarrow \mathbf{Z}[a_+, a_+^{-1}, a_-, a_-^{-1}, a_0, a_0^{-1}]$

such that a_+, a_-, a_0 *are skein coefficients. Then :*

1.) Γ *is a universal skein invariant.*

2.) Let δ *be a skein invariant taking values in* R *such that* $\delta(unlink^2) \notin \{\pm 1\}$*. Let* ϕ *and* ψ *be homomorphisms from* $\mathbf{Z}[a_+, a_+^{-1}, a_-, a_-^{-1}, a_0, a_0^{-1}]$ *to* R *satisfying* $\phi \circ \Gamma = \psi \circ \Gamma = \delta$*. Then there is a unit* $u \in R$ *such that* $u \cdot \phi = \psi$ *holds.*

3.) $\Gamma(L)$ *is a homogenous polynomial of total degree zero.*

4.) $\Gamma(L) = \Gamma(L^-)$ *, where* L^- *is obtained from* L *by simultaneously reversing the orientations of the components.*

5.) $\Gamma(K \coprod L) = \Gamma(K) \cdot \Gamma(L) \cdot \frac{-a_0}{a_+ + a_-}$

6.) $\Gamma(K \sharp L) = \Gamma(K) \cdot \Gamma(L)$

7.) $\Gamma(L^\times)(a_+, a_-, a_0) = \Gamma(L)(a_-, a_+, a_0)$

Proof : *1.) For a skein invariant* γ *with values in* R *and skein coefficients* $\alpha_+, \alpha_-, \alpha_0$*, define* $\Phi : \mathbf{Z}[a_+, a_+^{-1}, a_-, a_-^{-1}, a_0, a_0^{-1}] \longrightarrow R$ *by sending* a_+ *to* α_+*,* a_- *to* α_-*,* a_0 *to* α_0*. By induction over the complexity of a link one verifies* $\gamma = \phi \circ \Gamma$*. 2.) and 3.) are proven by induction over the complexity of a link using lemma 4.9. The following functions are skein invariants with values in* $\mathbf{Z}[a_+, a_+^{-1}, a_-, a_-^{-1}, a_0, a_0^{-1}]$ *and skein coefficients* a_+, a_-, a_0*.*

$L \mapsto \Gamma(L^-)$
$L \mapsto \Gamma(K \coprod L) \cdot \Gamma(K)^{-1} \cdot \Gamma(unlink^2)^{-1}$
$L \mapsto \Gamma(K \sharp L) \cdot \Gamma(K)^{-1}$
$L \mapsto \Gamma(L^\times)(a_-, a_+, a_0)$

Now the claims 4.) to 7.) follow from lemma 4.9 ■

Remark 4.12 If the skein invariant Γ exists, we derive from 4.8 that the trefoil and its mirrow image are not ambient isotopic. Notice that the Alexander polynomial cannot distinguish a knot from its mirrow image. This follows from remark 3.19 and the obvious fact that the (not refined) Alexander torsion is independent of the orientation of the knot. ■

In view of lemma 4.11 one may expect that there is 2-variable version of Γ*. This is, indeed, the case.*

Lemma 4.13 *Suppose there is a skein invariant*

$J : \{$ *isotopy classes of oriented links in* $S^3\} \longrightarrow \mathbf{Z}[l, l^{-1}, m, m^{-1}]$

such that l, l^{-1}, m *are skein coefficients. If* L *is any link, then* $\gamma(L)$ *is a sum of monomials* $r_{a,b} \cdot l^a \cdot m^b$ *for* $r_{a,b} \in R$ *such that* $a + b$ *is even. Hence we can define*

$$\Gamma(L) \in \mathbf{Z}[a_+, a_+^{-1}, a_-, a_-^{-1}, a_0, a_0^{-1}]$$

by

$$\Gamma(L)(a_+, a_-, a_0) := J(L)(a_+^{1/2} \cdot a_-^{-1/2}, a_0 \cdot a_+^{-1/2} \cdot a_-^{-1/2})$$

Then Γ *is a skein invariant with skein coefficients* a_+, a_-, a_0. ■

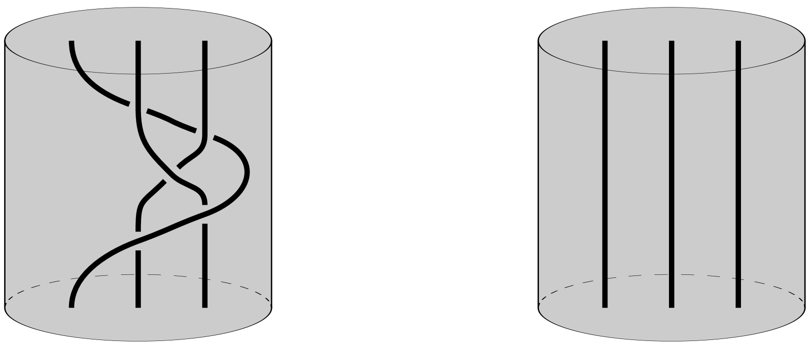
We will construct the skein invariant J *described in the lemma 4.13 above. Some preparations are needed.*

We begin with introducing the braid group B_n. *Let* P_n *be a set of* n *points in* D^2. *Let* $[n]$ *denote the set of* n *elements. A* braid with n strings *or shortly, a* n-braid, *is an embedding*

$$\beta : [n] \times [0, 1] \longrightarrow D^2 \times [0, 1]$$

sending (k, t) *to* $(\widehat{\beta_k}(t), t)$ *such that* $[n] \times \{i\}$ *is mapped to* $P_n \times \{i\}$. *If* $\widehat{\beta_k}$ *is constant, we obtain the trivial* n*-braid*

4.14 Nontrivial and trivial braid

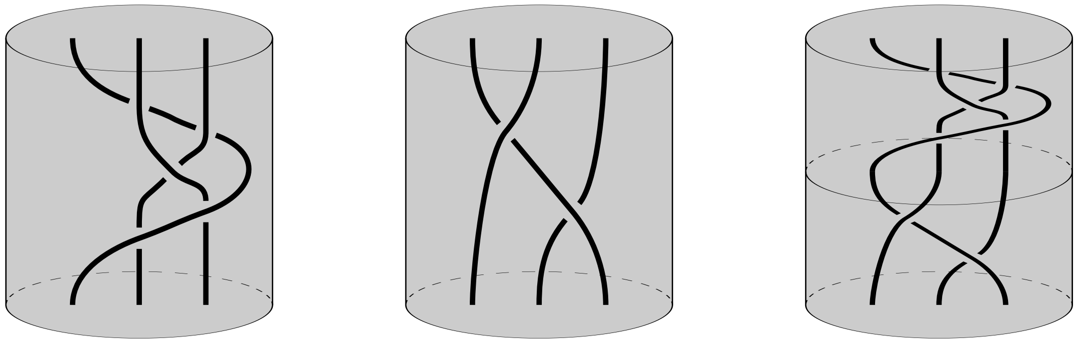


An isotopy I *of two* n*-braids* β *and* γ *is a map*

$$I : [n] \times [0, 1] \times [0, 1] \longrightarrow D^2 \times [0, 1]$$

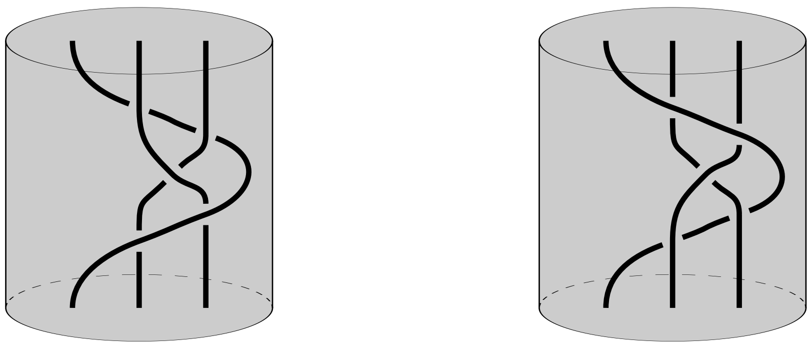
such that I_s, *given by the restriction of* I *to* $[n] \times [0, 1] \times \{s\}$, *is a* n*-braid for all* $s \in [0, 1]$. *Let* B_n *be the set of isotopy classes of* n*-braids. It inherits the structure of a group from the stacking operation*

4.15 Stacking of braids



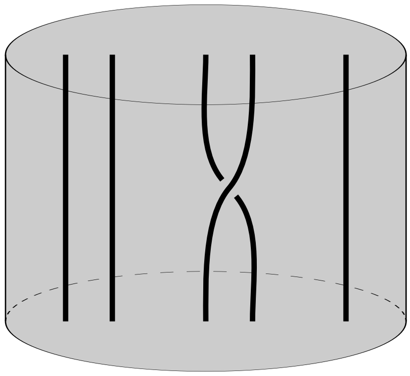
The trivial braid represents the unit element. The inverse of a class given by a braid β is represented by β^{-} which is obtained from β by reversing the t-direction

4.16 Inverse braid



Let $\sigma_i \in B_n$ for $1 \leq i < n$ be the following n-braid

4.17



Lemma 4.18 *The braid group B_n has the following presentation :*

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } 1 \leq i, j \leq n-1, \mid i-j \mid \geq 2 \\ \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \text{ for } 1 \leq i \leq n-2 \end{array} \right\rangle$$

A proof of this lemma can be found in Birman [4]. If we add the relation $\sigma_i\sigma_i = 1$, we get a presentation of the symmetric group Σ_n of permutations of $[n] = \{1, 2, \ldots n\}$. Hence there is an epimorphism

4.19 $p : B_n \longrightarrow \Sigma_n$

The image of a braid under p is the automorphism of P_n sending an element x of P_n to the element $p(x)$ which is connected to x by a string of the braid. Consider the epimorphism

4.20 $e : B_n \longrightarrow \mathbf{Z}$

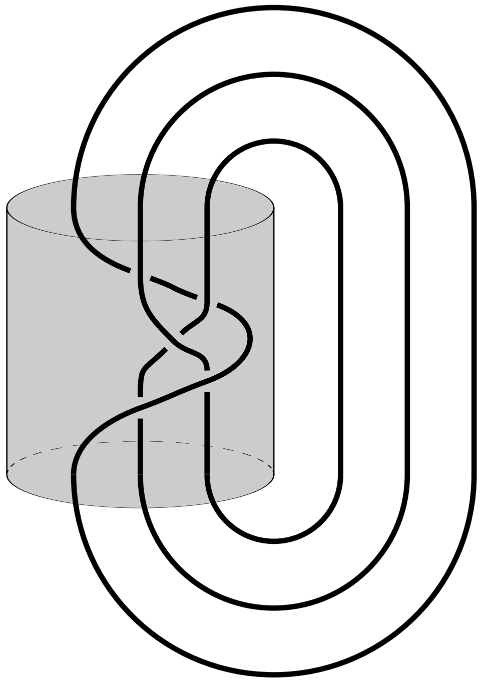
sending σ_i to 1. Given a picture of a braid β as in 4.14, $e(\beta)$ is the number of crossings counted with sign. One easily checks, that e is the abelianization of the braid group. There is an obvious inclusion

4.21 $i : B_n \longrightarrow B_{n+1}$

given by adding a trivial string.

Links and braids a related by the so called closure of a braid . *Namely, any braid β defines a link clos(β) by the construction indicated below.*

4.22 Closure of a braid



We obtain a map :

4.23 $clos : \coprod_{n\geq 1} B_n \longrightarrow \{$ isotopy classes of oriented links in $S^3\}$

Theorem 4.24 (Alexander) *The map clos is surjective* ∎

Let $\beta_n \in B_n$ and $\gamma_m \in B_m$ be braids. We say, that γ_m is obtained from β_n by a Markov operation of type 1 , *if $n = m$ and there is $\delta_n \in B_n$ satisfying $\gamma_n = \delta_n\beta_n\delta_n^{-1}$. If $m = n+1$ and $\gamma_{n+1} = \beta_n\sigma_n^\epsilon$ for some $\epsilon \in \{\pm 1\}$ holds, we say that γ_{n+1} is obtained from β_n by a Markov operation of type 2. Consider a sequence of braids $\delta_{n_1}^1$, $\delta_{n_2}^2$,... $\delta_{n_r}^r$ such that $\delta_i^{n_i}$ is obtained from $\delta_{i+1}^{n_{i+1}}$ by a Markov operation 1 or 2 or $\delta_{i+1}^{n_{i+1}}$ is obtained from $\delta_i^{n_i}$ by a Markov operation 1 or 2 for $1 \leq i \leq n-1$. Then we say that $\delta_{n_1}^1$ and $\delta_{n_r}^r$* are related by a sequence of Markov operations.

Theorem 4.25 (Markov) *Two braids $\beta_n \in B_n$ and $\gamma_m \in B_m$ have the same closure if and only if they are related by a sequence of Markov operations.* ∎

Proofs of the theorems of Alexander and Markov can be found in Birman [4]. These two results allow to translate constructions of invariants for links to the construction of invariants for braids. The main advantage of braids is that they build a group what is not true for links. Especially one can investigate representations of the braid group. We will see that this leads to a construction of the Jones polynomial. Before we go into the details, we explain how one can construct the Alexander polynomial out of the so called Burau representation of the braid group.

Let t be the generator of the group $\mathbf{Z}$*. The* Burau representation *is the homomorphism*

4.26 $\psi_n : B_n \longrightarrow Gl(n, \mathbf{Z}[\mathbf{Z}])$

sending the generator σ_i to the following matrix

$$A_i = \begin{pmatrix} E & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & E \end{pmatrix}$$

where E is the identity matrix and $(1-t)$ is the (i,i)-entry. This is well defined, because the matrices above satisfy the relations appearing in the presentation of the braid group given in lemma 4.18.

There is the following mechanical interpretation of the Burau representation. Suppose that we let particles travel along the strings of a braid β. We do not allow at a crossing that a particle moving along the undercrossing string jumps upwards to the overcrossing string, but a particle travelling on the overcrossing string has the probability t of falling down to the undercrossing string. Then the (i,j)-entry in $\psi(\beta)$ is the pobability that a particle starting at the i-th point will end up at the j-th point.

Let $\epsilon : \oplus_n \mathbf{Z}[\mathbf{Z}] \longrightarrow \mathbf{Z}[\mathbf{Z}]$ be the map sending $(x_1, x_2, \ldots, x_n)$ to $\sum_{i=1}^n x_i$. The homomorphism $\oplus_n \mathbf{Z}[\mathbf{Z}] \longrightarrow \oplus_n \mathbf{Z}[\mathbf{Z}]$ mapping x to xA_i leaves the kernel of ϵ invariant. Hence there is an induced representation $\widetilde{\psi}$ on the kernel of ϵ. This gives the reduced Burau representation

4.27 $\widetilde{\psi}_n : B_n \longrightarrow Gl(n-1, \mathbf{Z}[\mathbf{Z}])$

The proof of the following result can be found in Burde-Zieschang [6] proposition 10.20.

Lemma 4.28 *Let β_n be a n-braid and L the link in S^3 given by $L = clos(\beta_n)$. Then :*

$$\Delta_L(t) \cdot (1 + t + \ldots t^{n-1}) = det(1 - \widetilde{\psi}_n(\beta_n)(t))$$

The theorem of Alexander 4.24 and Markov 4.25 and the computation of the Alexander polynomial by the Burau representation in lemma 4.28 suggest the following strategy for *constructing link invariants. Find representations of the braid groups together with an invariant of a braid constructed out of this representation which is invariant under the Markov operations. Since the first Markov operation is given by conjugation, it is natural to use a trace. A natural candidate for such things are Hecke algebras together with a trace as described below.*

Let F be a field and $q \in F$ an element in this field. The Hecke algebra $H_n = H_n(F,q)$ *associated with F and q for $n \geq 2$ is the associative F-algebra with unit 1, generated by T_1,T_2 $\ldots T_{n-1}$ subject to the following relations.*

4.29

$$\begin{array}{ll} T_iT_j = T_jT_i & \text{for } 1 \leq i,j \leq n-1 \text{ and } \mid i-j \mid \geq 2 \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} & \text{for } 1 \leq i < n \\ T_i^2 = (1-q)T_i + q & \text{for } 1 \leq i < n \end{array}$$

If n is 1, we put $H_1(F,q) := F$. We see that there is a natural map $H_{n-1} \longrightarrow H_n$ of F-algebras. In particular H_n becomes a H_{n-1}-H_{n-1}-bimodule. An important example of a Hecke algebra is the group ring of the symmetric group Σ_n which is the Hecke algebra $H_n(F,1)$. The main technical result is the following

Lemma 4.30 *There is an isomorphism of H_n-H_n-bimodules*

$$\phi : H_n \oplus \left(H_n \otimes_{H_{n-1}} H_n\right) \longrightarrow H_{n+1}$$

sending $a + \sum_i b_i \otimes c_i$ to $a + \sum_i b_iT_nc_i$. ■

First we show that ϕ is well defined. We have to show for $u \in H_{n-1}$ and $b, c \in H_n$ that buT_nc and bT_nuc agree. But u is a linear combination of monomials in T_1,T_2,$\ldots T_{n-2}$ and all these elements commute with T_n.

It is also easy to see by induction on n that ϕ is surjective. It suffices to show by induction over n that any element $x \in H_{n+1}$ can be written as a linear combination of monomials in the T_i such that T_n occurs only once in it. We reduce the occurences of T_n as follows. Consider $x = y_1T_ny_2T_ny_3$ such that y_i does not contain T_n. If y_2 does not contain T_{n-1}, an application of the first and third relation in 4.29 reduces the number of occurences of T_n. By induction hypothesis we can assume that T_{n-1} occurs precisely one times in y_2. Because of the first relation we can assume $y_2 = T_{n-1}$. Now an application of the second relation reduces the number of occurences of T_n. This shows surjectivity.

Suppose that $dim_F(H_n) = n!$. Then the dimensions of the source and target of ϕ are equal and hence ϕ is an isomorphism. The proof of $dim_F(H_n) = n!$ can be found in de la Harpe, Kervaire and Weber [15], section 4. They construct an explicit set B in H_n and an algebra map $L : H_{n+1} \longrightarrow End_F(F[\Sigma_{n+1}])$ such that its composition with evaluation at 1 defines a F-linear map $H_n \longrightarrow F[\Sigma_{n+1}]$ sending B bijectively to Σ_n.

There is a natural representation

4.31 $\rho_{n+1} : B_{n+1} \longrightarrow H_{n+1} \qquad \sigma_i \mapsto T_i$

The idea is to construct a trace on the Hecke algebras and to define an invariant of braids by applying the trace to the image of the braid under ρ. This would take care of the first Markov operation, as a trace is invariant under conjugation. For the second Markov operation one needs a good control over the trace of aT_n for $a \in H_n$. In view of the lemma 4.30 it is reasonable to define inductively over n traces tr_n on H_n such that $tr_{n+1}(aT_nb)$ can be expressed by a,b and the trace tr_n. Indeed, the following is true.

Lemma 4.32 *Let F be a field and q, z elements in F. Let H_n be the Hecke algebra $H_n(F, q)$. Then there exists F-linear maps*

$$tr_n : H_n \longrightarrow F$$

with the following properties :

1.) tr_{n+1} restricted to H_n is tr_n.
2.) $tr_n(1) = 1$
3.) $tr_n(ab) = tr_n(ba)$
4.) $tr_{n+1}(aT_nb) = z tr_n(ab) \qquad$ for $a, b \in H_n$

The maps tr_n are uniquely determined by these properties. ∎

The elementary proof of this lemma can be found in de la Harpe, Kervaire and Weber [15], section 5.

Our first attempt to define an invariant for links is :

$\widehat{J}(\beta_n) = tr_n(\rho(\beta_n)) \qquad$ *for* $\beta \in B_n$

We have to check the transformation behaviour under the two Markov operations. The first one does not change the invariant :

$tr_n(\rho(\beta_n)) = tr_n(\rho(\gamma_n\beta_n\gamma_n^{-1}))$

In the second case we obtain for $\beta_n \in B_n$ and the generator $\sigma_n \in B_{n+1}$

$tr_{n+1}(\rho_{n+1}(\beta_n\sigma_n)) = z \cdot tr_n(\rho(\beta_n))$
$tr_{n+1}(\rho_{n+1}(\beta_n\sigma_n^{-1})) = w \cdot tr_n(\rho(\beta_n))$

where $w := \frac{1}{q}(z + q - 1)$. Notice that $T_i^{-1} = \frac{1}{q}(T_i + q - 1)$ holds because of the third relation in the definition of a Hecke algebra 4.29. Hence we modify our first attempt as follows. For not yet defined functions

$a_n : B_n \longrightarrow \mathbf{Z}$
$b_n : B_n \longrightarrow \mathbf{Z}$

we put

$\widehat{J}(\beta_n) = z^{a_n(\beta_n)} \cdot w^{b_n(\beta_n)} \cdot tr_n(\rho(\beta_n))$

Then $\widehat{J}$ is invariant under the Markov moves, if the following conditions are satisfied :

$a_n(\gamma_n\beta_n\gamma_n^{-1}) = a_n(\beta_n)$

$b_n(\gamma_n\beta_n\gamma_n^{-1}) = b_n(\beta_n)$
$a_{n+1}(\beta_n\sigma_n) = a_n(\beta_n) - 1$
$b_{n+1}(\beta_n\sigma_n) = b_n(\beta_n)$
$a_{n+1}(\beta_n\sigma_n^{-1}) = a_n(\beta_n)$
$b_{n+1}(\beta_n\sigma_n^{-1}) = b_n(\beta_n) - 1$

Our invariant is supposed to assign 1 to the trivial link. Since the trivial link is given by the braid σ_1, we also demand :

$a_2(\sigma_1) = 0$
$b_2(\sigma_1) = 0$

One easily finds out that the following functions have these properties :

$a_n(\beta_n) = \frac{1}{2} \cdot (-e(\beta_n) - n + 1)$
$b_n(\beta_n) = \frac{1}{2} \cdot (e(\beta_n) - n + 1$

Hence we can define for a be a link L in S^3 :

4.33 $\widehat{J}(L) = z^{-e(\beta_n)-n+1} \cdot w^{e(\beta)-n+1} \cdot tr_n(\rho_n(\beta_n))$

where β_n is any braid with L as closure.

