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Novikov-Shubin invariants for arbitrary group actions and their positivity

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Dedicated to Mel Rothenberg on the occasion of his 65th birthday

Abstract. We extend the notion of Novikov-Shubin invariant for free Γ-CW-complexes of finite type to spaces with arbitrary Γ-actions and prove some statements about their positivity. In particular we apply this to classifying spaces of discrete groups.

1. Introduction

In [**8**] the first author extended the notion of L^2-Betti number for free Γ-CW-complexes of finite type to topological spaces with arbitrary Γ-actions. The key ingredient there is the notion of a dimension function for arbitrary modules over a finite von Neumann algebra $\mathcal{A}$, extending the classical notion for finitely generated projective $\mathcal{A}$-modules defined in terms of the von Neumann trace of an associated projection. These notions turned out to be useful in particular in the case where Γ is amenable (see [**8**], [**9**]).

In this paper we carry out an analogous program for Novikov-Shubin invariants. So we will introduce for arbitrary $\mathcal{A}$-modules in Section 2 the equivalent notion of capacity (which is essentially the inverse of the Novikov-Shubin invariant and was introduced for finitely presented $\mathcal{A}$-modules in [**2**] and [**7**]) and study its main properties. This enables us to define the p-th capacity for a space X with an arbitrary action of a discrete group. Originally Novikov-Shubin invariants were defined in terms of the heat kernel of the universal covering of a compact Riemannian manifold in [**10**], [**11**].

We will use the extension to study Novikov-Shubin invariants of groups in Section 3. The key observation is that a group Γ which may have a model of finite type for $B\Gamma$ may contain interesting normal subgroups for which the classical definition does not apply because its classifying space is not even of finite type. We are in particular interested in the conjecture that for a regular covering of a CW-complex of finite type the Novikov-Shubin invariants are always positive, or equivalently, the capacities are always finite [**6**, Conjecture 7.1]. Our main results

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in this direction are Theorems 3.7 and 3.9 which say among other things that $\alpha_p(\Gamma) \geq 1$ for all $p \geq 1$ if Γ contains $\mathbb{Z}^n$ as normal subgroup for some $n \geq 1$ and that $\alpha_p(\Gamma) \geq 1$ for $p = 1, 2$ if Γ is elementary amenable and contains no infinite locally finite subgroup. In particular the first and second Novikov-Shubin invariants of the universal covering of a CW-complex X of finite type are greater or equal to 1 if $\pi_1(X)$ satisfies the condition above since they agree with the ones for $\pi_1(X)$.

We mention that α_p of a space or group is the Novikov-Shubin invariant associated to the p-th differential. Sometimes Novikov-Shubin invariants are also defined in terms of the Laplacian. If we denote the latter ones by $\widetilde{\alpha}_p$, the connection between these two invariants is $\widetilde{\alpha}_p = \min\{\alpha_{p+1}, \alpha_p\}$. Our normalization is such that $\alpha_1 = \widetilde{\alpha}_0$ of the universal covering of S^1 is 1. The p-th capacity will essentially be the inverse of the $(p+1)$-th Novikov-Shubin invariant.

2. Capacity of modules

There is the notion of the Novikov-Shubin invariant $\alpha(M) \in [0, \infty] \amalg \{\infty^+\}$ of a finitely presented $\mathcal{A}$-module M as defined in [**7**, Definition 3.1]. It is the Novikov-Shubin invariant of the spectral density function of a morphism of Hilbert $\mathcal{A}$-modules $f : l^2(\mathcal{A})^m \to l^2(\mathcal{A})^n$ which corresponds to a presentation matrix $A \in M(m, n, \mathcal{A})$ for $M = \operatorname{coker}(A : \mathcal{A}^m \to \mathcal{A}^n)$. We want to extend it to arbitrary $\mathcal{A}$-modules. For convenience we will use the notion of capacity (see [**2**, 4.8]) which is essentially the reciprocal of the Novikov-Shubin invariant and extend it to arbitrary $\mathcal{A}$-modules. In the sequel we will use the notion and properties of the dimension $\dim(M)$ of an $\mathcal{A}$-module introduced in [**7**, Theorem 0.6].

Definition 2.1. We call an $\mathcal{A}$-module M *measurable* if it is the quotient of a finitely presented $\mathcal{A}$-module L with $\dim(L) = 0$. We call an $\mathcal{A}$-module M *cofinal-measurable* if each finitely generated submodule is measurable. In particular this implies $\dim M = 0$. □

We will see later that the following definition of capacity is particularly well behaved on the the class of cofinal-measurable modules.

Definition 2.2. Let L be a finitely presented $\mathcal{A}$-module with $\dim(L) = 0$. Define its *capacity*

$$c(L) \in \{0^-\} \cup [0, \infty]$$

by

$$c(L) := \frac{1}{\alpha(L)},$$

where 0^- is a new formal symbol different from 0, and $0^{-1} = \infty$, $\infty^{-1} = 0$ and $(\infty^+)^{-1} = 0^-$. Let M be a measurable $\mathcal{A}$-module. Define

$$c'(M) \ := \ \inf\{c(L) \mid L \text{ finitely presented, } \dim(L) = 0, \ M \text{ quotient of } L\}.$$

Let N be an arbitrary $\mathcal{A}$-module. Define (see also Notation 2.5)

$$c''(N) \ := \ \sup\{c'(M) \mid M \text{ measurable, } M \subset N\}. \quad \square$$

Note that $\dim(N)$ is not necessarily zero. In fact $c''(N)$ measures the size of the largest zero-dimensional submodule of N (compare [**8**, 2.15]).

The invariants take value in $\{0^-\} \amalg [0, \infty]$. We define an order $<$ on this set by the usual one on $[0, \infty)$ and the rule $0^- < r < \infty$ for $r \in [0, \infty)$. For two elements

$r, s \in [0^-, \infty]$ we define another element $r + s = s + r$ in this set by the ordinary addition in $[0, \infty)$ and by

$$0^- + r \quad = \quad r \quad \text{and} \quad \infty + r \quad = \quad \infty \quad \text{for } r \in [0^-, \infty]$$

We have introduced $c(M)$ as the inverse of $\alpha(M)$ because then $c(M)$ becomes bigger if M becomes bigger and some of the formulas become nicer. Notice that for a finitely presented $\mathcal{A}$-module M with $\dim(M) = 0$ we have $c(M) = 0^-$ if and only if M is trivial. Hence a measurable $\mathcal{A}$-module M satisfies $c'(M) = 0^-$ if and only if it is trivial. A measurable $\mathcal{A}$-module is finitely generated but not necessarily finitely presented. We have for an arbitrary $\mathcal{A}$-module N that $c''(N) = 0^-$ if and only if any $\mathcal{A}$-map $f : M \longrightarrow N$ from a finitely presented $\mathcal{A}$-module M with $\dim(M) = 0$ to N is trivial. A cofinal-measurable $\mathcal{A}$-module M is trivial if and only if $c''(M) = 0^-$ (see also Example 2.10).

We will show that our different definitions of capacity coincide and prove their basic properties. For this we need

LEMMA 2.3. *Let $0 \to K \to P \to Q \to 0$ be an exact sequence of finitely presented $\mathcal{A}$-modules of dimension zero. Then*

$$c(K) \leq c(P), \qquad c(Q) \leq c(P)$$
$$c(P) \leq c(K) + c(Q).$$

PROOF. By [**7**, 3.4] we find resolutions $0 \to F_K \to F_K \to K \to 0$ and $0 \to F_Q \to F_Q \to Q \to 0$ with F_K and F_Q finitely generated free. Now we can construct a resolution $0 \to F \to F \to P \to 0$ with $F = F_K \oplus F_Q$ which fits into a short exact sequence of resolutions. Application of [**6**, Lemma 1.12] to this situation together with the equivalence of finitely generated Hilbert $\mathcal{A}$-modules and finitely generated projective algebraic $\mathcal{A}$-modules as in [**7**] gives the result. □

PROPOSITION 2.4. (1) *If M is a finitely presented $\mathcal{A}$-module which satisfies $\dim(M) = 0$, then M is measurable and*

$$c'(M) \;=\; c(M).$$

(2) *If M is a measurable $\mathcal{A}$-module, then M is cofinal-measurable and*

$$c''(M) \;=\; c'(M).$$

NOTATION 2.5. In view of Proposition 2.4 we sometimes will not distinguish between c, c' and c'' in the sequel.

PROOF OF PROPOSITION 2.4. If M and N are finitely presented with dimension zero and $p : N \to M$ is surjective, then $c(M) \leq c(N)$ by Lemma 2.3. We use that $\ker p$ is finitely presented by [**7**, Theorem 0.2]. The first statement follows.

