Topological Rigidity

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Outline

Conjecture (Borel Conjecture)

Two aspherical closed manifolds are homeomorphic if and only if their fundamental groups are isomorphic.

- Explain this conjecture. Put it into a general context. Report on its status.

Theorem (Bartels-Lück (2008))

The Borel Conjecture is true if the fundamental group is hyperbolic or CAT(0)*.*

Some basic notions

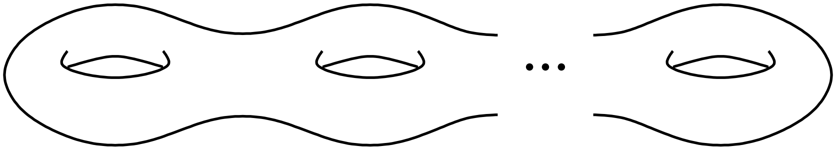
Definition (Homeomorphism)

A *homeomorphism* $f\colon X \to Y$ between topological spaces is a (continuous) map such that there exists a (continuous) map $g\colon Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Definition (Manifold)

An *n-dimensional manifold* M is a topological space which is locally homeomorphic to $\mathbb{R}^n$, i.e., for every point there is an open neighborhood which is homeomorphic to $\mathbb{R}^n$.
It is called *closed* if it is compact.

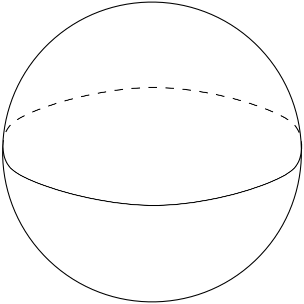
- A 2-dimensional orientable closed manifold is homeomorphic to the standard surface of genus g for precisely one g.



- Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map and $y \in \mathbb{R}^n$ be a regular value. Then the preimage $f^{-1}(y)$ is a manifold.

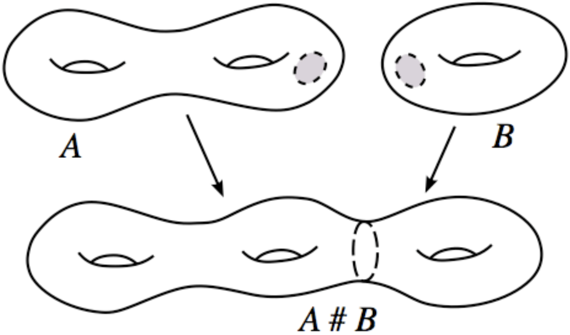
- An example is the *n-dimensional sphere*

$$S^n = \left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} x_i^2 = 1 \right\}$$



- Real or complex projective spaces are manifolds.
- The product of an m- and an n-dimensional manifold is a $(m+n)$-dimensional manifold.

- The connected sum $A \# B$ of two n-dimensional manifolds A and B is again one.



Definition (Homotopy)

Two maps $f_0, f_1 \colon X \to Y$ between topological spaces are called *homotopic* $f_0 \simeq f_1$, if and only if there is *homotopy* between them, i.e., a map

$$h \colon X \times [0,1] \to Y$$

satisfying $h(x,0) = f_0(x)$ and $h(x,1) = f_1(x)$ for all $x \in X$.

Definition (Homotopy equivalence)

A *homotopy equivalence* $f\colon X \to Y$ between topological spaces is a map such that there exists a map $g\colon Y \to X$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

- A homeomorphism is a homotopy equivalence.
- The converse is not true in general. For instance $\mathbb{R}^n$ and $\mathbb{R}^m$ are homotopy equivalent for all m, n, but they are homeomorphic if and only if $m = n$.

Definition (Contractible space)

A space is called *contractible* if the projection $X \to \{\bullet\}$ to the one-point-space is a homotopy equivalence.

- A space is contractible if and only if the identity is homotopic to a constant map.
- Any convex or star-shaped subset of $\mathbb{R}^n$ is contractible. In particular $\mathbb{R}^n$ is contractible.
- S^n is not contractible.
- A closed n-dimensional manifold is contractible if and only if it consists of one point.

Definition (Fundamental group)

Let (X, x) be a pointed space. Its *fundamental group* $\pi_1(X, x)$ has as elements pointed homotopy classes of loops with base point x, i.e., pointed maps
$(S^1, 1) \to (X, x)$.
The multiplication is given by concatenation of loops. The unit is given by the constant loop. The inverse is given by running around in a loop in the opposite direction.