Now we make a special choice for the field F. Let $\mathbf{C}(q, z)$ be the rational field over $\mathbf{C}$ in two independent variables q and z. Let K be the extension obtained by adjoining the square roots $\sqrt{q}$ and $\sqrt{z/w}$. Now we take H_n over K and let q and z be the elements in K given by the variables q and z.

Lemma 4.34 *Let (L_+, L_-, L_0) be a skein triple of oriented links in S^3. Define elements l and m in K by*

$$l = iz^{1/2}w^{-1/2}q^{-1/2} \qquad m = i(q^{1/2} - q^{-1/2})$$

Then we get

$$l \cdot \widehat{J}(L_+) + l^{-1} \cdot \widehat{J}(L_-) + m \cdot \widehat{J}(L_0) = 0$$

Proof : *We can find positive integers k and n and braids β and γ in B_n such that $k \leq n-1$ and the closure of the braids $\alpha_+ = \beta\sigma_k\gamma$, $\alpha_- = \beta\sigma_k^{-1}\gamma$ and $\alpha_0 = \beta\gamma$ is L_+, L_- and L_0. Now one computes on the level of Hecke algebras :*

$l \cdot z^{-e(\alpha_+)-n+1} \cdot w^{e(\alpha_+)-n+1} \cdot \rho_n(\alpha_+) + l^{-1} \cdot z^{-e(\alpha_-)-n+1} \cdot w^{e(\alpha_-)-n+1} \cdot \rho_n(\alpha_-) + m \cdot z^{-e(\alpha_0)-n+1} \cdot w^{e(\alpha_0)-n+1} \cdot \rho_n(\alpha_0)$

$= l \cdot z^{-1/2} \cdot w^{1/2} \cdot z^{-e(\alpha_0)-n+1} \cdot w^{e(\alpha_0)-n+1} \cdot \rho(\beta)T_k\rho(\gamma) + l^{-1} \cdot z^{1/2} \cdot w^{-1/2} \cdot z^{-e(\alpha_0)-n+1} \cdot w^{e(\alpha_0)-n+1} \cdot \rho(\beta)T_k^{-1}\rho(\gamma) + m \cdot z^{-e(\alpha_0)-n+1} \cdot w^{e(\alpha_0)-n+1} \cdot \rho(\beta)\rho(\gamma)$

$= \rho(\beta) \cdot \left(l \cdot z^{-1/2} \cdot w^{1/2} \cdot T_k + l^{-1} \cdot z^{1/2} \cdot w^{-1/2} \cdot q^{-1} \cdot (T_k + q - 1) + m\right) \rho(\gamma)$

This expression turns out to be zero because of the following easily verified equations :

$l \cdot z^{-1/2} \cdot w^{1/2} + l^{-1} \cdot z^{1/2} \cdot w^{-1/2} \cdot q^{-1} = 0$

$l^{-1} \cdot z^{1/2} \cdot w^{-1/2} \cdot q^{-1} \cdot (q-1) + m = 0$

Now apply tr_n to the equation above and the claim follows. ■

There is an embedding of algebras $\Psi : \mathbf{Z}[l, l^{-1}, m, m^{-1}] \longrightarrow K$ sending l to l and analogously for l^{-1},m and m^{-1}. If two of the elements $\widehat{J}(L_+)$, $\widehat{J}(L_-)$ and $\widehat{J}(L_0)$ lie in the image of ψ, then also the third by the lemma 4.34 above. Since $\widehat{J}(unknot)$ is 1 by construction, we show inductively over the complexity of a link that $\widehat{J}(L)$ lies in the image of Ψ for all oriented links L in S^3.

Definition 4.35 *Let L be an oriented link in S^3. The* Jones polynomial

$$J(L) \in \mathbf{Z}[l, l^{-1}, m, m^{-1}]$$

is defined by $\Psi(J(L)) = \widehat{J}(L)$. ■

We derive from lemma 4.13 and lemma 4.34.

Theorem 4.36 *The Jones polynomial is a skein invariant with skein coefficients l, l^{-1}, m and gives a universal skein invariant.* ■

Originally the Jones polynomials was introduced only in one variable. Namely ,the Jones polynomial as constructed by Jones [20] is a skein invariant with skein coefficients $t, -t^{-1}, (t^{1/2} - t^{-1/2})$. It came out of the investigation of the possible indices of subfactors of von Neumann algebras, where certain projections appear whose commuting relations are similar to the presentation of the braid group (see lemma 4.18). A few months later it was discovered independently by four different groups, that the one-variable polynomial of Jones could be generalized to the universal skein invariant as constructed above (see Freyd, P. and Yetter, D. ; Hoste, J. ; Lickorish, W.B.R. and Millet, K. ; Ocneanu, A. in [13]). The approach using Hecke algebras is due to Ocneanu. We also explain the construction of the one-variable Jones polynomial due to Kauffman which allows to compute it directly from a link diagram in a simple manner.

Let L be an unoriented *link diagram. For each crossings there are two choices of so called* markers *as indicated below.*

4.37

According to the choosen marker one may dissolve the crossing by connecting the two regions selected by the marker

4.38

A state S *for L is choice of marker at each crossings. Let $a(S)$ and $b(S)$ be the number of markers of type A resp. B. If we dissolve the crossings according to the state S as indicated above, we obtain a bunch of disjoint simple curves. Let $\mid S \mid$ be their number. Define an element*

$$< L > := \sum_S A^{a(S)} B^{b(S)} d^{|S|-1} \qquad \in \mathbf{Z}[A, B, d]$$

One easily checks :

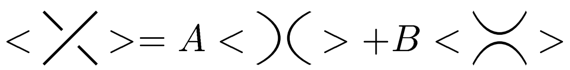
Lemma 4.39
1.) $< unknot > = 1$

2.) $< unknot \coprod L > = d \cdot < L >$, if L is non-empty

3.) If L is an oriented link diagram and $\times$ a crossing. Let L_A resp. L_B be the link diagram obtained from L by dissolving this crossing $\times$ according to the choice of marker A resp. B. Then we have :

$$< L > = A \cdot < L_A > + B \cdot < L_B > \qquad \blacksquare$$

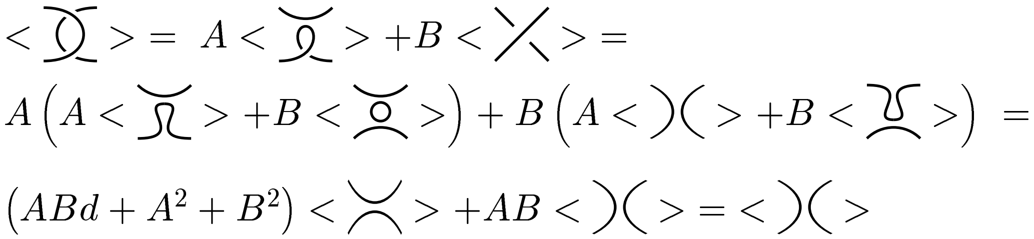
We will abbreviate the equation above by the following



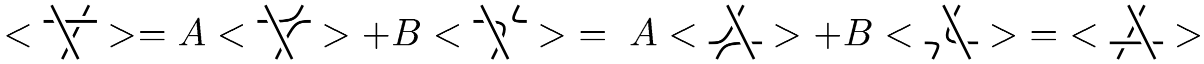
Recall that two link diagrams describe the same link if they can be obtained from one another by a sequence of Reidemeister moves **??**.

Lemma 4.40 *The invariant $< >$ is invariant under the first and third Reidemeister moves if that $B = A^{-1}$ and $d = -A^2 - A^{-2}$ holds.*

Proof : *This follows from the following calculations :*



and



■

It remains to treat the second Reidemeister move. Indeed, the invariant as it stands is not invariant under the second one.

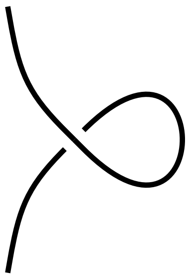
Definition 4.41 *The* writh number $w(L)$ *of an oriented link diagram is the sum of signs of the crossings* ■

Then Kauffman defines an invariant of an oriented link L in S^3 given by an oriented link diagram L :

4.42 $f_L(A) = (-A)^{-3w(L)} \cdot < L > \in \mathbf{Z}[A, A^{-1}]$

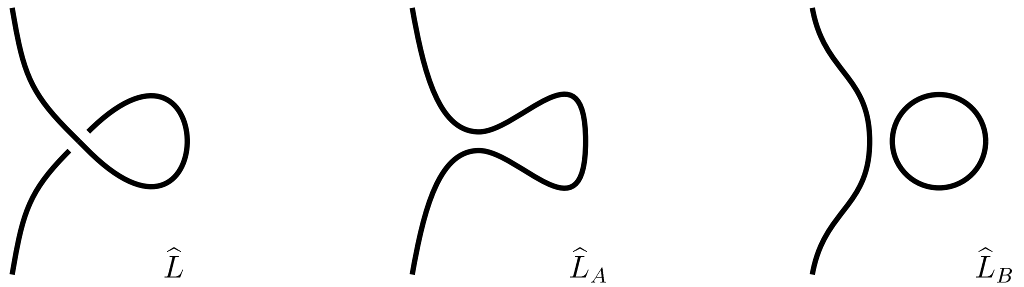
The writh number is not changed by the first and third Reidemeister move so that the invaraint f_L is invariant under these moves. Suppose that L is obtained from $\widehat{L}$ by removing the loop :

4.43



Then $\widehat{L}$, $\widehat{L}_A$ and $\widehat{L}_B$ look like

4.44



We get :

$< \widehat{L} >= A < L > + A^{-1} < L \coprod unknot >=$
$A < L > + A^{-1}(-A^2 - A^{-2}) < L >= (-A)^{-3} < L >$

Since $w(\widehat{L}) = w(L) - 1$ holds, we derive $f_{\widehat{L}} = f_L$. The proof for the other loop is similar. one easily checks using lemma 4.39 that f_L is a skein invariant with values in $\mathbf{Z}[A, A^{-1}]$ and skein coefficients $A^4, -A^{-4}, (A^2 - A^{-2})$. This shows :

Lemma 4.45 *We have for any oriented link L in S^3 :*

$$J_L(t) = f_L(t^{1/4})$$

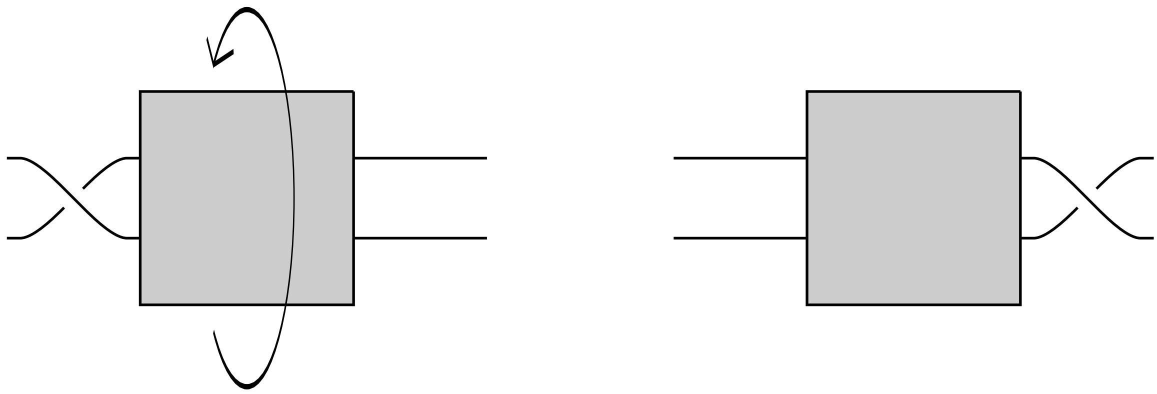
■

More details about this invariant and about a new invariant which is not a skein invariant can be found in the book by Kauffman [25], appendix.

Remark 4.46 The Alexander polynomial does not determine the one-variable Jones polynomial and vice versa. Namely, Conway's eleven crossings knot 11_{471} has trivial Alexander but non-trivial Jones polynomial, whereas the knots 4_1 and 11_{388} have the same Jones,but different Alexander polynomials. Moreover, the one-variable Jones polynomial and the Alexander polynomial together do not determine Jones polynomial defined in 4.35. Namely, 11_{388} and its mirrow image have the same Alexander and one variable Jones polynomial, but the Jones polynomial of 4.35 does distinguish them. ■

One of the striking properties of the Jones polynomial is that it can distinguish a knot from its mirrow image, what is not true for the Alexander polynomial. Another important applications are the proofs of Kauffman and Marasugi of the Taite conjectures. Taite is viewed as one of the founders of knot theory and he spelled out his conjectures 100 years ago. A survey about these conjectures is given in de la Harpe, Kervaire and Weber [15], *section 9. The main point is that the span of the one variable Jones polynomial gives a lower bound for the minimal number of crossings and is equal to it for alternating reduced link diagrams. This implies one of the Tait conjectures that for a prime alternating knot the minimal diagrams are exactly the alternating reduced ones. Reduced means that one cannot decrease the number of crossing by certain elementary moves and a link diagram is minimal if the number of crossing in any other link diagram presenting the same link is not smaller. A stronger still unproved versions says that two reduced alternating link diagrams determine the same link if and only if they can be obtained from one another by flyping, a special move indicated below*

4.47



Another now verified Tait conjecture says that the writh number for alternating reduced link diagrams depends only on the associated link. In particular the writh number of a reduced alternating link diagram describing an amphichiral link is zero. A link is called amphichiral if it is ambient isotopic to its mirror image. We mention the so called Perko pair of link diagrams which describe the same link, but have different writh numbers. We refer to Kauffman's book [25], appendix, for more information.

5.section
Quantum Field theory and the Jones polynomial

In this section we introduce the axioms of a quantum field theory in the sense of Segal and explain how one can construct the Jones polynomial out of a suitable quantum field theory. Some preparations are needed.

A (symmetric) monoidal category *is a 6-tuple* $\mathcal{C}, \coprod, \emptyset, S^1, S^2, S^3$ *consisting of*

- *a category* $\mathcal{C}$
- *a functor* $\coprod : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ *(product)*
- *an object* $\emptyset \in \mathcal{C}$ *(unit object)*
- *a natural transformation* $S^1(C, D, E) : (C \coprod D) \coprod E \longrightarrow C \coprod (D \coprod E)$ *(associativity)*
- *a natural transformation* $S^2(C, D) : C \coprod D \longrightarrow D \coprod C$ *(commutativity)*
- *a natural transformation* $S^3(C) : C \coprod \emptyset \longrightarrow C$ *(unit element)*

such that the obvious compatibility conditions are satisfied. We will often surpress the transformations and the unit element in the sequel. A functor of monoidal categories

$$(F, T_{\coprod}, \phi) : (\mathcal{C}, \coprod_{\mathcal{C}}) \longrightarrow (\mathcal{D}, \coprod_{\mathcal{D}})$$

consists of

- *a functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$
- *a natural transformation* $T_{\coprod}(C, D) : F(C \coprod_{\mathcal{C}} D) \longrightarrow F(C) \coprod_{\mathcal{D}} F(D)$
- *an isomorphism* $\phi : F(\emptyset_{\mathcal{C}}) \longrightarrow \emptyset_{\mathcal{D}}$

such that the obvious compatibility conditions are satisfied.

An involution (I, S) *on a monoidal category* $\mathcal{C}$ *is given by*

- *a contravariant functor* $I = (I, T_{\coprod}, \phi) : \mathcal{C} \longrightarrow \mathcal{C}$ *of monoidal categories*
- *a natural transformation* $S(C) : C \longrightarrow I \circ I(C)$

such that $S(I(C)) \circ I(S(C)) = id$ *holds for all objects* $C \in \mathcal{C}$ *and* $I(\phi) \circ S(\emptyset) = \phi$ *is true. We will often drop the natural transformation* S *in the notation. A* functor of monoidal categories with involution $(F, T_{\coprod}; T_I) : (\mathcal{C}, \coprod_{\mathcal{C}}, I_{\mathcal{C}}) \longrightarrow (\mathcal{D}, \coprod_{\mathcal{D}}, I_{\mathcal{D}})$ *consists of*

- *a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$*
- *a natural transformation $T_{\coprod}(C, D) : F(C \coprod_{\mathcal{C}} D) \longrightarrow F(C) \coprod_{\mathcal{D}} F(D)$*
- *a natural transformation $T_I(C) : F(I(C)) \longrightarrow I(F(C))$*

such that $T_I(\emptyset_{\mathcal{C}}) \circ F(\phi_{\mathcal{C}}) = \phi_{\mathcal{D}}$ and $(F, T_{\coprod}) : (\mathcal{C}, \coprod_{\mathcal{C}}) \longrightarrow (\mathcal{D}, \coprod_{\mathcal{D}})$ is a functor of monoidal categories. We will mainly deal with the following examples.

Convention 5.1 Let W be an oriented $d+1$-manifold for $d \geq 1$. Then the boundary inherits an orientation using the decomposition $TW\mid_{\partial W} = \nu(\partial W, W) \oplus T\partial W$ and the outward normal. This is consistent with the convention that the positive orientation on S^1 is given by the anti-clockwise direction and that the positive orientation on $\mathbf{R}^2$ is represented by the ordered bases $\{(1,0),(0,1)\}$. An orientation of a 0-dimensional manifold is a choice of $+$ or $-$ for each component. If W is an oriented 1-dimensional manifold, the induced orientation for a component of the boundary is $+$ resp. $-$, if the orientation at this component corresponds to the outward resp. inward normal. Notice that this means for an oriented path that he starts at $-$ and ends at $+$. ■

Example 5.2 Let M and N be oriented closed d-manifolds. A *bordism* from M to N is a 5-tuple $(W; W_0, W_1; f_0, f_1)$ consisting of an oriented $d+1$-dimensional manifold W with boundary ∂W and closed and open submanifolds W_0 and W_1 of the boundary satisfying $W_0 \cap W_1 = \emptyset$ and $W_0 \cup W_1 = \partial W$ together with orientation preserving diffeomorphisms $f_0 : M^- \longrightarrow W_0$ and $f_1 : W_1 \longrightarrow N$. The symbol M^- denotes the manifold N with the reversed orientation. We call two such bordisms $(W; W_0, W_1; f_0, f_1)$ and $(V; V_0, V_1; g_0, g_1)$ from M to N *equivalent* , if there is an orientation preserving diffeomorphism $F : W \longrightarrow V$ such that $F \circ f_0 = g_0$ and $g_1 \circ F\mid_{W_1} = f_1$ holds. If i_k denotes the obvious diffeomorphism, the trivial bordism $[0,1] \times M$ from M to M is represented by $([0,1] \times M; \{0\} \times M, \{1\} \times M; i_0, i_1)$. Notice that the orientation on $\{0\} \times M$ induced by the outward normal is the opposite of the orientation on M, whereas on $\{1\} \times M$ we get the orientation on M back. Let $(W; W_0, W_1; f_0, f_1)$ resp.$(V; V_0, V_1; g_0, g_1)$ be a bordism from L to M resp. M to N. We get a bordism $(W \cup_{g_0 \circ f_1} V; W_0, V_1; f_0, g_1)$ from L to N by glueing. This is compatible with the equivalence relation defined above.

Assume $d \geq 1$. Let $\mathcal{M}^{d,d+1}$ be the following monoidal category with involution. Objects are d-dimensional oriented closed manifolds M. Equivalence classes of bordisms from M to N build the set of morphisms from M to N. The identity morphism is given by the trivial bordism and the composition of morphisms given by the glueing process described above. The monoidal structure $\coprod_{\mathcal{M}^{d,d+1}}$ comes from the disjoint union. The unit object is the empty set. The involution $I_{\mathcal{M}^{d,d+1}}$ assigns to a morphism $W := (W; W_0, W_1; f_0, f_1)$ from M to N in $\mathcal{M}^{d,d+1}$ the morphism $\widehat{W} := (W; W_1, W_0; f_1^{-1}, f_0^{-1})$ from N^- to M^-. Notice that $\emptyset$ and $\emptyset^-$ are identical and the involution acts by the identity on the set of endomorphisms of $\emptyset$.

Let $f : M \longrightarrow N$ be an orientation preserving diffeomorphism of closed oriented d-dimensional manifolds. It determines a morphism, denoted by $f : M \longrightarrow N$, in $\mathcal{M}^{d,d+1}$ by the bordism $([0,1]\times M;\{0\}\times M,\{1\}\times M; id, f)$. Two diffeomorphisms $f_0, f_1 : M \longrightarrow N$ are called *pseudoisotopic* , if there is a diffeomorphism $F : M\times[0,1] \longrightarrow N\times[0,1]$ whose restriction to $M\times\{i\}$ is f_i for $i = 0,1$. One easily checks that f_0 and f_1 define the same morphism in $\mathcal{M}^{d,d+1}$ if and only if they are pseudoisotopic. If $W := (W; W_0, W_1; f_0, f_1)$ is a morphism from N to L and $f : M \longrightarrow N$ an orientation preserving diffeomorphism, the composition $W\circ f$ is represented by $(W; W_0, W_1; f_0\circ f, f_1)$ and similar for $f : L \longrightarrow K$ and $f\circ W$. ■

Example 5.3 Let $\mathcal{V}$ be the monoidal category with involution with finitely generated complex vector spaces as objects and linear maps as morphisms. The monoidal structure is induced by the tensor product. The unit element is given by $\mathbf{C}$. The involution sends V to the dual vector space V^*. The natural transformation $T(V) : V \longrightarrow V^{**}$ sends v to the homomorphism $V^* \longrightarrow \mathbf{C}$ mapping f to $f(v)$. Let $\phi : \mathbf{C} \longrightarrow \mathbf{C}^*$ send $z\in\mathbf{C}$ to the homomorphism $\mathbf{C} \longrightarrow \mathbf{C}$ sending u to $z\cdot u$. The inverse of ϕ is evaluation at 1. This definition makes also sense for finitely generated projective modules over any ring with involution.

