The Baum-Connes and the Farrell-Jones Conjectures in K- and L-Theory

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Summary. We give a survey of the meaning, status and applications of the Baum-Connes Conjecture about the topological K-theory of the reduced group C^*-algebra and the Farrell-Jones Conjecture about the algebraic K- and L-theory of the group ring of a (discrete) group G.

Key words: K- and L-groups of group rings and group C^*-algebras, Baum-Connes Conjecture, Farrell-Jones Conjecture.

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Introduction

This survey article is devoted to the Baum-Connes Conjecture about the topological K-theory of the reduced group C^*-algebra and the Farrell-Jones Conjecture about the algebraic K- and L-theory of the group ring of a discrete group G. We will present a unified approach to these conjectures hoping that it will stimulate further interactions and exchange of methods and ideas between algebraic and geometric topology on the one side and non-commutative geometry on the other.

Each of the above mentioned conjectures has already been proven for astonishingly large classes of groups using a variety of different methods coming from operator theory, controlled topology and homotopy theory. Methods have been developed for this purpose which turned out to be fruitful in other

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contexts. The conjectures imply many other well-known and important conjectures. Examples are the Borel Conjecture about the topological rigidity of closed aspherical manifolds, the Novikov Conjecture about the homotopy invariance of higher signatures, the stable Gromov-Lawson-Rosenberg Conjecture about the existence of Riemannian metrics with positive scalar curvature and the Kadison Conjecture about idempotents in the reduced C^*-algebra of a torsionfree discrete group G.

Formulation of the Conjectures

The Baum-Connes and Farrell-Jones Conjectures predict that for every discrete group G the following so called "assembly maps" are isomorphisms.

$$\begin{aligned} K_n^G(E_{\mathcal{FIN}}(G)) &\to K_n(C_r^*(G)); \\ H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) &\to K_n(RG); \\ H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) &\to L_n^{\langle -\infty \rangle}(RG). \end{aligned}$$

Here the targets are the groups one would like to understand, namely the topological K-groups of the reduced group C^*-algebra in the Baum-Connes case and the algebraic K- or L-groups of the group ring RG for R an associative ring with unit. In each case the source is a G-homology theory evaluated on a certain classifying space. In the Baum-Connes Conjecture the G-homology theory is equivariant topological K-theory and the classifying space $E_{\mathcal{FIN}}(G)$ is the classifying space of the family of finite subgroups, which is often called the classifying space for proper G-actions and denoted $\underline{E}G$ in the literature. In the Farrell-Jones Conjecture the G-homology theory is given by a certain K- or L-theory spectrum over the orbit category, and the classifying space $E_{\mathcal{VCY}}(G)$ is the one associated to the family of virtually cyclic subgroups. The conjectures say that these assembly maps are isomorphisms.

These conjectures were stated in [28, Conjecture 3.15 on page 254] and [111, 1.6 on page 257]. Our formulations differ from the original ones, but are equivalent. In the case of the Farrell-Jones Conjecture we slightly generalize the original conjecture by allowing arbitrary coefficient rings instead of $\mathbb{Z}$. At the time of writing no counterexample to the Baum-Connes Conjecture 2.3 or the Farrell-Jones Conjecture 2.2 is known to the authors.

One can apply methods from algebraic topology such as spectral sequences and Chern characters to the sources of the assembly maps. In this sense the sources are much more accessible than the targets. The conjectures hence lead to very concrete calculations. Probably even more important is the structural insight: to what extent do the target groups show a homological behaviour. These aspects can be treated completely analogously in the Baum-Connes and the Farrell-Jones setting.

However, the conjectures are not merely computational tools. Their importance comes from the fact that the assembly maps have geometric interpretations in terms of indices in the Baum-Connes case and in terms of surgery theory in the Farrell-Jones case. These interpretations are the key ingredient in applications and the reason that the Baum-Connes and Farrell-Jones Conjectures imply so many other conjectures in non-commutative geometry, geometric topology and algebra.

A User's Guide

A reader who wants to get specific information or focus on a certain topic should consult the detailed table of contents, the index and the index of notation in order to find the right place in the paper. We have tried to write the text in a way such that one can read small units independently from the rest. Moreover, a reader who may only be interested in the Baum-Connes Conjecture or only in the Farrell-Jones Conjecture for K-theory or for L-theory can ignore the other parts. But we emphasize again that one basic idea of this paper is to explain the parallel treatment of these conjectures.

A reader without much prior knowledge about the Baum-Connes Conjecture or the Farrell-Jones Conjecture should begin with Chapter 1. There, the special case of a torsionfree group is treated, since the formulation of the conjectures is less technical in this case and there are already many interesting applications. The applications themselves however, are not needed later. A more experienced reader may pass directly to Chapter 2.

Other (survey) articles on the Farrell-Jones Conjecture and the Baum-Connes Conjecture are [111], [128], [147], [225], [307].

Notations and Conventions

Here is a briefing on our main notational conventions. Details are of course discussed in the text. The columns in the following table contain our notation for: the spectra, their associated homology theory, the right hand side of the corresponding assembly maps, the functor from groupoids to spectra and finally the G-homology theory associated to these spectra valued functors.

$\mathbf{BU}$	$K_n(X)$	$K_n(C_r^*G)$	$\mathbf{K}^{\mathrm{top}}$	$H_n^G(X;\mathbf{K}^{\mathrm{top}})$
$\mathbf{K}(R)$	$H_n(X;\mathbf{K}(R))$	$K_n(RG)$	$\mathbf{K}_R$	$H_n^G(X;\mathbf{K}_R)$
$\mathbf{L}^{\langle j\rangle}(R)$	$H_n(X;\mathbf{L}^{\langle j\rangle}(R))$	$L_n^{\langle j\rangle}(RG)$	$\mathbf{L}_R^{\langle j\rangle}$	$H_n^G(X;\mathbf{L}_R^{\langle j\rangle})$

We would like to stress that $\mathbf{K}$ without any further decoration will always refer to the non-connective K-theory spectrum. $\mathbf{L}^{\langle j\rangle}$ will always refer to quadratic L-theory with decoration j. For a C^*- or Banach algebra A the symbol $K_n(A)$ has two possible interpretations but we will mean the topological K-theory.

A ring is always an associative ring with unit, and ring homomorphisms are always unital. Modules are left modules. We will always work in the category of compactly generated spaces, compare [295] and [330, I.4]. For our conventions concerning spectra see Section 6.2. Spectra are denoted with boldface letters such as $\mathbf{E}$.

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1 The Conjectures in the Torsion Free Case

In this chapter we discuss the Baum-Connes and Farrell-Jones Conjectures in the case of a torsion free group. Their formulation is less technical than in the general case, but already in the torsion free case there are many interesting and illuminating conclusions. In fact some of the most important consequences of the conjectures, like for example the Borel Conjecture (see Conjecture 1.27) or the Kadison Conjecture (see Conjecture 1.39), refer exclusively to the torsion free case. On the other hand in the long run the general case, involving groups with torsion, seems to be unavoidable. The general formulation yields a clearer and more complete picture, and furthermore there are proofs of the conjectures for torsion free groups, where in intermediate steps of the proof it is essential to have the general formulation available (compare Section 7.9).

The statement of the general case and further applications will be presented in the next chapter. The reader may actually skip this chapter and start immediately with Chapter 2.

We have put some effort into dealing with coefficient rings R other than the integers. A topologist may a priori be interested only in the case $R = \mathbb{Z}$ but other cases are interesting for algebraists and also do occur in computations for integral group rings.

1.1 Algebraic K-Theory - Low Dimensions

A ring R is always understood to be associative with unit. We denote by $K_n(R)$ *the algebraic K-group* of R for $n \in \mathbb{Z}$. In particular $K_0(R)$ is the Grothendieck group of finitely generated projective R-modules and elements in $K_1(R)$ can be represented by automorphisms of such modules. In this section we are mostly interested in the K-groups $K_n(R)$ with $n \le 1$. For definitions of these groups we refer to [221], [266], [286], [299], [323] for $n = 0, 1$ and to [22] and [268] for $n \le 1$.

For a ring R and a group G we denote by

$$A_0 = K_0(i)\colon K_0(R) \to K_0(RG)$$

the map induced by the natural inclusion $i\colon R \to RG$. Sending $(g, [P]) \in G \times K_0(R)$ to the class of the RG-automorphism

$$R[G] \otimes_R P \to R[G] \otimes_R P, \quad u \otimes x \mapsto ug^{-1} \otimes x$$

defines a map $\Phi\colon G_{\mathrm{ab}} \otimes_{\mathbb{Z}} K_0(R) \to K_1(RG)$, where G_{ab} denotes the abelianized group. We set

$$A_1 = \Phi \oplus K_1(i)\colon G_{\mathrm{ab}} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) \to K_1(RG).$$

We recall the notion of a regular ring. We think of modules as left modules unless stated explicitly differently. Recall that R is *Noetherian* if any submodule of a finitely generated R-module is again finitely generated. It is called *regular* if it is Noetherian and any R-module has a finite-dimensional projective resolution. Any principal ideal domain such as $\mathbb{Z}$ or a field is regular.

The Farrell-Jones Conjecture about algebraic K-theory implies for a torsion free group the following conjecture about the low dimensional K-theory groups.

Conjecture 1.1 (The Farrell-Jones Conjecture for Low Dimensional K-Theory and Torsion Free Groups). Let G be a torsion free group and let R be a regular ring. Then

$$K_n(RG) = 0 \quad \text{for} \quad n \leq -1$$

and the maps

$$\begin{aligned} K_0(R) &\xrightarrow{A_0} K_0(RG) \quad \text{and} \\ G_{\mathrm{ab}} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) &\xrightarrow{A_1} K_1(RG) \end{aligned}$$

are both isomorphisms.

Every regular ring satisfies $K_n(R) = 0$ for $n \leq -1$ [268, 5.3.30 on page 295] and hence the first statement is equivalent to $K_n(i)\colon K_n(R) \to K_n(RG)$ being an isomorphism for $n \leq -1$. In Remark 1.15 below we explain why we impose the regularity assumption on the ring R.

For a regular ring R and a group G we define $\mathrm{Wh}_1^R(G)$ as the cokernel of the map A_1 and $\mathrm{Wh}_0^R(G)$ as the cokernel of the map A_0. In the important case where $R = \mathbb{Z}$ the group $\mathrm{Wh}_1^{\mathbb{Z}}(G)$ coincides with the classical *Whitehead group* $\mathrm{Wh}(G)$ which is the quotient of $K_1(\mathbb{Z}G)$ by the subgroup consisting of the classes of the units $\pm g \in (\mathbb{Z}G)^{\mathrm{inv}}$ for $g \in G$. Moreover for every ring R we define the *reduced algebraic K-groups* $\widetilde{K}_n(R)$ as the cokernel of the natural map $K_n(\mathbb{Z}) \to K_n(R)$. Obviously $\mathrm{Wh}_0^{\mathbb{Z}}(G) = \widetilde{K}_0(\mathbb{Z}G)$.

Lemma 1.2. *The map A_0 is always injective. If R is commutative and the natural map $\mathbb{Z} \to K_0(R)$, $1 \mapsto [R]$ is an isomorphism, then the map A_1 is injective.*

Proof. The augmentation $\epsilon\colon RG \to R$, which maps each group element g to 1, yields a retraction for the inclusion $i\colon R \to RG$ and hence induces a retraction for A_0. If the map $\mathbb{Z} \to K_0(R)$, $1 \mapsto [R]$ induces an isomorphism and R is commutative, then we have the commutative diagram

$$\begin{tikzcd}
G_{\mathrm{ab}} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) \arrow[r, "A_1"] \arrow[dd, "\cong"'] & K_1(RG) \arrow[d] \\
& K_1(RG_{\mathrm{ab}}) \arrow[d, "(\det, K_1(\epsilon))"] \\
G_{\mathrm{ab}} \oplus K_1(R) \arrow[r] & RG_{\mathrm{ab}}^{\mathrm{inv}} \oplus K_1(R),
\end{tikzcd}$$

where the upper vertical arrow on the right is induced from the map $G \to G_{\mathrm{ab}}$ to the abelianization. Since RG_{ab} is a commutative ring we have the determinant $\det\colon K_1(RG_{\mathrm{ab}}) \to (RG_{\mathrm{ab}})^{\mathrm{inv}}$. The lower horizontal arrow is induced from the obvious inclusion of G_{ab} into the invertible elements of the group ring RG_{ab} and in particular injective. □

In the special case $R = \mathbb{Z}$ Conjecture 1.1 above is equivalent to the following conjecture.