Suppose M is measurable and $M_0 \subset M$ is finitely generated. We have to show that M_0 is measurable and $c'(M_0) \leq c'(M)$. Suppose L is finitely presented with $\dim(L) = 0$ and there is an epimorphism $f : L \longrightarrow M$. As M_0 is finitely generated, we can find a finitely generated module $K \subset L$ with $f(K) = M_0$. As K is finitely generated and L is finitely presented, L/K is finitely presented. Since the category of finitely presented $\mathcal{A}$-modules is abelian [**7**, Theorem 0.2], K is finitely presented. Hence M_0 is measurable. Moreover $c(K) \leq c(L)$ by Lemma 2.3. Therefore

$$c'(M_0) \leq \inf_{K \text{ as above}} c(K) \leq \inf_{L \text{ as above}} c(L) = c'(M).$$

□

Before we give a list of the basic properties of capacity, we state a simple lemma which will be used repeatedly during the proof.

Lemma 2.6. *Suppose M is a measurable $\mathcal{A}$-module and $p : F \to M$ a projection of a finitely generated free module onto M. Then*

$$c(M) = \inf\{c(F/K) |\, K \subset \ker p \text{ finitely generated with } \dim(F/K) = 0\}.$$

The set on the right hand side is nonempty if and only if M is measurable.

Proof. Let d denote the number on the right. Every F/K as above is finitely presented and projects onto M. From Definition 2.2 we get $d \geq c(M)$. On the other hand let $q : L \to M$ be an epimorphism with L finitely presented and $\dim L = 0$. We can lift p to a map f with $q \circ f = p$. Since the category of finitely presented modules is abelian [**7**, Theorem 0.2] the kernel of f is finitely generated. Moreover $\dim(F/\ker f) = \dim \operatorname{im} f \leq \dim L = 0$, $\ker f \subset \ker p$ and Lemma 2.3 implies $c(F/\ker f) \leq c(L)$. So for every L we found an F/K with $c(F/K) \leq c(L)$. This implies $d \leq c(M)$. □

Theorem 2.7.

(1) *Let $0 \longrightarrow M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \longrightarrow 0$ be an exact sequence of $\mathcal{A}$-modules. Then*
   (a) *$c(M_0) \leq c(M_1)$.*
   (b) *$c(M_2) \leq c(M_1)$, provided that M_1 is cofinal-measurable.*
   (c) *$c(M_1) \leq c(M_0) + c(M_2)$ if $\dim M_1 = 0$.*

(2) *Let $M = \bigcup_{i\in I} M_i$ be a directed union of submodules. Then*

$$c(M) = \sup\{c(M_i) \mid i \in I\}.$$

(3) *Let M be the colimit $\operatorname{colim}_{i\in I} M_i$ of a directed system of $\mathcal{A}$-modules with structure maps $\phi_{ij} : M_i \to M_j$. Then*

$$c(M) \quad \leq \quad \liminf_{i\in I} c(M_i) \quad \left(:= \sup_{i\in I}\{\inf_{j\geq i} c(M_j)\}\right).$$

If every M_i is measurable and ϕ_{ij} is surjective for every $j \geq i$, then

$$c(M) = \inf_{i\in I} c(M_i).$$

(4) *Let $\{M_i \mid i \in I\}$ be a family of $\mathcal{A}$-modules. Then*

$$c\left(\oplus_{i\in I} M_i\right) \quad = \quad \sup\{c(M_i) \mid i \in I\}.$$

Remark 2.8. Because zero-dimensional modules will be most important for us, we give a reminder of basic properties of the dimension, which are stated in or follow from [**8**]:

(1) If $0 \to M_0 \to M_1 \to M_2 \to 0$ is an exact sequence of $\mathcal{A}$-modules, then

$$\dim(M_1) = 0 \quad \Longleftrightarrow \quad \dim(M_0) = \dim(M_2) = 0.$$

(2) $\dim(\oplus_{i\in I}(M_i)) = 0 \iff \dim(M_i) = 0$ for all $i \in I$.
(3) If $M = \bigcup_{i\in I} M_i$, then $\dim M = 0 \iff \dim M_i = 0$ for all $i \in I$.
(4) If $M = \operatorname{colim}_{i\in I} M_i$ is the colimit of a directed system, then

$$\dim M \leq \liminf_{i\in I} \dim M_i.$$

Proof of Theorem 2.7. 1a) Every measurable submodule of M_0 is a measurable submodule of M_1, therefore $c''(M_0) \leq c''(M_1)$.

1b) For every measurable submodule M of M_2 we find a finitely generated submodule $N_M \subset M_1$ which projects onto M. If M_1 is cofinal-measurable, then N_M is measurable and $c'(N_M) \geq c'(M)$ since on the left we take the infimum over a smaller set of numbers. Therefore

$$c''(M_1) \geq \sup_{M \subset M_2 \text{ measurable}} c'(N_M) \geq \sup_{M \subset M_2 \text{ measurable}} c'(M) = c''(M_2).$$

1c) Step 1: We prove $c(M_1) \leq c(M_0) + c(M_2)$ if M_0 is measurable and M_2 is finitely presented. We will also see, that this implies, that M_1 is measurable:

By [**7**, Lemma 3.4] there is a finitely generated free $\mathcal{A}$-module F_2 and an exact sequence $0 \longrightarrow F_2 \xrightarrow{i_2} F_2 \longrightarrow M_2 \longrightarrow 0$. Let $q : F_0 \to M_0$ be a projection of a finitely generated free $\mathcal{A}$-module onto M_0.

We get the following commutative diagram with exact rows and columns where the F_i are finitely generated free:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M_0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F_0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K_0 & \longrightarrow & K_1 & \longrightarrow & i_2(F_2) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let $K_0' \subset K_0$ be a finitely generated submodule with $\dim F_0/K_0' = 0$ (Lemma 2.6). We can consider K_0' also as submodule of K_1. Let $s : i_2(F_2) \longrightarrow K_1$ be a section of the epimorphism $K_1 \longrightarrow i_2(F_2)$. Let K_1' be the (finitely generated!) submodule of K_1 generated by K_0' and the image of s. We obtain the exact sequence

$$0 \to K_0' \to K_1' \to i_2(F_2) \to 0.$$

Going to the quotients, we obtain a commutative diagram with epimorphisms as vertical maps whose lower row is an exact sequence of finitely presented $\mathcal{A}$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & F_0/K_0' & \longrightarrow & F_1/K_1' & \longrightarrow & M_2 & \longrightarrow & 0.
\end{array}$$

Note that this implies M_1 to be measurable. By Lemma 2.3

$$c(M_1) \overset{1b)}{\leq} c(F_1/K_1') \leq c(F_0/K_0') + c(M_2).$$

This holds for every K_0' as above, therefore also for the infimum $c(M_0)$ (by Lemma 2.6) in place of $c(F_0/K_0')$.

Step 2: We prove the inequality if M_0 and M_1 are arbitrary and M_2 is finitely presented: Choose $N_1 \subset M_1$ measurable. Let $N_0 := i^{-1}(N_1)$ and $N_2 := p(N_1)$. Let L_1 be a finitely presented module with $\dim(L_1) = 0$ projecting onto N_1. We get a commutative diagram with exact rows and surjective columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow \mathrm{id} & & \\ 0 & \longrightarrow & K_0 & \longrightarrow & L_1 & \longrightarrow & N_2 & \longrightarrow & 0. \end{array}$$

N_2 is the image of the composition $L_1 \to M_2$ and K_0 is the kernel of this map. Since L_1 and M_2 are finitely presented, by [**7**, Theorem 0.2] the same is true for N_2 and K_0. In particular N_0 is measurable. Therefore by Step 1

$$c(M_1) = \sup_{N_1 \subset M_1 \text{ measurable}} c(N_1) \leq \sup(c(N_0) + c(N_2)) \overset{1a)}{\leq} c(M_0) + c(M_2).$$

Step 3: We prove the inequality if M_1 is measurable: Obviously M_2 is also measurable. Let $f : L_2 \to M_2$ be a projection with L_2 finitely presented and $\dim(L_2) = 0$. By the pull back construction we obtain a commutative diagram with exact rows and epimorphisms as vertical arrows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & 0 \\ & & \mathrm{id}\uparrow & & \uparrow & & f\uparrow & & \\ 0 & \longrightarrow & M_0 & \longrightarrow & X_1 & \longrightarrow & L_2 & \longrightarrow & 0. \end{array}$$

Note that X_1 as a submodule of $L_2 \oplus M_1$ is cofinal-measurable by the proof of Proposition 2.4.2. Then by the last step and 1b)

$$c(M_1) \overset{1b)}{\leq} c(X_1) \leq c(M_0) + c(L_2).$$

This holds for every L_2 as above, therefore also for the infimum $c(M_2)$.