Let $\mathcal{H}$ be the monoidal category with involution with Hilbert spaces as objects and bounded linear operators as morphisms. The tensor product and the dual space consisting of bounded linear operators with $\mathbf{C}$ as target yield the product and the involution. The transformation T(H), the unit element and ϕ are defined as above. Notice that one does not need a Hilbert structure but the structure of a reflexive Banach space. ■

Definition 5.4 *A d-d+1-*quantum field theory *is a functor of monoidal categories with involution*

$$H : \mathcal{M}^{d,d+1} \longrightarrow \left\{ \begin{array}{l} \mathcal{V} \\ \mathcal{H} \end{array} \right. \qquad ■$$

Let W be an oriented $d+1$-dimensional manifold. Let $W : \emptyset \longrightarrow \partial W$ be the morphism given by $(W;\emptyset,\partial W; id, id)$. Since $H(\emptyset) = \mathbf{C}$, we have the element $1\in H(\emptyset)$. We define :

5.5 $Z(W) := H(W)(1) \in H(\partial W)$

Notice that $Z(W)$ is a complex number, if W is closed. Built into the definition of a quantum field theory is a kind of glueing formula. Let V and W be $d+1$-dimensional oriented manifolds and $f : \partial V \longrightarrow \partial W^-$ an orientation preserving diffeomorphism. We obtain a pairing

5.6 $<\,,\,>_f :\ H(\partial V)\otimes H(\partial W) \longrightarrow \mathbf{C}$

by the composition

$$H(\partial V)\otimes H(\partial W) \xrightarrow{H(f)\otimes id} H(\partial W^-)\otimes H(\partial W) \xrightarrow{T_I(\partial W)\otimes id} H(\partial W)^*\otimes H(\partial W) \xrightarrow{ev} \mathbf{C}$$

We obtain a closed oriented $d+1$-dimensional manifold $V\cup_f W$ by glueing and hence a complex number $Z(V\cup_f W)$.

Lemma 5.7 $Z(V \cup_f W) =< Z(V), Z(W) >_f$ ■

Proof : *Recall that* $H(\emptyset)$ *is* $\mathbf{C}$. *Consider the following diagram*

$$\begin{tikzcd}
H(\emptyset) \otimes H(\emptyset) \arrow[r, "H(V) \otimes H(W)"] \arrow[d, "id"] & H(\partial V) \otimes H(\partial W) \arrow[d, "H(f) \otimes id"] \\
H(\emptyset) \otimes H(\emptyset) \arrow[r, "H(f \circ V) \otimes H(W)"] \arrow[d, "T_I(\emptyset) \otimes id"] & H(\partial W^-) \otimes H(\partial W) \arrow[d, "T_I(\partial W) \otimes id"] \\
H(\emptyset)^* \otimes H(\emptyset) \arrow[r, "H(\widehat{V} \circ \widehat{f})^* \otimes H(W)"] \arrow[d, "ev"] & H(\partial W)^* \otimes H(\partial W) \arrow[d, "ev"] \\
H(\emptyset) \arrow[r, "H((V \cup_f W))"] & H(\emptyset)
\end{tikzcd}$$

We derive from the definition of a functor of monoidal categories with involution that the diagram commutes and that $T_I(\emptyset) : H(\emptyset) \longrightarrow H(\emptyset)^*$ *is just the map* $\phi : \mathbf{C} \longrightarrow \mathbf{C}^*$. *Consider* $1 \otimes 1 \in H(\emptyset) \otimes H(\emptyset)$. *Sending it from the left upper corner to the right lower corner in the clockwise direction gives* $< Z(V), Z(W) >_f$ *and in the anti-clockwise direction gives* $Z(V \cup_f W)$ *by the definitions.* ■

Next we give the most elementary non-trivial example of a d-$d+1$-*quantum field theory for* d *even. Denote by* $\chi(W)$ *the Euler characteristic and by* $\sigma(W)$ *the signature of an oriented manifold. Recall that the signature is defined to be zero, if the dimension is not divisible by four, and to be the signature of the intersection pairing in dimensions divisible by* 4. *These invariants satisfy the following additivity formulas. Given* $d+1$- *dimensional manifolds* V *and* W *and an orientation preserving diffeomorphism* $f : V_1 \longrightarrow W_0$ *between disjoint unions of components of the boundaries of* V *and* W, *we get :*

$$\chi(V \cup_h W^-) = \chi(V) + \chi(W) - \chi(V_1)$$

and

$$\sigma(V \cup_f W^-) = \sigma(V) - \sigma(W)$$

Notice for odd d that $\chi(V_0)$ is zero by Poincaré duality so that we get for odd d :

$$\chi(V \cup_h W^-) = \chi(V) + \chi(W)$$

We mention the following conclusions $\sigma(V^-) = -\sigma(V)$ and $\chi(V^-) = \chi(V)$. These invariants are characterized by the following property.

A SK-invariant *for closed oriented m-dimensional manifolds is a function ρ assigning to any closed oriented m-dimensional manifold W an element*

5.8 $\rho(W) \in A$

in an abelian group A such that the following holds :

- *If V and W are oriented diffeomorphic, then $\rho(V) = \rho(W)$*
- $\rho(V \coprod W) = \rho(V) + \rho(W)$
- *Let $f, g : \partial W_1 \longrightarrow \partial W_2$ be two orientation preserving diffeomorphisms. Then we have $\rho(W_1 \cup_f W_2^-) = \rho(W_1 \cup_g W_2^-)$*

For SK-invariants in general and a proof of the following theorem we refer to Karras-Kreck-Neumann-Ossa [23]. See also Jänich [17] and [18]. A SK-group (A, ρ) is called universal, if for any SK-group A', ρ' there is a homomorphism $\phi : A \longrightarrow A'$ uniquely determined by the property $\rho' = \phi \circ \rho$

Lemma 5.9 *The universal SK-invariant is given by*

$(\chi(W) - \sigma(W))/2, \sigma(W) \in \mathbf{Z} \oplus \mathbf{Z}$	*,if $m \equiv 0 \bmod 4$*
$\chi(W)/2 \in \mathbf{Z}$	*,if $m \equiv 2 \bmod 4$*

and is zero for odd m ■

Example 5.10 Let d be even. Let r be a positive real number and z be an element in $S^1 \subset \mathbf{C}$. The d-$d+1$-dimensional quantum field theory

$$H(r, z) = H : \mathcal{M}^{d,d+1} \longrightarrow \mathcal{H}$$

assigns $\mathbf{C}$ with the standard Hilbert structure to all objects and the map $r^{\chi(W)} \cdot z^{\sigma(W)}$ to a morphism $(W; W_0, W_1; f_0, f_1)$. We have to check the axioms. We obtain functoriality from the additivity formulas. Obviously H is compatible with the involutions. We will later introduce two further axioms 5.13 and 5.14. Both are satified by $H(r, z)$. ■

Notice for the quantum field theory above that the group $Diff^+(M)$ of self diffeomorphisms of the oriented closed d-manifold M acts trivially on $H(M)$. Under this assumption we cannot expect other invariants $Z(W)$ for closed orientable $d+1$-dimensional manifolds W than the Euler characteristic and the signature. Namely, we get as a corollary of lemma 5.7 and lemma 5.9

Lemma 5.11 *Let H be a d-$d+1$-quantum field theory such that for any orientable closed d-manifold the group $Diff^+(M)$ acts trivially on $H(M)$ and $Z(W) \in \mathbf{C}$ is not zero for all closed orientable $d+1$-manifolds W. Then we get for any closed orientable $d+1$-manifold W :*

$$Z(W) = Z(S^{d+1})^{(\chi(W)-\sigma(W))/2} \cdot \left(Z(\mathbf{CP^{d+1}}) \cdot Z(S^{d+1})^{-(d+1)/4}\right)^{\sigma(W)} \qquad \text{if } d+1 \equiv 0 \bmod 4$$

$$Z(W) = Z(S^{d+1})^{\chi(W)/2} \qquad \text{if } d+1 \equiv 2 \bmod 4$$

$$Z(W) = 1 \qquad \text{if } d+1 \text{ is odd} \qquad \blacksquare$$

Remark 5.12 Notice that a m-braid may be viewed as an automorphism of (S^2, m) given by S^2 with m points with positive orientation. Hence the braid group B_m embedds into the group of automorphisms of (S^2, m). Thus we obtain a representation of B_m.

There are the following two additional axioms one may or may not require. Both will be satisfied in the situations we will study, but in particular the second one is not fulfilled in other interesting cases e.g, the 3-4-quantum field theory given by the Donaldson polynomial and Floer homology.

5.13 Given three objects M, L and N in $\mathcal{M}^{d,d+1}$, there is a natural bijection

$$\tau_L : Mor(M \cup L, N) \longrightarrow Mor(M, L^- \cup N)$$

sending $(W; W_M \amalg W_L, W_N; f_M \cup f_L; f_N)$ to $(W; W_M; W_L \amalg W_N; f_M, f_L^{-1} \amalg f_N)$. Analogously, for three objects A, B and C in $\mathcal{V}$ resp.$\mathcal{H}$, there is a natural bijection

$$\tau_B : Hom(A \otimes B, C) \longrightarrow Hom(A, B^* \otimes C)$$

induced from the natural isomorphisms

$$Hom(A, Hom(B, C)) \longrightarrow Hom(A \otimes B, C)$$

and

$$B^* \otimes C \longrightarrow Hom(B, C)$$

. The axiom says that the functor H is compatible with these maps. In other words, the following square commutes

$$\begin{tikzcd}
Mor(M \cup L, N) \arrow[r, "\tau_L"] \arrow[d, "H"] & Mor(M, L^- \cup N) \arrow[d, "H"'] \\
Hom(H(M) \otimes H(L), H(N)) \arrow[r, "\tau_{H(L)}"] & Hom(H(M), H(L)^* \otimes H(N))
\end{tikzcd}$$

This axiom says in particular that it suffices to treat morphisms with the empty set as source only.

The map induced by the involution from $Mor(M,N)$ to $Mor(N^-,M^-)$ sending W to $\widehat{W}$ is the composition of τ_N, $S^2(M^-,N)$ and τ_M and analogously in the category $\mathcal{V}$ and $\mathcal{H}$, so that this axiom is an extension of the axiom that H is compatible with the involutions.

If this axiom holds, there is an obvious extension of lemma 5.7 in the case, where only a disjoint union of components of the boundaries and not necessarily the whole boundaries are glued together. $\blacksquare$

5.14 This axiom makes sense only if the quantum field theory takes values in the category $\mathcal{H}$ of Hilbert spaces. For a morphism $W := (W; W_0, W_1; f_0, f_1)$ from M to N in $\mathcal{M}^{d,d+1}$ define the morphism $W^- : M^- \longrightarrow N^-$ by $W^- := (W^-; W_0^-, W_1^-; f_0, f_1)$. Given an object M, define the isomorphism $\tau_M : H(M^-) \longrightarrow H(M)$ by the composition of the inverse of the isomorphism $H(M) \longrightarrow H(M)^*$ coming from the Hilbert structure and the natural isomorphism $T_I(M) : H(M^-) \longrightarrow H(M)^*$. The axiom requires the commutativity of the following diagram

$$\begin{tikzcd} H(M^-) \arrow[r, "H(W^-)"] \arrow[d, "\tau_M"] & H(N^-) \arrow[d, "\tau_N"'] \\ H(M) \arrow[r, "H(W)"] & H(N) \end{tikzcd}$$

Suppose that this axiom is satisfied. Then for any morphism $W : M \longrightarrow N$ the adjoint of $H(W) : H(M) \longrightarrow H(N)$ is given by $H(\widehat{W^-}) : H(N) \longrightarrow H(M)$. In particular we get for any closed oriented $d+1$-dimensional manifold W because of $\widehat{W^-} = W^-$

$$Z(W^-) = \overline{Z(W)}$$

If $f : M \longrightarrow N$ is an orientation preserving diffeomorphism of closed oriented d-manifolds, we get $\widehat{f^-} = f^{-1}$. Hence $H(f) : H(M) \longrightarrow H(N)$ is an isometry. $\blacksquare$

We will always assume that also these two axioms are satisfied unless explicitely stated differently.

Example 5.15 We look at all 0-1-quantum field theories $H : \mathcal{M}^{0,1} \longrightarrow \mathcal{H}$. Let V be the complex Hilbert space associated to the object p given by a point with a fixed orientation. Then $H(p^-)$ must be V^* and we get in general :

$$H\left(\left(\coprod_n p\right) \coprod \left(\coprod_m p^-\right)\right) = V^{\otimes n} \otimes V^{*\otimes m}$$

There is precisely one morphism $w : p \longrightarrow p^-$ and the only morphism from p^- to p is $\widehat{w} = w^-$. The induced maps $H(w) : H(p) \longrightarrow H(p^-)$ and $H(\widehat{w}) : H(p^-) \longrightarrow H(p)$ are to another inverse isometries if we equip V^* with the Hilbert structure coming from V. We use them as identifications. Now we get

$$H\left(\left(\coprod_n p\right) \coprod \left(\coprod_m p^-\right)\right) = V^{\otimes(n+m)}$$

and T_I becomes trivial. A morphism in $\mathcal{M}^{0,1}$ is a permutation of the set $\{1, 2, \cdots, n+m\}$. The induced map on $V^{\otimes n+m}$ is just given by the permutation itself. Hence a 0-1-quantum field theory is up to natural equivalence given by a complex Hilbert space V. $\blacksquare$

In order to define invariants for links, we have to enlarge our category $\mathcal{M}^{d,d+1}$ considerably. We will only consider the dimensions, we are interested in. The generalization to other dimensions is obvious. In the sequel we denote by $\underline{\mathbf{R}}$ the trivial bundle with fibre $\mathbf{R}$. Given a framing α of $\underline{\mathbf{R}} \oplus \xi$, we denote by α^- the framing obtained by composition with the bundle automorphism $(-id) \oplus id$ of $\underline{\mathbf{R}} \oplus \xi$.

5.16 We will consider the following category $\mathcal{M}$.

- An object $(P, M, \alpha_M; i, \alpha_i)$ consists of
  - an oriented 0-dimensional manifold P
  - a 2-dimensional manifold M together with framing α_M of $\underline{\mathbf{R}} \oplus TM$
  - an embedding $i : P \longrightarrow M$ together with a framing α_i of the normal bundle $\nu(i)$ such that for all $x \in P$ the framings α_M and α_i induce the same orientation on TM_x, if x has the positive orientation, and opposite orientations on TM_x otherwise.

  We will often abbreviate $(P, M, \alpha_M; i, \alpha_i)$ by (M, P). We denote by $(M, P)^-$ the object obtained from (M, L) by substituting the framing α_M by $\alpha_M{}^-$ and reversing the orientation of P. The framing α_i is unchanged.

- A morphism $(L, W, \alpha_W; k, \alpha_k; W_0, W_1; f_0, f_1)$ from $(P, M, \alpha_M; i, \alpha_i)$ to $(Q, N, \alpha_N; j, \alpha_j)$ is given by :
  - a 1-dimensional oriented manifold L
  - a 3-dimensional manifold W together with a framing α_W of the tangent bundle
  - open and closed submanifolds W_0 and W_1 of ∂W such that $W_0 \cup W_1 = \partial W$ and $W_0 \cap W_1 = \emptyset$ holds. Notice that $\underline{\mathbf{R}} \oplus T\partial W$ inherits a framing from TW using the outward normal.
  - open and closed submanifolds L_0 and L_1 of ∂L with $L_0 \cup L_1 = \partial L$ and $L_0 \cap L_1 = \emptyset$.

- diffeomorphisms $f_0 : M \longrightarrow W_0$ and $f_1 : W_1 \longrightarrow N$ such that $\alpha_M^- = f_0^* \alpha_{W_0}$ and $\alpha_{W_1} = f_1^* \alpha_N$ holds
- an embedding $k : L \longrightarrow W$ together with a framing α_k of the normal bundle $\nu(k)$ with the following properties: The orientations of TL_x, $\nu(k)_x$ and TW_x given by assumption or by the framings match up for all $x \in L$. The map f_0 satisfies $f_0^* \alpha_k = \alpha_P$ and induces an orientation preserving diffeomorphism from $i(P)$ to $k(L) \cap W_0$ and analogously at the other end W_1.

If $(L, W, \alpha_W; k, \alpha_k; W_0, W_1; f_0, f_1)$ and $(L', W', \alpha_{W'}; k', \alpha_{k'}; W_0', W_1'; f_0', f_1')$ are two morphisms from $(P, M, \alpha_M; i, \alpha_i)$ to $(Q, N, \alpha_N; j, \alpha_j)$, they will be identified if the following exists :

- diffeomorphisms $g : L \longrightarrow L'$ and $G : W \longrightarrow W'$ with $G \circ k = k' \circ g$, $F \circ f_0 = f_0'$ and $f_1' \circ F \mid_{W_1} = f_1$
- an ambient isotopy relative boundary ι between the embeddings $G \circ k$ and $k' \circ f$.
- an isotopy relative boundary of the framings α_W and $G^* \alpha_{W'}$
- There is an isotopy relative boundary of the framings α_k and $\iota^* \alpha_{k'}$ of the embedding k, where $\iota^* \alpha_{k'}$ comes from g,G and ι and the framing $\alpha_{k'}$

We will often write shortly (W, L) for a morphism.

- Composition is given by glueing. The identity morphism of $(P, M, \alpha_M; i, \alpha_i)$ is defined by crossing with the unit interval
- the monoidal structure is given by the disjoint union and the involution sending a morphism $(W, L) : (M, P) \longrightarrow (N, Q)$ to $\widehat{(W, L)} : (N, Q)^- \longrightarrow (M, P)^-$ is given by reversing the bordism

■

Notation 5.17 *In the sequel* a quantum field theory *will be a functor of monoidal categories with involution*

$$H : \mathcal{M} \longrightarrow \mathcal{H}$$

such that the analogues of the axioms 5.13 and 5.14 are satisfied. ■

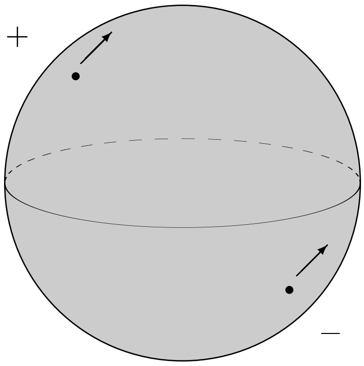
Remark 5.18 Our goal is to construct an invariant for oriented links in oriented homology 3-spheres using a quantum field theory so that it is natural to invoke links in the category $\mathcal{M}$. The choice of an isotopy class of framings for the links is needed in the explicit construction of a quantum field theory. In the comparitively easy case of an abelian theory certain integrals appear, which just give the linking number of two disjoint knots (see lemma **??**). But the self linking number of a knot is only defined, if one has specified an isotopy class of framings. Since the links have to have isotopy classes of framings, the 3-manifolds appearing in a morphism should also come with an isotopy class of framings. As composition is given by glueing, we are forced to put on the objects an actual framing and not only an isotopy class. Namely, it is not true for 3 manifolds W and V with $\partial W = \partial V$ that isotopy classes of framings on V and W which restrict to the same isotopy class of stable framings on the common boundary determine an isotopy class of framings on $W \cup V$. The reason is that the homotopy category of spaces has no push outs. Here is a counterexample. Take V and W to be the lower and upper hemisphere in $S^3 = V \cup W$. There is precisely one isotopy class of framings on the contractible spaces V and W, but the isotopy classes of framings on S^3 are in bijective correspondence to the set of homotopy classes of maps from S^3 to $GL(3, \mathbf{R})$ which is $\mathbf{Z} \times \{\pm 1\}$. ■

We will now discuss which properties a quantum field theory has to satisfy in order to give a skein invaraint. We specify the following objects and morphisms in $\mathcal{M}$*. In the sequel we equip* S^3 *and* D^3 *with the standard framings. We put on* $S^2 = \partial D^3$ *the induced stable framing of* $\underline{\mathbf{R}} \oplus TS^2$.