Conjecture 1.3 (Vanishing of Low Dimensional K-Theory for Torsionfree Groups and Integral Coefficients). For every torsion free group G we have

$$K_n(\mathbb{Z}G) = 0 \text{ for } n \leq -1, \quad \widetilde{K}_0(\mathbb{Z}G) = 0 \quad \text{and} \quad \mathrm{Wh}(G) = 0.$$

Remark 1.4 (Torsionfree is Necessary). In general $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ do not vanish for finite groups. For example $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/23]) \cong \mathbb{Z}/3$ [221, page 29, 30] and $\mathrm{Wh}(\mathbb{Z}/p) \cong \mathbb{Z}^{\frac{p-3}{2}}$ for p an odd prime [70, 11.5 on page 45]. This shows that the assumption that G is torsion free is crucial in the formulation of Conjecture 1.1 above.

For more information on $\widetilde{K}_0(\mathbb{Z}G)$ and Whitehead groups of finite groups see for instance [22, Chapter XI], [79], [220], [231], [232] and [299].

1.2 Applications I

We will now explain the geometric relevance of the groups whose vanishing is predicted by Conjecture 1.3.

1.2.1 The s-Cobordism Theorem and the Poincaré Conjecture

The Whitehead group $\mathrm{Wh}(G)$ plays a key role if one studies manifolds because of the so called s-Cobordism Theorem. In order to state it, we explain the notion of an h-cobordism.

Manifold always means smooth manifold unless it is explicitly stated differently. We say that W or more precisely $(W; M^-, f^-, M^+, f^+)$ is an n-dimensional *cobordism* over M^- if W is a compact n-dimensional manifold together with the following: a disjoint decomposition of its boundary ∂W into two closed $(n-1)$-dimensional manifolds $\partial^- W$ and $\partial^+ W$, two closed $(n-1)$-dimensional manifolds M^- and M^+ and diffeomorphisms $f^- \colon M^- \to \partial^- W$ and $f^+ \colon M^+ \to \partial^+ W$. The cobordism is called an *h-cobordism* if the inclusions $i^- \colon \partial^- W \to W$ and $i^+ \colon \partial^+ W \to W$ are both homotopy equivalences. Two cobordisms $(W; M^-, f^-, M^+, f^+)$ and $(W'; M^-, f'^-, M'^+, f'^+)$ over M^- are *diffeomorphic relative* M^- if there is a diffeomorphism $F \colon W \to W'$ with $F \circ f^- = f'^-$. We call a cobordism over M^- *trivial*, if it is diffeomorphic relative M^- to the trivial h-cobordism given by the cylinder $M^- \times [0,1]$ together with the obvious inclusions of $M^- \times \{0\}$ and $M^- \times \{1\}$. Note that "trivial" implies in particular that M^- and M^+ are diffeomorphic.

The question whether a given h-cobordism is trivial is decided by the Whitehead torsion $\tau(W; M^-) \in \mathrm{Wh}(G)$ where $G = \pi_1(M^-)$. For the details of the definition of $\tau(W; M^-)$ the reader should consult [70], [220] or Chapter 2 in [200]. Compare also [266].

Theorem 1.5 (s-Cobordism Theorem). *Let M^- be a closed connected oriented manifold of dimension $n \geq 5$ with fundamental group $G = \pi_1(M^-)$. Then*

(i) An h-cobordism W over M^- is trivial if and only if its Whitehead torsion $\tau(W, M^-) \in \mathrm{Wh}(G)$ vanishes.
(ii) Assigning to an h-cobordism over M^- its Whitehead torsion yields a bijection from the diffeomorphism classes relative M^- of h-cobordisms over M^- to the Whitehead group $\mathrm{Wh}(G)$.

The s-Cobordism Theorem is due to Barden, Mazur and Stallings. There are also topological and PL-versions. Proofs can be found for instance in [173], [176, Essay III], [200] and [272, page 87-90].

The s-Cobordism Theorem tells us that the vanishing of the Whitehead group (as predicted in Conjecture 1.3 for torsion free groups) has the following geometric interpretation.

Consequence 1.6. *For a finitely presented group G the vanishing of the Whitehead group $\mathrm{Wh}(G)$ is equivalent to the statement that each h-cobordism over a closed connected manifold M^- of dimension $\dim(M^-) \geq 5$ with fundamental group $\pi_1(M^-) \cong G$ is trivial.*

Knowing that all h-cobordisms over a given manifold are trivial is a strong and useful statement. In order to illustrate this we would like to discuss the case where the fundamental group is trivial.

Since the ring $\mathbb{Z}$ has a Gaussian algorithm, the determinant induces an isomorphism $K_1(\mathbb{Z}) \xrightarrow{\cong} \{\pm 1\}$ (compare [268, Theorem 2.3.2]) and the Whitehead group $\mathrm{Wh}(\{1\})$ of the trivial group vanishes. Hence any h-cobordism over a simply connected closed manifold of dimension ≥ 5 is trivial. As a consequence one obtains the Poincaré Conjecture for high dimensional manifolds.

Theorem 1.7 (Poincaré Conjecture). *Suppose $n \geq 5$. If the closed manifold M is homotopy equivalent to the sphere S^n, then it is homeomorphic to S^n.*

Proof. We only give the proof for $\dim(M) \geq 6$. Let $f\colon M \to S^n$ be a homotopy equivalence. Let $D^n_- \subset M$ and $D^n_+ \subset M$ be two disjoint embedded disks. Let W be the complement of the interior of the two disks in M. Then W turns out to be a simply connected h-cobordism over ∂D^n_-. Hence we can find a diffeomorphism

$$F\colon (\partial D^n_- \times [0,1]; \partial D^n_- \times \{0\}, \partial D^n_- \times \{1\}) \to (W; \partial D^n_-, \partial D^n_+)$$

which is the identity on $\partial D^n_- = \partial D^n_- \times \{0\}$ and induces some (unknown) diffeomorphism $f^+\colon \partial D^n_- \times \{1\} \to \partial D^n_+$. By the *Alexander trick* one can extend $f^+\colon \partial D^n_- = \partial D^n_- \times \{1\} \to \partial D^n_+$ to a homeomorphism $\overline{f^+}\colon D^n_- \to D^n_+$. Namely, any homeomorphism $f\colon S^{n-1} \to S^{n-1}$ extends to a homeomorphism $\overline{f}\colon D^n \to D^n$ by sending $t \cdot x$ for $t \in [0,1]$ and $x \in S^{n-1}$ to $t \cdot f(x)$. Now define a homeomorphism $h\colon D^n_- \times \{0\} \cup_{i_-} \partial D^n_- \times [0,1] \cup_{i_+} D^n_- \times \{1\} \to M$ for the canonical inclusions $i_k\colon \partial D^n_- \times \{k\} \to \partial D^n_- \times [0,1]$ for $k = 0,1$ by $h|_{D^n_- \times \{0\}} = \mathrm{id}$, $h|_{\partial D^n_- \times [0,1]} = F$ and $h|_{D^n_- \times \{1\}} = \overline{f^+}$. Since the source of h is obviously homeomorphic to S^n, Theorem 1.7 follows. □

The Poincaré Conjecture (see Theorem 1.7) is at the time of writing known in all dimensions except dimension 3. It is essential in its formulation that one concludes M to be homeomorphic (as opposed to diffeomorphic) to S^n. The Alexander trick does not work differentiably. There are *exotic spheres*, i.e. smooth manifolds which are homeomorphic but not diffeomorphic to S^n [218].

More information about the Poincaré Conjecture, the Whitehead torsion and the s-Cobordism Theorem can be found for instance in [50], [70], [86], [131], [132], [141], [173], [200], [219], [220], [266] and [272].

1.2.2 Finiteness Obstructions

We now discuss the geometric relevance of $\widetilde{K}_0(\mathbb{Z}G)$.

Let X be a CW-complex. It is called *finite* if it consists of finitely many cells. It is called *finitely dominated* if there is a finite CW-complex Y together with maps $i\colon X \to Y$ and $r\colon Y \to X$ such that $r \circ i$ is homotopic to the identity on X. The fundamental group of a finitely dominated CW-complex is always finitely presented.

While studying existence problems for spaces with prescribed properties (like for example group actions), it happens occasionally that it is relatively easy to construct a finitely dominated CW-complex within a given homotopy type, whereas it is not at all clear whether one can also find a homotopy equivalent *finite* CW-complex. *Wall's finiteness obstruction*, a certain obstruction element $\widetilde{o}(X) \in \widetilde{K}_0(\mathbb{Z}\pi_1(X))$, decides the question.

Theorem 1.8 (Properties of the Finiteness Obstruction). *Let X be a finitely dominated CW-complex with fundamental group $\pi = \pi_1(X)$.*

(i) The space X is homotopy equivalent to a finite CW-complex if and only if $\widetilde{o}(X) = 0 \in \widetilde{K}_0(\mathbb{Z}\pi)$.
(ii) Every element in $\widetilde{K}_0(\mathbb{Z}G)$ can be realized as the finiteness obstruction $\widetilde{o}(X)$ of a finitely dominated CW-complex X with $G = \pi_1(X)$, provided that G is finitely presented.
(iii) Let Z be a space such that $G = \pi_1(Z)$ is finitely presented. Then there is a bijection between $\widetilde{K}_0(\mathbb{Z}G)$ and the set of equivalence classes of maps $f\colon X \to Z$ with X finitely dominated under the equivalence relation explained below.

The equivalence relation in (iii) is defined as follows: Two maps $f\colon X \to Z$ and $f'\colon X' \to Z$ with X and X' finitely dominated are equivalent if there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[drr, "f"'] & X_1 \arrow[r, "h"] \arrow[dr, "f_1"] & X_2 \arrow[d, "f_2"] & X_3 \arrow[l, "h'"'] \arrow[dl, "f_3"'] & X' \arrow[l, "j'"'] \arrow[dll, "f'"] , \\
& & Z & &
\end{tikzcd}$$

where h and h' are homotopy equivalences and j and j' are inclusions of subcomplexes for which X_1, respectively X_3, is obtained from X, respectively X', by attaching a finite number of cells.

The vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ as predicted in Conjecture 1.3 for torsion free groups hence has the following interpretation.

Consequence 1.9. *For a finitely presented group G the vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ is equivalent to the statement that any finitely dominated CW-complex X with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex.*

For more information about the finiteness obstruction we refer for instance to [125], [126], [196], [224], [257], [266], [308], [317] and [318].

1.2.3 Negative *K*-Groups and Bounded h-Cobordisms

One possible geometric interpretation of negative K-groups is in terms of bounded h-cobordisms. Another interpretation will be explained in Subsection 1.4.2 below.

We consider *manifolds W parametrized over $\mathbb{R}^k$*, i.e. manifolds which are equipped with a surjective proper map $p\colon W \to \mathbb{R}^k$. We will always assume that the fundamental group(oid) is bounded, compare [239, Definition 1.3]. A map $f\colon W \to W'$ between two manifolds parametrized over $\mathbb{R}^k$ is *bounded* if $\{p' \circ f(x) - p(x) \mid x \in W\}$ is a bounded subset of $\mathbb{R}^k$.

