On hyperbolic groups with spheres as boundary

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Preview of the main result

Conjecture (Gromov (1994))

Let G be a hyperbolic group whose boundary is a sphere S^{n-1}. Then there is a closed aspherical manifold M with $\pi_1(M) \cong G$.

Theorem (Bartels-Lück-Weinberger (2009))

The Conjecture is true for $n \geq 6$.

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Hyperbolic spaces and hyperbolic groups

Definition (Hyperbolic space)

A δ-hyperbolic space X is a geodesic space whose geodesic triangles are all δ-thin.

A geodesic space is called hyperbolic if it is δ-hyperbolic for some $\delta > 0$.

- A geodesic space with bounded diameter is hyperbolic.
- A tree is 0-hyperbolic.
- A simply connected complete Riemannian manifold M with $\sec(M) \leq \kappa$ for some $\kappa < 0$ is hyperbolic.
- $\mathbb{R}^n$ is hyperbolic if and only if $n \leq 1$.

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- Two geodesic rays $c_1, c_2 \colon [0, \infty) \to X$ are called equivalent if there exists $C > 0$ satisfying $d_X\big(c_1(t), c_2(t)\big) \le C$ for $t \in [0, \infty)$.

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Let X be a hyperbolic space. Define its boundary ∂X to be the set of equivalence classes of geodesic rays. Put

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Lemma

There is a topology on $\overline{X}$ *with the properties:*

- $\overline{X}$ *is compact and metrizable;*
- *The subspace topology* $X \subseteq \overline{X}$ *is the given one;*
- X *is open and dense in* $\overline{X}$.

- Let M be a simply connected complete Riemannian manifold M with $\sec(M) \leq \kappa$ for some $\kappa < 0$. Then M is hyperbolic and $\partial M = S^{\dim(M)-1}$.

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Definition (Quasi-isometry)

A map $f\colon X \to Y$ of metric spaces is called a quasi-isometry if there exist real numbers $\lambda, C > 0$ satisfying:

- The inequality

$$\lambda^{-1} \cdot d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq \lambda \cdot d_X(x_1, x_2) + C$$

  holds for all $x_1, x_2 \in X$;
- For every y in Y there exists $x \in X$ with $d_Y(f(x), y) < C$.

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Lemma (Švarc-Milnor Lemma)

Let X be a geodesic space. Suppose that G acts properly, cocompactly and isometrically on X. Choose a base point $x \in X$. Then the map

$$f\colon G \to X, \quad g \mapsto gx$$

is a quasiisometry.

Lemma (Quasi-isometry invariance of the Cayley graph)

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Lemma (Quasi-isometry invariance of being hyperbolic)

The property "hyperbolic" is a quasi-isometry invariant of geodesic spaces.

Lemma (Quasi-isometry invariance of the boundary)

A quasi-isometry $f \colon X_1 \to X_2$ of hyperbolic spaces induces a homeomorphism

$$\partial X_1 \xrightarrow{\cong} \partial X_2.$$

Definition (Hyperbolic group)

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- Let M be a closed Riemannian manifold with $\sec(M) < 0$. Then $\pi_1(M)$ is hyperbolic with $S^{\dim(M)-1}$ as boundary.
- A subgroup of a hyperbolic group is either virtually cyclic or contains $\mathbb{Z} * \mathbb{Z}$ as subgroup.
- $\mathbb{Z}^2$ is not a subgroup of a hyperbolic group.
- If the boundary of a hyperbolic groups contains an open subset homeomorphic to $\mathbb{R}^n$, then the boundary is homeomorphic to S^n.
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Gromov's Conjecture in low dimensions

Theorem (Casson-Jungreis (1994), Freden (1995), Gabai (1991))

A hyperbolic group has S^1 as boundary if and only if it is a Fuchsian group.

Conjecture (Cannon's Conjecture)

A hyperbolic group G has S^2 as boundary if and only if it acts properly, cocompactly and isometrically on $\mathbb{H}^3$.

Theorem (Bestvina-Mess (1991))

Let G be an infinite hyperbolic group which is the fundamental group of a closed irreducible 3-manifold M. Then M is hyperbolic and G satisfies Cannon's Conjecture.

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- In dimension four the only hyperbolic groups which are known to be good in the sense of Freedman are virtually cyclic.
- Possibly our results hold also in dimension 5.

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ANR-homology manifolds

Definition (Absolute neighborhood retract (ANR))

A topological space X is called absolute neighborhood retract (ANR) if it is normal and for every normal space Z, closed subset $Y \subseteq Z$ and map $f: Y \to X$ there is an open neighborhood $U \subseteq Z$ of Y and a map $F: U \to X$ extending f.