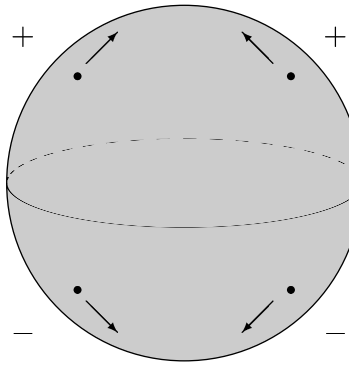
We use from now on the following convention. Let $k : A \longrightarrow B$ *be an embedding of an oriented manifold into a framed manifold such that* $dim(B) = dim(A) + 2$ *holds. Then a choice of a non-vanishing section of the normal bundle* $\nu(k)$ *determines a framing of it and vice versa. Given a framing* $\alpha : \underline{\mathbf{R}}^2 \longrightarrow \nu(k)$*, we obtain a section of* $\nu(k)$ *by composing the constant section of* $\underline{\mathbf{R}}^2$ *with value the first element of the standard bases. Given a section* s *of* $\nu(k)$*, we obtain a second linearly independent section* $\widehat{s}$ *and hence a framing by the following construction. The framing on* TB *induces an orientation and a Riemannian metric on* TB *and by means of the orientation of* A *also on* $\nu(k)$*. Given any vector* v *in the fibre* $\nu(k)_x$ *at* x*, there is precisely one vector* $\widehat{v}$*, such that the norm of* $\widehat{v}$ *is* 1*,* v *and* $\widehat{v}$ *are orthogonal and* $\{v, \widehat{v}\}$ *agrees with the orientation. Now define* $\widehat{s}(x)$ *for* $x \in A$ *by* $\widehat{s(x)}$*. This gives a bijective correspondence between the isotopy classes of non-vanishing sections and isotopy classes of framings of the normal bundles* $\nu(k)$*. We will illustrate framings of points in* 2*-manifolds resp. links in* 3*-manifolds in pictures by drawing a tangent vector resp. a parallel curve which indicates a non-vanishing section.*

Next we define two objects $(S^2, 2)$ *and* $(S^2, 4)$ *in* $\mathcal{M}$*. The underlying stably framed* 2*-manifold is in both cases the* 2*-dimensional sphere* S^2*. The embeddings of* 0*-dimensional manifolds are given by* 2 *resp.* 4 *oriented points on* S^2 *together with explicit framings of the normal bundles. The objects are illustrated by the following pictures using the conventions above.*

5.19 Objects



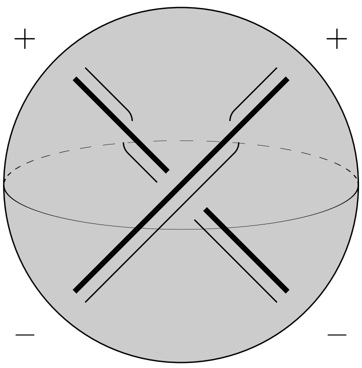
$(S^2, 2)$



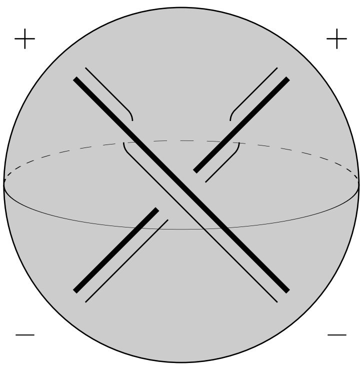
$(S^2, 4)$

Next we define three morphism D^3_+, D^3_- and D^3_0 from $\emptyset$ to $(S^2, 4)$ in $\mathcal{M}$. The underlying framed bordism is D^3 in all cases. The embedded 1-dimensional manifolds together with the framing of their normal bundles is indicated below

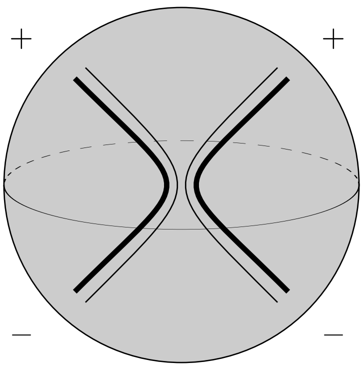
5.20 Morphisms



D^3_+



D^3_-



D^3_0

The following observation will be important for the sequel. We define diffeomorphisms

5.21 $\omega : S^2 \longrightarrow S^2$
$\Omega : D^3 \longrightarrow D^3$

satisfying $\omega = \Omega \mid_{S^2}$. The diffeomorphism Ω is height preserving and is given by a rotation about the angle $\phi(t)$ on the level of height $t \in [1, 1]$, where $\phi(t)$ is zero for $t \geq 0$, is $-2\pi t$

for $-1/2 \leq t \leq 0$ and is π for $-1 \leq t \leq -1/2$. The framings of the normal bundles at the four points specified in the definition of the object S^2 are respected by the differential of ω. The pull back of the standard framing of $TS^2 \oplus \mathbf{R}$ is isotopic to its pull back with ω by an explicit isotopy induced from the obvious isotopy between id and ω. Such an isotopy may be viewed as an (unstable) framing on the trivial bordism from $S^2 \times [0,1]$. These data define a morphism with the trivial bordism as underlying bordism

5.22 $\tilde{\omega} : (S^2, 4) \longrightarrow (S^2, 4)$

One easily checks using the extension Ω of ω

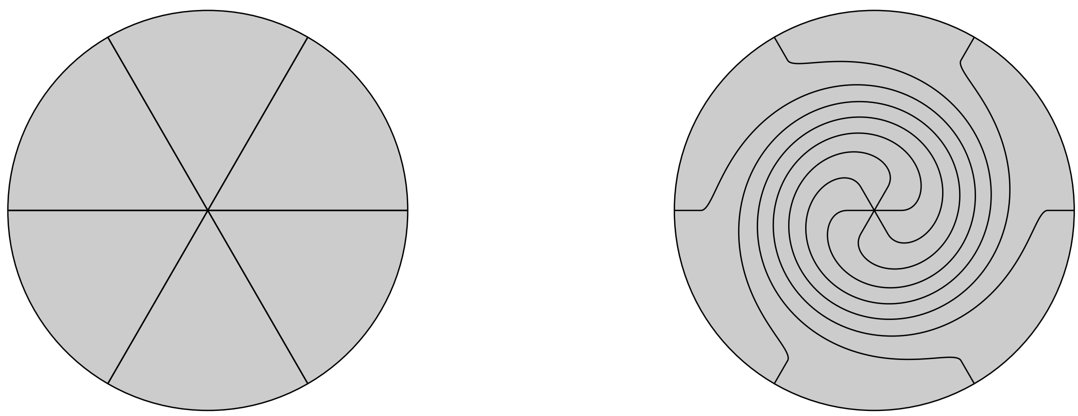
Lemma 5.23 *We have the following identities of morphisms from $\emptyset$ to $(S^2, 4)$:*
$\tilde{\omega} \circ D^3_+ = D^3_0$
and
$\tilde{\omega} \circ D^3_0 = D^3_-$ ■

Define a diffeomorphism

5.24 $\delta : S^2 \longrightarrow S^2$

by a local Dehn twist at the positive point of the two marked points in S^2. This leaves this point and the complement of a small neighbourhood of it fixed and looks within this neighbourhood as indicated below

5.25 Local Dehn twist



Notice that the differential of δ at the positive point is the identity, so that δ respects the framing of the normal bundles. Moreover, there is an explicit isotopy relative the positive point (but not relative to the positive point and the differential at this point) between δ and id. It induces an explicit isotopy between the standard framing of $TS^2 \oplus \mathbf{R}$ and its pull back with δ. Hence we obtain a morphism

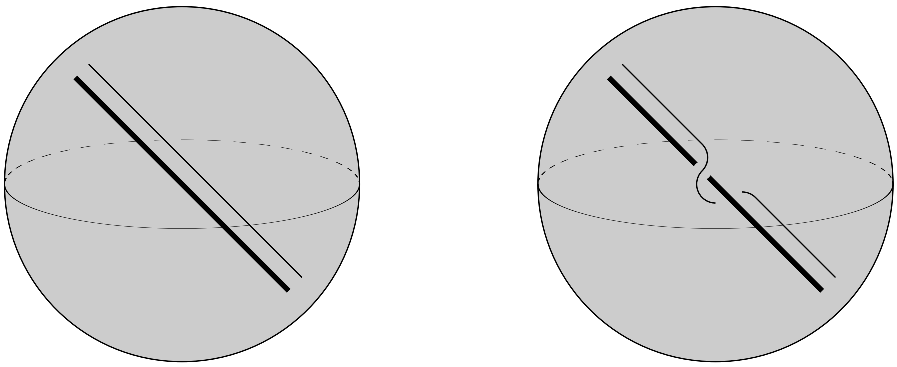
5.26 $\widetilde{\delta} : (S^2, 2) \longrightarrow (S^2, 2)$

There are morphisms from $\emptyset$ to $(S^2, 2)$ for $i = 0, 1$

5.27 $\gamma_i : \emptyset \longrightarrow (S^2, 2)$

indicated by the pictures below

5.28



γ_1 γ_2

One easily checks

Lemma 5.29 *We have $\widetilde{\delta} \circ \gamma_1 = \gamma_2$* ■

Now we make the following assumption on our quantum field theory

Assumption 5.30 *$dim(H(S^2, 2)) = 1$ and $dim(H(S^2, 4)) = 2$* ■

Let L be a link in a framed 3-manifold W together with a framing (of its normal bundle) meeting the boundary in an appropiate way. Then we obtain a morphism

$$(W, L) : \emptyset \longrightarrow (\partial W, L \cap \partial W)$$

and hence an invariant 5.5

$$Z(W, L) \in H(\partial W, L \cap \partial W)$$

Recall that this is just a complex number and depends only on the isotopy class of the framings, provided that W is closed. Now suppose we have three such pairs (W, L_+), (W, L_-) and (W, L_0) and there is an embedded 3-ball D^3 in W such that the framed links are identically outside of D^3 and look in D^3 as in the definition 5.20 of D^3_+, D^3_- and D^3_0.

Lemma 5.31 *Let det and trace be the determinant and the trace of the linear endomorphism $H(\omega)$ of $H(S^2, 4)$. Then we get :*

$$det \cdot Z(W, L_+) + Z(W, L_-) - trace \cdot Z(W, L_0) = 0 \qquad \blacksquare$$

Proof : *We derive from lemma 5.7, if (W^c, L^c) is the morphism from $\emptyset$ to $(S^2, 4)$ given by the complement of D^3 in W :*

$< Z(W^c, L^c), Z(W, L_+) >_{id} = Z(W, L_+)$
$< Z(W^c, L^c), Z(W, L_-) >_{id} = Z(W, L_-)$
$< Z(W^c, L^c), Z(W, L_0) >_{id} = Z(W, L_0)$

Since the dimension of $H(S^2, 4)$ is two by assumption 5.30, we get for the characteristic polynomial p of the endomorphism $H(\omega)$

$p(x) = x^2 - trace \cdot x + det$

We get zero, if we put $H(\omega)$ into its characteristic polynomial. We derive from lemma 5.23

$Z(W, L_-) - trace \cdot Z(W, L_0) + det \cdot Z(W, L_+) = 0$

and the claim follows. $\blacksquare$

Next we have to check the dependency on the framing. Let L be an oriented link in 3-manifold M with two framings α_L and α'_L such that the orientations of TL_x, $\nu(L \subset M)_x$ and TM_x match up for all $x \in L$. The composition $\alpha_L^{-1} \circ \alpha'_L$ is a framing of the trivial bundle $\underline{\mathbf{R}}$ over L compatible with the standard orientation. Isotopy classes of such framings are in bijective correspondence with homotopy classes of maps $L \longrightarrow GL(2, \mathbf{R})^+$. Since $\pi_1(GL(2, \mathbf{R})^+)$ is $\mathbf{Z}$ and L is oriented, this can be identified with $\mathbf{Z}^{r(L)}$, where $r(L)$ is the number of components of L. The sum of the components is the total relative framing number *and denoted by :*

5.32 $d(\alpha_L, \alpha'_L) \in \mathbf{Z}$

Lemma 5.33 *Let L be a link with framings α_L and α'_L in a closed 3-manifold W with framing α_W. Then the complex number $Z(W, L)$ depends only on the isotopy classes of framings. Let c be the complex number for which the endomorphism $H(\widetilde{\delta})$ of the 1-dimensional vector space $H(S^2, 2)$ is given by $c \cdot id$. Then we get :*

$$Z(W, \alpha_W, L, \alpha'_L) = c^{d(\alpha_L, \alpha_{L'})} \cdot Z(W, \alpha_W, L, \alpha_L)$$

Proof : *Suppose that α'_L is obtained from α_L in the following way. There is an embedded disk D^3 in W such that the intersection of L and D^3 looks like γ_1 as indicated in 5.28. Now α'_L is obtained from α_L by taking out γ_1 and plugging in γ_2. We get from lemma 5.7:*

$< Z(W^c, L^c), Z(\gamma_1) >= Z(W, L, \alpha_L)$
$< Z(W^c, L^c), Z(\gamma_2) >= Z(W, L, \alpha'_L)$

Then we get from lemma 5.29 :

$Z(\gamma_2) = c \cdot Z(\gamma_1)$

This implies the claim in this particular case. The general case is obtained by an iteration of this special case, because any two isotopy classes of framings of L can be transformed into one another by a sequence of operations of the type above. ■

Next we consider framed links in a framed oriented homology 3-sphere M. Let S be a Seifert surface for L. Then the outward normal field of S at the boundary induces a framing on L. We call such a framing a Seifert framing . *We claim that the isotopy class of this framing is independent of the choice of the Seifert surfaces. Recall that isotopy classes of framings of L are in bijective correspondence with $\mathbf{Z}^{r(L)}$. Namely, a framing determines a non-vanishing section of the normal bundles and hence a push-off L^p of the link L into the link complement. For each component L_i we obtain an integer by the linking number $n(L_i) := link(L_i^p, L_i)$. Let $l(i,j)$ be the linking number $link(L_i, L_j)$. Consider the following* linking matrix

5.34

$$A_{link} = \begin{pmatrix} n(1) & l(1,2) & l(1,3) & \cdots & l(1,r) \\ l(1,1) & n(2) & l(2,3) & \cdots & l(2,r) \\ l(3,1) & l(3,2) & n(3) & \cdots & l(3,r) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l(r,1) & l(r,2) & l(r,3) & \cdots & n(r) \end{pmatrix}$$

Lemma 5.35 *The sum of the elements in any column of the linking matrix is zero*

Proof : *Fix $1 \leq i \leq r$. Let S be a Seifert surface for L. Then L_i^p does not meet the Seifert surface. Hence its linking number with L, i.e. $\sum_i^r link(L_i^p, L_i)$ vanishes. But $link(L_i^p, L_j)$ is $n(i)$ for $i = j$ and $l(i,j)$ otherwise.* ■

Let L be an oriented link in S^3. Equip S^3 with the standard framing and L with a Seifert framing. Then we obtain a morphism $(S^3, L) : \emptyset \longrightarrow \emptyset$. Define :

5.36 $\gamma(L) = Z(S^3, L) \in \mathbf{C}$

Notice that $\gamma(L)$ is an invariant of the ambient isotopy class of the link L in S^3. We claim that this is a skein invariant.

Lemma 5.37 *Let det and trace be the determinant and the trace of the linear endomorphism $H(\omega)$ of the 2-dimensional vector space $H(S^2, 4)$. Denote by c the complex number for which the endomorphism $H(\widetilde{\delta})$ of the 1-dimensional vector space $H(S^2, 2)$ is given by $c \cdot id$. Define*

$l(H) = c \cdot det^{1/2}$
$m(H) = -trace \cdot det^{-1/2}$

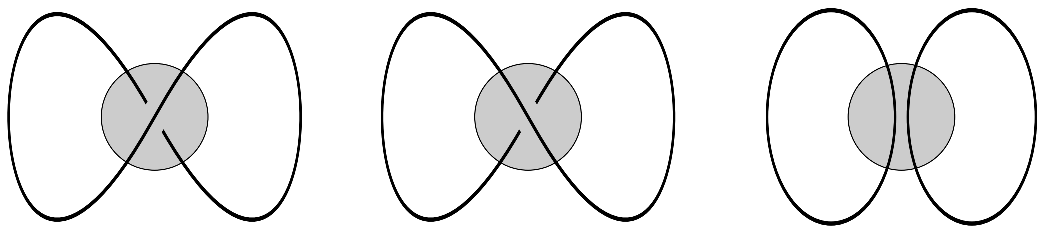
Then γ defined in 5.36 is a skein invariant with skein coefficients $l(H), l(H)^{-1}, m(H)$.

Proof : *Consider a skein triple (L_+, L_-, L_0) of oriented links in the oriented homology 3-sphere M. Equip them with framings α_+, α_- and α_0 such that α_- is a Seifert framing and the assumptions in lemma 5.31 are satisfied, i.e. there is an embedded 3-ball D^3 in W such that the framed links are identically outside of D^3 and look in D^3 as in the definition 5.20 of D^3_+, D^3_- and D^3_0. Hence we have :*

$$det \cdot Z(W, L_+) + Z(W, L_-) - trace \cdot Z(W, L_0) = 0$$

Denote by α^s_+, α^s_- and α^s_0 Seifert framings of L_+,L_- and L_0. We want to determine the total relative framing numbers $f(\alpha_+, \alpha^s_+)$, $f(\alpha_-, \alpha^s_-)$ and $f(\alpha_0, \alpha^s_0)$ defined in 5.32. We have by assumption $\alpha_- = \alpha^s_-$ so that $f(\alpha_-, \alpha^s_-)$ is zero. We may suppose that we have Seifert surfaces S_+, S_- and S_0 for L_+,L_- and L_0 which agree outside the embedded D^3. Hence we can assume that the Seifert framings of the links agree outside D^3. By assumption the framings α_+, α_- and α_0 agree outside D^3. Hence the differences $f(\alpha_+, \alpha^s_+) - f(\alpha_-, \alpha^s_-)$ and $f(\alpha_0, \alpha^s_0) - f(\alpha_-, \alpha^s_-)$ do not depend on the links outside D^3. Therefore we can compute these numbers for one specific example and get the common answer for all cases. We derive from

5.38



the relations :

$f(\alpha_+, \alpha^s_+) - f(\alpha_-, \alpha^s_-) = 2$
$f(\alpha_0, \alpha^s_0) - f(\alpha_-, \alpha^s_-) = 1$

Hence γ is a skein invariant with skein coefficients $c^2 \cdot det, 1, -c \cdot trace$ by lemma 5.31 and lemma 5.33. Now the claim follows. ∎

Theorem 5.39 *Suppose that there are quantum field theories $H(n, k)$ indexed by positive integers n and k such that any complex polynomial $p(x, y)$ in two variables with the property* *that* $p(l(H(n,k)),m(H(n,k)))$ *vanishes for all* k *and* n*, is zero and assumption 5.30 is satisfied. Then there is a skein invariant as described in lemma 4.13,namely*

$J : \{$ *isotopy classes of oriented links in* $S^3\} \longrightarrow \mathbf{Z}[l,l^{-1},m,m^{-1}]$

with l,l^{-1},m *as skein coefficients.*

Proof : *One proves by induction over the complexity of a link in* S^3 *the unique existence of a polynomial* $J(L)(l,m)$ *such that* $J(L)(l(H(n,k)),m(H(n,k))) = \gamma_{H(n,k)}(L)$ *holds for all* n,k. $\blacksquare$

6.section
The construction of quantum field theories

In this section we sketch the construction of appropiate quantum field theories $H(n,k)$ *which give the Jones polynomial as described in lemma 5.37 and theorem 5.39. The index* $k \geq 2$ *will be called the* level *and* n *will parametrize the underlying family of Lie groups, namely, the family* $SU(n)$*.*

We have to enlarge our category $\mathcal{M}$ *defined in 5.16 as follows. Namely, we require for an object that to each element in the* 0*-dimensional manifold* P *we have assigned a* $SU(n)$*-representation* V*. Similarly we demand for morphism that we have attached to each component of the link* L *a representation* V *which agrees with the given representation at the positive end and with the dual of the given representation at the negative end. In all explicite objects appearing in the last section we require that we have assigned the* n*-dimensional canonical representation* $\mathbf{C}^n$ *with the obvious* $SU(n)$*-action to points with positive orientation and the dual representation* $(\mathbf{C}^n)^*$ *to points with negative orientation. The representations attached to the components of the links appearing in the explicite morphisms are always* $\mathbf{C}^n$*. In view of example 5.15 this means that we are coupling a* 0-1*-quantum field theory and a* 2-3*-quantum field theory in the sequel.*

We start with the construction of the Hilbert space $H(M)$ *assigned to an object in the case, where the* 0*-dimensional submanifold* P *is empty. Then the object consists of a closed* 2*-manifold with a framing of* $\underline{\mathbf{R}} \oplus TM$*. The construction is done in several steps summarized as follows.*

6.1

1. The framing of $\underline{\mathbf{R}} \oplus TM$ induces a Riemannian metric and an orientation on M.
2. The Riemannian metric on M defines a conformal structure on M.
3. The conformal structure determines an almost complex structure on TM.
4. The almost complex structure induces a holomorphic structure on M by the theorem of Nirenberg and Neulander.
5. Given the holomorphic structure on M, there is a moduli space $\mathcal{MODB}$ of stable holomorphic $SU(n)$ bundles which are topologically trivial. This space turns out to be a complex Kähler manifold.
6. There is a family of $\overline{\partial}$-operators parametized by $\mathcal{MODB}$.
7. Associated to such family is the determinant line bundle $det(\overline{\partial})$ constructed by Quillen. This is a holomorphic vector bundle over $\mathcal{MODB}$. It possess a Riemannian metric.
8. Define $H(M)$ to be the finite-dimensional vector space of holomorphic sections of $\otimes_k det(\overline{\partial})$. ■

We make some comments on the items of the list 6.1 :

1.) A framing induces a Riemannian metric and an orientation and a subbundle inherits a Riemannian metric and an orientation.

2.) A conformal structure *on a manifold is an equivalence class of Riemannian metrics. Two Riemannian metrics $\langle\ ,\ \rangle^1$ and $\langle\ ,\ \rangle^2$ are conformally equivalent if there is a function $f : M \longrightarrow \mathbf{R}$ such that for all $x \in M$ and $v, w \in TM_x$ we have :*

$$\langle v, w\rangle_x^1 = f(x) \cdot \langle v, w\rangle_x^2$$

Hence two Riemannian metrics give the same conformal structure if and only if they define the same angles between tangent vectors.