A *bounded cobordism* $(W; M^-, f^-, M^+, f^+)$ is defined just as in Subsection 1.2.1 but compact manifolds are replaced by manifolds parametrized over

$\mathbb{R}^k$ and the parametrization for $M^\pm$ is given by $p_W \circ f^\pm$. If we assume that the inclusions $i^\pm \colon \partial^\pm W \to W$ are homotopy equivalences, then there exist deformations $r^\pm \colon W \times I \to W$, $(x,t) \mapsto r_t^\pm(x)$ such that $r_0^\pm = \mathrm{id}_W$ and $r_1^\pm(W) \subset \partial^\pm W$.

A bounded cobordism is called a *bounded h-cobordism* if the inclusions $i^\pm$ are homotopy equivalences and additionally the deformations can be chosen such that the two sets

$$S^\pm = \{p_W \circ r_t^\pm(x) - p_W \circ r_1^\pm(x) \mid x \in W, t \in [0,1]\}$$

are bounded subsets of $\mathbb{R}^k$.

The following theorem (compare [239] and [327, Appendix]) contains the s-Cobordism Theorem 1.5 as a special case, gives another interpretation of elements in $\widetilde{K}_0(\mathbb{Z}\pi)$ and explains one aspect of the geometric relevance of negative K-groups.

Theorem 1.10 (Bounded h-Cobordism Theorem). *Suppose that M^- is parametrized over $\mathbb{R}^k$ and satisfies $\dim M^- \geq 5$. Let π be its fundamental group(oid). Equivalence classes of bounded h-cobordisms over M^- modulo bounded diffeomorphism relative M^- correspond bijectively to elements in $\kappa_{1-k}(\pi)$, where*

$$\kappa_{1-k}(\pi) = \begin{cases} \mathrm{Wh}(\pi) & \textit{if } k = 0, \\ \widetilde{K}_0(\mathbb{Z}\pi) & \textit{if } k = 1, \\ K_{1-k}(\mathbb{Z}\pi) & \textit{if } k \geq 2. \end{cases}$$

More information about negative K-groups can be found for instance in [8], [22], [57], [58], [113], [213], [238], [239], [252], [259], [268] and [327, Appendix].

1.3 Algebraic K-Theory - All Dimensions

So far we only considered the K-theory groups in dimensions ≤ 1. We now want to explain how Conjecture 1.1 generalizes to higher algebraic K-theory. For the definition of higher algebraic K-theory groups and the (connective) K-theory spectrum see [35], [52], [158], [249], [268], [292], [315] and [323]. We would like to stress that for us $\mathbf{K}(R)$ will always denote the *non-connective algebraic K-theory spectrum* for which $K_n(R) = \pi_n(\mathbf{K}(R))$ holds for all $n \in \mathbb{Z}$. For its definition see [52], [194] and [237].

The Farrell-Jones Conjecture for algebraic K-theory reduces for a torsion free group to the following conjecture.

Conjecture 1.11 (Farrell-Jones Conjecture for Torsion Free Groups and K-Theory). Let G be a torsion free group. Let R be a regular ring. Then the assembly map

$$H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$

is an isomorphism for $n \in \mathbb{Z}$.

Here $H_n(-;\mathbf{K}(R))$ denotes the homology theory which is associated to the spectrum $\mathbf{K}(R)$. It has the property that $H_n(\mathrm{pt};\mathbf{K}(R))$ is $K_n(R)$ for $n \in \mathbb{Z}$, where here and elsewhere pt denotes the space consisting of one point. The space BG is the *classifying space of the group* G, which up to homotopy is characterized by the property that it is a CW-complex with $\pi_1(BG) \cong G$ whose universal covering is contractible. The technical details of the construction of $H_n(-;\mathbf{K}(R))$ and the assembly map will be explained in a more general setting in Section 2.1.

The point of Conjecture 1.11 is that on the right-hand side of the assembly map we have the group $K_n(RG)$ we are interested in, whereas the left-hand side is a homology theory and hence much easier to compute. For every homology theory associated to a spectrum we have the Atiyah-Hirzebruch spectral sequence, which in our case has $E^2_{p,q} = H_p(BG;K_q(R))$ and converges to $H_{p+q}(BG;\mathbf{K}(R))$.

If R is regular, then the negative K-groups of R vanish and the spectral sequence lives in the first quadrant. Evaluating the spectral sequence for $n = p+q \leq 1$ shows that Conjecture 1.11 above implies Conjecture 1.1.

Remark 1.12 (Rational Computation). Rationally an Atiyah-Hirzebruch spectral sequence collapses always and the homological Chern character gives an isomorphism

$$\mathrm{ch}\colon \bigoplus_{p+q=n} H_p(BG;\mathbb{Q}) \otimes_{\mathbb{Q}} (K_q(R) \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\cong} H_n(BG;\mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The Atiyah-Hirzebruch spectral sequence and the Chern character will be discussed in a much more general setting in Chapter 8.

Remark 1.13 (Separation of Variables). We see that the left-hand side of the isomorphism in the previous remark consists of a group homology part and a part which is the rationalized K-theory of R. (Something similar happens before we rationalize at the level of spectra: The left hand side of Conjecture 1.11 can be interpreted as the homotopy groups of the spectrum $BG_+ \wedge \mathbf{K}(R)$.) So essentially Conjecture 1.11 predicts that the K-theory of RG is built up out of two independent parts: the K-theory of R and the group homology of G. We call this principle *separation of variables*. This principle also applies to other theories such as algebraic L-theory or topological K-theory. See also Remark 8.11.

Remark 1.14 (K-Theory of the Coefficients). Note that Conjecture 1.11 can only help us to explicitly compute the K-groups of RG in cases where we know enough about the K-groups of R. We obtain no new information about the K-theory of R itself. However, already for very simple rings the computation of their algebraic K-theory groups is an extremely hard problem.

It is known that the groups $K_n(\mathbb{Z})$ are finitely generated abelian groups [248]. Due to Borel [39] we know that

$$K_n(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } n = 0; \\ \mathbb{Q} & \text{if } n = 4k+1 \text{ with } k \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbb{Z}$ is regular we know that $K_n(\mathbb{Z})$ vanishes for $n \leq -1$. Moreover, $K_0(\mathbb{Z}) \cong \mathbb{Z}$ and $K_1(\mathbb{Z}) \cong \{\pm 1\}$, where the isomorphisms are given by the rank and the determinant. One also knows that $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$, $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ [189] and $K_4(\mathbb{Z}) \cong 0$ [264]. Finite fields belong to the few rings where one has a complete and explicit knowledge of all K-theory groups [247]. We refer the reader for example to [177], [226], [265], [322] and Soulé's article in [193] for more information about the algebraic K-theory of the integers or more generally of rings of integers in number fields.

Because of Borel's calculation the left hand side of the isomorphism described in Remark 1.12 specializes for $R = \mathbb{Z}$ to

$$H_n(BG;\mathbb{Q}) \oplus \bigoplus_{k=1}^{\infty} H_{n-(4k+1)}(BG;\mathbb{Q}) \tag{1}$$

and Conjecture 1.11 predicts that this group is isomorphic to $K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Next we discuss the case where the group G is infinite cyclic.

Remark 1.15 (Bass-Heller-Swan Decomposition). The so called *Bass-Heller-Swan-decomposition*, also known as the *Fundamental Theorem of algebraic K-theory*, computes the algebraic K-groups of $R[\mathbb{Z}]$ in terms of the algebraic K-groups and Nil-groups of R:

$$K_n(R[\mathbb{Z}]) \;\cong\; K_{n-1}(R) \oplus K_n(R) \oplus NK_n(R) \oplus NK_n(R).$$

Here the group $NK_n(R)$ is defined as the cokernel of the split injection $K_n(R) \to K_n(R[t])$. It can be identified with the cokernel of the split injection $K_{n-1}(R) \to K_{n-1}(\mathcal{N}il(R))$. Here $K_n(\mathcal{N}il(R))$ denotes the K-theory of the exact category of nilpotent endomorphisms of finitely generated projective R-modules. For negative n it is defined with the help of Bass' notion of a contracting functor [22] (see also [57]). The groups are known as *Nil-groups* and often denoted $\mathrm{Nil}_{n-1}(R)$.

For proofs of these facts and more information the reader should consult [22, Chapter XII], [25], [135, Theorem on page 236], [249, Corollary in §6 on page 38], [268, Theorems 3.3.3 and 5.3.30], [292, Theorem 9.8] and [300, Theorem 10.1].

If we iterate and use $R[\mathbb{Z}^n] = R[\mathbb{Z}^{n-1}][\mathbb{Z}]$ we see that a computation of $K_n(RG)$ must in general take into account information about $K_i(R)$ for all $i \leq n$. In particular we see that it is important to formulate Conjecture 1.11 with the non-connective K-theory spectrum.

Since S^1 is a model for $B\mathbb{Z}$, we get an isomorphism

$$H_n(B\mathbb{Z};\mathbf{K}(R)) \;\cong\; K_{n-1}(R) \oplus K_n(R)$$

and hence Conjecture 1.11 predicts

$$K_n(R[\mathbb{Z}]) \;\cong\; K_{n-1}(R) \oplus K_n(R).$$

This explains why in the formulation of Conjecture 1.11 the condition that R is regular appears. It guarantees that $NK_n(R) = 0$ [268, Theorem 5.3.30 on page 295]. There are weaker conditions which imply that $NK_n(R) = 0$ but "regular" has the advantage that R regular implies that $R[t]$ and $R[\mathbb{Z}] = R[t^{\pm 1}]$ are again regular, compare the discussion in Section 2 in [23].

The Nil-terms $NK_n(R)$ seem to be hard to compute. For instance $NK_1(R)$ either vanishes or is infinitely generated as an abelian group [95]. In Subsection 4.2.3 we will discuss the Isomorphism Conjecture for NK-groups. For more information about Nil-groups see for instance [73], [74], [146], [324] and [325].

1.4 Applications II

1.4.1 The Relation to Pseudo-Isotopy Theory

Let I denote the unit interval $[0,1]$. A topological *pseudoisotopy* of a compact manifold M is a homeomorphism $h\colon M \times I \to M \times I$, which restricted to $M \times \{0\} \cup \partial M \times I$ is the obvious inclusion. The space $P(M)$ of pseudoisotopies is the (simplicial) group of all such homeomorphisms. Pseudoisotopies play an important role if one tries to understand the homotopy type of the space $\mathrm{Top}(M)$ of self-homeomorphisms of a manifold M. We will see below in Subsection 1.6.2 how the results about pseudoisotopies discussed in this section combined with surgery theory lead to quite explicit results about the homotopy groups of $\mathrm{Top}(M)$.

There is a stabilization map $P(M) \to P(M \times I)$ given by crossing a pseudoisotopy with the identity on the interval I and the stable pseudoisotopy space is defined as $\mathcal{P}(M) = \operatorname{colim}_k P(M \times I^k)$. In fact $\mathcal{P}(-)$ can be extended to a functor on all spaces [144]. The natural inclusion $P(M) \to \mathcal{P}(M)$ induces an isomorphism on the i-th homotopy group if the dimension of M is large compared to i, see [43] and [157].

Waldhausen [314], [315] defines the algebraic K-theory of spaces functor $\mathbf{A}(X)$ and the functor $\mathbf{Wh}^{PL}(X)$ from spaces to spectra (or infinite loop spaces) and a fibration sequence

$$X_+ \wedge \mathbf{A}(\mathrm{pt}) \to \mathbf{A}(X) \to \mathbf{Wh}^{PL}(X).$$

Here $X_+ \wedge \mathbf{A}(\mathrm{pt}) \to \mathbf{A}(X)$ is an assembly map, which can be compared to the algebraic K-theory assembly map that appears in Conjecture 1.11 via a commutative diagram

$$\begin{array}{ccc} H_n(X;\mathbf{A}(\mathrm{pt})) & \longrightarrow & \pi_n(\mathbf{A}(X)) \\ \big\downarrow & & \big\downarrow \\ H_n(B\pi_1(X);\mathbf{K}(\mathbb{Z})) & \longrightarrow & K_n(\mathbb{Z}\pi_1(X)). \end{array}$$

In the case where $X \simeq BG$ is aspherical the vertical maps induce isomorphisms after rationalization for $n \geq 1$, compare [314, Proposition 2.2]. Since $\Omega^2 \operatorname{Wh}^{PL}(X) \simeq \mathcal{P}(X)$ (a guided tour through the literature concerning this and related results can be found in [90, Section 9]), Conjecture 1.11 implies rational vanishing results for the groups $\pi_n(\mathcal{P}(M))$ if M is an aspherical manifold. Compare also Remark 4.9.