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Definition (Homology ANR-manifold)

A homology ANR-manifold X is an ANR satisfying:

- X has a countable basis for its topology;
- The topological dimension of X is finite;
- X is locally compact;
- for every $x \in X$ we have for the singular homology

$$H_i(X, X - \{x\}; \mathbb{Z}) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$$

If X is additionally compact, it is called a closed ANR-homology manifold.

There is also the notion of a compact ANR-homology manifold with boundary.

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- Every closed topological manifold is a closed ANR-homology manifold.
- Let M be homology sphere with non-trivial fundamental group. Then its suspension ΣM is a closed ANR-homology manifold but not a topological manifold.

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Definition (Disjoint Disk Property (DDP))

A homology ANR-manifold M has the disjoint disk property (DDP), if for any $\epsilon > 0$ and maps $f, g \colon D^2 \to M$, there are maps $f', g' \colon D^2 \to M$ so that f' is ϵ-close to f, g' is ϵ-close to g and $f'(D^2) \cap g'(D^2) = \emptyset$

- A topological manifold of dimension ≥ 5 is a closed ANR-homology manifold, which has the DDP by transversality.

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Poincaré duality groups

Definition (Poincaré duality group)

A Poincaré duality group G of dimension n is a finitely presented group satisfying:

- G is of type FP;
- $H^i(G;\mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

Lemma

Let X be a closed aspherical ANR-homology manifold of dimension n. Then its fundamental group is a Poincaré duality group of dimension n.

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Theorem (Poincaré duality groups and ANR-homology manifolds Bartels-Lück-Weinberger (2009))

Let G be a torsion-free group. Suppose that its satisfies the K- and L-theoretic Farrell-Jones Conjecture. Consider $n \geq 6$.

Then the following statements are equivalent:

1. *G is a Poincaré duality group of dimension n;*
2. *There exists a closed aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$;*
3. *There exists a closed aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$ which has the DDP.*

If the first statements holds, then the homology ANR-manifold M appearing above is unique up to s-cobordism of ANR-homology manifolds.

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The proof of the result above relies on:

- Surgery theory as developed by Browder, Novikov, Sullivan, Wall for smooth manifolds and its extension to topological manifolds using the work of Kirby-Siebenmann.
- The algebraic surgery theory of Ranicki
- The surgery theory for ANR-manifolds due to Bryant-Ferry-Mio-Weinberger and basic ideas of Quinn.

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Theorem (Bartels-Lück-Reich(2008), Bartels-Lück(2009), Bartels-Farrell-Lück-Reich (2010))

Let $\mathcal{FJ}$ be the class of groups for which both the K-theoretic and the L-theoretic Farrell-Jones Conjectures hold with coefficients in any additive G-category (with involution). It has the following properties:

- *Hyperbolic group and virtually nilpotent groups belongs to $\mathcal{FJ}$;*
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- *If we demand for the K-theory version only that the assembly map is 1-connected and keep the full L-theory version, then the properties above remain valid and the class $\mathcal{FJ}$ contains also all* CAT(0)*-groups;*
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A hyperbolic G is a Poincaré duality group of dimension n if and only if its boundary and S^{n-1} have the same Čech cohomology.

Corollary

Let G be a torsion-free word-hyperbolic group. Let $n \geq 6$. Then the following statements are equivalent:

1. *The boundary ∂G has the integral Čech cohomology of S^{n-1};*
2. *G is a Poincaré duality group of dimension n;*
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Quinn's resolution obstruction

Theorem (Quinn (1987))

There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ for homology ANR-manifolds with the following properties:

- *if $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;*
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Question

Does the Quinn obstruction always vanishes for aspherical closed homology ANR-manifolds?

- If the answer is yes, we can replace "closed ANR-homology manifold" by "closed topological manifold" in the theorem above.
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Let G_1 and G_2 be torsionfree hyperbolic groups.

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Let G be a torsion-free hyperbolic group and let n be an integer ≥ 6. Then the following statements are equivalent:

1. *The boundary ∂G is homeomorphic to S^{n-1};*
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Exotic Examples

By hyperbolization techniques due to Charney, Davis,Januskiewicz one can find the following examples:

Examples (Exotic universal coverings)

There are aspherical closed topological manifolds M with hyperbolic fundamental group $G = \pi_1(M)$ satisfying:

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For every $k \geq 2$ there exists a torsionfree hyperbolic group G with $\partial G \cong S^{4k-1}$ such that there is no aspherical closed smooth manifold M with $\pi_1(M) \cong G$. In particular G is not the fundamental group of a closed smooth Riemannian manifold with $\sec(M) < 0$.

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