Step 4: Finally M_0, M_1 and M_2 are arbitrary $\mathcal{A}$-modules. Suppose $N_1 \subset M_1$ is measurable. We get the exact sequence $0 \to N_0 \to N_1 \to N_2 \to 0$ as above with $N_i \subset M_i$. Then by the previous step and 1a)

$$c(N_1) \leq c(N_0) + c(N_2) \overset{1a)}{\leq} c(M_0) + c(M_2).$$

This holds for all N_1 as above and passing to the supremum yields the desired inequality.

2.) This follows from 1a) and the fact, that each finitely generated and in particular each measurable submodule $L \subset M$ is contained in some M_i.

3.) Let $N \subset M$ be a measurable submodule. It suffices to show

$$c(N) \leq \liminf_{i \in I} c(M_i).$$

Since N is finitely generated and M is the colimit of a directed system, we find i_0 and a finitely generated $N_{i_0} \subset M_{i_0}$ which projects onto N. For $i \geq i_0$ set $N_i := \phi_{i_0 i}(N_{i_0}) \subset M_i$. Then $c(N_i) \leq c(M_i)$ by 1a) and (since I is directed) $\liminf_{i \geq i_0} c(N_i) \leq \liminf_{i \in I} c(M_i)$. Observe that N_i projects onto N for all $i \geq i_0$. We will show that there is $i_1 \in I$ such that N_i is measurable for $i \geq i_1$. Then by 1b) $c(N) \leq c(N_i)$ for $i \geq i_1$ and therefore also $c(N) \leq \liminf_{i \geq i_1} c(N_i)$ which implies the assertion.

Let $p_{i_0} : F \to N_{i_0}$ be a projection of a finitely generated free module onto N_{i_0} and let $p : F \to N$ be the composed projection. By Lemma 2.6, since N is measurable, we find a finitely generated submodule $K \subset \ker(p)$ with $\dim(F/K) = 0$. Because K is finitely generated and I is a directed system, we find $i_1 \geq i_0$ such that $\phi_{i_0 i_1} p_{i_0}(K) = 0$. Then $\phi_{i_0 i} p_{i_0}$ induces a projection of the finitely presented zero-dimensional module F/K onto N_i for every $i \geq i_1$. Therefore these N_i are measurable and the inequality follows.

For the second part suppose that every M_i is measurable and every ϕ_{ij} is surjective. 1b) implies that $\liminf c(M_i) = \inf c(M_i)$. It remains to prove $c(M) \geq \inf c(M_i)$. Choose some $a \in I$ and a projection $p_a : F \to M_a$ with F finitely generated free. By composition we get a projection $p : F \to M$. Let $K \subset \ker(p)$ be finitely generated with $\dim(F/K) = 0$. Since I is a directed system, we find $b \geq a$ so that with $p_b := \phi_{ab} p_a$ already $p_b(K) = 0$. Since ϕ_{ab} is surjective, $p_b : F \to M_b$ is onto. Therefore by 1b)

$$c(F/K) \geq c(M_b) \geq \inf c(M_i).$$

This holds for any finitely generated K as above, therefore also for the infimum in place of $c(F/K)$ which by Lemma 2.6 is $c(M)$.

4.) Since $\oplus_{i \in I} M_i = \bigcup_{\substack{J \subset I \\ J \text{ finite}}} \oplus_{i \in J} M_i$ and because of 2.) we may assume I is finite. By induction we restrict to the case $\oplus_{i \in I} M_i = M_0 \oplus M_1$. Because of 1a) it remains to prove

$$c\left(M_0 \oplus M_1\right) \quad \leq \quad \sup\{c(M_0), c(M_1)\}. \tag{2.9}$$

If M_0 and M_1 are finitely presented (2.9) follows from [**6**, Lemma 1.12] in the same way as Lemma 2.3 does. If M_0 and M_1 are measurable, we can choose epimorphisms $L_0 \to M_0$ and $L_1 \to M_1$ with L_0 and L_1 finitely presented and of dimension zero. Note that $L_0 \oplus L_1$ is finitely presented and therefore the result for the finitely presented case implies $c(M_0 \oplus M_1) \leq c(L_0 \oplus L_1) \leq \sup\{c(L_0), c(L_1)\}$. Since this holds for every choice of L_0 and L_1 we can pass to the infimum and get (2.9). Now let M_0 and M_1 be arbitrary modules. Then every measurable submodule $N \subset M_0 \oplus M_1$ is contained in $N_0 \oplus N_1$ where N_0 and N_1 are the images of N under projection to M_0 and M_1. In particular they are measurable as quotients of a measurable module. Applying the result in the measurable case we get $c(N) = c(N_0 \oplus N_1) \leq \sup\{c(N_0), c(N_1)\} \leq \sup\{c(M_0), c(M_1)\}$. Passing to the supremum on the left yields (2.9) for arbitrary modules M_0 and M_1. This finishes the proof of 2.7. □

There are examples showing that the inequality 1c) in Theorem 2.7 is sharp. Moreover the assumption on cofinality in 1b) is necessary by the following example.

Example 2.10. We construct a non-trivial finitely generated $\mathcal{A}$-module M with $\dim(M) = 0$ which contains no non-trivial measurable $\mathcal{A}$-submodule. In particular M is not cofinal-measurable and $c(M) = 0^-$. Moreover, we construct a quotient module N of M with $c(N) > c(M)$.

Take $\mathcal{A} = L^\infty(S^1)$ which can be identified with the group von Neumann algebra $\mathcal{N}(\mathbb{Z})$. Let χ_n be the characteristic function of the subset $\{\exp(2\pi i t) \mid t \in [1/n, 1 - 1/n]\}$ of S^1. Let P_n be the submodule in $P = L^\infty(S^1)$ generated by χ_n. It is a direct summand. Hence the quotient P/P_n is a finitely generated projective $\mathcal{A}$-module of dimension $2/n$. Projectivity implies $c(P/P_n) = 0^-$. Define $I := \bigcup_{n \in \mathbb{N}} P_n \subset L^\infty(S^1)$ and put $M = L^\infty(S^1)/I$. Then $M = \operatorname{colim}_{n \in \mathbb{N}} P/P_n$ is a finitely generated $\mathcal{A}$-module with $\dim(M) = 0$ and $c(M) = 0^-$ by Theorem 2.7.3 and Remark 2.8

Observe that the same argument applies to any von Neumann algebra $\mathcal{A}$ with a directed system $P_i \subset P$ of direct summands of a projective $\mathcal{A}$-module P such that $\dim P_i < \dim P$ but $\dim P = \sup_{i \in I} \dim P_i$.

For the quotient example, put $N_n = L^\infty(S^1)/((z-1)^n)$ for positive integers $n \geq 0$, where $((z-1)^n)$ is the ideal generated by the function $S^1 \longrightarrow \mathbb{C}$ sending z to $(z-1)^n$. Obviously $I \subset ((z-1)^n)$ so that N_n is a quotient of M. The $\mathcal{A}$-module N_n is finitely presented with $\alpha(N_n) = 1/n$ [**7**, Example 4.3]. Hence we get

$$c(M) = 0^-; \quad \text{but} \quad c(N_n) = n.$$

□

LEMMA 2.11. (1) *A finitely generated submodule of a measurable $\mathcal{A}$-module is again measurable.*
(2) *A quotient module of a measurable $\mathcal{A}$-module is again measurable.*
(3) *An $\mathcal{A}$-module M is cofinal-measurable if and only if it is the union of its measurable submodules.*
(4) *Submodules and quotient modules of cofinal-measurable $\mathcal{A}$-modules are again cofinal-measurable.*
(5) *Let $0 \longrightarrow M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \longrightarrow 0$ be an exact sequence of $\mathcal{A}$-modules. If M_0 and M_2 are cofinal-measurable, then M_1 is cofinal-measurable.*
(6) *The full subcategory of the abelian category of all $\mathcal{A}$-modules consisting of cofinal-measurable modules is abelian and closed under colimits over directed systems. Given $r \in \{0^-\} \amalg [0, \infty]$, this is also true for the full subcategory of cofinal-measurable $\mathcal{A}$-modules M with $c(M) \leq r$.*
(7) *If C_* is an $\mathcal{A}$-chain complex of cofinal-measurable $\mathcal{A}$-modules, then its homology $H_p(C_*)$ is cofinal-measurable for all p. Moreover $c(H_p(C)) \leq c(C_p)$.*