Consequence 1.16. *Suppose M is a closed aspherical manifold and Conjecture 1.11 holds for $R = \mathbb{Z}$ and $G = \pi_1(M)$, then for all $n \geq 0$*

$$\pi_n(\mathcal{P}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

Similarly as above one defines smooth pseudoisotopies and the space of stable smooth pseudoisotopies $\mathcal{P}^{\mathrm{Diff}}(M)$. There is also a smooth version of the Whitehead space $\operatorname{Wh}^{\mathrm{Diff}}(X)$ and $\Omega^2 \operatorname{Wh}^{\mathrm{Diff}}(M) \simeq \mathcal{P}^{\mathrm{Diff}}(M)$. Again there is a close relation to A-theory via the natural splitting $\mathbf{A}(X) \simeq \Sigma^\infty(X_+) \vee \mathbf{Wh}^{\mathrm{Diff}}(X)$, see [316]. Here $\Sigma^\infty(X_+)$ denotes the suspension spectrum associated to X_+. Using this one can split off an assembly map $H_n(X; \mathbf{Wh}^{\mathrm{Diff}}(\mathrm{pt})) \to \pi_n(\mathbf{Wh}^{\mathrm{Diff}}(X))$ from the A-theory assembly map. Since for every space $\pi_n(\Sigma^\infty(X_+)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(X; \mathbb{Q})$ Conjecture 1.11 combined with the rational computation in (1) yields the following result.

Consequence 1.17. *Suppose M is a closed aspherical manifold and Conjecture 1.11 holds for $R = \mathbb{Z}$ and $G = \pi_1(M)$. Then for $n \geq 0$ we have*

$$\pi_n(\mathcal{P}^{\mathrm{Diff}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{k=1}^{\infty} H_{n-4k+1}(M; \mathbb{Q}).$$

Observe that the fundamental difference between the smooth and the topological case occurs already when G is the trivial group.

1.4.2 Negative *K*-Groups and Bounded Pseudo-Isotopies

We briefly explain a further geometric interpretation of negative K-groups, which parallels the discussion of bounded h-cobordisms in Subsection 1.2.3.

Let $p \colon M \times \mathbb{R}^k \to \mathbb{R}^k$ denote the natural projection. The space $P_b(M; \mathbb{R}^k)$ of bounded pseudoisotopies is the space of all self-homeomorphisms $h \colon M \times \mathbb{R}^k \times I \to M \times \mathbb{R}^k \times I$ such that restricted to $M \times \mathbb{R}^k \times \{0\}$ the map h is the inclusion and such that h is bounded, i.e. the set $\{p \circ h(y) - p(y) \mid y \in M \times \mathbb{R}^k \times I\}$ is a bounded subset of $\mathbb{R}^k$. There is again a stabilization map $P_b(M; \mathbb{R}^k) \to P_b(M \times I; \mathbb{R}^k)$ and a stable bounded pseudoisotopy space $\mathcal{P}_b(M; \mathbb{R}^k) = \operatorname{colim}_j P_b(M \times I^j; \mathbb{R}^k)$. There is a homotopy equivalence $\mathcal{P}_b(M; \mathbb{R}^k) \to \Omega \mathcal{P}_b(M; \mathbb{R}^{k+1})$ [144, Appendix II] and hence the sequence of spaces $\mathcal{P}_b(M; \mathbb{R}^k)$ for $k = 0, 1, \ldots$ is an Ω-spectrum $\mathbf{P}(M)$. Analogously one defines the differentiable bounded pseudoisotopies $\mathcal{P}_b^{\mathrm{diff}}(M; \mathbb{R}^k)$ and an Ω-spectrum $\mathbf{P}^{\mathrm{diff}}(M)$. The negative homotopy groups of these spectra have an interpretation in terms of low and negative dimensional K-groups. In terms of unstable homotopy groups this is explained in the following theorem which is closely related to Theorem 1.10 about bounded h-cobordisms.

Theorem 1.18 (Negative Homotopy Groups of Pseudoisotopies). *Let $G = \pi_1(M)$. Suppose n and k are such that $n + k \geq 0$, then for $k \geq 1$ there are isomorphisms*

$$\pi_{n+k}(\mathcal{P}_b(M;\mathbb{R}^k)) = \begin{cases} \mathrm{Wh}(G) & \text{if } n = -1, \\ \widetilde{K}_0(\mathbb{Z}G) & \text{if } n = -2, \\ K_{n+2}(\mathbb{Z}G) & \text{if } n < -2 \end{cases}$$

The same result holds in the differentiable case.

Note that Conjecture 1.11 predicts that these groups vanish if G is torsion-free. The result above is due to Anderson and Hsiang [8] and is also discussed in [327, Appendix].

1.5 L-Theory

We now move on to the L-theoretic version of the Farrell-Jones Conjecture. We will still stick to the case where the group is torsion free. The conjecture is obtained by replacing K-theory and the K-theory spectrum in Conjecture 1.11 by 4-periodic L-theory and the L-theory spectrum $\mathbf{L}^{\langle -\infty \rangle}(R)$. Explanations will follow below.

Conjecture 1.19 (Farrell-Jones Conjecture for Torsion Free Groups and L-Theory). Let G be a torsion free group and let R be a ring with involution. Then the assembly map

$$H_n(BG;\mathbf{L}^{\langle -\infty \rangle}(R)) \to L_n^{\langle -\infty \rangle}(RG)$$

is an isomorphism for $n \in \mathbb{Z}$.

To a ring with involution one can associate (decorated) symmetric or quadratic algebraic L-groups, compare [44], [45], [256], [259] and [332]. We will exclusively deal with the quadratic algebraic L-groups and denote them by $L_n^{\langle j \rangle}(R)$. Here $n \in \mathbb{Z}$ and $j \in \{-\infty\} \amalg \{j \in \mathbb{Z} \mid j \leq 2\}$ is the so called *decoration*. The decorations $j = 0, 1$ correspond to the decorations p, h and $j = 2$ is related to the decoration s appearing in the literature. Decorations will be discussed in Remark 1.21 below. The L-groups $L_n^{\langle j \rangle}(R)$ are 4-periodic, i.e. $L_n^{\langle j \rangle}(R) \cong L_{n+4}^{\langle j \rangle}(R)$ for $n \in \mathbb{Z}$.

If we are given an involution $r \mapsto \overline{r}$ on a ring R, we will always equip RG with the involution that extends the given one and satisfies $\overline{g} = g^{-1}$. On $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ we always use the trivial involution and on $\mathbb{C}$ the complex conjugation.

One can construct an *L-theory spectrum* $\mathbf{L}^{\langle j \rangle}(R)$ such that $\pi_n(\mathbf{L}^{\langle j \rangle}(R)) = L_n^{\langle j \rangle}(R)$, compare [258, § 13]. Above and in the sequel $H_n(-;\mathbf{L}^{\langle j \rangle}(R))$ denotes the homology theory which is associated to this spectrum. In particular we have $H_n(\mathrm{pt}; \mathbf{L}^{\langle j\rangle}(R)) = L_n^{\langle j\rangle}(R)$. We postpone the discussion of the assembly map to Section 2.1 where we will construct it in greater generality.

Remark 1.20 (The Coefficients in the L-Theory Case). In contrast to K-theory (compare Remark 1.14) the L-theory of the most interesting coefficient ring $R = \mathbb{Z}$ is well known. The groups $L_n^{\langle j\rangle}(\mathbb{Z})$ for fixed n and varying $j \in \{-\infty\} \amalg \{j \in \mathbb{Z} \mid j \leq 2\}$ are all naturally isomorphic (compare Proposition 1.23 below) and we have $L_0^{\langle j\rangle}(\mathbb{Z}) \cong \mathbb{Z}$ and $L_2^{\langle j\rangle}(\mathbb{Z}) \cong \mathbb{Z}/2$, where the isomorphisms are given by the signature divided by 8 and the Arf invariant, and $L_1^{\langle j\rangle}(\mathbb{Z}) = L_3^{\langle j\rangle}(\mathbb{Z}) = 0$, see [41, Chapter III], [256, Proposition 4.3.1 on page 419].

Remark 1.21 (Decorations). L-groups are designed as obstruction groups for surgery problems. The decoration reflects what kind of surgery problem one is interested in. All L-groups can be described as cobordism classes of suitable quadratic Poincaré chain complexes. If one works with chain complexes of finitely generated free based R-modules and requires that the torsion of the Poincaré chain homotopy equivalence vanishes in $\widetilde{K}_1(R)$, then the corresponding L-groups are denoted $L_n^{\langle 2\rangle}(R)$. If one drops the torsion condition, one obtains $L_n^{\langle 1\rangle}(R)$, which is usually denoted $L^h(R)$. If one works with finitely generated projective modules, one obtains $L^{\langle 0\rangle}(R)$, which is also known as $L^p(R)$.

The L-groups with negative decorations can be defined inductively via the Shaneson splitting, compare Remark 1.26 below. Assuming that the L-groups with decorations j have already been defined one sets

$$L_{n-1}^{<j-1>}(R) = \operatorname{coker}(L_n^{<j>}(R) \to L_n^{<j>}(R[\mathbb{Z}])).$$

Compare [259, Definition 17.1 on page 145]. Alternatively these groups can be obtained via a process which is in the spirit of Subsection 1.2.3 and Subsection 1.4.2. One can define them as L-theory groups of suitable categories of modules parametrized over $\mathbb{R}^k$. For details the reader could consult [55, Section 4]. There are forgetful maps $L_n^{\langle j+1\rangle}(R) \to L_n^{\langle j\rangle}(R)$. The group $L_n^{\langle -\infty\rangle}(R)$ is defined as the colimit over these maps. For more information see [254], [259].

For group rings we also define $L_n^s(RG)$ similar to $L_n^{\langle 2\rangle}(RG)$ but we require the torsion to lie in $\operatorname{im} A_1 \subset \widetilde{K}_1(RG)$, where A_1 is the map defined in Section 1.1. Observe that $L_n^s(RG)$ really depends on the pair (R, G) and differs in general from $L_n^{\langle 2\rangle}(RG)$.

Remark 1.22 (The Interplay of K- and L-Theory). For $j \leq 1$ there are forgetful maps $L_n^{\langle j+1\rangle}(R) \to L_n^{\langle j\rangle}(R)$ which sit inside the following sequence, which is known as the *Rothenberg sequence* [256, Proposition 1.10.1 on page 104], [259, 17.2].

$$\ldots \to L_n^{\langle j+1\rangle}(R) \to L_n^{\langle j\rangle}(R) \to \widehat{H}^n(\mathbb{Z}/2;\widetilde{K}_j(R)) \\ \to L_{n-1}^{\langle j+1\rangle}(R) \to L_{n-1}^{\langle j\rangle}(R) \to \ldots . \quad (2)$$

Here $\widehat{H}^n(\mathbb{Z}/2;\widetilde{K}_j(R))$ is the Tate-cohomology of the group $\mathbb{Z}/2$ with coefficients in the $\mathbb{Z}[\mathbb{Z}/2]$-module $\widetilde{K}_j(R)$. The involution on $\widetilde{K}_j(R)$ comes from the involution on R. There is a similar sequence relating $L_n^s(RG)$ and $L_n^h(RG)$, where the third term is the $\mathbb{Z}/2$-Tate-cohomology of $\mathrm{Wh}_1^R(G)$. Note that Tate-cohomology groups of the group $\mathbb{Z}/2$ are always annihilated by multiplication with 2. In particular $L_n^{\langle j\rangle}(R)[\frac{1}{2}] = L_n^{\langle j\rangle}(R)\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{2}]$ is always independent of j.