3.) An almost complex structure *on TM is a bundle isomorphism $J : TM \longrightarrow TM$ over $id : M \longrightarrow M$ such that $J \circ J = -id$ holds and for each $v \in TM_x$ for $x \in M$ the set $\{v, J(v)\}$ is a bases consistent with the orientation of M. Given a Riemannian metric $\langle\ ,\ \rangle$ and $v \in TM_x$ for $x \in M$, define $J(v) \in TM_x$ to be the tangent vector uniquely determined by the properties that $\{v, J(v)\}$ is a orthogonal bases of TM_x corresponding to the orientation of M and $\langle v, v\rangle$ and $\langle J(v), J(v)\rangle$ agree. Then J is an almost complex structure and depends only on the conformal structure determined by the Riemannian metric $\langle\ ,\ \rangle$.*

Suppose, we are given an almost complex structure J. Fix a covering $U = \{U_i \mid i \in I\}$ of M such that TM restricted to any U_i is trivial. Choose for any $i \in I$ a nowhere-vanishing section s of $TM\mid_{U_i}$. Let $\langle\ ,\ \rangle_i$ be the Riemannian metric on $TM\mid_{U_i}$ for which $\{s(x), J(s(x))\}$ is an orthonormal bases of TM_x for all $x \in M$. Choose a partition $\{e_i \mid i \in I\}$ of unity subordinate to the open covering U. Let $\langle\ ,\ \rangle$ be the Riemannian metric $\sum_{i\in I} e_i \cdot \langle\ ,\ \rangle_i$. One easily checks that the conformal class of this Riemannian metric does only depend on J and that these two constructions give to another inverse bijections between the set of conformal structures on M and the set of almost complex structures on M.

4.) Let J be an almost complex structure on M. Extend J to an automorphism of $TM \otimes \mathbf{C}$, also denoted by J. As J^2 is $-id$, the eigenspaces of the eigenvalues i and $-i$ of J give a decomposition

$$TM \otimes \mathbf{C} = TM' \oplus TM''$$

This decomposition is orthogonal with respect to any unitary Riemannian metric, i.e. a Riemannian metric for which J is isometric. We obtain a decomposition of the complexified dual tangent bundle

$$T^*M \otimes \mathbf{C} = \Lambda^{1,0}M \oplus \Lambda^{0,1}M$$

and thus a decomposition :

$$\Lambda^n T^*M \otimes \mathbf{C} = \oplus_{p+q=n} \Lambda^{p,q}M$$

if we put $\Lambda^{p,q}M = \Lambda^p(\Lambda^{1,0}) \otimes \Lambda^q(\Lambda^{0,1}M)$ for $n \geq 0$. The exterior differental d induces :

$$\begin{aligned} \partial^{p,q} &: C^\infty(\Lambda^{p,q}M) \longrightarrow C^\infty(\Lambda^{p+1,q}M) \\ \overline{\partial}^{p,q} &: C^\infty(\Lambda^{p,q}M) \longrightarrow C^\infty(\Lambda^{p,q+1}M) \end{aligned}$$

Now suppose that M is holomorphic. In local coordinates $z_1, z_2, \ldots z_m$ define for $z_i = x_i + i \cdot y_i$:

$$\frac{\partial}{\partial z_i} = \frac{1}{2} \cdot \left(\frac{\partial}{x_i} - \frac{\partial}{y_i} \right) \qquad \frac{\partial}{\partial \overline{z_i}} = \frac{1}{2} \cdot \left(\frac{\partial}{x_i} + i \cdot \frac{\partial}{y_i} \right)$$

$$dz_i = \frac{1}{2} \cdot (dx_i - i \cdot dy_i) \qquad d\overline{z}_i = \frac{1}{2} \cdot (dx_i + i \cdot dy_i)$$

For $f : M \longrightarrow \mathbf{C}$ we define :

$$\partial(f) = \sum_i \frac{\partial f}{\partial z_i} \cdot dz_i$$

$$\overline{\partial}(f) = \sum_i \frac{\partial f}{\partial \overline{z}_i} \cdot d\overline{z}_i$$

Define complex subbundles of $T^*M \otimes \mathbf{C}$:

$$TM' = span\left\{ \frac{\partial}{\partial z_i} \right\} \qquad TM'' = span\left\{ \frac{\partial}{\partial \overline{z}_i} \right\}$$

$$\Lambda^{1,0} M = span\{dz\} \qquad \Lambda^{0,1} M = span\{d\overline{z}_i\}$$

Moreover we get operators :

$$\partial : C^\infty(M) \longrightarrow C^\infty(\Lambda^{1,0} M) \qquad \overline{\partial} : C^\infty(M) \longrightarrow C^\infty(\Lambda^{0,1} M)$$

These bundle maps are invariantly defined and independent of the coordinate charts. An almost complex structure on TM is given by :

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} \qquad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}$$

Notice that the definitions of TM', TM'', $\Lambda^{1,0}M$, $\Lambda^{0,1}M$, $\partial : C^\infty(M) \longrightarrow C^\infty(\Lambda^{1,0}M)$ and $\overline{\partial} : C^\infty(M) \longrightarrow C^\infty(\Lambda^{0,1}M)$ for holomorphic M agree with the one for J, if J is the almost complex structure induced from the holomorphic structure.

Theorem 6.2 (Nirenberg-Neulander) *The following assertions are equivalent for an almost complex structure J on TM :*

1. *J arises from a holomorphic structure.*
2. $d = \partial + \overline{\partial}$
3. $\overline{\partial} \circ \overline{\partial} = 0$

If M is 2-dimensional, $\overline{\partial} \circ \overline{\partial} = 0$ holds always for dimension reasons. Hence any almost complex structure J comes from a holomorphic structure on M. If two holomorphic structures define the same almost complex structure, they agree. This follows from the fact that a diffeomorphism $f : U \longrightarrow V$ of open subsets of $\mathbf{C}^n$ is holomorphic if and only if its differential is compatible with the almost complex structures on U and V coming from the standard almost complex structure on $\mathbf{C}^n$. Hence for surfaces we get a bijective correspondence between almost complex structures and holomorphic structures.

5.) Let M be a surface with holomorphic structure. Let ξ be a holomorphic vector bundle. It is called stable if for any proper subbundles η we have :

$$\frac{rank(\xi)}{degree(\xi)} < \frac{rank(\eta)}{degree(\eta)}$$

The degree of ξ is the evaluation of the first chern class of the line bundle $\Lambda^{rank(\xi)}\xi$ on the fundamental class of M. Let $\mathcal{MODB}$ be the moduli space of stable holomorphic $SU(n)$-bundles which are topologically trivial. This space is a projective algebraic variety. In particular it is a holomorphic manifold and is Kähler, *i.e. there is a unitary metric on M such that the associated the Kähler 2-form*

$$\Omega(v,w) = -\langle v, J(w)\rangle$$

is closed. We mention that a holomorphic manifold M is Hodge, *if and only if it admits a Kähler metric such that the Kähler form Ω is $c_1(\xi)$ for some line bundle ξ. Obviuosly $\mathbf{C}P^n$ is Hodge and hence any algebraic variety is Hodge. The converse is also true (see Gilkey [14] remark 3.6.3.)*

The moduli space $\mathcal{MODB}$ can be identified with the moduli space $\mathcal{MODC}$ of flat connections modulo gauge transformations on the trivial $SU(n)$-bundle over M.

6.) We have introduced the notion of the $\overline{\partial}$ operator on a holomorphic manifold M. If ξ is a holomorphic vector bundle over M, we can consider a ξ-twisted version :

$$\overline{\partial} : C^\infty(\xi) \longrightarrow C^\infty(\Lambda^{0,1}TM \otimes \xi)$$

If ξ is trivial, $C^\infty(\xi)$ is a direct sum of copies $C^\infty(M)$ and $C^\infty(\Lambda^{0,1}M \otimes \xi)$ is a direct sum of copies of $C^\infty(\Lambda^{0,1}M)$. Define for trivial ξ the operator $\overline{\partial}$ by a matrix of operators $\overline{\partial} : C^\infty(M) \longrightarrow C^\infty\Lambda^{0,1}M$. This generalizes to arbitrary ξ using local cooodinates.

Hence the moduli space of stable holomorphic topologically trivial vector bundles $\mathcal{MODB}$ of rank n parametrizes a family of operators

$$\overline{\partial} : C^\infty(\underline{\mathbf{C}^n}) \longrightarrow C^\infty(\Lambda^{0,1}M \otimes \underline{\mathbf{C}^n})$$

if $\underline{\mathbf{C}^n}$ denotes in this case the trivial C^∞-vector bundle over M. The kernel of $\overline{\partial}$ is just the vector space of holomorphic sections of the corresponding holomorphic bundle.

7.) Given such an operator

$$\overline{\partial} : C^\infty(\underline{\mathbf{C}^n}) \longrightarrow C^\infty(\Lambda^{0,1}M \otimes \underline{\mathbf{C}^n})$$

define its determinant line $det(\overline{\partial})$ to be

$$det(\overline{\partial}) = \Lambda^{max} ker(\overline{\partial})^* \otimes \Lambda^{max} coker(\overline{\partial})$$

where $\Lambda^{max}V$for V a finite-dimensional vector space V is $\lambda^{dim(V)}V$. This definition makes sense as $\overline{\partial}$ is an Fredholm operator. Thus we get for any element in the moduli space $\mathcal{MODB}$ a complex line. These fit together to the determinant line bundle *of the family of $det(\overline{\partial})$ indexed by $\mathcal{MODB}$:*

$$det(\overline{\partial}) \downarrow \mathcal{MODB}$$

Next we make some comments in the case where the 0-dimensional submanifold P is non-trivial. Then one has to modify the moduli space in such a way that the holomorphic vector bundles are related at the marked points to the specified representations. The representations at the marked points determine so called parabolic structures on the bundles under consideration and one divides out not the full gauge group, but the group of gauge transformation respecting these extra structures. The corresponding moduli spaces have been developed by Seshradi. In terms of representation theory resp. flat connections one studies representations $\pi_1(N - \{p_1, \dots p_r\}) \longrightarrow G$ whose monodromy around the marked points, i.e. the image of a small loop around the marked point, lie in a given conjugacy class of elements of order k in G which are given by the representations attached to the marked points.

The effect one wants to have is the following :

Lemma 6.3 *Consider S^2 with a positively oriented points p_1^+, p_2^+, $\dots p_a^+$ and b negatively oriented points p_1^-, p_2^-, $\dots p_b^-$ and representations V_1^+, V_2^+, $\dots V_a^+$ and V_1^-, V_2^- $\dots V_b^-$. Then the Hilbert space assigned to the object determined by this data is, where for $a = 0$ or $b = 0$ the tensor product over an empty set is defined to be $\mathbf{C}$:*

$$\left(V_1^+ \otimes V_2^+ \otimes \dots V_a^+ \otimes V_1^- \otimes V_2^- \otimes \dots V_b^-\right)^G \qquad \blacksquare$$

Lemma 6.4 *We obtain for the dimension $dim = dim(H(X))$ of the Hilbert space assigned to the following objects :*

1. *S^2 with no marked points :*

   $dim = 1$

2. *S^2 with two marked points p^+ and p^- and irreducible representations V and W :*

   $dim = 1$ *if V and W^* are linearly isomorphic*
   $dim = 0$ *otherwise*

3. *S^2 with two positively oriented points and two negatively oriented points and representation $\mathbf{C}^n$ in the positive and $(\mathbf{C}^n)^*$ in the negative case :*

   $dim = 2$

Proof : *1.) The moduli space reduces to a point and hence the determinant line bundle becomes the Hilbert space $\mathbf{C}$ over a point.*

2.) We get from lemma 6.3 that the Hilbert space is given by :

$$Hom_{SU(n)}(V, W^*)$$

By Schur's lemma this is zero if V and W^* are not linearly isomorphic, and a skew field over $\mathbf{C}$ and hence $\mathbf{C}$ itself if V and W^* are linearly isomorphic.

3.) We get from lemma 6.3 that the Hilbert space is given by :

$$Hom_{SU(n)}(\mathbf{C}^n \otimes \mathbf{C}^n, \mathbf{C}^n \otimes \mathbf{C}^n)$$

Now we have the decomposition :

$$\mathbf{C}^n \otimes \mathbf{C}^n = Sym(\mathbf{C}^n) \oplus Alt(\mathbf{C}^n)$$

As $Sym(\mathbf{C}^n)$ and $Alt(\mathbf{C}^n)$ are irreducible and not isomorphic, we get from Schur's lemma :

$Hom_{SU(n)}(\mathbf{C}^n \otimes \mathbf{C}^n, \mathbf{C}^n \otimes \mathbf{C}^n) =$

$Hom_{SU(n)}(Sym(\mathbf{C}^n), Sym(\mathbf{C}^n)) \oplus Hom_{SU(n)}(Alt(\mathbf{C}^n), Alt(\mathbf{C}^n)) = \mathbf{C} \oplus \mathbf{C}$

and the claim follows. ■

Now we have shown that the quantum field theory constructed above satisfies the asumption 5.30. Further explicit computations prove that the condition appearing in theorem 5.39 are satisfied.

Next we deal with morphisms in the category $\mathcal{M}$ and the maps they induce on the associated Hilbert spaces. We only consider the case of a morphism from $\emptyset$ to $\emptyset$. Hence we have to assign a complex number to a closed framed 3-manifold W together with a framed link $L \subset W$ together with a choice of representations for each component of the link. Some preparations are needed.

Let G be a compact Lie group and $p : E \longrightarrow B$ a G-principal bundle over a manifold B. A connection *on p is a 1-form on E with values in the Lie algebra LG of G*

$$\Theta \in \Lambda^1(E; LG)$$

with the following properties

- *For all $x \in E$ we have $\Theta_x \circ \nu_x = id$, where $\nu_x : LG = T_1G \longrightarrow T_xE$ is the differential at 1 of the map $G \longrightarrow E$ sending g to gx.*
- *$R_g^*\Theta = ad(g)_*\Theta$, where $ad : G \longrightarrow End(LG)$ is the adjoint representation.*

Notice that we obtain a horizontal subspace $H_x \in T_xE$ for $x \in E$ by $ker(\Theta_x)$ because of the first condition. Horizontal means that $T_xp : T_xE \longrightarrow T_{p(x)}B$ induces an isomorphism $H_x \longrightarrow T_{p(x)}B$. The second condition ensures that $R_g^*H_x = H_{gx}$ holds for $g \in G$ and $x \in E$. Thus a connection is the infinitesimal version of parallel transport *. Namely, for any path $w : I \longrightarrow B$ and $v \in E_{w(0)}$ the connection defines a lift $\widetilde{w} : I \longrightarrow E$ of w satisfying $\widetilde{w}(0) = v$. Hence we obtain an isomorphism, the* parallel transport along *w :*

$$tp_\Theta(w) : E_{w(0)} \longrightarrow E_{w(1)}$$

The curvature *$\Omega = \Omega_\Theta$ is the 2-form with values in LG defined by :*

$$\Omega = d\Theta + \frac{1}{2} \cdot [\Theta, \Theta]$$

It satisfies the Bianchi identity *:*

$$d\Omega = [\Omega, \Theta]$$

This form is equivariant and horizontal. It is in particular determined by its values on horizontal tangent vectors. The curvature can be interpreted as follows. Given two tangent vectors v and w in T_xE, we may project them down to $T_{p(x)}B$. Then tv and tw determine an infinitesimal parallelogram. The parallel transport along this parallelogram determines an automorphism of E_x and thus an element $g(t)$ in G. Consider the path $g(t)$ in G for small t. It determines an element in LG, which is by definition $\Omega(v,w)$. A connection is called flat, *if its curvature vanishes. This is equivalent to the statement that the parallel transport along a path w depends only on the homotopy class relative endpoints of the path. In particular a flat connection determines a homomorphism $\pi_1(B) \longrightarrow G$. Hence flat connections are in bijective correspondence to representations of the fundamental group of B into G.*

Now consider a closed 3-manifold W and the trivial $SU(n)$-bundle $E \downarrow W$ for $n \geq 2$. Let $\mathcal{A}$ be the space of connections A on $E \downarrow W$. The difference of two connections is an invariant horizontal 1-form on E with coefficients in LG and hence a 1-form on W with coefficients in LG. Hence $\mathcal{A}$ is an affine space modelled on $\Lambda^1(B; LG)$. In particular it makes sense to speak of the tangent space of $\mathcal{A}$ at Θ, it can be identified with $\Lambda^1(B; LG)$. A 1-form on $\mathcal{A}$ is given by a family of linear maps $\Lambda^1(B; LG) \longrightarrow \mathbf{R}$ parametrized by $\mathcal{A}$. We get a 1-form $curv$ on $\mathcal{A}$, by the following construction :

$$curv_\Theta : \Lambda^1(B; LG) \longrightarrow \mathbf{R} \qquad\qquad \omega \mapsto \int_W tr(\omega \wedge \Omega_\Theta)$$

This 1-form turns out to be closed. It turns out that it is exact, i.e. there is a function $\mathcal{L}$ on $\mathcal{A}$ satisfying $d\mathcal{L} = Curv$. This function is the so called Chern-Simons functional *(see Chern-Simons [7]) :*

$$\mathcal{L} : \mathcal{A} \longrightarrow \mathbf{R} \qquad\qquad A \mapsto \frac{1}{4\pi} \cdot \int_M tr(A \wedge dA + \frac{2}{3} \cdot A \wedge A \wedge A)$$

Let $\mathcal{G}$ be the gauge group of $E \downarrow W$, i.e. the group of bundle automorphisms of $E \downarrow W$ over the identity on W. As $E \downarrow W$ is trivial, $\mathcal{G}$ is just $map(W, G)$. We will be interested in $\mathcal{A}/\mathcal{G}$. Notice that $curv$ is $\mathcal{G}$-invariant and hence defines also a 1-form on $\mathcal{A}/\mathcal{G}$. The function $\mathcal{L}$ is at least invariant under the action of the component of the identity $\mathcal{G}^0$ of $\mathcal{G}$. As W is 3-dimensional and $SU(n)$ is 2-connected, we get :

$$\pi_0(\mathcal{G}) = \mathcal{G}/\mathcal{G}^0 = [W, SU(N)] = H^3(W; \pi_3(SU(n))) = H^3(W; \mathbf{Z}) = \mathbf{Z}$$

The action of $\pi_0(\mathcal{G}) = \mathbf{Z}$ on $\mathcal{L}$ is given by adding a certain integer to $\mathcal{L}$. Hence $\mathcal{L}$ induces a function :

$$\overline{\mathcal{L}} : \mathcal{A}/\mathcal{G} \longrightarrow \mathbf{R}/\mathbf{Z}$$

Hence we obtain a well-defined function :

$$\mathcal{A}/\mathcal{G} \longrightarrow \mathbf{C} \qquad\qquad A \cdot \mathcal{G} \mapsto exp(ik\mathcal{L}(A))$$

Now Witten defines a complex number

$$Z(W) = \int_{\mathcal{A}/\mathcal{G}} exp(ik\mathcal{L}(A)) \; d\mathcal{A}/\mathcal{G}$$

Of couse the real meaning of this integral is not clear, as one is integrating over a very big space and no explicite measure is known.

Finally, we explain, how one takes links L in W together with representations for V_i for each component L_i into account. Given a connection A , let the Wilson line *be defined by :*

$$W_{L_i}(A) = char_{V_i}(tp_A(L_i))$$

where $tp_A(L_i) \in G$ is the parallel transport along L_i given by A and $char_{V_i} : G \longrightarrow \mathbf{C}$ the character of the representation V_i. Then one defines :

$$Z(W, L) = \int_{\mathcal{A}/\mathcal{G}} exp(ik\mathcal{L}(A)) \cdot \prod_i W_{L_i}(A)\, d\mathcal{A}/\mathcal{G}$$

7.section
Basic facts about 3-manifolds

In this section we collect some basic facts about 3-manifolds. We begin with the proof of the following theorem due to Stiefel.