PROOF. 1.) This follows from the proof of Proposition 2.4.
2.) This is obvious.
3.) Since M is the union of its finitely generated submodules, it is the union of its measurable submodules provided that M is cofinal-measurable. Suppose that M is the union of its measurable submodules and $L \subset M$ is finitely generated. There are finitely many measurable submodules $K_1, K_2, \ldots, K_r$ such that L is contained in the submodule K generated by $K_1, K_2, \ldots, K_r$. Obviously K and therefore L is measurable by assertions 1.).
4.) follows from assertionss 1.) and 2.).
5.) We have to show for a finitely generated submodule $M_1' \subset M_1$ that it is measurable. Let $M_2' \subset M_2$ be the finitely generated submodule $p(M_1')$ and $M_0' \subset M_0$ be $i^{-1}(M_1')$. Since M_2 is cofinal-measurable, M_2' is measurable. Choose a finitely presented $\mathcal{A}$-module M_2'' with $\dim(M_2'') = 0$ together with an epimorphism $f : M_2'' \longrightarrow M_2'$. The pull back construction yields a commutative square with exact rows and epimorphisms as vertical arrows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_0' & \longrightarrow & M_1' & \longrightarrow & M_2' & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \overline{f}}\uparrow & & {\scriptstyle f}\uparrow & & \\
0 & \longrightarrow & M_0' & \xrightarrow{i''} & M_1'' & \xrightarrow{p''} & M_2'' & \longrightarrow & 0.
\end{array}$$

Since M_1' is finitely generated, there is a finitely generated submodule $M_1''' \subset M_1''$ such that $\overline{f}(M_1''') = M_1'$. Let $M_2''' \subset M_2''$ be the finitely generated submodule $p''(M_1''')$ and $M_0''' \subset M_0'$ be $(i'')^{-1}(M_1''')$. We obtain an exact sequence

$$0 \longrightarrow M_0''' \longrightarrow M_1''' \longrightarrow M_2''' \longrightarrow 0.$$

Since M_2''' is a finitely generated submodule of the finitely presented $\mathcal{A}$-module M_2'', the quotient M_2''/M_2''' is finitely presented. Since the category of finitely presented $\mathcal{A}$-modules is abelian [**7**, Theorem 0.2], the $\mathcal{A}$-module M_2''' is finitely presented. Since M_0''' is the kernel of an epimorphism of the finitely generated $\mathcal{A}$-module M_1''' onto the finitely presented $\mathcal{A}$-module M_2''', M_0''' is finitely generated. As M_0''' is a submodule of the cofinal-measurable $\mathcal{A}$-module M_0, the $\mathcal{A}$-module M_0''' is measurable. Since M_0''' is measurable and M_2''' finitely presented, M_1''' is measurable as follows from the first step of the proof of Theorem 2.7.1c). Hence M_1' is measurable by assertions 2.), since it is a quotient of M_1'''.
6.) If $M = \operatorname{colim}_{i\in I} M_i$ is the colimit of a directed system with $\psi_i : M_i \to M$, then M is the directed union of $\psi_i(M_i)$. If all M_i are cofinal-measurable, then their quotients $\psi_i(M_i)$ are cofinal-measurable by assertions 4.) and the same is true for M by assertions 3.). The assertions now follows from 4.), 5.) and Theorem 2.7.1.
7.) follows from 4.) and Theorem 2.7.1. □

Finally we discuss the behaviour of these notions under induction and restriction for subgroups $i : \Delta \to \Gamma$. The functor i_* was already investigated in [**8**, Theorem 3.3] where it is shown that i_* is exact and $\dim_{\mathcal{N}(\Delta)}(M) = \dim_{\mathcal{N}(\Gamma)}(i_*M)$.

Lemma 2.12. *Let $i : \Delta \longrightarrow \Gamma$ be an inclusion of groups, then i induces an inclusion $i : \mathcal{N}(\Delta) \to \mathcal{N}(\Gamma)$.*

(1) *If M is a measurable or cofinal-measurable respectively $\mathcal{N}(\Delta)$-module, then the $\mathcal{N}(\Gamma)$-module $i_*M := \mathcal{N}(\Gamma) \otimes_{\mathcal{N}(\Delta)} M$ is measurable or cofinal-measurable respectively and*

$$c_{\mathcal{N}(\Delta)}(M) \quad = \quad c_{\mathcal{N}(\Gamma)}(i_*M).$$

For an arbitrary $\mathcal{N}(\Delta)$-module N we have $c(N) \leq c(i_*N)$.

(2) *If the index of Δ in Γ is finite and M is an arbitrary $\mathcal{N}(\Delta)$-module, then*

$$c_{\mathcal{N}(\Delta)}(M) = c_{\mathcal{N}(\Gamma)}(i_*M).$$

(3) *If the index of Δ in Γ is finite, N is an $\mathcal{N}(\Gamma)$-module and i^*N the $\mathcal{N}(\Delta)$-module obtained by restriction, then*

$$c_{\mathcal{N}(\Delta)}(i^*N) = c_{\mathcal{N}(\Gamma)}(N)$$

and i^*N is measurable or cofinal-measurable if and only if N has the same property.

Proof. 1.) First suppose M is a finitely presented zero-dimensional $\mathcal{N}(\Delta)$-module. Choose a resolution $0 \to F \xrightarrow{f} F \to M \to 0$ with a finitely generated free module F as in [**7**, Lemma 3.4]. Apply the proof of [**6**, Lemma 3.6] to f, taking into account the equivalence of free Hilbert $\mathcal{N}\Gamma$-modules and free algebraic $\mathcal{N}\Gamma$-modules of [**7**]. It follows that i_*M is a finitely presented $\mathcal{N}(\Gamma)$-module with $\dim_{\mathcal{N}(\Gamma)}(i_*M) = 0$ and $c_{\mathcal{N}(\Delta)}(M) = c_{\mathcal{N}(\Gamma)}(i_*M)$.

Next let M be a measurable $\mathcal{N}(\Delta)$-module. Let $p : F \to M$ be a projection of a finitely generated free $\mathcal{N}(\Delta)$-module onto M. Set $K := \ker(p)$. Then $i_*p : i_*F \to i_*M$ is surjective with kernel i_*K (since i_* is exact). If $K_1 \subset K$ is finitely generated with $\dim F/K_1 = 0$ (such a module exists by Lemma 2.6), then $i_*K_1 \subset i_*K$ is also finitely generated with

$$\dim(i_*F/i_*K_1) \overset{\text{exactness}}{=} \dim i_*(F/K_1) = \dim F/K_1 = 0.$$

Since i_*F/i_*K_1 is finitely presented and projects onto i_*M the latter module is measurable. Moreover by Lemma 2.6 and the first step applied to F/K_1

$$c(i_*M) \leq \inf_{K_1} c(i_*F/i_*K_1) = \inf_{K_1} c(F/K_1) = c(M).$$

Choose on the other hand a finitely generated $\mathcal{N}(\Gamma)$-submodule $L \subset i_*K$ with $\dim(i_*F)/L = 0$. For L we find finitely many generators $\sum u_i \otimes k_i \in i_*K = \mathcal{N}(\Gamma) \otimes_{\mathcal{N}(\Delta)} K$. Let L' be the submodule of K generated by all the k_i. Then $L \subset i_*L'$, therefore

$$0 \leq \dim F/L' = \dim i_*F/i_*L' \leq \dim i_*F/L = 0$$

and (since F/L' is finitely presented and by 2.7.1b))

$$c(M) \leq c(F/L') = c(i_*F/i_*L') \leq c(i_*F/L).$$

Since this holds for arbitrary L as above, Lemma 2.6 implies $c(M) \leq c(i_*M)$.

If M is cofinal-measurable, then it is the union $\cup_{i\in I} M_i$ over the directed system of its measurable $\mathcal{N}(\Delta)$-submodules. Since i_* is exact, the $\mathcal{N}(\Gamma)$-module i_*M is the union $\bigcup_{i\in I} i_*M_i$ over the directed system of measurable $\mathcal{N}(\Gamma)$-submodules i_*M_i. We conclude from Theorem 2.7.2 and the previous step that i_*M is cofinal-measurable and

$$c_{\mathcal{N}(\Delta)}(M) = \sup_{i\in I}\{c_{\mathcal{N}(\Delta)}(M_i)\} = \sup_{i\in I}\{c_{\mathcal{N}(\Gamma)}(i_*M_i))\} = c_{\mathcal{N}(\Gamma)}(i_*M).$$

Last let M be an arbitrary $\mathcal{N}(\Delta)$-module. Since every measurable $\mathcal{N}(\Delta)$-submodule of M induces an $\mathcal{N}(\Gamma)$-submodule of i_*M of the same capacity, $c(M) \leq c(i_*M)$ by Definition 2.2.