Let us formulate explicitly what we obtain from the above sequences for $R=\mathbb{Z}G$.

Proposition 1.23. *Let G be a torsion free group, then Conjecture 1.3 about the vanishing of* $\mathrm{Wh}(G)$, $\widetilde{K_0}(\mathbb{Z}G)$ *and* $K_{-i}(\mathbb{Z}G)$ *for* $i\geq 1$ *implies that for fixed n and varying $j\in\{-\infty\}\amalg\{j\in\mathbb{Z}\mid j\leq 1\}$ the L-groups $L_n^{\langle j\rangle}(\mathbb{Z}G)$ are all naturally isomorphic and moreover $L_n^{\langle 1\rangle}(\mathbb{Z}G)=L_n^h(\mathbb{Z}G)\cong L_n^s(\mathbb{Z}G)$.*

Remark 1.24 (Rational Computation). As in the K-theory case we have an Atiyah-Hirzebruch spectral sequence:

$$E^2_{p,q} = H_p(BG;L_q^{\langle-\infty\rangle}(R)) \quad\Rightarrow\quad H_{p+q}(BG;\mathbf{L}^{\langle-\infty\rangle}(R)).$$

Rationally this spectral sequence collapses and the homological Chern character gives for $n\in\mathbb{Z}$ an isomorphism

$$\mathrm{ch}\colon \bigoplus_{p+q=n} H_p(BG;\mathbb{Q})\otimes_{\mathbb{Q}}\left(L_q^{\langle-\infty\rangle}(R)\otimes_{\mathbb{Z}}\mathbb{Q}\right) \\ \xrightarrow{\cong} H_n(BG;\mathbf{L}^{\langle-\infty\rangle}(R))\otimes_{\mathbb{Z}}\mathbb{Q}. \quad (3)$$

In particular we obtain in the case $R=\mathbb{Z}$ from Remark 1.20 for all $n\in\mathbb{Z}$ and all decorations j an isomorphism

$$\mathrm{ch}\colon \bigoplus_{k=0}^{\infty} H_{n-4k}(BG;\mathbb{Q}) \xrightarrow{\cong} H_n(BG;\mathbf{L}^{\langle j\rangle}(\mathbb{Z}))\otimes_{\mathbb{Z}}\mathbb{Q}. \quad (4)$$

This spectral sequence and the Chern character above will be discussed in a much more general setting in Chapter 8.

Remark 1.25 (Torsion Free is Necessary). If G is finite, $R=\mathbb{Z}$ and $n=0$, then the rationalized left hand side of the assembly equals $\mathbb{Q}$, whereas the right hand side is isomorphic to the rationalization of the real representation ring. Since the group homology of a finite group vanishes rationally except in dimension 0, the previous remark shows that we need to assume the group to be torsion free in Conjecture 1.19

Remark 1.26 (Shaneson splitting). The Bass-Heller-Swan decomposition in K-theory (see Remark 1.15) has the following analogue for the algebraic L-groups, which is known as the *Shaneson splitting* [284]

$$L_n^{\langle j \rangle}(R[\mathbb{Z}]) \cong L_{n-1}^{\langle j-1 \rangle}(R) \oplus L_n^{\langle j \rangle}(R). \tag{5}$$

Here for the decoration $j = -\infty$ one has to interpret $j-1$ as $-\infty$. Since S^1 is a model for $B\mathbb{Z}$, we get an isomorphisms

$$H_n(B\mathbb{Z}; \mathbf{L}^{\langle j \rangle}(R)) \cong L_{n-1}^{\langle j \rangle}(R) \oplus L_n^{\langle j \rangle}(R).$$

This explains why in the formulation of the L-theoretic Farrell-Jones Conjecture for torsion free groups (see Conjecture 1.19) we use the decoration $j = -\infty$.

As long as one deals with torsion free groups and one believes in the low dimensional part of the K-theoretic Farrell-Jones Conjecture (predicting the vanishing of $\mathrm{Wh}(G)$, $\widetilde{K}_0(\mathbb{Z}G)$ and of the negative K-groups, see Conjecture 1.3) there is no difference between the various decorations j, compare Proposition 1.23. But as soon as one allows torsion in G, the decorations make a difference and it indeed turns out that if one replaces the decoration $j = -\infty$ by $j = s, h$ or p there are counterexamples for the L-theoretic version of Conjecture 2.2 even for $R = \mathbb{Z}$ [123].

Even though in the above Shaneson splitting (5) there are no terms analogous to the Nil-terms in Remark 1.15 such Nil-phenomena do also occur in L-theory, as soon as one considers amalgamated free products. The corresponding groups are the UNil-groups. They vanish if one inverts 2 [49]. For more information about the UNil-groups we refer to [15] [46], [47], [74], [77], [96], [260].

1.6 Applications III

1.6.1 The Borel Conjecture

One of the driving forces for the development of the Farrell-Jones Conjectures is still the following topological rigidity conjecture about closed aspherical manifolds, compare [107]. Recall that a manifold, or more generally a CW-complex, is called *aspherical* if its universal covering is contractible. An aspherical CW-complex X with $\pi_1(X) = G$ is a model for the classifying space BG. If X is an aspherical manifold and hence finite dimensional, then G is necessarily torsionfree.

Conjecture 1.27 (Borel Conjecture). Let $f\colon M \to N$ be a homotopy equivalence of aspherical closed topological manifolds. Then f is homotopic to a homeomorphism. In particular two closed aspherical manifolds with isomorphic fundamental groups are homeomorphic.

Closely related to the Borel Conjecture is the conjecture that each aspherical finitely dominated Poincaré complex is homotopy equivalent to a closed topological manifold. The Borel Conjecture 1.27 is false in the smooth category, i.e. if one replaces topological manifold by smooth manifold and homeomorphism by diffeomorphism [106].

Using surgery theory one can show that in dimensions ≥ 5 the Borel Conjecture is implied by the K-theoretic vanishing Conjecture 1.3 combined with the L-theoretic Farrell-Jones Conjecture.

Theorem 1.28 (The Farrell-Jones Conjecture Implies the Borel Conjecture). *Let G be a torsion free group. If $\mathrm{Wh}(G)$, $\widetilde{K}_0(\mathbb{Z}G)$ and all the groups $K_{-i}(\mathbb{Z}G)$ with $i \geq 1$ vanish and if the assembly map*

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \to L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$$

is an isomorphism for all n, then the Borel Conjecture holds for all orientable manifolds of dimension ≥ 5 whose fundamental group is G.

The Borel Conjecture 1.27 can be reformulated in the language of surgery theory to the statement that the topological structure set $\mathcal{S}^{\mathrm{top}}(M)$ of an aspherical closed topological manifold M consists of a single point. This set is the set of equivalence classes of homotopy equivalences $f\colon M' \to M$ with a topological closed manifold as source and M as target under the equivalence relation, for which $f_0\colon M_0 \to M$ and $f_1\colon M_1 \to M$ are equivalent if there is a homeomorphism $g\colon M_0 \to M_1$ such that $f_1 \circ g$ and f_0 are homotopic.

The surgery sequence of a closed orientable topological manifold M of dimension $n \geq 5$ is the exact sequence

$$\begin{aligned} \ldots \to \mathcal{N}_{n+1}(M \times [0,1], M \times \{0,1\}) \xrightarrow{\sigma} L^s_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial} \mathcal{S}^{\mathrm{top}}(M) \\ \xrightarrow{\eta} \mathcal{N}_n(M) \xrightarrow{\sigma} L^s_n(\mathbb{Z}\pi_1(M)), \end{aligned}$$

which extends infinitely to the left. It is the basic tool for the classification of topological manifolds. (There is also a smooth version of it.) The map σ appearing in the sequence sends a normal map of degree one to its surgery obstruction. This map can be identified with the version of the L-theory assembly map where one works with the 1-connected cover $\mathbf{L}^s(\mathbb{Z})\langle 1 \rangle$ of $\mathbf{L}^s(\mathbb{Z})$. The map $H_k(M; \mathbf{L}^s(\mathbb{Z})\langle 1 \rangle) \to H_k(M; \mathbf{L}^s(\mathbb{Z}))$ is injective for $k = n$ and an isomorphism for $k > n$. Because of the K-theoretic assumptions we can replace the s-decoration with the $\langle -\infty \rangle$-decoration, compare Proposition 1.23. Therefore the Farrell-Jones Conjecture 1.19 implies that the maps $\sigma\colon \mathcal{N}_n(M) \to L^s_n(\mathbb{Z}\pi_1(M))$ and $\mathcal{N}_{n+1}(M \times [0,1], M \times \{0,1\}) \xrightarrow{\sigma} L^s_{n+1}(\mathbb{Z}\pi_1(M))$ are injective respectively bijective and thus by the surgery sequence that $\mathcal{S}^{\mathrm{top}}(M)$ is a point and hence the Borel Conjecture 1.27 holds for M. More details can be found e.g. in [127, pages 17,18,28], [258, Chapter 18].

For more information about surgery theory we refer for instance to [41], [44], [45], [121], [122], [167], [178], [253], [294], [293], and [320].

1.6.2 Automorphisms of Manifolds

If one additionally also assumes the Farrell-Jones Conjectures for higher K-theory, one can combine the surgery theoretic results with the results about pseudoisotopies from Subsection 1.4.1 to obtain the following results.

Theorem 1.29 (Homotopy Groups of $\operatorname{Top}(M)$**).** *Let M be an orientable closed aspherical manifold of dimension > 10 with fundamental group G. Suppose the L-theory assembly map*

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \to L_n^{\langle -\infty \rangle}(\mathbb{Z}G)$$

is an isomorphism for all n and suppose the K-theory assembly map

$$H_n(BG; \mathbf{K}(\mathbb{Z})) \to K_n(\mathbb{Z}G)$$

is an isomorphism for $n \leq 1$ and a rational isomorphism for $n \geq 2$. Then for $1 \leq i \leq (\dim M - 7)/3$ one has

$$\pi_i(\operatorname{Top}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \operatorname{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1, \\ 0 & \text{if } i > 1 \end{cases}$$

In the differentiable case one additionally needs to study involutions on the higher K-theory groups. The corresponding result reads:

Theorem 1.30 (Homotopy Groups of $\operatorname{Diff}(M)$**).** *Let M be an orientable closed aspherical differentiable manifold of dimension > 10 with fundamental group G. Then under the same assumptions as in Theorem 1.29 we have for $1 \leq i \leq (\dim M - 7)/3$*

$$\pi_i(\operatorname{Diff}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \operatorname{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ \bigoplus_{j=1}^{\infty} H_{(i+1)-4j}(M; \mathbb{Q}) & \text{if } i > 1 \text{ and } \dim M \text{ odd}; \\ 0 & \text{if } i > 1 \text{ and } \dim M \text{ even}. \end{cases}$$

See for instance [97], [109, Section 2] and [120, Lecture 5]. For a modern survey on automorphisms of manifolds we refer to [329].

1.7 The Baum-Connes Conjecture in the Torsion Free Case

We denote by $K_*(Y)$ the complex K-homology of a topological space Y and by $K_*(C_r^*(G))$ the (topological) K-theory of the reduced group C^*-algebra. More explanations will follow below.

Conjecture 1.31 (Baum-Connes Conjecture for Torsion Free Groups). Let G be a torsion free group. Then the Baum-Connes assembly map

$$K_n(BG) \to K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

Complex K-homology $K_*(Y)$ is the homology theory associated to the topological (complex) K-theory spectrum $\mathbf{K}^{\mathrm{top}}$ (which is is often denoted $\mathbf{BU}$) and could also be written as $K_*(Y) = H_*(Y; \mathbf{K}^{\mathrm{top}})$. The *co*homology theory associated to the spectrum $\mathbf{K}^{\mathrm{top}}$ is the well known complex K-theory defined in terms of complex vector bundles. Complex K-homology is a 2-periodic theory, i.e. $K_n(Y) \cong K_{n+2}(Y)$.