Theorem 7.1 *Any orientable 3-manifold is parallizable, i.e. its tangent bundle is trivial.* ■

The proof needs some preparation. Let

$$w(M) := w(TM) \in H^*(M; \mathbf{Z}/2)$$

be the total Stiefel-Whitney class *of M. The Stiefel-Whitney classes of a n-dimensional vector bundle $\xi \downarrow X$ are defined as follows. The cohomology ring with $\mathbf{Z}/2$-coefficients of the classifying space $BO(n)$ of such bundles is a free polynomial algebra*

$$H^*(BO(n); \mathbf{Z}/2) = \mathbf{Z}/2[w_1, w_2, \ldots w_n]$$

where the degree of w_i is i. Let $f_\xi : X \longrightarrow BSO(n)$ be the classifying map of ξ, i.e., the map uniquely determined up to homotopy by the property that $f_\xi^* \gamma^n$ is isomorphic to ξ where $\gamma \downarrow BO(n)$ is the universal bundle. Then the i-th Stiefel-Whitney class $w_i(\xi)$ is defined by $f_\xi^* w_i$. The total Wu-class

$$v(M) \in H^*(M; \mathbf{Z}/2)$$

is uniquely defined by the property that for all total cohomology classes x we have :

$$\langle x \cup v(M), [M] \rangle = \langle Sq(x), [M] \rangle$$

if $Sq^i : H^*(X) \longrightarrow H^{*+i}(X)$ is the cohomology operation given by the Steenrod squares. The Wu formula *says :*

$$w(M) = Sq(v(M))$$

The Steenrod squares satisfy $Sq^i(x) = 0$ for any j-dimensional cohomology class x if $j < i$ holds. Hence the Wu class of a n-dimensional manifold satisfies $v_k(M) = 0$ for $k < n - k$. In particular we get for a 3-manifold M :

$$v(M) = 1 + v_1(M)$$

We derive from the Wu formula $w_1(M) = v_1(M)$. As M is supposed to be orientable, $w_1(M)$ vanishes. Hence we get

Lemma 7.2 *The total Wu class and the total Stiefel Whitney class of an orientable 3-manifold are trivial.* ■

Next we deal with the lower homotopy groups of $SO(3)$ *and* $BSO(3)$. *Let* $\mathbf{H} \cong \mathbf{R}^4$ *be the Lie group of quaternions. Denote by* S^3 *the unit sphere. We obtain an operation by conjugation :*

$$q : \mathbf{H} - \{0\} \times \mathbf{H} \longrightarrow \mathbf{H} \qquad\qquad (x, y) \mapsto xyx^{-1}$$

The center of $\mathbf{H}$ *is given by* $\mathbf{R}$. *Let* V *be the orthogonal complement of* $\mathbf{R}$ *in* $\mathbf{H}$. *We obtain an induced operation by orientation preserving isometries :*

$$S^3 \times V \longrightarrow V$$

As V *is isometric to* $\mathbf{R}^3$ *as real vector space, we obtain an exact sequence of Lie groups :*

7.3 $\mathbf{Z}/2 \longrightarrow S^3 \xrightarrow{p} SO(3)$

In particular p *is a covering. An alternative description of* p *is given by the adjoint representation of* $SU(2)$ *and an identification of Euclidean spaces between the Lie algebra* $LSU(2)$ *and* $\mathbf{R}^3$. *We derive from elementary homotopy theory :*

Lemma 7.4

$$\pi_i(BSO(3)) = \pi_{i-1}(SO(3)) = \begin{cases} \{0\} & i = 0, 1, 3 \\ \mathbf{Z}/2 & i = 2 \\ \mathbf{Z} & i = 4 \end{cases} \qquad \blacksquare$$

Hence the obvious map $j : BSO(3) \longrightarrow K(\mathbf{Z}/2, 2)$ *into the Eilenberg-MacLane space* $K(\mathbf{Z}/2, 2)$ *is* 4*-connected. As a* 3*-dimensional manifold* M *has the homotopy type of a* 3*-dimensional CW-complex, we obtain a bijection :*

$$j_* : [M; BSO(3)] \longrightarrow [M, K(\mathbf{Z}/2, 2)]$$

There is a natural isomorphism :

$$\phi : [M, K(\mathbf{Z}/2, 2)] \longrightarrow H^2(M; \mathbf{Z}/2)$$

The cohomology ring $H^*(BSO(n); \mathbf{Z}/2)$ *is a free polynomial algebra*

$$H^*(BSO(n); \mathbf{Z}/2) = \mathbf{Z}/2[\widetilde{w_2}, \ldots \widetilde{w_n}] \qquad\qquad , deg(\widetilde{w_i}) = i$$

and for the canonical map $q : SO(3) \longrightarrow O(3)$ *we get for* $2 \leq i \leq n$*:*

$$Bq^i(w_i) = \widetilde{w_i}$$

This shows for an oriented 3*-manifold* M *:*

$$\phi \circ j_*(f_{TM}) = w_2(M)$$

As $\phi \circ j_*$ *is an isomorphism, theorem 7.1 follows from lemma 7.2. More details can be found in Milnor-Stasheff [34].*

Next we deal with Heegaard decompositions of an oriented closed 3*-manifold.*

Notation 7.5

Let $W_g = W$ be the standard modell of the 3-dimensional handle body of genus g. Namely W is the g-fold connected sum of $S^1 \times D^2$.

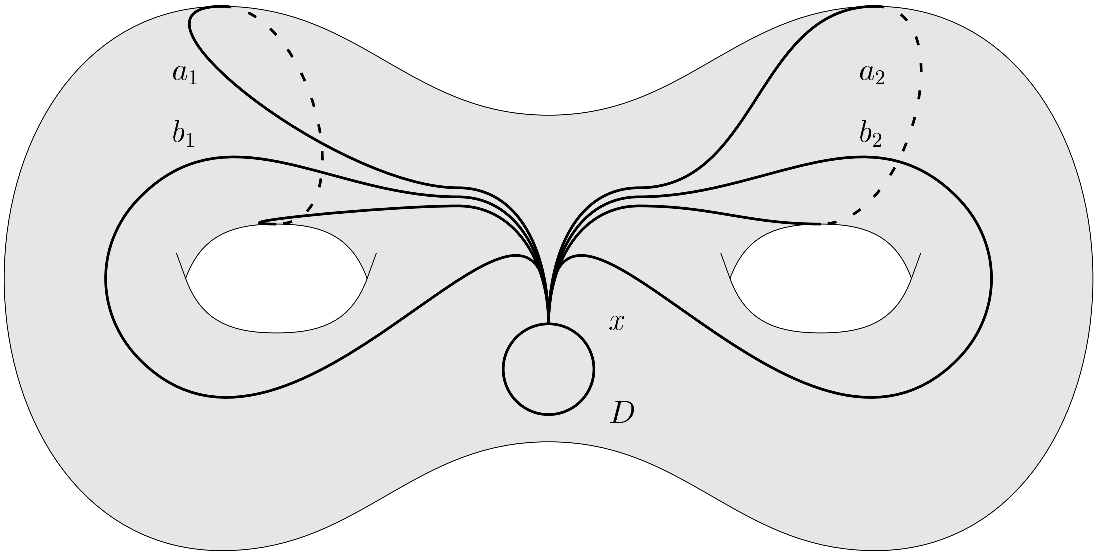
Denote by F the boundary of W. This is the surface of genus g, or in other words, the g-fold connected sum of $S^1 \times S^1$.

Let $D \subset F$ be a fixed embedded 2-disk.

Put $F^* := F - D$ and $S^1 := \partial D$.

Fix a base point $x \in D$ ■

7.6



The standard orientation of $\mathbf{R}^3$ induces an orientation on W. Then F, F^*, D and S^1 inherits orientations by the general agreement that an oriented manifold induces an orientation on its boundary using the decomposition $\nu(\partial M, M) \oplus T\partial M = TM$ and the outward normal field.

Definition 7.7 (Heegard modell) *If $h : (F, D, x) \longrightarrow (F, D, x)$ is an orientation reversing homeomorphism, define the* Heegard modell of h *by*

$$| (W, h) | := W \cup_h W \qquad ■$$

Definition 7.8 (Heegard splitting) *Let M be an oriented closed* 3*-manifold. A* Heegard splitting *of M is a pair (W_1, W_2) consisting of submanifolds $W_1, W_2 \subset M$ of codimension* 0 *satisfying*

$$W_1 \cup W_2 = M \qquad \partial W_1 = W_1 \cap W_2 = \partial W_2 \qquad W_1 \cong W_2 \qquad ■$$

Lemma 7.9 *Any closed* 3*-manifold admits a Heegard decomposition.*

Proof : *Choose a handlebody decomposition of M with exactly one* 0*-handle and one* 3*-handle. Let W_1 be the union of* 0*- and* 1*-handles and W_2 the union of all* 2*- and* 3*-handles. Then W_1 resp. W_2 is diffeomorphic to the standard model of the* 3*-dimensional handle body whose genus is the number of* 1 *resp* 3*-handles. The number of* 1*- resp* 3*-handles has to agree, as the Euler characteristic of a closed* 3*-manifold is zero by Poincare duality and can be computed from a handlebody decomposition by $\sum_{i=0}^{i=3}(-1)^i \cdot h_i$, where h_i denotes the number of i-dimensional handles. This shows that W_1 and W_2 are diffeomorphic. Obviously $M = W_1 \cup W_2$ and $\partial M = W_1 \cap W_2$ holds.* ■

Another proof of lemma 7.9 is done as follows. A triangulation (T, h) *of M consists of a finite simplicial complex T together with a homeomorphism $h : \mid T \mid \longrightarrow M$. Two triangulations (T_1, h_1) and (T_2, h_2) are* compatible , *if $h_2^{-1} \circ h_1$ is piecewise linear. The* star *of a simplex σ is the subcomplex of T consisting of all simplices of K, which meet σ, together with all their faces. The* link *is the subcomplex of all simplices which do not meet K, but which are faces of some simplex of K containing σ. A triangulation is called* combinatorial *if for each vertex v of T the link $link(v)$ is PL-homeomorphic to an $n-1$-simplex or the boundary of an n-simplex according to $h(v) \in \partial M$ or $h(v) \in int(M)$. A* PL-structure *on M is a maximal, non-empty collection of compatible combinatorial triangulations of M. Define for a subcomplex L of T:*

$$N(L, T) = \cup_{\sigma \in T}\ star(\sigma, T)$$

If there are finite subcomplexes K and L of T such that K collapses down to L, then $N = h(K)$ is a regular neighbourhood of $P = h(L) = \mid L \mid$. Such regular neighbourhoods N of P are in a certain sense unique, i.e. there is a PL-homeomorphism from N_1 to N_2 which is the identity on P, if N_1 and N_2 are regular neighbourhoods of P satisfying $P \subset int(N_i)$. A regular neighbourhood of $\mid L \mid$ in $\mid T \mid$ is given by $N(K'', T'')$.

Given a triangulation of M, let Γ_1 be the 1*-skeleton and Γ_2 be the dual* 1*-skeleton, i.e. a maximal* 1*-subcomplex of the barycentric subdivision T' disjoint from Γ_1. Put $V_i = N(\Gamma_i, T'')$. Then V_i turns out to be a regular neighbourhood of Γ_i. Moreover, (V_1, V_2) is a Heegaard decomposition of M (see Hempel [16], page 17).*

Two Heegard decompositions (W_1, W_2) and (V_1, V_2) of M are called isotopic

$$(W_1, W_2) \approx (V_1, V_2)$$

if there is an ambient isotopy of M taking W_1 to V_1 and W_2 to V_2. Given a Heegard decomposition (W_1, W_2) of genus g, we define a new Heegard decomposition of genus $g+1$, the suspension, $\Sigma(W_1, W_2)$ *as follows. Choose an unknotted handle H in W_2, i.e. an embedding of $D^2 \times [0, 1]$ in W_2 such that $H \cap \partial W_2 = D^2 \times \partial[0, 1]$ holds and there is an embedded disk B^2 in W_2 such that the union of $B^2 \cap \partial W_2$ and $\{0\} \times [0, 1]$ is the boundary of B^2. Define $\Sigma(W_1, W_2)$ to be $(W_1 \cup H, clos(W_2 - H))$. Then the isotopy class of the suspension depends only on the isotopy class of (W_1, W_2). Two Heegard decompositions (W_1, W_2) and (V_1, V_2) are called* stably equivalent ,

$$(W_1, W_2) \sim (V_1, V_2)$$

if there are non-negative integers a and b satisfying :

$$\Sigma^a(W_1, W_2) \approx \Sigma^b(V_1, V_2)$$

It may happen that two Heegard decompositions of the same 3-manifold M are not isotopic. However, we have :

Theorem 7.10 (Singer) *Two Heegard decompositions of the same 3-manifold M are stably equivalent.*

Proof : *We give only a sketch of a proof. In the first step one verifies for an arbitrary Heegard decomposition (W_1, W_2) the existence of a triangulation (T, h) such that :*

$$(W_1, W_2) \sim (N(\Gamma_1), clos(M - (N(\Gamma_1)))$$

We describe at least the triangulation T. Think of W_i as a 3-ball with g 1-handles attached. Now choose a triangulation (T_0, h_0) of $F = \partial W_1 \cap \partial W_2$ such that $\partial D^2 \times \partial[0,1]$ for any 1-handle $D^2 \times [0,1]$ is a subgraph of $h(T_0^1)$. Extend (T_0, h_0) over $D^2 \times \partial[0,1]$ by coning over the centers yielding (T^1, h^1). Next extend (T^1, h^1) to (T^2, h^2) by coning to the center of the 0-handle. Then extend (T^2, h^2) to the desired triangulation (T, h) by coning to the centers of the 1-handles. Then W_1 is a regular neighbourhood of a certain subgraph Γ of (T, H) and their is a sequence of subgraphs $\Gamma = \Gamma_1,\ \Gamma_2,\ \ldots \Gamma_n = h(T)$ such that Γ_{i+1} is obtained from Γ_i by attaching an edge in a specific way. One shows that the Heegard decompositions given by Γ_i and Γ_{i+1} are stably equivalent.

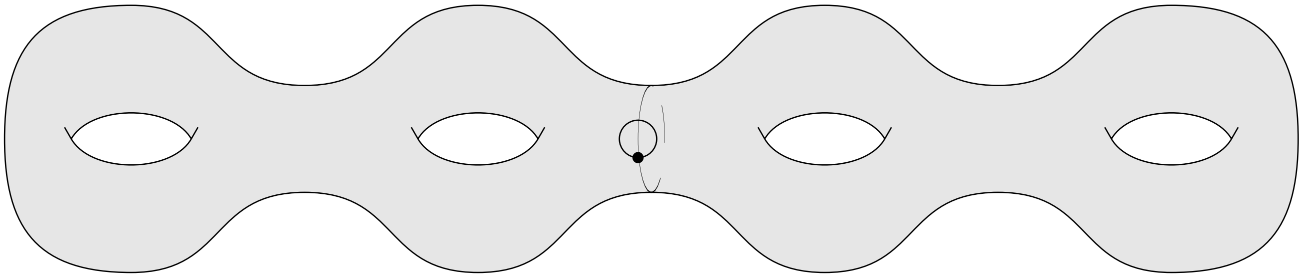
In the second step one verifies that the Heegard decompositions given by a subdivision of (T, h) and of (T, h) itself are stably equivalent. As two triangulations have a common subdivision, the claim follows. ■

7.11 Let (W_1, W_2) and (V_1, V_2) be Heegard decompositions of M and N. Choose 3-balls B and C in M and N such that $B \cap \partial W_1$ is a 2-disk with boundary $\partial B \cap \partial W_1$ holds and a similar statement for C. Taking the boundary connected sums of the handle bodies

$$(W_1, W_2)\sharp(V_1, V_2) = (W_1 \sharp_\partial V_1, W_2 \sharp_\partial V_2)$$

yields a Heegard decomposition of the connected sum $M \sharp N$. ■

7.12



The genus one Heegard decomposition of S^3 is given by the Heegard model $(S^1 \times S^2, f)$ where $f : \partial(S^1 \times S^2) \longrightarrow \partial(S^1 \times S^2)$ is given by the flip map on $S^1 \times S^1 = \partial(S^1 \times D^2)$.

Given a Heegard splitting (W_1, W_2) of M, we obtain a diagram of inclusions of spaces

7.13

$$\begin{tikzcd} & & & W_1 \arrow[dr] & \\ \partial F^* \arrow[r] & F^* \arrow[r] & F \arrow[ur] \arrow[dr] & & M \\ & & & W_2 \arrow[ur] & \end{tikzcd}$$

Applying the fundamental group with respect to the base point x gives a diagram of groups

7.14

$$\begin{tikzcd} & & & \pi_1(W_1) \arrow[dr] & \\ \pi_1(\partial F^*) \arrow[r] & \pi_1(F^*) \arrow[r] & \pi_1(F) \arrow[ur] \arrow[dr] & & \pi_1(M) \\ & & & \pi_1(W_2) \arrow[ur] & \end{tikzcd}$$

That all the maps in the diagram 7.14 are epimorphisms except for the first one, follows from the following presentations of the fundamental groups if M is the Heegard modell $\mid (W, h) \mid$. The paths a_i and b_i on F^* are indicated in diagram 7.6 and $i : F \longrightarrow W$ is the inclusion.

7.15 $\pi_1(F^*, x) =< a_1, a_2, ..., a_g, b_1, b_2, ...b_g >$
$\pi_1(F, x) =< a_1, a_2, ..., a_g, b_1, b_2, ...b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 >$
$\pi_1(W_1, x) =< a_1, a_2, ..., a_g, b_1, b_2, ...b_g \mid b_1 = b_2 = ... = b_g = 1 >$
$\pi_1(W_2, x) =< a_1, a_2, ..., a_g, b_1, b_2, ...b_g \mid (i \circ h^{-1})_*(b_j) = 1 \quad 1 \leq j \leq g >$
$\pi_1(M, x) =< a_1, a_2, ..., a_g, b_1, b_2, ..., b_g \mid b_j = 1, (i \circ h)_*^{-1}(b_j) = 1 \quad 1 \leq j \leq g >$

Next we compute the first homology of M from the homeomorphism h appearing in the Heegard model (W,h) of M. As F is oriented, we have the intersection pairing. Its matrix with respect to the bases of $H_1(F;\mathbf{Z})$ given by $\{a_1,\dots,a_g,b_1,\dots b_g\}$ is :

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Let $h : F \longrightarrow F$ be an orientation reversing diffeomorphism. Then $h_* = H_1(h;\mathbf{Z})$ respects the intersection pairing up to a sign. This is equivalent to

$$h_*^{-1} = -J^{-1} h_*^{tr} J$$

If we write h_* with respect to the bases above :

$$h_* = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we get :

$$h_*^{-1} = \begin{pmatrix} D^{tr} & -B^{tr} \\ -C^{tr} & A^{tr} \end{pmatrix}$$

Because of the presentations of $\pi_1(M)$ in 7.15 we obtain presentation matrices for $H_1(M;\mathbf{Z})$ by both B and the following matrix P:

7.16 $P = \begin{pmatrix} 0 & B^{tr} \\ I & -A^{tr} \end{pmatrix}$

In particular we conclude

Lemma 7.17 *If we write h_* as*

$$h_* = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we get :

1. *$H_1(M)$ is different from zero, if and only if $det(B) = det(P)$ is different from zero.*
2. *Suppose that $det(B)$ is different from zero. Then we get*

$$\mid det(B) \mid = \mid det(P) \mid = \mid H_1(M) \mid \qquad \blacksquare$$

Next we want to deal with the Kirby-calculus. For this purpose we recall the basic notions of surgery along framed links.

A framed link L in an oriented 3-manifold M is a link L together with a framing of its normal bundle, i.e. an isomorphism of vector bundles $\mathbf{R}^2 \times L \longrightarrow \nu(L,M)$ over the identity, such that the induced orientation on $\nu(L,M)$ and the given orientations on L and M are *compatible with the obvious isomorphism* $\nu(L,M) \oplus TL \longrightarrow TM\mid_L$. *Notice that* $\nu(L,M)$ *is stably trivial and hence for dimension reasons trivial because of theorem 7.1. In an oriented homology* 3*-sphere we can describe a framing of a link by an element* $n \in H^0(L) = \mathbf{Z}^{r(L)}$, *where* $r(L)$ *is the number of components of the link. Define a map*

$$n : \{\textit{isotopy classes of framings on } L\} \longrightarrow \mathbf{Z}^r$$

as follows. Given a framing $f : \mathbf{R}^2 \times L \longrightarrow \nu(L,M)$, *we obtain from the standard section of* $\mathbf{R}^2 \times L$ *given by the first standard basic vector in* $\mathbf{R}^2$ *and the exponential map* $exp : \nu(L,M) \longrightarrow N(L)$ *onto an open neighbourhood of* L *a parallel link* L'. *Choose an orientation on* L *and equip* L' *with the induced orientation. Let* $n(L)$ *be given by the sequence of integers* $link(L_i, L_i')$, *where* i *runs over the components of the link. Notice that this is independent of the choice of orientation on* L, *as a change of the orientation of* L_i *causes also a change of orientation on* L_i'.

Lemma 7.18 *Let* M *be an oriented homology* 3*-sphere and* $L \subset M$ *be an oriented link. Then*

$$n : \{\textit{isotopy classes of framings on } L\} \longrightarrow \mathbf{Z}^r$$

is a bijection.