3.) We begin with studying the restriction. Here $i^*\mathcal{N}(\Gamma) = \oplus_{i=1}^{[\Gamma:\Delta]}\mathcal{N}(\Delta)$ since the same holds for $\mathbb{C}\Gamma$ as a $\mathbb{C}\Delta$-module and $\mathcal{N}(\Gamma) = \mathcal{N}(\Delta) \otimes_{\mathbb{C}\Delta} \mathbb{C}\Gamma$. This observation and the proof of [**6**, Lemma 3.6] imply that if N is a finitely presented $\mathcal{N}(\Gamma)$-module, then i^*N is finitely presented as $\mathcal{N}(\Delta)$-module and

$$\dim_{\mathcal{N}(\Delta)}(i^*N) = [\Gamma : \Delta] \dim_{\mathcal{N}(\Gamma)}(N); \qquad c_{\mathcal{N}(\Delta)}(i^*N) = c_{\mathcal{N}(\Gamma)}(N).$$

If N is arbitrary and $L \to N$ a projection of a finitely presented zero-dimensional $\mathcal{N}(\Gamma)$-module, then $i^*L \to i^*N$ is a corresponding projection and $c(L) = c(i^*L)$. If on the other hand L' is a zero-dimensional $\mathcal{N}(\Delta)$-module projecting onto i^*N, then $i_*L' = \mathcal{N}(\Gamma)\otimes_{\mathcal{N}(\Delta)} L'$ is a finitely presented $\mathcal{N}(\Gamma)$-module naturally projecting onto N with the same dimension and capacity. In particular N is measurable if and only if i^*N is measurable and by Definition 2.2 the capacities coincide in this case.

Any measurable submodule of an $\mathcal{N}(\Gamma)$-module N restricts to a measurable $\mathcal{N}(\Delta)$-submodule of i^*N with the same capacity. On the other hand, if $U \subset i^*N$ is a measurable $\mathcal{N}(\Delta)$-submodule and V is the $\mathcal{N}(\Gamma)$-module generated by U, then V is a quotient of i_*U and U a submodule of V, therefore by assertions 1.) and Theorem 2.7.1 V is measurable and

$$c(U) = c(i_*U) \geq c(V) = c(i^*V) \geq c(U).$$

Definition 2.2 implies $c(N) = c(i^*N)$. The cofinal-measurable case is proven as above.
2.) Let M be an $\mathcal{N}(\Delta)$-module. Then $i^*i_*M \cong \oplus_{i=1}^{[\Gamma:\Delta]} M$. By 1.) and 3.)

$$c(M) \leq c(i_*M) = c(i^*i_*M) = c(\oplus_{i=1}^{[\Gamma:\Delta]} M) = c(M).$$

□

3. Capacity of groups

In this section we apply the concepts developed so far to define and study Novikov-Shubin invariants respectively capacities for arbitrary spaces with an arbitrary action of a discrete group. In particular we will apply them to arbitrary groups G via the classifying space EG.

Definition 3.1. Let X be a topological space with an action of the discrete group Γ. Let $H_p^\Gamma(X;\mathcal{N}(\Gamma))$ be the $\mathcal{N}(\Gamma)$-module given by the p-th homology of the chain complex $\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_*^{\text{sing}}(X)$, where $C_*^{\text{sing}}(X)$ is the singular chain complex of X. Define the *p-th capacity* of X

$$c_p(X;\mathcal{N}(\Gamma)) \ := c(H_p^\Gamma(X;\mathcal{N}(\Gamma))).$$

Define the *p-th capacity* of the group Γ by

$$c_p(\Gamma) \ := \ c_p(E\Gamma;\mathcal{N}(\Gamma)),$$

where $E\Gamma$ is any universal free Γ-space.

A group is *locally* finite, nilpotent, abelian, ... respectively, if any finitely generated subgroup is finite, nilpotent, abelian, ... respectively. A group is *virtually* nilpotent, abelian, ... respectively, if it contains a subgroup of finite index which is nilpotent, abelian, ... respectively.

If S is a finite set of generators for the group G, let $b_S(k)$ be the number of elements in G which can be written as a word in k letters of $S \cup S^{-1} \cup \{1\}$. The group G has *polynomial growth of degree not greater than* n if there is C with $b_S(k) \leq C \cdot k^d$ for all $k \geq 1$. This property is a property of G and independent of the choice of the finite set S of generators. We say that G has *polynomial growth* if it has polynomial growth of degree not greater than n for some $n > 0$. A finitely generated group Γ is *nilpotent* if Γ possesses a finite *lower central series*

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_s = \{1\} \qquad\qquad \Gamma_{k+1} = [\Gamma, \Gamma_k].$$

Let n_i be the rank of the finitely generated abelian group Γ_i/Γ_{i+1} and let n be the integer $\sum_{i\geq 1} i \cdot n_i$. Suppose that $\overline{\Gamma}$ contains Γ as subgroup of finite index. Then for any finite set S of generators of $\overline{\Gamma}$ there is a constant $C > 0$ such that $C^{-1} \cdot k^n \leq b_S(k) \leq C \cdot k^n$ holds for any $k \geq 1$ and in particular $\overline{\Gamma}$ has polynomial growth precisely of degree n [**1**, page 607 and Theorem 2 on page 608].

A famous result of Gromov [**3**] says that a finitely generated group has polynomial growth if and only if it is virtually nilpotent.

Proposition 3.2. (1) *If* Γ *is not locally finite, then* $H_0^\Gamma(E\Gamma;\mathcal{N}(\Gamma))$ *is measurable and*

$$c_0(\Gamma) = \inf\{c_0(\Gamma') |\ \Gamma' < \Gamma \textit{ infinite finitely generated}\}.$$

If Γ *is locally finite, then* $c_0(\Gamma) = 0^-$ *but* $H_0^\Gamma(E\Gamma;\mathcal{N}(\Gamma))$ *is non-trivial and is in particular not cofinal-measurable.*

(2) *Suppose that Γ is finitely generated. Then*

$$c_0(\Gamma) = \begin{cases} 0^-; & \text{if } \Gamma \text{ is finite or non-amenable,} \\ 1/n; & \text{if } \Gamma \text{ has polynomial growth of degree } n, \\ 0; & \text{if } \Gamma \text{ is infinite and amenable, but not virtually nilpotent.} \end{cases}$$

This computes c_0 for every finitely generated group.

(3) *Let Γ be an arbitrary group. Then*

$$\Gamma \text{ is locally finite or non-amenable} \iff c_0(\Gamma) = 0^-.$$

If Γ is locally virtually nilpotent but not locally finite, then

$$c_0(\Gamma) = \inf\{c_0(\Gamma') |\, \Gamma' < \Gamma \text{ infinite finitely generated nilpotent}\}.$$

If Γ is amenable and contains a subgroup which is finitely generated but not virtually nilpotent, then

$$c_0(\Gamma) = 0.$$

Note that every group belongs to one of the categories and that $c_0(\Gamma) > 0$ implies that Γ is locally virtually nilpotent but not locally finite.

The above also applies to c_0 of arbitrary path-connected Γ-spaces by [**8**, 4.10]. Observe that $H_0^\Gamma(E\Gamma; \mathcal{N}(\Gamma))$ is not cofinal-measurable if Γ is finite or locally finite. This is responsible for the clumsiness of some of the statements below because Theorem 2.7 1b) becomes false without the condition cofinal-measurable as shown in Example 2.10.

Proof. Remember that $H_0^\Gamma(E\Gamma; \mathcal{N}(\Gamma)) = \mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}$ which has dimension zero if and only if Γ is infinite and is trivial if and only if Γ is nonamenable by [**8**, 4.10]. Moreover $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}$ is finitely presented if Γ is finitely generated.