Also the *topological K-groups* $K_n(A)$ of a C^*-algebra A are 2-periodic. Whereas $K_0(A)$ coincides with the algebraically defined K_0-group, the other groups $K_n(A)$ take the topology of the C^*-algebra A into account, for instance $K_n(A) = \pi_{n-1}(GL(A))$ for $n \geq 1$.

Let $\mathcal{B}(l^2(G))$ denote the bounded linear operators on the Hilbert space $l^2(G)$ whose orthonormal basis is G. The *reduced complex group C^*-algebra* $C_r^*(G)$ is the closure in the norm topology of the image of the regular representation $\mathbb{C}G \to \mathcal{B}(l^2(G))$, which sends an element $u \in \mathbb{C}G$ to the (left) G-equivariant bounded operator $l^2(G) \to l^2(G)$ given by right multiplication with u. In particular one has natural inclusions

$$\mathbb{C}G \subseteq C_r^*(G) \subseteq \mathcal{B}(l^2(G))^G \subseteq \mathcal{B}(l^2(G)).$$

It is essential to use the reduced group C^*-algebra in the Baum-Connes Conjecture, there are counterexamples for the version with the maximal group C^*-algebra, compare Subsection 4.1.2. For information about C^*-algebras and their topological K-theory we refer for instance to [37], [71], [80], [154], [188], [228], [279] and [321].

Remark 1.32 (The Coefficients in the Case of Topological K-Theory). If we specialize to the trivial group $G = \{1\}$, then the complex reduced group C^*-algebra reduces to $C_r^*(G) = \mathbb{C}$ and the topological K-theory is well known: by periodicity it suffices to know that $K_0(\mathbb{C}) \cong \mathbb{Z}$, where the homomorphism is given by the dimension, and $K_1(\mathbb{C}) = 0$. Correspondingly we have $K_q(\mathrm{pt}) = \mathbb{Z}$ for q even and $K_q(\mathrm{pt}) = 0$ for odd q.

Remark 1.33 (Rational Computation). There is an Atiyah-Hirzebruch spectral sequence which converges to $K_{p+q}(BG)$ and whose E^2-term is $E^2_{p,q} = H_p(BG; K_q(\mathrm{pt}))$. Rationally this spectral sequence collapses and the homological Chern character gives an isomorphism for $n \in \mathbb{Z}$

$$\begin{aligned} \mathrm{ch}\colon \bigoplus_{k\in\mathbb{Z}} H_{n-2k}(BG;\mathbb{Q}) \;=\; \bigoplus_{p+q=n} H_p(BG;\mathbb{Q}) \otimes_{\mathbb{Q}} (K_q(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q}) \\ \xrightarrow{\cong} K_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q}. \quad (6) \end{aligned}$$

Remark 1.34 (Torsionfree is Necessary). In the case where G is a finite group the reduced group C^*-algebra $C_r^*(G)$ coincides with the complex group ring $\mathbb{C}G$ and $K_0(C_r^*(G))$ coincides with the complex representation ring of G. Since the group homology of a finite group vanishes rationally except in dimension 0, the previous remark shows that we need to assume the group to be torsion free in Conjecture 1.31.

Remark 1.35. (Bass-Heller-Swan-Decomposition for Topological K-Theory) There is an analogue of the Bass-Heller-Swan decomposition in algebraic K-theory (see Remark 1.15) or of the Shaneson splitting in L-theory (see Remark 1.26) for topological K-theory. Namely we have

$$K_n(C_r^*(G \times \mathbb{Z})) \;\cong\; K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)),$$

see [243, Theorem 3.1 on page 151] or more generally [244, Theorem 18 on page 632]. This is consistent with the obvious isomorphism

$$K_n(B(G \times \mathbb{Z})) = K_n(BG \times S^1) \cong K_{n-1}(BG) \oplus K_n(BG).$$

Notice that here in contrast to the algebraic K-theory no Nil-terms occur (see Remark 1.15) and that there is no analogue of the complications in algebraic L-theory coming from the varying decorations (see Remark 1.26). This absence of Nil-terms or decorations is the reason why in the final formulation of the Baum-Connes Conjecture it suffices to deal with the family of finite subgroups, whereas in the algebraic K- and L-theory case one must consider the larger and harder to handle family of virtually cyclic subgroups. This in some sense makes the computation of topological K-theory of reduced group C^*-algebras easier than the computation of $K_n(\mathbb{Z}G)$ or $L_n(\mathbb{Z}G)$.

Remark 1.36 (Real Version of the Baum-Connes Conjecture). There is an obvious *real version of the Baum-Connes Conjecture*. It says that for a torsion free group the real assembly map

$$KO_n(BG) \to KO_n(C_r^*(G;\mathbb{R}))$$

is bijective for $n \in \mathbb{Z}$. We will discuss in Subsection 4.1.1 below that this real version of the Baum-Connes Conjecture is implied by the complex version Conjecture 1.31.

Here $KO_n(C_r^*(G;\mathbb{R}))$ is the topological K-theory of the *real reduced group C^*-algebra $C_r^*(G;\mathbb{R})$*. We use KO instead of K as a reminder that we work with real C^*-algebras. The topological real K-theory $KO_*(Y)$ is the homology theory associated to the spectrum $\mathbf{BO}$, whose associated *co*homology theory is given in terms of real vector bundles. Both, topological K-theory of a real C^*-algebra and KO-homology of a space are 8-periodic and $KO_n(\mathrm{pt}) = K_n(\mathbb{R})$ is $\mathbb{Z}$, if $n = 0, 4$ (8), is $\mathbb{Z}/2$ if $n = 1, 2$ (8) and is 0 if $n = 3, 5, 6, 7$ (8).

More information about the K-theory of real C^*-algebras can be found in [281].

1.8 Applications IV

We now discuss some consequences of the Baum-Connes Conjecture for Torsion Free Groups 1.31.

1.8.1 The Trace Conjecture in the Torsion Free Case

The assembly map appearing in the Baum-Connes Conjecture has an interpretation in terms of index theory. This is important for geometric applications.

It is of the same significance as the interpretation of the L-theoretic assembly map as the map σ appearing in the exact surgery sequence discussed in Section 1.5. We proceed to explain this.

An element $\eta \in K_0(BG)$ can be represented by a pair (M, P^*) consisting of a cocompact free proper smooth G-manifold M with Riemannian metric together with an elliptic G-complex P^* of differential operators of order 1 on M [29]. To such a pair one can assign an index $\mathrm{ind}_{C_r^*(G)}(M, P^*)$ in $K_0(C_r^*(G))$ [223] which is the image of η under the assembly map $K_0(BG) \to K_0(C_r^*(G))$ appearing in Conjecture 1.31. With this interpretation the surjectivity of the assembly map for a torsion free group says that any element in $K_0(C_r^*(G))$ can be realized as an index. This allows to apply index theorems to get interesting information.

Here is a prototype of such an argument. The *standard trace*

$$\mathrm{tr}_{C_r^*(G)} \colon C_r^*(G) \to \mathbb{C} \tag{7}$$

sends an element $f \in C_r^*(G) \subseteq \mathcal{B}(l^2(G))$ to $\langle f(1), 1 \rangle_{l^2(G)}$. Applying the trace to idempotent matrices yields a homomorphism

$$\mathrm{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}.$$

Let $\mathrm{pr} \colon BG \to \mathrm{pt}$ be the projection. For a group G the following diagram commutes

$$\begin{tikzcd} K_0(BG) \arrow[r, "A"] \arrow[d, "K_0(\mathrm{pr})"'] & K_0(C_r^*(G)) \arrow[r, "\mathrm{tr}_{C_r^*(G)}"] & \mathbb{R} \\ K_0(\mathrm{pt}) \arrow[r, "\cong"] & K_0(\mathbb{C}) \arrow[r, "\cong", "\mathrm{tr}_{\mathbb{C}}"'] & \mathbb{Z}. \arrow[u, "i"'] \end{tikzcd} \tag{8}$$

Here $i \colon \mathbb{Z} \to \mathbb{R}$ is the inclusion and A is the assembly map. This non-trivial statement follows from Atiyah's L^2-index theorem [12]. Atiyah's theorem says that the L^2-index $\mathrm{tr}_{C_r^*(G)} \circ A(\eta)$ of an element $\eta \in K_0(BG)$, which is represented by a pair (M, P^*), agrees with the ordinary index of $(G\backslash M; G\backslash P^*)$, which is $\mathrm{tr}_{\mathbb{C}} \circ K_0(\mathrm{pr})(\eta) \in \mathbb{Z}$.

The following conjecture is taken from [27, page 21].

Conjecture 1.37 (Trace Conjecture for Torsion Free Groups). For a torsion free group G the image of

$$\mathrm{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$$

consists of the integers.

The commutativity of diagram (8) above implies

Consequence 1.38. *The surjectivity of the Baum-Connes assembly map*

$$K_0(BG) \to K_0(C_r^*(G))$$

implies Conjecture 1.37, the Trace Conjecture for Torsion Free Groups.

1.8.2 The Kadison Conjecture

Conjecture 1.39 (Kadison Conjecture). If G is a torsion free group, then the only idempotent elements in $C_r^*(G)$ are 0 and 1.

Lemma 1.40. *The Trace Conjecture for Torsion Free Groups 1.37 implies the Kadison Conjecture 1.39.*

Proof. Assume that $p \in C_r^*(G)$ is an idempotent different from 0 or 1. From p one can construct a non-trivial projection $q \in C_r^*(G)$, i.e. $q^2 = q$, $q^* = q$, with $\mathrm{im}(p) = \mathrm{im}(q)$ and hence with $0 < q < 1$. Since the standard trace $\mathrm{tr}_{C_r^*(G)}$ is faithful, we conclude $\mathrm{tr}_{C_r^*(G)}(q) \in \mathbb{R}$ with $0 < \mathrm{tr}_{C_r^*(G)}(q) < 1$. Since $\mathrm{tr}_{C_r^*(G)}(q)$ is by definition the image of the element $[\mathrm{im}(q)] \in K_0(C_r^*(G))$ under $\mathrm{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$, we get a contradiction to the assumption $\mathrm{im}(\mathrm{tr}_{C_r^*(G)}) \subseteq \mathbb{Z}$. $\square$

Recall that a ring R is called an *integral domain* if it has no non-trivial zero-divisors, i.e. if $r, s \in R$ satisfy $rs = 0$, then r or s is 0. Obviously the Kadison Conjecture 1.39 implies for $R \subseteq \mathbb{C}$ the following.

Conjecture 1.41 (Idempotent Conjecture). Let R be an integral domain and let G be a torsion free group. Then the only idempotents in RG are 0 and 1.

The statement in the conjecture above is a purely algebraic statement. If $R = \mathbb{C}$, it is by the arguments above related to questions about operator algebras, and thus methods from operator algebras can be used to attack it.

1.8.3 Other Related Conjectures

We would now like to mention several conjectures which are not directly implied by the Baum-Connes or Farrell-Jones Conjectures, but which are closely related to the Kadison Conjecture and the Idempotent Conjecture mentioned above.

The next conjecture is also called the *Kaplansky Conjecture*.

Conjecture 1.42 (Zero-Divisor-Conjecture). Let R be an integral domain and G be a torsion free group. Then RG is an integral domain.

Obviously the Zero-Divisor-Conjecture 1.42 implies the Idempotent Conjecture 1.41. The Zero-Divisor-Conjecture for $R = \mathbb{Q}$ is implied by the following version of the Atiyah Conjecture (see [202, Lemma 10.5 and Lemma 10.15]).

Conjecture 1.43 (Atiyah-Conjecture for Torsion Free Groups). Let G be a torsion free group and let M be a closed Riemannian manifold. Let $\overline{M} \to M$ be a regular covering with G as group of deck transformations. Then all L^2-Betti numbers $b_p^{(2)}(\overline{M}; \mathcal{N}(G))$ are integers.