- If X is path-connected, the fundamental group is independent of the choice of the base point up to group isomorphism.
- If two spaces are homotopy equivalent, their fundamental groups are isomorphic.
- Sending $n \in \mathbb{Z}$ to the loop $S^1 \to S^1,\ z \mapsto z^n$ induces an isomorphism

$$\mathbb{Z} \xrightarrow{\cong} \pi_1(S^1).$$

- For $n \geq 2$ the n-dimensional sphere S^n is simply-connected, i.e., it is path-connected and its fundamental group is trivial.
- Fix $n \geq 4$ and a finitely presented group G. Then there exists an n-dimensional closed manifold M with $\pi_1(M) \cong G$.

- Let X be a path-connected space. Its universal covering $\widetilde{X} \to X$ is the unique covering for which the total space is simply-connected.
- Its group of deck transformations can be identified with $\pi_1(X)$. In particular we rediscover X from $\widetilde{X}$ by

$$X = \widetilde{X}/\pi_1(X).$$

- The universal covering of S^1 is

$$\mathbb{R} \to S^1, \quad r \mapsto e^{2\pi i r}.$$

Aspherical manifolds

Definition (Aspherical)

A path connected space is called *aspherical* if the total space of its universal covering is contractible.

- A path-connected space is aspherical if and only if all its higher homotopy groups vanish.
- The fundamental group of an aspherical closed manifold is torsionfree.
- S^n is aspherical if and only if $n = 1$.

- The orientable closed surface of genus g is aspherical if and only if $g \geq 1$.
- An orientable closed 3-manifold is aspherical if and only if its fundamental group is torsionfree, prime and not isomorphic to $\mathbb{Z}$.
- A closed Riemannian manifold with non-positive sectional curvature is aspherical.

- Let L be a connected Lie group. Let $K \subseteq L$ be a maximal compact subgroup. Let $G \subseteq L$ be a torsionfree discrete subgroup.
  Then the double coset space
  $$M := G\backslash L/K$$
  is an aspherical manifold.
- A simply connected closed manifold is aspherical if and only if it consists of one point.
- Slogan: A "random" closed manifold is expected to be aspherical.

Topological rigidity and the Borel Conjecture

Definition (Topologically rigid)

A closed topological manifold M is called *topologically rigid* if any homotopy equivalence $N \longrightarrow M$ with some manifold N as source and M as target is homotopic to a homeomorphism.

- The Poincaré Conjecture in dimension n is equivalent to the statement that S^n is topologically rigid.

Theorem (Kreck-Lück (to appear in 2009))

- *Suppose that $k + d \neq 3$. Then $S^k \times S^d$ is topologically rigid if and only if both k and d are odd.*
- *Every closed 3-manifold with torsionfree fundamental group is topologically rigid.*
- *Let M and N be closed manifolds of the same dimension $n \geq 5$ with torsionfree fundamental groups. If both M and N are topologically rigid, then the same is true for their connected sum $M \# N$.*

Theorem (Chang-Weinberger (2003))

Let M^{4k+3} be a closed oriented smooth manifold for $k \geq 1$ whose fundamental group has torsion. Then M is not topologically rigid.

- Hence in most cases the fundamental group of a topologically rigid manifold is torsionfree.

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts that a closed aspherical manifold M with $\pi_1(M) \cong G$ *is topologically rigid.*

- Two aspherical manifolds are homotopy equivalent if and only if their fundamental groups are isomorphic.
- The Borel Conjecture predicts that two aspherical manifolds have isomorphic fundamental groups if and only if they are homeomorphic.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity.
- One version of Mostow rigidity says that any homotopy equivalence between hyperbolic closed Riemannian manifolds is homotopic to an isometric diffeomorphism.
- In particular they are isometrically diffeomorphic if and only if their fundamental groups are isomorphic.

- The Borel Conjecture becomes definitely false if one replaces homeomorphism by diffeomorphism.
- For instance, there are smooth manifolds M which are homeomorphic to T^n but not diffeomorphic to T^n.
- The Borel Conjecture is true in dimensions 1 and 2 by classical results. It is true in dimension 3 by Perelman's proof of Thurston's Geometrization Conjecture.