1.) If Γ' is finitely generated infinite $\mathcal{N}(\Gamma') \otimes_{\mathbb{C}\Gamma'} \mathbb{C}$ is measurable and by Lemma 2.12.1 $\mathcal{N}(\Gamma) \otimes_{\mathcal{N}(\Gamma')} \mathcal{N}(\Gamma') \otimes_{\mathbb{C}\Gamma'} \mathbb{C}$ is measurable and $c(\mathcal{N}(\Gamma) \otimes_{\mathcal{N}(\Gamma')} \mathcal{N}(\Gamma') \otimes_{\mathbb{C}\Gamma'} \mathbb{C}) = c(\mathcal{N}(\Gamma') \otimes_{\mathbb{C}\Gamma'} \mathbb{C})$. If Γ is not locally finite the system of infinite finitely generated subgroups is cofinal and therefore

$$\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C} = \operatorname{colim} \mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma'} \mathbb{C},$$

where the colimit is taken over the directed system of infinite finitely generated subgroups. Now the second part of Theorem 2.7.3 yields the claim if Γ is not locally finite.

The proof of Theorem 3.7.3 shows for locally finite Γ

$$H_0^\Gamma(E\Gamma; \mathcal{N}(\Gamma)) \;=\; \operatorname{colim}_{\Delta \subset \Gamma} H_0^\Delta(E\Delta; \mathcal{N}(\Gamma)) \;=\; \operatorname{colim}_{\Delta \subset \Gamma} H_0(E\Delta) \otimes_{\mathbb{C}\Gamma} \mathcal{N}(\Gamma)),$$

where $\Delta \subset \Gamma$ runs though the finite subgroups. For finite Δ, the $\mathbb{C}\Delta$-module $H_0(E\Delta)$ is projective and hence the $\mathcal{N}(\Gamma)$-module $H_0^\Delta(E\Delta; \mathcal{N}(\Gamma))$ is projective which implies $c(H_0^\Delta(E\Delta; \mathcal{N}(\Gamma))) = 0^-$. By Theorem 2.7.3

$$c(\Gamma) = c(H_0^\Gamma(E\Gamma; \mathcal{N}(\Gamma))) = 0^-.$$

Since Γ is non-amenable, $H_0^\Gamma(E\Gamma; \mathcal{N}(\Gamma))$ is non-trivial and hence cannot be cofinal-measurable.

2.) If Γ is finite or non-amenable, then $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}$ is finitely generated projective or trivial and therefore $c_0(\Gamma) = 0^-$. If Γ is infinite and amenable, then $H_0^\Gamma(E\Gamma; \mathcal{N}(\Gamma))$ is finitely presented and zero-dimensional but non-trivial, therefore $c_0(\Gamma) \geq 0$. The rest is the content of Lemma 3.3.

3.) This follows by combining the above results. □

Lemma 3.3. *Suppose Γ is a finitely generated infinite group. Then $c_0(\Gamma) = 1/n$ if Γ has polynomial growth precisely of rate n. If Γ is not virtually nilpotent, then $c_0(\Gamma) \leq 0$.*

Proof. First observe that Γ has polynomial growth of precisely degree n if and only if the recurrency probablity $p(k)$ of the natural random walk on Γ decreases polynomially with exponent $n/2$, i.e. there is a constant $C > 0$ with $C^{-1} \cdot k^{-n/2} \leq p(n) \leq C \cdot k^{-n/2}$ for $k \geq 1$ and it is not virtually nilpotent if and only if for any $n > 0$ there is $C(n) > 0$ satisfying $p(k) \leq C \cdot k^{-n}$ for $k \geq 1$. These are results of Varopoulos [**12**], compare [**13**, 6.6 and 6.7].

Now we have to translate this statement to information about the spectrum of the Laplacian. We will prove the following result from which our lemma follows: the finitely generated group Γ has polynomially decreasing recurrence probability with exponent $n/2$ if and only if $c_0(\Gamma) = 1/n$.

This can be deduced from [**12**]. We will give a self-contained and simple proof.

If S is a finite set of generators of Γ, then

$$H_0^{\Gamma}(E\Gamma; \mathcal{N}(\Gamma)) = \operatorname{coker}(\oplus_{s\in S}\mathcal{N}(\Gamma) \xrightarrow{d_0 = \oplus(s-1)} \mathcal{N}(\Gamma)).$$

Therefore $c_0(\Gamma)$ is the inverse of the Novikov-Shubin invariant $\alpha_0(\Gamma)$ of d_0, which we compute from the combinatorial Laplacian $\Delta_0 = 1 - P$. Here P is the transition operator $P(g) = (1/|S|) \cdot \sum_{s\in S} sg$. The recurrence probability is given by

$$p(k) := (P^k(e), e) = \operatorname{tr}(P^k),$$

and the spectral density function of Δ_0 by

$$F(\lambda) = \operatorname{tr}(\chi_{[1-\lambda,1]}(P)).$$

All of the operators in question are positive and therefore for $k \in \mathbb{N}$ and $0 < \lambda < 1$

$$(1-\lambda)^k \chi_{[1-\lambda,1]}(P) \leq P^k \leq (1-\lambda)^k \chi_{[0,1-\lambda]}(P) + \chi_{[1-\lambda,1]}(P) \leq 1. \tag{3.4}$$

Application of the trace to these inequalities gives

$$(1-\lambda)^k F(\lambda) \leq p(k) \leq (1-\lambda)^k + F(\lambda) \qquad \text{for all } 0 < \lambda < 1. \tag{3.5}$$

The first inequality implies if $0 < \lambda < 1$

$$\frac{\ln F(\lambda)}{\ln \lambda} \geq \frac{\ln p(k)}{\ln \lambda} - k\frac{\ln(1-\lambda)}{\ln \lambda}.$$

If $p(k) \leq Ck^{-a}$ for $C > 0$, then, putting $k = [\lambda^{-1}]$ (the largest integer not larger than λ^{-1}) we see

$$\alpha_1(\Gamma) = 2 \liminf_{\lambda\to 0^+} \frac{\ln F(\lambda)}{\ln \lambda} \geq 2 \lim_{\lambda \to 0} \left(a\frac{\ln \lambda}{\ln \lambda} + \frac{\ln C}{\ln \lambda} - \frac{\ln(1-\lambda)}{\lambda \ln \lambda} \right) = 2a.$$

Suppose now that $p(k) \geq Ck^{-a}$. Choose $\epsilon > 0$. Putting $k := [\lambda^{-(1+\epsilon)}] + 1$ the second part of (3.5) implies

$$F(\lambda) \geq C\lambda^{a(1+\epsilon)} \left(\frac{[\lambda^{-1-\epsilon}] + 1}{\lambda^{-1-\epsilon}} \right)^{-a} - (1-\lambda)^{[\lambda^{-(1+\epsilon)}]+1}.$$

Using lemma 3.6 below with $\delta = C/2$ and $a(1+\epsilon)$ instead of a this implies

$$(1-\lambda)^{[\lambda^{-1-\epsilon}]+1} \overset{1-\lambda<1}{\leq} (1-\lambda)^{-1-\epsilon} \leq \frac{C}{2}\lambda^{a(1+\epsilon)}$$
$$\Longrightarrow \frac{\ln F(\lambda)}{\ln \lambda} \leq \underbrace{\frac{\ln(C\left(([\lambda^{-1-\epsilon}]+1)/\lambda^{-1-\epsilon}\right)^{-a} - C/2)}{\ln \lambda}}_{\to 0 \text{ if } \lambda \to 0} + a(1+\epsilon)\frac{\ln \lambda}{\ln \lambda}.$$

Since the inequality is true for arbitrary $\epsilon > 0$ we conclude

$$\alpha_0(\Gamma) = 2 \liminf_{\lambda \to 0^+} \frac{\ln F(\lambda)}{\ln \lambda} \leq 2a.$$

Now Lemma 3.3 follows. □

Lemma 3.6. *For arbitrary $\epsilon, \delta, a > 0$ one finds $\lambda_0 > 0$ so that*

$$(1-\lambda)^{\lambda^{-1-\epsilon}} \leq \delta\lambda^a \qquad \text{for all } 0 < \lambda < \lambda_0.$$

Proof. Note that for $0 < \lambda < 1$ the stated inequality is equivalent to

$$\lambda^{-1-\epsilon}\ln(1-\lambda) \leq \ln\delta + a\ln\lambda$$
$$\Longleftrightarrow 1 \geq \frac{(\ln\delta + a\ln\lambda)\lambda^{1+\epsilon}}{\ln(1-\lambda)}.$$

For $\lambda \to 0$ the right hand side tends to 0 which can be seen using l'Hospital's rule. □

Theorem 3.7. (1) *Let $\Delta \subset \Gamma$ be a subgroup of finite index. Then*

$$c_n(\Gamma) \quad = \quad c_n(\Delta).$$

Moreover $H_n^\Delta(E\Delta; \mathcal{N}(\Delta))$ is cofinal-measurable if and only if $H_n^\Gamma(E\Gamma; \mathcal{N}(\Gamma))$ is cofinal-measurable.