For the precise definition and more information about L^2-Betti numbers and the group von Neumann algebra $\mathcal{N}(G)$ we refer for instance to [202], [205].

This more geometric formulation of the Atiyah Conjecture is in fact implied by the following more operator theoretic version. (The two would be equivalent if one would work with rational instead of complex coefficients below.)

Conjecture 1.44 (Strong Atiyah-Conjecture for Torsion Free Groups). Let G be a torsion free group. Then for all (m,n)-matrices A over $\mathbb{C}G$ the von Neumann dimension of the kernel of the induced G-equivariant bounded operator

$$r_A^{(2)} \colon l^2(G)^m \to l^2(G)^n$$

is an integer.

The Strong Atiyah-Conjecture for Torsion Free Groups implies both the Atiyah-Conjecture for Torsion Free Groups 1.43 [202, Lemma 10.5 on page 371] and the Zero-Divisor-Conjecture 1.42 for $R = \mathbb{C}$ [202, Lemma 10.15 on page 376].

Conjecture 1.45 (Embedding Conjecture). Let G be a torsion free group. Then $\mathbb{C}G$ admits an embedding into a skewfield.

Obviously the Embedding Conjecture implies the Zero-Divisor-Conjecture 1.42 for $R = \mathbb{C}$. If G is a torsion free amenable group, then the Strong Atiyah-Conjecture for Torsion Free Groups 1.44 and the Zero-Divisor-Conjecture 1.42 for $R = \mathbb{C}$ are equivalent [202, Lemma 10.16 on page 376]. For more information about the Atiyah Conjecture we refer for instance to [192], [202, Chapter 10] and [261].

Finally we would like to mention the Unit Conjecture.

Conjecture 1.46 (Unit-Conjecture). Let R be an integral domain and G be a torsion free group. Then every unit in RG is trivial, i.e. of the form $r \cdot g$ for some unit $r \in R^{\mathrm{inv}}$ and $g \in G$.

The Unit Conjecture 1.46 implies the Zero-Divisor-Conjecture 1.42. For a proof of this fact and for more information we refer to [187, Proposition 6.21 on page 95].

1.8.4 L^2-Rho-Invariants and L^2-Signatures

Let M be a closed connected orientable Riemannian manifold. Denote by $\eta(M) \in \mathbb{R}$ the *eta-invariant* of M and by $\eta^{(2)}(\widetilde{M}) \in \mathbb{R}$ the L^2-*eta-invariant* of the $\pi_1(M)$-covering given by the universal covering $\widetilde{M} \to M$. Let $\rho^{(2)}(M) \in \mathbb{R}$ be the L^2-*rho-invariant* which is defined to be the difference $\eta^{(2)}(\widetilde{M}) - \eta(M)$. These invariants were studied by Cheeger and Gromov [64], [65]. They show that $\rho^{(2)}(M)$ depends only on the diffeomorphism type of M and is in contrast to $\eta(M)$ and $\eta^{(2)}(\widetilde{M})$ independent of the choice of Riemannian metric on M. The following conjecture is taken from Mathai [214].

Conjecture 1.47 (Homotopy Invariance of the L^2-Rho-Invariant for Torsionfree Groups). If $\pi_1(M)$ is torsionfree, then $\rho^{(2)}(M)$ is a homotopy invariant.

Chang-Weinberger [62] assign to a closed connected oriented $(4k-1)$-dimensional manifold M a Hirzebruch-type invariant $\tau^{(2)}(M) \in \mathbb{R}$ as follows. By a result of Hausmann [145] there is a closed connected oriented $4k$-dimensional manifold W with $M = \partial W$ such that the inclusion $\partial W \to W$ induces an injection on the fundamental groups. Define $\tau^{(2)}(M)$ as the difference $\operatorname{sign}^{(2)}(\widetilde{W}) - \operatorname{sign}(W)$ of the L^2-signature of the $\pi_1(W)$-covering given by the universal covering $\widetilde{W} \to W$ and the signature of W. This is indeed independent of the choice of W. It is reasonable to believe that $\rho^{(2)}(M) = \tau^{(2)}(M)$ is always true. Chang-Weinberger [62] use $\tau^{(2)}$ to prove that if $\pi_1(M)$ is not torsionfree there are infinitely many diffeomorphically distinct manifolds of dimension $4k+3$ with $k \geq 1$, which are tangentially simple homotopy equivalent to M.

Theorem 1.48 (Homotopy Invariance of $\tau^{(2)}(M)$ and $\rho^{(2)}(M)$). *Let M be a closed connected oriented $(4k-1)$-dimensional manifold M such that $G = \pi_1(M)$ is torsionfree.*

(i) If the assembly map $K_0(BG) \to K_0(C^*_{\max}(G))$ for the maximal group C^*-algebra is surjective (see Subsection 4.1.2), then $\rho^{(2)}(M)$ is a homotopy invariant.
(ii) Suppose that the Farrell-Jones Conjecture for L-theory 1.19 is rationally true for $R = \mathbb{Z}$, i.e. the rationalized assembly map

$$H_n(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \to L_n^{\langle -\infty \rangle}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism for $n \in \mathbb{Z}$. Then $\tau^{(2)}(M)$ is a homotopy invariant. If furthermore G is residually finite, then $\rho^{(2)}(M)$ is a homotopy invariant.

Proof. (i) This is proved by Keswani [174], [175].

(ii) This is proved by Chang [61] and Chang-Weinberger [62] using [210]. □

Remark 1.49. Let X be a $4n$-dimensional Poincaré space over $\mathbb{Q}$. Let $\overline{X} \to X$ be a normal covering with *torsion-free* covering group G. Suppose that the assembly map $K_0(BG) \to K_0(C^*_{\max}(G))$ for the maximal group C^*-algebra is surjective (see Subsection 4.1.2) or suppose that the rationalized assembly map for L-theory

$$H_{4n}(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \to L_{4n}^{\langle -\infty \rangle}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. Then the following L^2-signature theorem is proved in Lück-Schick [211, Theorem 2.3]

$$\operatorname{sign}^{(2)}(\overline{X}) = \operatorname{sign}(X).$$

If one drops the condition that G is torsionfree this equality becomes false. Namely, Wall has constructed a finite Poincaré space X with a finite G covering $\overline{X} \to X$ for which $\operatorname{sign}(\overline{X}) \neq |G| \cdot \operatorname{sign}(X)$ holds (see [258, Example 22.28], [319, Corollary 5.4.1]).

Remark 1.50. Cochran-Orr-Teichner give in [69] new obstructions for a knot to be slice which are sharper than the Casson-Gordon invariants. They use L^2-signatures and the Baum-Connes Conjecture 2.3. We also refer to the survey article [68] about non-commutative geometry and knot theory.

1.9 Applications V

1.9.1 Novikov Conjectures

The Baum-Connes and Farrell-Jones Conjectures discussed so far imply obviously that for torsion free groups the rationalized assembly maps

$$\begin{aligned} H_*(BG;\mathbf{K}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} &\to K_*(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \\ H_*(BG;\mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} &\to L_*^{\langle -\infty \rangle}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \\ K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} &\to K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q} \end{aligned}$$

are injective. For reasons that will be explained below these "rational injectivity conjectures" are known as "Novikov Conjectures". In fact one expects these injectivity results also when the groups contain torsion. So there are the following conjectures.

Conjecture 1.51 (K- and L-theoretic Novikov Conjectures). For every group G the assembly maps

$$\begin{aligned} H_*(BG;\mathbf{K}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} &\to K_*(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \\ H_*(BG;\mathbf{L}^p(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} &\to L_*^p(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \\ K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} &\to K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q} \end{aligned}$$

are injective.

Observe that, since the $\mathbb{Z}/2$-Tate cohomology groups vanish rationally, there is no difference between the various decorations in L-theory because of the Rothenberg sequence. We have chosen the p-decoration above.

1.9.2 The Original Novikov Conjecture

We now explain the Novikov Conjecture in its original formulation.

Let G be a (not necessarily torsion free) group and $u\colon M \to BG$ be a map from a closed oriented smooth manifold M to BG. Let $\mathcal{L}(M) \in \prod_{k \geq 0} H^k(M;\mathbb{Q})$ be the *L-class of M*, which is a certain polynomial in the Pontrjagin classes and hence depends a priori on the tangent bundle and hence on the differentiable structure of M. For $x \in \prod_{k\geq 0} H^k(BG;\mathbb{Q})$ define the *higher signature of M associated to x and u* to be

$$\operatorname{sign}_x(M,u) := \langle \mathcal{L}(M) \cup u^*x, [M]\rangle \qquad \in \mathbb{Q}. \tag{9}$$

The Hirzebruch signature formula says that for $x = 1$ the signature $\operatorname{sign}_1(M,u)$ coincides with the ordinary signature $\operatorname{sign}(M)$ of M, if $\dim(M) = 4n$, and is zero, if $\dim(M)$ is not divisible by four. Recall that for $\dim(M) = 4n$ the *signature* $\operatorname{sign}(M)$ of M is the signature of the non-degenerate bilinear symmetric pairing on the middle cohomology $H^{2n}(M;\mathbb{R})$ given by the intersection pairing $(a,b) \mapsto \langle a \cup b, [M]\rangle$. Obviously $\operatorname{sign}(M)$ depends only on the oriented homotopy type of M. We say that sign_x for $x \in H^*(BG;\mathbb{Q})$ is *homotopy invariant* if for two closed oriented smooth manifolds M and N with reference maps $u\colon M \to BG$ and $v\colon N \to BG$ we have

$$\operatorname{sign}_x(M,u) = \operatorname{sign}_x(N,v)$$

if there is an orientation preserving homotopy equivalence $f\colon M \to N$ such that $v \circ f$ and u are homotopic.

Conjecture 1.52 (Novikov Conjecture). Let G be a group. Then sign_x is homotopy invariant for all $x \in \prod_{k\geq 0} H^k(BG;\mathbb{Q})$.

By Hirzebruch's signature formula the Novikov Conjecture 1.52 is true for $x = 1$.

1.9.3 Relations between the Novikov Conjectures

Using surgery theory one can show [260, Proposition 6.6 on page 300] the following.

Proposition 1.53. *For a group G the original Novikov Conjecture 1.52 is equivalent to the L-theoretic Novikov Conjecture, i.e. the injectivity of the assembly map*

$$H_*(BG;\mathbf{L}^p(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \to L_*^p(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In particular for torsion free groups the L-theoretic Farrell-Jones Conjecture 1.19 implies the Novikov Conjecture 1.52. Later in Proposition 3.19 we will prove in particular the following statement.

Proposition 1.54. *The Novikov Conjecture for topological K-theory, i.e. the injectivity of the assembly map*

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the L-theoretic Novikov Conjecture and hence the original Novikov Conjecture.

For more information about the Novikov Conjectures we refer for instance to [38], [52], [55], [81], [120], [127], [179], [258] and [269].

1.9.4 The Zero-in-the-Spectrum Conjecture

The following Conjecture is due to Gromov [136, page 120].

Conjecture 1.55 (Zero-in-the-spectrum Conjecture). Suppose that $\widetilde{M}$ is the universal covering of an aspherical closed Riemannian manifold M (equipped with the lifted Riemannian metric). Then zero is in the spectrum of the minimal closure

$$(\Delta_p)_{\min}\colon L^2\Omega^p(\widetilde{M}) \supset \mathrm{dom}(\Delta_p)_{\min} \to L^2\Omega^p(\widetilde{M}),$$

for some $p \in \{0, 1, \ldots, \dim M\}$, where Δ_p denotes the Laplacian acting on smooth p-forms on $\widetilde{M}$.

Proposition 1.56. *Suppose that M is an aspherical closed Riemannian manifold with fundamental group G, then the injectivity of the assembly map*

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the Zero-in-the-spectrum Conjecture for $\widetilde{M}$.