Proof : *The orientations of* L *and* M *induce an orientation on* $\nu(L,M)$. *Choose a Riemannian metric on* $\nu(L,M)$. *Given* $v \neq 0 \in \nu(L,M)$, *let* $\widehat{v} \in \nu(L,M)$ *be the vector uniquely determined by the property that* v *and* $\widehat{v}$ *are orthogonal,* $\widehat{v}$ *has norm* 1 *and the bases* $\{v, \widehat{v}\}$ *is compatible with the orientation. If* s *is a nowhere-vanishing section of* $\nu(L,M)$, *we obtain another section* $\widehat{s}$ *by requiring* $\widehat{s}(x) := \widehat{s(x)}$ *for* $x \in L$. *Notice that* $s, \widehat{s}$ *determines a framing of* $\nu(L,M)$, *denoted by* $f(s)$. *Moreover, the isotopy class of* $f(s)$ *is independent of the choice of Riemannian metric on* $\nu(L,M)$ *and depends only on the isotopy class of the nowhere vanishing section* s. *One easily checks that we obtain a bijection*

$$f : \{\textit{isotopy classes of nowhere vanishing sections in } \nu(L,M)\} \longrightarrow \{\textit{isotopy classes of framings of } \nu(L,M)\}$$

As the isotopy classes of framings of nowhere vanishing sections in the trivial bundle $\mathbf{R}^2 \times L$ *correspond bijectively to the homotopy classes of maps* $L \longrightarrow \mathbf{R}^2 - \{0\}$, *the claim follows.*
■

7.19 Let L be an oriented link in an oriented homology 3-sphere M. Suppose, we are given for any component L_i an element in $r_i \in \mathbf{Q} \cup \{\infty\}$. Choose integers p_i and q_i such that $r_i = p_i/q_i$ holds and p_i and q_i are prime, provided that $r_i \in \mathbf{Q}$. If r_i is ∞ put $p = 0$ and $q = 1$. Denote by $N(L_i)$ a tubular neighbourhood of L_i and by $N(L)$ their union. Let $M(L_i)$ be $M - int(N(L_i))$ and $M(L)$ be $M - N(L)$. Choose classes μ_i and λ_i in $H_1(\partial N(L_i))$ such that μ_i resp. λ_i lies in the kernel of the homomorphism $H_1(\partial N(L_i)) \longrightarrow H_1(N(L_i))$ resp. $H_1(\partial N(L_i)) \longrightarrow H_1(M(L_i))$ induced from the inclusion and the intersection number of μ_i and λ_i in $\partial N(L_i)$ with respect to the orientation induced from the one on M is 1. Notice that the pair (p,q) resp. (μ, λ) is not unique, there is exactly one other choice, namely, $(-p,-q)$ resp. $(-\mu,-\lambda)$. Since p and q are relatively prime, we can choose a homeomorphism $\sigma_i : S^1 \times D^2 \longrightarrow N(L_i)$ such that $H_1(\sigma_i)$ sends the class of $\{1\} \times \partial D^2$ to $p\mu + q\lambda$. Let $i : \partial N(L) \longrightarrow M(L)$ be the inclusion. Then define the result of Dehn surgery $M(L, r_1, \ldots r_{r(L)})$ by the push out

$$\begin{tikzcd}
\coprod_i S^1 \times \partial D^2 \arrow[r, "i \circ \coprod_i \sigma_i"] \arrow[d] & M(L) \arrow[d] \\
\coprod_i S^1 \times D^2 \arrow[r] & M(L, r_1, \ldots r_{r(L)})
\end{tikzcd}$$

We have already indicated in the first section that this construction depends up to oriented homeomorphism only on the isotopy class of L and the elements $r_1, \ldots, r_{r(L)}$. ∎

7.20 Let L be a framed link in an oriented 3-manifold M. The framing together with the exponential map induce an homeomorphism $\Phi : \coprod_i S^1 \times D^2 \longrightarrow N(L)$ onto a tubular neighbourhood of L in M. Let $M(L)$ be $M - N(L)$. Let $\phi : \coprod_i S^1 \times S^1 \longrightarrow \partial M(L)$ be given by the restriction of ϕ and the inclusion $\partial M(L) \longrightarrow M(L)$. Now define the result under surgery along the framed link L by the push out :

$$\begin{tikzcd}
\coprod_i S^1 \times S^1 \arrow[r, "\phi"] \arrow[d] & M(L) \arrow[d] \\
\coprod_i D^2 \times S^1 \arrow[r] & M_L
\end{tikzcd}$$

∎

7.21 Let V be a an oriented 4-manifold and L be a framed link in ∂V. As above the framing and the exponential map determine an homeomorphism $\Phi : \coprod_i S^1 \times D^2 \longrightarrow N(L)$ onto a tubular neighbourhood of L in ∂V. Now define the oriented 4-manifold V_L by the push out :

$$\begin{tikzcd}
\coprod S^1 \times D^2 \arrow[r, "\Phi"] \arrow[d] & \partial V \arrow[d] \\
\coprod D^2 \times D^2 \arrow[r] & V_L
\end{tikzcd}$$

■

These types of surgeries are related as follows :

Lemma 7.22

1. *Let L be a framed link in a oriented homology 3-sphere M. Let the integers $n_1, \ldots,$ n_r be given by the framings. Then the result under Dehn surgery $M(L, n_1, \ldots, n_{r(L)})$ defined in 7.19 and the result under surgery M_L defined in 7.20 agree.*

2. *Let V be an oriented 4-manifold and L be a framed link in $M = \partial V$. Then we get :*

$$\partial(V_L) = M_L \qquad \blacksquare$$

Let L be a oriented framed link in a oriented homology 3-sphere M. We have defined its linking matrix *in 5.34*

$$A_{link} = \begin{pmatrix} n(1) & l(1,2) & l(1,3) & \cdots & l(1,r) \\ l(1,1) & n(2) & l(2,3) & \cdots & l(2,r) \\ l(3,1) & l(3,2) & n(3) & \cdots & l(3,r) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l(r,1) & l(r,2) & l(r,3) & \cdots & n(r) \end{pmatrix}$$

where $l(i,j)$ denotes the linking number of L_i and L_j.

Lemma 7.23 *Let V be an oriented homology 4-ball. Then its boundary $M = \partial V$ is an oriented homology 3-sphere . Let L be an oriented framed link in M. Then :*

1. *$H_i(V_L)$ vanishes for $i = 1, 3$.*

2. *$H_2(M_L)$ is a finitely generated free abelian group and their is a canonical bases coming from the link. With respect to this bases the intersection pairing is described by the linking matrix A_L of L.*

3. *A_L is a presentation matrix for $H_1(M_L)$.*

4. M_L *is a rational* 3 *homology sphere if and only if* $det(A_L)$ *is different from zero.*

5. M_L *is a homology* 3*-sphere if and only if* $det(A_L)$ *is* ± 1.

Proof : *1.)* V_L *has no* 1*-handles.*

2.) Let S_i *and* S_j *for* $1 \leq i, j \leq r(L)$ *,* $i \neq j$ *be Seifert surfaces of* L_i *and* L_j *in* M*. As* M *has a collar in* V*, one may find a surface* S'_j *in* V *by pushing off* S_j *a little bit, such that* $S'_j \cap M = L_j$*. Denote by* F_i *and* F_j *the core of the corresponding handle* H_i *and* H_j*. Notice that* $F_i \cap \partial H_i = F_i \cap M = L_i$ *holds. Hence* $S_i \cup_{L_i} F_i$ *and* $S'_j \cup_{L_j} F_j$ *are closed embedded surfaces in* V_L *representing the canonical bases. Now the* (i,j)*-entry in the intersection matrix is given by counting elements in their intersection with signs. By construction this is the same as counting the intersection of* S_i *and* L_j *in* M *with signs, what is just the linking number of* L_i *and* L_j*. A similar argument shows that the* (i,i)*-entry in the intersection matrix is just the framing number* $n(L_i)$*.*

3.) The intersection pairing is described by the following composition:

$$H_2(V_L) \xrightarrow{i_*} H_2(V_L, \partial V_L) \xrightarrow{\cap[V_L]} H^2(V_L) \xrightarrow{\mu} Hom(H_2(V_L; \mathbf{Z})$$

where i_* *is induced from the inclusion. Notice that* $\cap[V_L]$ *is the Poincaré isomorphism and the canonical map* μ *is an isomorphism by the universal coefficient theorem, since* $H_1(V_L)$ *is zero. The long homology sequence of the pair* (V_L, M_L) *gives an exact sequence :*

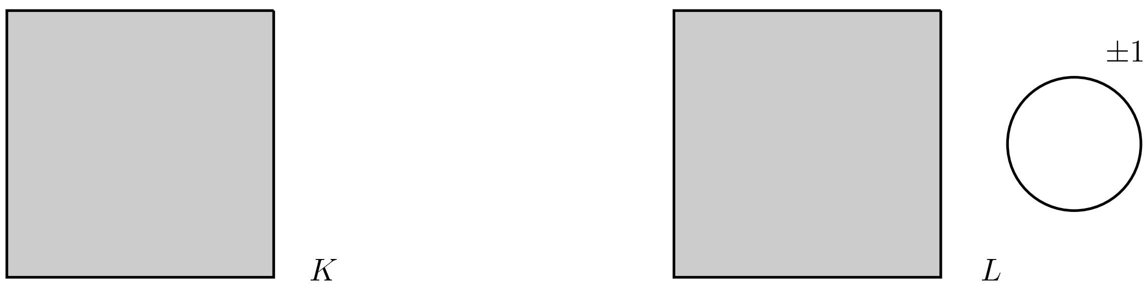
$$H_2(V_L) \xrightarrow{i_*} H_2(V_L, \partial V_L) \longrightarrow H_1(M_L) \longrightarrow \{0\}$$

Now the claim 3. follows from 1.

The other assertions are now easy consequences of claim 3. ■

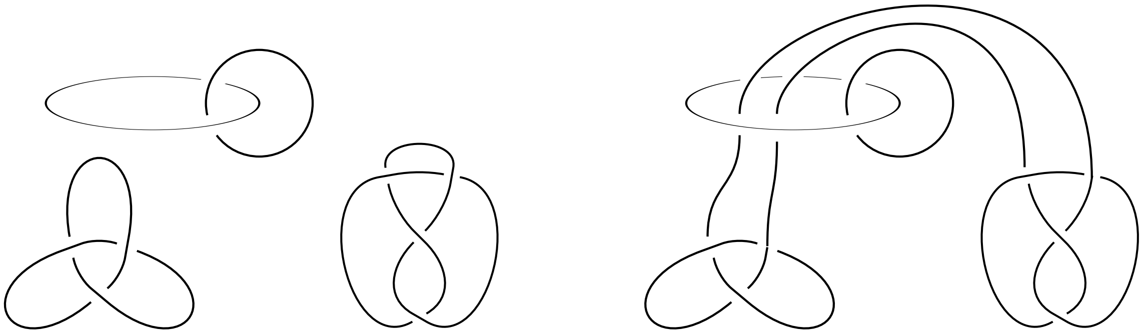
The Kirby calculus deals with the question, when S^3_K *and* S^3_L *resp.* D^4_K *and* D^4_L *for framed links* K *and* L *in* S^3 *are oriented homeomorphic. We recall that homeomorphic implies diffeomorphic for* 3*-manifolds. We define two so called Kirby moves for framed links* K *and* L *in* S^3*. Recall that the isotopy classes of framings are specified by attaching an integer to each of the components (see lemma 7.18). We say that* L *is obtained from* K *by a Kirby move of type* 1*, if* L *is obtained from* K *by the disjoint union with a trivial knot with framing* ± 1*.*

7.24 Kirby move of type 1



We say that L is obtained from K by a Kirby move of type 2, if the following is true. There are two different components L_i and L_j and a band w from some arc on L_i to some arc on L_j' where L_j' is a parallel curve of L_j corresponding to the framing $n(L_j)$ of L_j. Recall that the parallel curve L_j' is given by the exponential map of the normal bundles and a nowhere vanishing section of the normal bundle such that $link(L_j, L_j')$ is $n(j)$. Let $L_i \sharp L_j'$ be the connected sum of L_i and L_j' along w. Then the link K has the same components as L except for the component L_i, which is substituted by $L_i \sharp L_j'$.

7.25 Kirby move of type 2



K L

The framings of K are determined by the following property of the linking matrix. Choose orientations on each components of L. Let ϵ be 1, if the connecting band is compatible with the orientations, and -1 otherwise. Then the linking matrices A_L and A_K satisfy :

7.26 $A_K = E_{i,j}(\epsilon) A_L E_{i,j}(\epsilon)^{tr}$

where $E_{i,j}(a)$ is the elementary matrix with a as (i,j)-entry. Notice that the framings on K do not depend on the choice of orientations.

Recall that all entries in $E_{i,j}(a)$ off the diagonal are zero except the (i,j)-entry which is a, and all diagonal entries are 1. We mention the behaviour of the linking matrix under the first Kirby move :

7.27 $A_K = \begin{pmatrix} A_L & 0 \\ 0 & \pm 1 \end{pmatrix}$

Now we have :

Theorem 7.28

1. *For any closed oriented 3-manifold M there is a framed link L in S^3 such that M is oriented homeomorphic to S^3_L.*

2. *Let K and L be framed links in S^3. Then S^3_K and S^3_L are oriented homeomorphic if and only if one can obtain K from L by a sequence of Kirby moves of type 1 or 2 or their inverses.* ■

A proof of this theorem can be found in Kirby [26]. A corollary of this theorem is that any oriented closed 3-manifold is the boundary of an oriented 4-manifold.

Lemma 7.29 *Let M be an oriented 3-manifold. Then there is an oriented framed link L in S^3 such that M is oriented homeomorphic to S^3_L and the linking matrix A_L is a diagonal matrix with ± 1 as diagonal entries.*

Proof : *Because of theorem 7.28 we can find a framed link L in S^3 such that M is S^3_L. By Kirby moves of type 1 we obtain a framed link L' such that M is $S^3_{L'}$ and the linking matrix of L' looks like :*

$$A_{L'} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ A_L & 0 & 0 \end{pmatrix}$$

Hence $A_{L'}$ is a unimodular, indefinite and odd symmetric bilinear form over $\mathbf{Z}$. Recall that unimodular means that $A_{L'}$ is invertible, indefinite means that there are x and y satisfying $xA_{L'}x^{tr} > 0$ and $yA_{L'}y^{tr} < 0$, and odd means that $xA_{L'}x^{tr}$ is odd for some x. By the result of Milnor and Husemoller [33] there is an invertible matrix U such that $UA_{L'}U^{tr}$ is diagonal. We may suppose that $\det(U)$ is 1 otherwise multiply U with an appropiate diagonal matrix. As $\mathbf{Z}$ is a principal domain, U can be written as a product of elementary matries $E_{i,j}(n)$. Since $E_{i,j}(a) \cdot E_{i,j}(b) = E_{i,j}(a+b)$ holds, we may even suppose that U is a product of elementary matries $E_{i,j}(\pm 1)$. Now the claim follows from lemma 7.26. ■

Corollary 7.30

1. *Let M be an oriented homology 3-sphere. Then there is a sequence of oriented homology 3-spheres M_0, M_1 , ..., M_n such that M_i is obtained from M_{i-1} by ± 1-Dehn surgery on a knot in M_{i-1}, M_0 is M and M_n is S^3.*

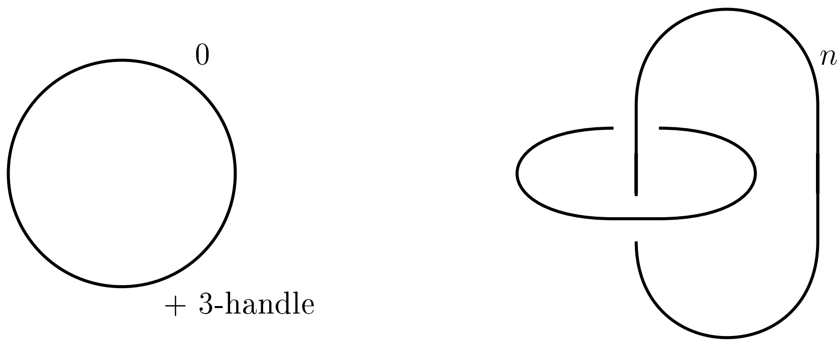
2. *The Casson invariant is uniquely determined by the surgery formula and the condition $\lambda(S^3) = 1$.* ■

Next we deal with the Kirby calculus for 4-manifolds. The Kirby calculus describing D^4_L, which we have developed so far, deals only with 2-handles. In order to take 1-handles into account, we extend our notion of framed link to the notion of extended framed link. Namely, an extended framed link is a link such that any component either has a framing given by an integer or is the trivial knot and has a dot on it. In other words, we have framed components *as before and have additionally unframed trivial components, which are distinguished from the others by a dot. The latter ones correspond to* 1*-handles and the framed components to* 2*-handles. Here is the precise description of D^4_L for an extended framed link L in S^3. Attach for any non-framed component a* 1*-handles. Call the result V. Consider the framed link L' in V obtained from L in D^4 by the following construction. Whenever an arc of a framed component L_i of L runs through the trivial knot representing a* 1*-handles, let the arc go over the* 1 *handle. Now let L' be the union of all framed components of L modified in the above way. Now the problem is to specify the attached* 3*-handles by indicating embeddings of S^2. Fortunately, this is not necessary, provided that we deal with closed* 4*-manifolds. Namely, we have the result of Montesinos [36] (see also Trace [42]).*

Theorem 7.31 *Let M be a closed orientable* 4*-manifold with a handle body decomposition $M = H^0 \cup aH^1 \cup bH^2 \cup cH^3 \cup H^4$. Then the oriented homeomorphism type of M is completely determined by $H^0 \cup aH^1 \cup bH^2$ and the number c of* 3*-handles.* ■

Thus the way the 3*- and* 4*-handles are attached does not matter, provided that M is closed. Given an extended framed link L in D^4 and a non-negative integer c, let $D^4_{(L,c)}$ be the closed* 4*-manifold obtained from D^4_L by attaching c* 3*-handles and one* 1*-handle. It may happen that we cannot get a closed manifold this way and then $D^4_{(L,c)}$ is not defined. Now we consider the following third Kirby move on an extended framed link. It consists of introducing*

7.32 Kirby move of type 3

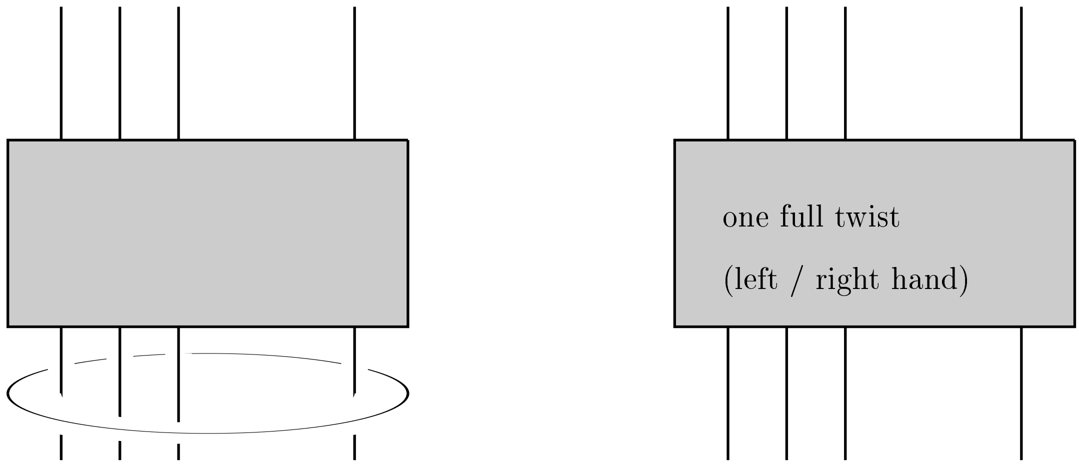


The proof of the following theorem is given in Kirby [26].

Theorem 7.33 *Let L and K be extended framed links in D^4 and c_K and c_L be non-negative numbers. Suppose that $D^4(K, c_K)$ and $D^4_{(L,c_L)}$ are defined. Then they are oriented homeomorphic if and only if one can obtain (L, c_L) from (K, c_K) by a sequence of Kirby moves of type* 2 *and type* 3 *or their inverses.* ■

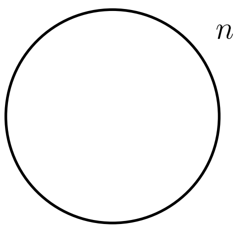
There are other kind of more convenient moves which can be used to decide whether two (extended) framed links describe the same manifold. For example :

7.34

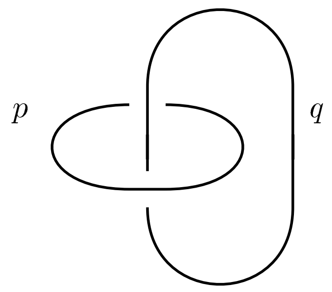


Here are some examples of closed oriented 3*-manifolds and their representations in the Kirby calculus :*

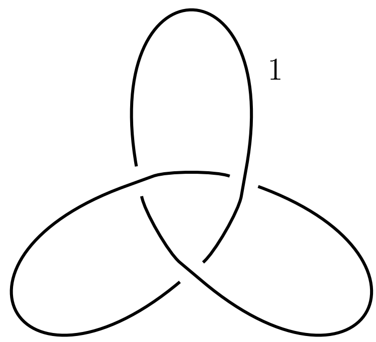
7.35 Lens space $L(n, 1, 1)$



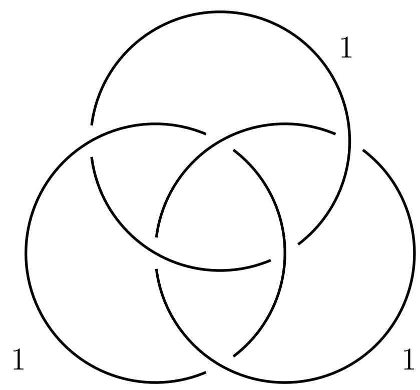
7.36 Lens space $L(pq-q,p,q) = L(pq-1,q,p)$



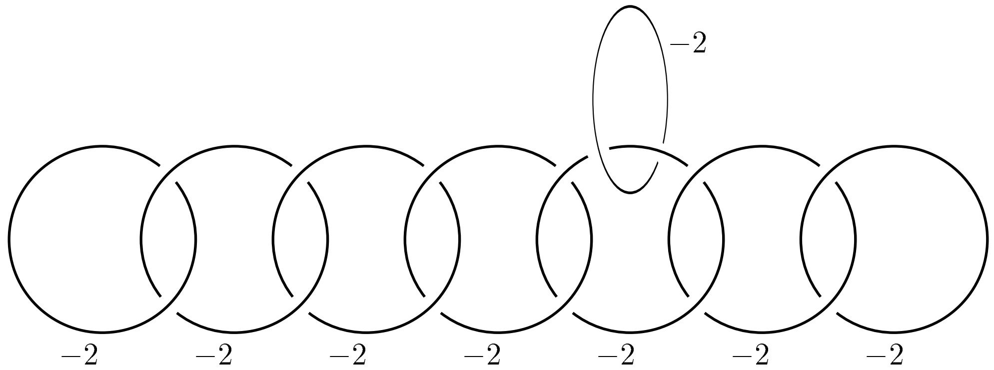
7.37 Poincaré sphere



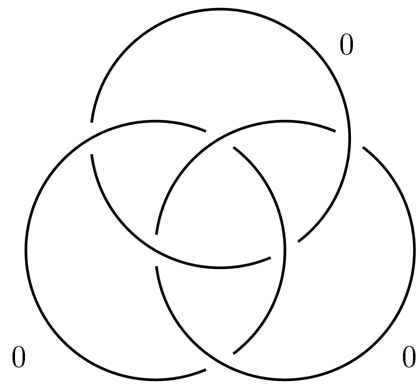
7.38 Poincaré sphere



7.39 Poincaré sphere



7.40 Torus



More information about the Kirby calculus can be found in Fenn-Rourke [10] Kirby [26] and [27] and Mandelbaum [29].

8.section
The construction of the Casson invariant

This section is devoted to the construction of the Casson invariant. We have already explained in section 1 what the main properties and applications of the Casson invariant are. We are following the expositions of Akbulut and McCarthy [1] and of Marin [30].