(2) *Let $\Delta \subset \Gamma$ be a normal subgroup. Suppose that $H_q^\Delta(E\Delta; \mathcal{N}(\Delta))$ is cofinal-measurable for $q \leq n$. Then we get for $p = 0, 1, \ldots, n$ that $H_p^\Gamma(E\Gamma; \mathcal{N}(\Gamma))$ is cofinal-measurable and*

$$c_p(\Gamma) \quad \leq \quad \sum_{q=0}^{p} c_q(\Delta).$$

(3) *If there is a cofinal system of subgroups $\Delta \subset \Gamma$ with $H_p^\Delta(E\Delta; \mathcal{N}(\Delta))$ cofinal-measurable, then $H_p^\Gamma(E\Gamma; \mathcal{N}(\Gamma))$ is cofinal-measurableand*

$$c_n(\Gamma) \leq \liminf\{c_n(\Delta)\},$$

where Δ runs over the cofinal system.

(4) *If $n \geq 1$ and $\Gamma = \mathbb{Z}^n$ or $\mathbb{Z}^\infty = \oplus_{i=1}^\infty \mathbb{Z}$, then $H_p^\Gamma(E\Gamma; \mathcal{N}(\Gamma))$ is cofinal-measurable for every p and*

$$c_p(\mathbb{Z}^n) = \begin{cases} 1/n & \text{if } 0 \leq p \leq n-1; \\ 0^- & \text{if } p \geq n; \end{cases}$$
$$c_p(\mathbb{Z}^\infty) \leq 0 \quad \text{if } p \geq 0.$$

(Remark: It is possible to show $c_p(\mathbb{Z}^\infty) = 0$ for all $p \geq 0$.)

(5) *Suppose that* Γ *is a finitely generated virtually nilpotent group but not finite. Then* $H_p^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma))$ *is cofinal-measurable for* $p \geq 0$ *and*

$$\begin{aligned} c_0(\Gamma) + c_1(\Gamma) &\leq 1; \\ c_p(\Gamma) &\leq 1 \quad \text{for } p \geq 1. \end{aligned}$$

The same holds for Γ *locally virtually nilpotent but not locally finite.*

Proof. 1.) Let $i : \mathcal{N}(\Delta) \longrightarrow \mathcal{N}(\Gamma)$ be the ring homomorphism induced by the inclusion. Since $\mathcal{N}(\Delta) \otimes_{\mathbb{Z}\Delta} \mathbb{Z}\Gamma$ is isomorphic to $\mathcal{N}(\Gamma)$ as $\mathcal{N}(\Delta)$-$\mathbb{Z}\Gamma$-bimodule and since $E\Gamma$ viewed as Δ-space is a model for $E\Delta$, we get $i^* H_n^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma)) = H_n^{\Delta}(E\Delta;\mathcal{N}(\Delta))$. The statement now follows from Lemma 2.12.
2.) There is a spectral sequence converging to $H_{p+q}^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma))$ whose E_1-term is given by

$$E_{p,q}^1 = H_q^{\Delta}(E\Delta;\mathcal{N}(\Gamma)) \otimes_{\mathbb{Z}\pi} C_p(E\pi) = \oplus_{I_p} i_* H_q^{\Delta}(E\Delta;\mathcal{N}(\Delta)),$$

where $i : \Delta \longrightarrow \Gamma$ is the inclusion and I_p the set of p-cells of $B\pi$. We conclude from Theorem 2.7.4 and Lemma 2.12.1 that $E_{p,q}^1$ is cofinal-measurable for $q \leq n$ and $c(E_{p,q}^1) = c_q(\Delta)$. We conclude from Lemma 2.11.7 that $E_{p,q}^{\infty}$ is cofinal-measurable for $q \leq n$ and $c(E_{p,q}^{\infty}) \leq c_q(\Delta)$. Theorem 2.7.1 and 2.11.5 implies that $H_q^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma))$ is cofinal-measurable for $q \leq n$ and $c_q(\Gamma) \leq \sum_{p=0}^{q} c_p(\Delta)$ for $0 \leq p \leq q$.
3.) Since Γ is the union of the subgroups Δ, one can choose a model for $E\Gamma$ such that for each subgroup Δ the model $E\Delta$ is a subcomplex of $E\Gamma$ and $E\Gamma$ is the union of all the $E\Delta$'s. For instance take as model for $E\Gamma$ the infinite join $\Gamma * \Gamma * \ldots$. Hence

$$\begin{aligned} E\Gamma &= \operatorname{colim}_{\Delta \subset \Gamma} \Gamma \times_{\Delta} E\Delta; \\ H_p^{\Gamma}(E\Gamma;\mathcal{N}(\Gamma)) &= \operatorname{colim}_{\Delta \subset \Gamma} H_p^{\Delta}(E\Delta;\mathcal{N}(\Gamma)), \end{aligned}$$

where $\Delta \subset \Gamma$ runs through the finitely generated subgroups. Since $H_p^{\Delta}(E\Delta;\mathcal{N}(\Gamma))$ is $\mathcal{N}(\Gamma)$-isomorphic to $\mathcal{N}(\Gamma) \otimes_{\mathcal{N}(\Delta)} H_p^{\Delta}(E\Delta;\mathcal{N}(\Delta))$ the claim follows from Theorem 2.7.3 and Lemma 2.12.1.
4.) A direct computation shows the result for $\Gamma = \mathbb{Z}^n$. For $\mathbb{Z}^{\infty}$ apply assertions 3.
5.) By 3.) we can assume that Γ is finitely generated infinite and virtually nilpotent. We claim that such a group Γ is either virtually abelian or contains a normal subgroup Δ such that there exists a central extension $1 \longrightarrow \mathbb{Z} \longrightarrow \Delta \longrightarrow \mathbb{Z}^2 \longrightarrow 1$. This is proven as follows.

Recall that subgroups and quotient groups of nilpotent groups are nilpotent again, any nilpotent group contains a normal torsionfree group of finite index and the center of a non-trivial nilpotent group is non-trivial. Now choose a normal torsionfree subgroup Γ_0 of Γ of finite index and inspect the exact sequence $1 \longrightarrow \operatorname{cent}(\Gamma_0) \longrightarrow \Gamma_0 \longrightarrow \Gamma_0/\operatorname{cent}(\Gamma_0) \longrightarrow 1$. If $\Gamma_0/\operatorname{cent}(\Gamma_0)$ is finite, Γ is virtually abelian. Suppose that $\Gamma_0/\operatorname{cent}(\Gamma_0)$ is infinite. By inspecting the analogous sequence for a normal torsionfree subgroup $\Gamma_1 \subset \Gamma_0/\operatorname{cent}(\Gamma_0)$ of finite index and using the fact that $\Gamma_1/\operatorname{cent}(\Gamma_1)$ is either finite or contains $\mathbb{Z}$ as normal subgroup, one sees that $\Gamma_0/\operatorname{cent}(\Gamma_0)$ contains $\mathbb{Z}$ as subgroup of finite index or contains both $\mathbb{Z}$ and $\mathbb{Z}^2$ as normal subgroups. Since $\operatorname{cent}(\Gamma_0)$ has at least rank 1, the claim follows for Γ_0 and hence for Γ.

If Γ is virtually abelian the assertion 5.) follows from 1.) and 4.). Suppose that Δ is a normal subgroup of Γ and that there is a central extension $1 \longrightarrow \mathbb{Z} \longrightarrow$

$\Delta \longrightarrow \mathbb{Z}^2 \longrightarrow 1$. Because of 2.) $c_0(\Gamma) + c_1(\Gamma) \leq 2 \cdot c_0(\Delta) + c_1(\Delta)$. Hence it remains to show

$$2 \cdot c_0(\Delta) + c_1(\Delta) \leq 1.$$

One can realize Δ as the fundamental group of a closed 3-manifold M which is a principal S^1-bundle over T^2. Hence M is a Seifert manifold whose base orbifold has Euler characteristic $\chi = 0$ and the computation in [**6**, Theorem 4.1] shows $c_0(\Delta) = c_1(\Delta) = 1/3$ if the Euler class $e(M) = 0$ and $c_0(\Delta) = 1/4$ and $c_1(\Delta) = 1/2$ if $e(M) \neq 0$. This finishes the proof of Theorem 3.7. □

Definition 3.8. Given a class of groups $\mathcal{X}$, let $L\mathcal{X}$ be the class of groups Γ for which any finitely generated subgroup Δ belongs to $\mathcal{X}$. Given classes of groups $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{X}\mathcal{Y}$ be the class of groups Γ which contain a normal subgroup $\Delta \subset \Gamma$ with $\Delta \in \mathcal{X}$ and $\Gamma/\Delta \in \mathcal{Y}$. The *class $\mathcal{E}$ of elementary amenable groups* is defined as the smallest class of groups which contains all abelian and all finite groups, is closed under extensions, taking subgroups, forming quotient groups and under directed unions. The class of finite groups is denoted by $\mathcal{F}$.