Other prominent Conjectures

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Conjecture (Reduced projective class group)

If R is a principal ideal domain and G is torsionfree, then $\widetilde{K}_0(RG) = 0$.

Conjecture (Serre)

If G is of type FP, then G is already of type FF.

Conjecture (Whitehead group)

If G is torsionfree, then the Whitehead group $\mathrm{Wh}(G)$ *vanishes.*

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M that its higher signatures over BG are homotopy invariants.

Conjecture (K-theoretic Farrell-Jones Conjecture for regular rings and torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- There is an L-theoretic version of the Farrell-Jones Conjecture.
- Both the K-theoretic and the L-theoretic Farrell-Jones Conjecture can be formulated for arbitrary groups G and arbitrary rings R allowing also a G-twist on R.

Theorem (The Farrell-Jones Conjecture implies (nearly) everything)

If G satisfies both the K-theoretic and L-theoretic Farrell-Jones Conjecture (for any additive G-category as coefficients), then all the conjectures mentioned above (and further conjectures) will follow for G.

- The Borel Conjecture (for dim ≥ 5),
- Kaplansky Conjecture (for R a field of characteristic zero),
- Vanishing of $\widetilde{K}_0(RG)$ and $\mathrm{Wh}(G)$,
- Serre's Conjecture,
- Novikov Conjecture (for dim ≥ 5),
- other conjecture, e.g., the ones due to Bass and Moody, the one about Poincaré duality groups (for dim ≥ 5) and the one about the homotopy invariance of L^2-torsion.

The status of the Farrell-Jones Conjecture

Theorem (Bartels-Lück (preprint will be available in the beginning of 2009))

Let $\mathcal{FJ}$ be the class of groups for which both the K-theoretic and the L-theoretic Farrell-Jones Conjectures holds (in his most general form, namely with coefficients in any additive G-category) has the following properties:

- *Hyperbolic groups,* CAT(0)*-groups and virtually nilpotent groups belongs to $\mathcal{FJ}$;*

Theorem (Continued)

- *If* G_1 *and* G_2 *belong to* $\mathcal{FJ}$*, then* $G_1 \times G_2$ *and* $G_1 * G_2$ *belong to* $\mathcal{FJ}$*;*
- *If* H *is a subgroup of* G *and* $G \in \mathcal{FJ}$*, then* $H \in \mathcal{FJ}$*;*
- *Let* $\{G_i \mid i \in I\}$ *be a directed system of groups (with not necessarily injective structure maps) such that* $G_i \in \mathcal{FJ}$ *for* $i \in I$*. Then* $\operatorname{colim}_{i \in I} G_i$ *belongs to* $\mathcal{FJ}$*.*

- Limit groups in the sense of Zela are CAT(0)-groups (Alibegovic-Bestvina (2005)).
- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.
- One example is the construction of groups with expanders due to Gromov. These yield counterexamples to the Baum-Connes Conjecture with coefficients (see Higson-Lafforgue-Skandalis (2002)).

- However, our results show that these groups do satisfy the Farrell-Jones Conjecture in its most general form and hence also the other conjectures mentioned above.

- Mike Davis (1983) has constructed exotic closed aspherical manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.
- However, in all cases the universal coverings are CAT(0)-spaces and hence the fundamental groups are CAT(0)-groups.
- Hence by our main theorem they satisfy the Farrell-Jones Conjecture and hence the Borel Conjecture in dimension ≥ 5.

- There are still many interesting groups for which the Farrell-Jones Conjecture in its most general form is open. Examples are:
  - Amenable groups;
  - $Sl_n(\mathbb{Z})$ for $n \geq 3$;
  - Mapping class groups;
  - $\mathrm{Out}(F_n)$;
  - Thompson groups.
- If one looks for a counterexample, there seems to be no good candidates which do not fall under our main theorems.

Boundaries of hyperbolic groups

Theorem (Bartels-Lück-Weinberger (2009?))

Let G be a torsionfree hyperbolic group and let n be an integer ≥ 6. Then the following statements are equivalent:

- *The boundary ∂G is homeomorphic to S^{n-1};*
- *There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.*