Let M be an oriented homology 3-sphere. We use the notation 7.5 and choose a Heegard splitting (W_1, W_2) as defined in 7.8. This defines a diagram of spaces 7.13 and by applying the functor "fundamental group" a diagram of groups 7.14. If G is a connected compact Lie group and π a discrete group, denote by $R(\pi, G)$ the space of homorphisms from π to G equipped with the topology induced from the inclusion $R(\pi, G) \subset map(\pi, G)$, where $map(\pi, G)$ gets the compactly generated topology coming from the compact-open topology. If G is $SU(2)$, we write briefly $R(\pi)$ instead of $R(\pi, SU(2))$. Notice that G acts on $R(\pi, G)$ by composition with the conjugation homorphism $c(g) : G \longrightarrow G$ which sends h to $g^{-1}hg$. As the center of G acts trivially, we obtain an induced $G/center(G)$-action. Thus $R(?, G)$ becomes a contravariant functor from the category of discrete groups to the category of $G/center(G)$-spaces. Notice for $G = SU(2)$ that $SU(2)$ can be identified with the unit sphere S^3 in the quaternions $\mathbf{H}$ by the Lie group isomorphism

$$S^3 \subset \mathbf{C}^2 \longrightarrow SU(2) \qquad\qquad (a,b) \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

The center of $SU(2)$ is $\pm id$. In particular we get from the exact sequence 7.3 an identification of $SU(2)/center(SU(2)) = SO(3)$. Hence we obtain a contravariant functor:

8.1 $R(?) : \{ \text{ discrete groups } \} \longrightarrow \{ \; SO(3)\text{-spaces } \}$

There are the following natural maps :

8.2 $\psi : H^1(\pi) \longrightarrow H_3(R(\pi, G))$

given by the following composition :

$H^1(\pi) \longrightarrow Hom(\pi, \mathbf{Z}) \longrightarrow map(R(\mathbf{Z}, G), R(\pi, G)) \longrightarrow$

$map(G, R(\pi, G)) \longrightarrow Hom(H_n(G), H_n(R(\pi, G))) \xrightarrow{ev} H_n(R(\pi, G))$

where n is the dimension of G and ev evaluation at the fundamental class $[G] \in H_n(G)$. Define

8.3 $\phi : T_{\rho_0} R(\pi, G) \longrightarrow H^1(\pi; LG)$

as follows. Consider an element in the tangent space $T_{\rho_0}R(\pi, G)$ of $R(\pi, G)$ at the trivial representation ρ_0 given by a derivation D on the germ of functions on $R(\pi, G)$ at ρ_0. Let $ev_w : R(\pi, G) \longrightarrow G$ be the homomorphism given by evaluation at $w \in \pi$. Pulling back the derivation D with ev_w defines a derivation D_w sending a germ f of functions on G at 1 to $D(f \circ ev_w)$. This defines a homomorphism from π to $T_1G = LG$. Let its class in $H^1(\pi; LG)$ be the image of D under ϕ.

Lemma 8.4

1. *The homomorphisms ψ and ϕ defined in 8.2 and 8.3 are natural in π.*
2. *The homomorphisms ψ and ϕ defined in 8.2 and 8.3 are compatible with amalgamation of groups.*
3. *Suppose that π is a finitely generated free group and G is S^3. Then the homomorphisms ψ and ϕ defined in 8.2 and 8.3 are isomorphisms.*
4. *Suppose that π is a finitely generated free group and G is S^3. Then we obtain an isomorphism, natural in π*

$$\Psi : H^*(R(\pi, S^3) \longrightarrow \bigwedge^{*} H_1(\pi)$$

Proof : *3.) Choose a bases s_1, s_2 $\ldots s_r$ of π. Then we get a natural identification :*

$$R(\pi) = \prod_{i=1}^{r} S^3$$

Because of 2.) the following square commutes

$$\begin{tikzcd}
\oplus_i^r H^1(\mathbf{Z}\langle s_i\rangle) \arrow[r, "\oplus_i^r \psi \mathbf{Z}\langle s_i\rangle"] \arrow[d] & \oplus_i^r H_3(R(\mathbf{Z}\langle s_i\rangle)) \arrow[d] \\
H^1(*_{i=1}^r \mathbf{Z}\langle s_i\rangle) \arrow[r, "\psi(*_{i=1}^r \mathbf{Z}\langle s_i\rangle)"] & H_3(R(*_{i=1}^r \mathbf{Z}\langle s_i\rangle))
\end{tikzcd}$$

where the right vertical arrow comes from the Künneth formula and is an isomorphism because $H_i(S^3)$ is $\mathbf{Z}$ for $i = 0, 3$ and zero otherwise. Also the left vertical arrow is a bijection. Hence it suffices to prove the claim in the special case $\pi = \mathbf{Z}$ what is easily done.

4.) Define Ψ by the following composition of isomorphisms resp. their inverses :

$$\Psi : H^*(R(\pi)) \xleftarrow{\cup} \wedge^* H^3(R(\pi)) \longleftarrow \wedge^* Hom(H_3(R(\pi)), \mathbf{Z})$$

$\xrightarrow{\bigwedge^* \psi^*} \bigwedge^* Hom(H^1(\pi), \mathbf{Z}) \longleftarrow \bigwedge^* H_1(\pi)$ ∎

Recall that a SU(2)- representation is reducible *if it contains a proper invariant linear subspace and* irreducible *otherwise.*

Lemma 8.5 *Let M be an orientable closed 3-manifold. Then the following assertions are equivalent :*

1. *M is a homology sphere.*
2. *The fundamental group $\pi_1(M)$ is perfect.*
3. *$H_1(M) = H_1(M; \mathbf{Z})$ is zero.*
4. *There are no non-trivial reducible $SO(3)$-representations of $\pi_1(M)$.* ∎

Proof : *1.) and 3.) are equivalent by Poincare duality and the universal coefficient theorem. Since $H_1(M)$ is the abelianization of $\pi_1(M)$, the assertions 2.) and 3.) are equivalent. If $H_1(M)$ is not trivial, one easily constructs a non-trivial representation of $H_1(M)$ and hence of $\pi_1(M)$. It remains to prove that 4.) implies 3.)*

Suppose that ρ is a non-trivial reducible representation of $\pi_1(M)$. Hence ρ is the direct sum of two 1-dimensional unitary representations ρ_1 and ρ_2. But these are given by homomorphisms from $\pi_1(M)$ to S^1. As ρ is non-trivial, ρ_1 or ρ_2 is non-trivial. Hence there is a non-trivial homomorphism from $\pi_1(M)$ to the abelian group S^1. This implies that $H_1(M)$ is non-trivial. ∎

Lemma 8.6 *A representation ρ of the discrete group π into $SU(n)$ is irreducible, if and only if its isotropy group under the $SU(n)$-operation on $R(\pi, SU(n))$ by conjugation is the center of $SU(n)$.*

Proof : *Let A be an element of the isotropy group of ρ, i.e. $A \cdot \rho(w) \cdot A^{-1} = \rho(w)$ holds for all $w \in \pi$. Let λ be a complex number and $E_\lambda(A)$ be the eigenspace of A for the eigenvalue λ. As $A \cdot \rho(w) = \rho(w) \cdot A$ holds for all $w \in \pi$, $E_\lambda(A)$ is a ρ-invariant subspace of $\mathbf{C}^n$. Notice that A has at least one eigenvalue λ. Then $E_\lambda(A)$ is a non-trivial subspace of $\mathbf{C}^n$. Suppose that the representation ρ is irreducible. Then $E_\lambda(A)$ is the whole space $\mathbf{C}^n$ and A lies in the center of $SU(n)$. Suppose that ρ is reducible. If V is a proper ρ- invariant subspace in $\mathbf{C}^n$, we get a decomposition of ρ into $V \oplus V^\perp$. Let A be a matrix acting on V by the identity and on $V^\perp$ by $\lambda \cdot id$ for some complex number $\lambda \neq 1$. Then A belongs to the isotropy group of ρ, but not to the center of $SU(n)$.* ∎

If we apply the functor $R(?)$ to the diagram 1.23 and define :

Notation 8.7 $R_\partial := R(\pi_1(\partial F^*))$
$R^* := R(\pi_1(F^*))$
$R := R(\pi_1(F))$
$Q_i := R(\pi_1(W_i))$ *for* i = *0,1* ■

we obtain :

8.8

$$\begin{tikzcd} & & & Q_1 \arrow[dl] & \\ R_\partial & R^* \arrow[l, "\partial"'] & R \arrow[l] & & R(\pi_1(M)) \arrow[ul] \arrow[dl] \\ & & & Q_2 \arrow[ul] & \end{tikzcd}$$

Notice that all maps in this diagram are inclusions except the map $\partial : R^* \longrightarrow R_\partial$ *which is an epimorphism. We derive from the presentation of the fundamental groups 7.15*

8.9 $R^* = \prod_{j=1}^{2g} S^3$
$Q_i = \prod_{j=1}^{g} S^3$

Therefore the intersection number $< Q_1, Q_2 >_{R^*}$ *of* Q_1 *and* Q_2 *in* R^* *is defined.*

Proposition 8.10 *Let* M *be an oriented* 3*-manifold. Then :*

1. $< Q_1, Q_2 >_{R^*}$ *is different from* 0 *if and only if* M *is a rational homology sphere.*
2. *If* M *is a rational homology sphere, then*
$$|< Q_1, Q_2 >_{R^*}| = | H_1(M; \mathbf{Z}) |$$
3. $< Q_1, Q_2 >_{R^*}$ *is* ± 1 *if and only if* M *is an integral homology sphere.*
4. Q_1 *and* Q_2 *intersect at* 1 *transversely if and only if* M *is a rational homology sphere.*

Proof : *Consider the following composition of isomorphisms :*

$$\bigwedge^* (H_1(\pi_1(W_1)) \oplus H_1(\pi_1(W_2))) \longrightarrow (\bigwedge^* (H_1(\pi_1(W_1))) \otimes (\bigwedge^* H_1(\pi_1(W_2))) \xrightarrow{\Psi \otimes \Psi}$$

$$H^*(R(\pi_1(W_1))) \otimes H^*(R(\pi_1(W_2))) \longrightarrow H^*(Q_1) \otimes H^*(Q_2) \longrightarrow H^*(Q_1 \times Q_2)$$

As $\pi_1(F^*)$ is free of rank g, there is a Lie group structure on $R^* = \prod_{i=1}^{g} S^3$. Define the map :

$$m : Q_1 \times Q_2 \longrightarrow R^*$$

by $m(q_1, q_2) = q_1 \cdot q_2$. Then the intersection number of Q_1 and Q_2 in R^* is just the degree of m. The following square commutes :

$$\begin{tikzcd}
\bigwedge^* H_1(\pi_1(F^*)) \arrow[r, "\bigwedge^* (H_1(i_1) \oplus H_1(i_2))"] \arrow[d, "\psi"] & \bigwedge^* (H_1(\pi_1(W_1)) \oplus H_1(\pi_1(W_2))) \arrow[d, "\psi"] \\
H^*(R^*) \arrow[r, "H^*(m)"] & H^*(Q_1 \times Q_2)
\end{tikzcd}$$

The upper horizontal arrow can be identified with the first arrow in the following sequence, whose exactness follows from the Mayer-Vietoris sequence :

$$H_1(F^*) \stackrel{H_1(i_1) \oplus H_1(i_2)}{\longrightarrow} H_1(W_1) \oplus H_1(W_2) \longrightarrow H_1(M) \longrightarrow \{0\}$$

The cokernel of $H_1(i_1) \oplus H_1(i_2)$ is finite, if and only if $\bigwedge^{2g} (H_1(i_1) \oplus H_1(i_2))$ is different from zero. If this cokernel is finite, its cardinality is the cardinality of the cokernel of $\bigwedge^{2g} (H_1(i_1) \oplus H_1(i_2))$. Now the assertions 1.),, 2.) and 3.) follow.

It remains to prove 4.). Notice that Q_1 and Q_2 intersect tranversely at 1, if and only if the following map is an epimorphism :

$$T_1 Q_1 \oplus T_1 Q_2 \longrightarrow T_1 R^*$$

Now the claim follows from lemma 8.4 applied to the map ϕ defined in 8.3. ∎

Next we examine the orbit spaces under the $SO(3)$-action on the representation spaces.

8.11 $S = S(\pi_1(F^*, x)) := \{\rho \in R^* = R(\pi_1(F^*, x)) \mid \rho \text{ is reducible}\}$

Proposition 8.12

1. *The map ∂ is surjective.*
2. *The set of critical points is the set S of reducible representations.*
3. *$S(\pi_1(F^*, x)) = S(\pi_1(F, x))$*
4. *$R = \partial^{-1}(1)$*

5. *$R - S$ is an open smooth manifold of dimension $6g-3$ and carries a free proper $SO(3)$-action.*

Proof : *1.) and 4.) The functor $R(?,G)$ turns push outs of groups into pull backs of spaces. Now apply the theorem of Seifert-von Kampen.*

3.) Any reducible $SU(2)$-representation of π factorizes through the abelianization of π. Hence any reducible $SU(2)$-representation of $\pi_1(F^*,x)$ factorizes through $\pi_1(F,x)$.

5.) The action is free by lemma 8.6. Since $SO(3)$ is compact, the action is proper.

We omitt the proof of 2.), as 2.) is not used in the construction of the Casson invariant. ∎

Notation 8.13 $\hat{R} := (R-S)/SO(3) \qquad \hat{Q}_i := (Q_i - S)/SO(3)$ ∎

Proposition 8.14

1. *$\hat{R}$ is a smooth open manifold of dimension $6g-6$.*
2. *$\hat{Q}_i$ is a properly embedded open submanifold of dimension $3g-3$ in $\hat{R}$.*
3. *$\hat{Q}_1 \cap \hat{Q}_2$ is compact.*

Proof : *We derive 1.) and 2.) directly from proposition 8.12. It remains to prove 3.) that $\hat{Q}_1 \cap \hat{Q}_2$ is compact.*

Since $\pi_1(F^*,x) \longrightarrow \pi_1(M,x)$ is an epimorphism, we get :

$$S(\pi_1(M,x)) = S \cap R(\pi_1(M,x))$$

By the theorem of Seifert-von Kampen the square in the diagram of groups 7.14 is a push out of groups. As the functor $R(?,G)$ turns push outs into pull backs, we conclude :

$$R(\pi_1(M,x)) = Q_1 \cap Q_2$$

This implies :

$$((Q_1 - S) \cap (Q_2 - S)) \coprod \{1\} = Q_1 \cap Q_2$$

Since Q_1 and Q_2 intersect at 1 transversely, $\{1\}$ is an open subset in $Q_1 \cap Q_2$. Since $(Q_1 - S) \cap (Q_2 - S)$ is a closed subset of the compact set $Q_1 \cap Q_2$, $(Q_1 - S) \cap (Q_2 - S)$ and hence its quotient under the $SO(3)$- action $\hat{Q}_1 \cap \hat{Q}_2$ is compact. ∎

If one has fixed orientations on $\hat{R}$ and $\hat{Q}_i$, then the intersection number $< \hat{Q}_1, \hat{Q}_2 >_{\hat{R}}$ is defined by proposition 8.14. One can find an isotopy of $\widehat{Q_i}$ which is constant outside a compact set containing $\hat{Q}_1 \cap \hat{Q}_2$ such that the intersection of $\hat{Q}_1$ and $\hat{Q}_2$ consists of finitely *many points where $\hat{Q}_1$ and $\hat{Q}_2$ meet transversely. Then the intersection number is the sum of these finitely many intersection points counted with a sign which depends on the local orientations.*

The orientation on M induces an orientation on W_1 and W_2 by restriction. Then F from W_1, F^* from F and ∂F^* from F^* inherit orientations by the general conventions for boundaries of oriented manifolds resp. by restriction. The orientation on ∂F^* determines a generator in $\pi_1(\partial F^*)$ and thus an orientation on R_∂ . Fix any orientation on R^*. As $R - S$ sits in the preimage of 1 of the map $\partial : R^* \longrightarrow R_\partial$, we obtain a short exact sequence

$$0 \longrightarrow T_x(R-S) \longrightarrow T_xR^* \xrightarrow{T_x\partial} T_1R_\partial \longrightarrow 0$$

Thus the oreintations of R^* and R_∂ induce an orientation on $R - S$. This determines also on orientation on $\hat{R}$ using the exact sequence

$$0 \longrightarrow T_1SO(3) \longrightarrow T_x(R-S) \longrightarrow T_1\hat{R} \longrightarrow 0$$

All in all we have explained, how an orientation of M induces an orientation on $\hat{R}$, if we have fixed an orientation on R^*. Choose any orientations on Q_1 and Q_2. This induces orientations on $\hat{Q}_1$ and $\hat{Q}_2$. Now we define

Definition 8.15 (Casson invariant)

Let M be a oriented homology 3-sphere . Define :

$$\lambda(M) := \frac{(-1)^g \cdot < \hat{Q}_1, \hat{Q}_2 >_{\hat{R}}}{2 \cdot < Q_1, Q_2 >_{R^*}}$$

Obviously this is independent of the choice of orientation of R^*, Q_1 and Q_2 because a change of one of these orientations changes the sign in the nominator and denominator in the fraction defining the Casson invariant simultaneously. The condition that M is a rational homology 3-sphere guarantees that $< Q_1, Q_2 >_{R^*}$ is not zero (see 8.10). We have to divide out this term to ensure that the choice of orientation on R^*, Q_1 and Q_2 do not matter. If we neglect this choice, the Casson invariant would reduce to a number mod 2 and hence just to the Rohlin invariant. But we even need that M is an integral homology sphere because then the only reducible $SO(3)$-representation of $\pi_1(M; \mathbf{Z})$ is the trivial one (see Lemma 8.5). This is crucial for the proof that the intersection of $\hat{Q}_1$ and $\hat{Q}_2$ in $\hat{R}$ is defined (see Proposition 8.14).

We have to show that the Casson-invariant is independent of the choice of Heegard-splitting. We begin with verifying, that we get the same invariant, if we interchange the order of the W_i-s to (W_2, W_1). If we keep all orientations as in (W_1, W_2), but interchange Q_1 and Q_2, we get :

$$< \hat{Q}_1, \hat{Q}_2 >_{\hat{R}} = (-1)^{(3g-3)\cdot(3g-3)} \cdot < \hat{Q}_2, \hat{Q}_1 >_{\hat{R}}$$
$$< Q_1, Q_2 >_{R^*} = (-1)^{3g \cdot 3g} < Q_2, Q_1 >_{R^*}$$

This implies :

$$\frac{(-1)^g \cdot < \hat{Q}_1, \hat{Q}_2 >_{\hat{R}}}{2\cdot < Q_1, Q_2 >_{R^*}} = -\frac{(-1)^g \cdot < \hat{Q}_2, \hat{Q}_1 >_{\hat{R}}}{2\cdot < Q_2, Q_1 >_{R^*}}$$

If we interchange W_1 and W_2, F^* and hence R_∂ and R get the reversed orientations, whereas we can assume that the orientations on Q_1, Q_2 and R^* are unchanged. Hence the order of the W_i-s does not matter.

In order to show that the choice of Heegard splitting does not matter, it suffices because of theorem 7.10 to analyse what happens under suspension. Then the genus of the Heegard decomposition is increased by one and the corresponding diagram of representation spaces can be identified with :

8.16

$$\begin{tikzcd}
 & & & Q_1 \times S^3 \times \{1\} \arrow[dl] & \\
R'_\partial & R^* \times S^3 \times S^3 \arrow[l, "\partial'"'] & R' \arrow[l] & & R(\pi_1(M)) \arrow[ul] \arrow[dl] \\
 & & & Q_2 \times \{1\} \times S^3 \arrow[ul] &
\end{tikzcd}$$

We compute for the intersection number $< Q'_1, Q'_2 >_{R^{*\prime}}$, where $< \ , \ >$ denotes both the intersection and the Kronecker pairing and $[\ ]$ denotes the images of the fundamental classes or the Poincar'e duals of them in the homology resp. cohomology of R^* resp. $R^* \times S^3 \times S^3$:

$< Q'_1, Q'_2 >_{R^{*\prime}} =$

$< Q_1 \times S^3 \times \{1\}, Q_2 \times \{1\} \times S^3 >_{R^* \times S^3 \times S^3} =$

$< [Q_1 \times S^3 \times \{1\}] \cup [Q_2 \times \{1\} \times S^3], [R^* \times S^3 \times S^3] > =$

$(-1)^g \cdot < [Q_1] \cup [Q_2] \cup [S^3 \times \{1\}] \cup [\{1\} \times S^3], [R^*] \cup [S^3 \times S^3] > =$

$(-1)^g \cdot < [Q_1] \cup [Q_2], [R^*] > \cdot < [S^3 \times \{1\}] \cup [\{1\} \times S^3], [S^3 \times S^3] > =$

$(-1)^g \cdot < Q_1, Q_2 >_{R^*}$

We get on the quotient level :

$$< \hat{Q}_1{}', \hat{Q}_2{}' >_{\hat{R}'} = \ (-1)^{g-1} < \hat{Q}_1, \hat{Q}_2 >_{\hat{R}}$$

To prove this, one purtubates $Q_1 \times S^3 \times \{1\}$ within $R^{*\prime}$ to P_1 relative to a compact subset such that $\hat{P}_1$ and $(Q_2 \times \{1\} \times S^3)\hat{}$ are transverse in $\hat{R}'$. Then $P_1 \cap R^* \times \{1\} \times \{1\}$ lies in R and is on the quotient level a pertubation of $\hat{Q}_1$ which is transverse to $\hat{Q}_2$ in $\hat{R}$. Now the set of intersection points of the two relevant sets agree, but the signs of the intersection points differ by a sign $(-1)^{g-1}$, because the dimension of $\hat{Q}_2$ is $3g-3$. This shows that suspending the Heegard decomposition does not affect the number appearing in the definition of the Casson invariant. This shows that the Casson invariant is well-defined.

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