Theorem 3.9. *Let $\mathcal{C}$ be the class of groups Γ for which $H_p^\Gamma(E\Gamma;\mathcal{N}(\Gamma))$ is cofinal-measurable for all $p \geq 0$ and $c_0(\Gamma) + c_1(\Gamma) \leq 1$ holds.*

1. *Finite and locally finite groups do* **not** *belong to $\mathcal{C}$.*
2. *If the infinite elementary amenable group Γ contains no infinite locally finite subgroup, then Γ belongs to $\mathcal{C}$.*
3. *If Γ contains a normal subgroup Δ which belongs to $\mathcal{C}$, then Γ belongs to $\mathcal{C}$.*
4. *Let Γ be the amalgamated product $\Gamma_0 *_\Delta \Gamma_1$ for a common subgroup Δ of Γ_0 and Γ_1. Suppose that Δ, Γ_0 and Γ_1 belong to $\mathcal{C}$ and that $c_0(\Delta) \leq 0$, then Γ belongs to $\mathcal{C}$.*
5. *$L(\mathcal{C} \cup \mathcal{F}) = \mathcal{C} \cup L\mathcal{F}$.*
6. *Suppose for the group Γ that $c_0(\Gamma) > 0$ or that it contains $\mathbb{Z}^n$ for some $n \geq 1$ as a normal subgroup. Then Γ belongs to $\mathcal{C}$ and moreover*

$$c_p(\Gamma) \leq 1 \qquad \text{also holds for } p \geq 1.$$

Proof. 1.) $H_0^\Gamma(E\Gamma;\mathcal{N}(\Gamma))$ is not cofinal-measurable if Γ is (locally) finite.
5.) We have to show for a group $\Gamma \in L(\mathcal{C} \cup \mathcal{F})$ which is not locally finite that Γ belongs to $\mathcal{C}$. Since any infinite finitely generated subgroup $\Gamma' \subset \Gamma$ belongs to $\mathcal{C}$ by assumption and these groups form a cofinal system of subgroups, we get from Theorem 3.7.3 that $H_p^\Gamma(E\Gamma;\mathcal{N}(\Gamma))$ is cofinal-measurable. and

$$c_1(\Gamma) = \liminf\{c_1(\Gamma')\} \leq 1.$$

If $c_0(\Gamma) \leq 0$ we are done, otherwise we know from 3.2.3 that Γ is locally virtually nilpotent but not locally finite and the result follows from 3.7.5.
3.) We get from Theorem 3.7.2 that $H_p^\Gamma(E\Gamma;\mathcal{N}(\Gamma))$ is cofinal-measurable for $p \geq 0$ and

$$c_1(\Gamma) \leq c_0(\Delta) + c_1(\Delta) \leq 1. \tag{3.10}$$

It remains to show $c_0(\Gamma) + c_1(\Gamma) \leq 1$. If $c_0(\Gamma) \leq 0$, this follows from (3.10). If $c_0(\Gamma) > 0$ we are again in the case, where Γ is locally virtually nilpotent but not locally finite and can apply 3.7.5.

2.) We use the following description of the class of elementary-amenable groups [**4**, Lemma 3.1]. Let $\mathcal{B}$ be the class of all groups which are finitely generated and virtually free abelian. Define for each ordinal α

$$\begin{array}{rcll} \mathcal{E}_0 & = & \{1\} & ; \\ \mathcal{E}_\alpha & = & (L\mathcal{E}_{\alpha-1})\mathcal{B} & \text{, if } \alpha \text{ is a successor ordinal} \\ \mathcal{E}_\alpha & = & \cup_{\beta<\alpha}\mathcal{E}_\beta & \text{, if } \alpha \text{ is a limit ordinal.} \end{array}$$

Then $\mathcal{E} = \cup_{\alpha\geq 0}\mathcal{E}_\alpha$ is the class of elementary amenable groups. For any elementary amenable group Γ there is a least ordinal α with $\Gamma \in \mathcal{E}_\alpha$ and we use transfinite induction to show that Γ belongs to $\mathcal{C}$, provided that Γ is infinite and contains no infinite locally finite subgroup.

The induction begin $\alpha = 0$ is obvious. If α is a limit ordinal, the induction step is clear. Suppose that α is a successor ordinal. Then there is an extension $1 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \pi \longrightarrow 1$ such that $\Delta \in L\mathcal{E}_{\alpha-1}$ and $\pi \in \mathcal{B}$. Every finitely generated subgroup $\Delta' \subset \Delta$ belongs to $\mathcal{E}_{\alpha-1}$. By assumption Γ and therefore Δ and Δ' contain no infinite locally finite subgroup. The induction hypothesis implies that Δ' is finite or belongs to $\mathcal{C}$. From assertions 5.) we get $\Delta \in L(\mathcal{C}\cup\mathcal{F}) = \mathcal{C}\cup L\mathcal{F}$. Since Δ is not infinite locally finite either Δ is finite or $\Delta \in \mathcal{C}$. In the second case apply 3.). If Δ is finite π must be infinite. Since $\mathcal{F}\mathcal{B} = \mathcal{B}$ we have Γ infinite virtually free abelian. In this case the statement follows from Theorem 3.7.1 and 3.7.5.
4.) This follows from the long Mayer-Vietoris sequence, Theorem 2.7.1, Lemma 2.11 and Lemma 2.12. One has to argue as above for the case $c_0(\Gamma) > 0$.
6.) Suppose $c_0(\Gamma) > 0$. Then Γ is locally virtually nilpotent but not locally finite and the statement is 3.7.5. Now suppose Γ contains $\mathbb{Z}^n$ as a normal subgroup for $n \geq 1$. Then the result follows from 3.7.4 and 3.7.2. □

4. Final Remarks

Remark 4.1. We mention that the proof for Theorem 3.9.2 goes also through if one enlarges the class $\mathcal{E}$ as defined by transfinite induction in the proof by substituting the class $\mathcal{B}$ of virtually abelian groups by any bigger class $\mathcal{B}'$ with the properties that $\mathcal{B}' \subset \mathcal{C}$, and $\mathcal{F}\mathcal{B}' = \mathcal{B}'$. □

Remark 4.2. Let $1 \longrightarrow \Delta \xrightarrow{i} \Gamma \xrightarrow{i} \mathbb{Z} \longrightarrow 1$ be an extension of groups. Suppose that Δ is locally finite. Then $H_p(E\Gamma;\mathcal{N}(\Gamma))$ is trivial for $p \geq 2$ and for $p = 1$ equal to the kernel K of the $\mathcal{N}(\Gamma)$-map

$$\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Delta} \mathbb{C} \longrightarrow \mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Delta} \mathbb{C} : \quad u \otimes n \mapsto u(t-1) \otimes n$$

for some $t \in \mathcal{N}(\Gamma)$ which maps to a generator of $\mathbb{Z}$ under p. If we would know that K is trivial, then Γ would belong to $\mathcal{C}$ and it would suffice in Theorem 3.9.2 to assume that Γ itself is not locally finite instead of assuming that Γ contains no infinite locally finite subgroup. □

Remark 4.3. So far we know no counterexample to the following statement: If Γ is elementary amenable and not locally-finite, then $H_p(E\Gamma;\mathcal{N}(\Gamma))$ is cofinal-measurable for all $p \geq 0$ and

$$c_p(\Gamma) \leq 1; \qquad \text{for } p \geq 0. \tag{4.4}$$

To prove this, it suffices to show inequality 4.4 for any group Γ such that there is an extension $1 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \mathbb{Z} \longrightarrow 1$ with a group Δ which already satisfies inequality 4.4. Then the proof of Theorem 3.9.2 would go through.

On the other hand one can construct a fiber bundle $F \longrightarrow E \longrightarrow S^1$ of closed manifolds with simply-connected fiber F such that $c_p(\widetilde{E})$ is arbitrary large. This follows from the observation that in this case one can read off $c_p(\widetilde{E})$ from the automorphism of $H_p(F)$ induced by the monodromy map $F \longrightarrow F$ and one can realize any element in $GL(n, \mathbb{Z})$ as the automorphism induced on $H_3(\prod_{i=1}^n S^3)$ by an automorphism of $\prod_{i=1}^n S^3$. □

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