Proof. We give a sketch of the proof. More details can be found in [195, Corollary 4]. We only explain that the assumption that in every dimension zero is not in the spectrum of the Laplacian on $\widetilde{M}$, yields a contradiction in the case that $n = \dim(M)$ is even. Namely, this assumption implies that the $C_r^*(G)$-valued index of the signature operator twisted with the flat bundle $\widetilde{M} \times_G C_r^*(G) \to M$ in $K_0(C_r^*(G))$ is zero, where $G = \pi_1(M)$. This index is the image of the class $[S]$ defined by the signature operator in $K_0(BG)$ under the assembly map $K_0(BG) \to K_0(C_r^*(G))$. Since by assumption the assembly map is rationally injective, this implies $[S] = 0$ in $K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q}$. Notice that M is aspherical by assumption and hence $M = BG$. The homological Chern character defines an isomorphism

$$K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q} = K_0(M) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \geq 0} H_{2p}(M; \mathbb{Q})$$

which sends $[S]$ to the Poincaré dual $\mathcal{L}(M) \cap [M]$ of the Hirzebruch L-class $\mathcal{L}(M) \in \bigoplus_{p \geq 0} H^{2p}(M; \mathbb{Q})$. This implies that $\mathcal{L}(M) \cap [M] = 0$ and hence $\mathcal{L}(M) = 0$. This contradicts the fact that the component of $\mathcal{L}(M)$ in $H^0(M; \mathbb{Q})$ is 1. □

More information about the Zero-in-the-spectrum Conjecture 1.55 can be found for instance in [195] and [202, Section 12].

2 The Conjectures in the General Case

In this chapter we will formulate the Baum-Connes and Farrell-Jones Conjectures. We try to emphasize the unifying principle that underlies these conjectures. The point of view taken in this chapter is that all three conjectures are conjectures about specific equivariant homology theories. Some of the technical details concerning the actual construction of these homology theories are deferred to Chapter 6.

2.1 Formulation of the Conjectures

Suppose we are given

- A discrete group G;
- A family $\mathcal{F}$ of subgroups of G, i.e. a set of subgroups which is closed under conjugation with elements of G and under taking finite intersections;
- A G-homology theory $\mathcal{H}_*^G(-)$.

Then one can formulate the following Meta-Conjecture.

Meta-Conjecture 2.1. *The assembly map*

$$A_{\mathcal{F}} \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(\mathrm{pt})$$

which is the map induced by the projection $E_{\mathcal{F}}(G) \to \mathrm{pt}$*, is an isomorphism for* $n \in \mathbb{Z}$.

Here $E_{\mathcal{F}}(G)$ is the classifying space of the family $\mathcal{F}$, a certain G-space which specializes to the universal free G-space EG if the family contains only the trivial subgroup. A G-homology theory is the "obvious" G-equivariant generalization of the concept of a homology theory to a suitable category of G-spaces, in particular it is a functor on such spaces and the map $A_{\mathcal{F}}$ is simply the map induced by the projection $E_{\mathcal{F}}(G) \to \mathrm{pt}$. We devote the Subsections 2.1.1 to 2.1.4 below to a discussion of G-homology theories, classifying spaces for families of subgroups and related things. The reader who never encountered these concepts should maybe first take a look at these subsections.

Of course the conjecture above is not true for arbitrary G, $\mathcal{F}$ and $\mathcal{H}_*^G(-)$, but the Farrell-Jones and Baum-Connes Conjectures state that for specific G-homology theories there is a natural choice of a family $\mathcal{F} = \mathcal{F}(G)$ of subgroups for every group G such that $A_{\mathcal{F}(G)}$ becomes an isomorphism for all groups G.

Let R be a ring (with involution). In Proposition 6.7 we will describe the construction of G-homology theories which will be denoted

$$H_n^G(-;\mathbf{K}_R), \quad H_n^G(-;\mathbf{L}_R^{\langle -\infty \rangle}) \quad \text{and} \quad H_n^G(-;\mathbf{K}^{\mathrm{top}}).$$

If G is the trivial group, these homology theories specialize to the (non-equivariant) homology theories with similar names that appeared in Chapter 1, namely to

$$H_n(-;\mathbf{K}(R)), \quad H_n(-;\mathbf{L}^{\langle -\infty\rangle}(R)) \quad \text{and} \quad K_n(-).$$

Another main feature of these G-homology theories is that evaluated on the one point space pt (considered as a trivial G-space) we obtain the K- and L-theory of the group ring RG, respectively the topological K-theory of the reduced C^*-algebra (see Proposition 6.7 and Theorem 6.9 (iii))

$$\begin{aligned} K_n(RG) &\cong H_n^G(\mathrm{pt};\mathbf{K}_R), \\ L_n^{\langle -\infty\rangle}(RG) &\cong H_n^G(\mathrm{pt};\mathbf{L}_R^{\langle -\infty\rangle}) \quad \text{and} \\ K_n(C_r^*(G)) &\cong H_n^G(\mathrm{pt};\mathbf{K}^{\mathrm{top}}). \end{aligned}$$

We are now prepared to formulate the conjectures around which this article is centered. Let $\mathcal{FIN}$ be the family of finite subgroups and let $\mathcal{VCY}$ be the family of virtually cyclic subgroups.

Conjecture 2.2 (Farrell-Jones Conjecture for K- and L-theory). Let R be a ring (with involution) and let G be a group. Then for all $n \in \mathbb{Z}$ the maps

$$\begin{aligned} A_{\mathcal{VCY}}\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) &\to H_n^G(\mathrm{pt};\mathbf{K}_R) \cong K_n(RG); \\ A_{\mathcal{VCY}}\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) &\to H_n^G(\mathrm{pt};\mathbf{L}_R^{\langle -\infty\rangle}) \cong L_n^{\langle -\infty\rangle}(RG), \end{aligned}$$

which are induced by the projection $E_{\mathcal{VCY}}(G) \to \mathrm{pt}$, are isomorphisms.

The conjecture for the topological K-theory of C^*-algebras is known as the Baum-Connes Conjecture and reads as follows.

Conjecture 2.3 (Baum-Connes Conjecture). Let G be a group. Then for all $n \in \mathbb{Z}$ the map

$$A_{\mathcal{FIN}}\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathrm{top}}) \to H_n^G(\mathrm{pt};\mathbf{K}^{\mathrm{top}}) \cong K_n(C_r^*(G))$$

induced by the projection $E_{\mathcal{FIN}}(G) \to \mathrm{pt}$ is an isomorphism.

We will explain the analytic assembly map $\mathrm{ind}_G\colon K_n^G(X) \to K_n(C_r^*(G))$, which can be identified with the assembly map appearing in the Baum-Connes Conjecture 2.3 in Section 7.2.

Remark 2.4. Of course the conjectures really come to life only if the abstract point of view taken in this chapter is connected up with more concrete descriptions of the assembly maps. We have already discussed a surgery theoretic description in Theorem 1.28 and an interpretation in terms index theory in Subsection 1.8.1. More information about alternative interpretations of assembly maps can be found in Section 7.2 and 7.8. These concrete interpretations of the assembly maps lead to applications. We already discussed many such applications in Chapter 1 and encourage the reader to go ahead and browse through Chapter 3 in order to get further ideas about these more concrete aspects.

Remark 2.5 (Relation to the "classical" assembly maps). The value of an equivariant homology theory $\mathcal{H}_*^G(-)$ on the universal free G-space $EG = E_{\{1\}}(G)$ (a free G-CW-complex whose quotient EG/G is a model for BG) can be identified with the corresponding non-equivariant homology theory evaluated on BG, if we assume that $\mathcal{H}_*^G$ is the special value of an equivariant homology theory $\mathcal{H}_*^?$ at $? = G$. This means that there exists an induction structure (a mild condition satisfied in our examples, compare Section 6.1), which yields an identification

$$\mathcal{H}_n^G(EG) \cong \mathcal{H}_n^{\{1\}}(BG) = \mathcal{H}_n(BG).$$

Using these identifications the "classical" assembly maps, which appeared in Chapter 1 in the versions of the Farrell-Jones and Baum-Connes Conjectures for torsion free groups (see Conjecture 1.11, 1.19 and 1.31),

$$\begin{gathered} H_n(BG;\mathbf{K}(R)) \cong H_n^G(EG;\mathbf{K}_R) \to H_n^G(\mathrm{pt};\mathbf{K}_R) \cong K_n(RG); \\ H_n(BG;\mathbf{L}^{\langle -\infty\rangle}(R)) \cong H_n^G(EG;\mathbf{L}_R^{\langle -\infty\rangle}) \to H_n^G(\mathrm{pt};\mathbf{L}_R^{\langle -\infty\rangle}) \cong L_n^{\langle -\infty\rangle}(RG); \\ \text{and} \quad K_n(BG) \cong H_n^G(EG;\mathbf{K}^{\mathrm{top}}) \to H_n^G(\mathrm{pt};\mathbf{K}^{\mathrm{top}}) \cong K_n(C_r^*(G)), \end{gathered}$$

correspond to the assembly maps for the family $\mathcal{F} = \{1\}$ consisting only of the trivial group and are simply the maps induced by the projection $EG \to \mathrm{pt}$.

Remark 2.6 (The choice of the right family). As explained above the Farrell-Jones and Baum-Connes Conjectures 2.2 and 2.3 can be considered as special cases of the Meta-Conjecture 2.1. In all three cases we are interested in a computation of the right hand side $\mathcal{H}_n^G(\mathrm{pt})$ of the assembly map, which can be identified with $K_n(RG)$, $L_n^{\langle -\infty\rangle}(RG)$ or $K_n(C_r^*(G))$. The left hand side $\mathcal{H}_n^G(E_{\mathcal{F}}(G))$ of such an assembly map is much more accessible and the smaller $\mathcal{F}$ is, the easier it is to compute $\mathcal{H}_n^G(E_{\mathcal{F}}(G))$ using homological methods like spectral sequences, Mayer-Vietoris arguments and Chern characters.

In the extreme case where $\mathcal{F} = \mathcal{ALL}$ is the family of all subgroups the assembly map $A_{\mathcal{ALL}} \colon \mathcal{H}_n^G(E_{\mathcal{ALL}}(G)) \to \mathcal{H}_n^G(\mathrm{pt})$ is always an isomorphism for the trivial reason that the one point space pt is a model for $E_{\mathcal{ALL}}(G)$ (compare Subsection 2.1.3) and hence the assembly map is the identity. The goal however is to have an isomorphism for a family which is as small as possible.

We have already seen in Remark 1.4, Remark 1.25 and Remark 1.34 that in all three cases the classical assembly map, which corresponds to the trivial family, is not surjective for finite groups. This forces one to include at least the family $\mathcal{FIN}$ of finite groups. The K- or L-theory of the finite subgroups of the given group G will then enter in a computation of the left hand side of the assembly map similar as the K- and L-theory of the trivial subgroup appeared on the left hand side in the classical case, compare e.g. Remark 1.14. In the Baum-Connes case the family $\mathcal{FIN}$ seems to suffice. However in the case of algebraic K-theory we saw in Remark 1.15 that already the simplest torsion free group, the infinite cyclic group, causes problems because of the Nil-terms that appear in the Bass-Heller-Swan formula. The infinite dihedral group is a "minimal counterexample" which shows that the family $\mathcal{FIN}$ is not sufficient in the $L_{\mathbb{Z}}^{\langle -\infty \rangle}$-case. There are non-vanishing UNil-terms, compare 2.2.6 and 2.2.7. Also the version of the L-theoretic Farrell-Jones Conjecture with the decoration s, $h = \langle 1 \rangle$ or $p = \langle 0 \rangle$ instead of $\langle -\infty \rangle$ is definitely false. Counterexamples are given in [123]. Recall that there were no Nil-terms in the topological K-theory context, compare Remark 1.35.

The choice of the family $\mathcal{VCY}$ of virtually cyclic subgroups in the Farrell-Jones conjectures pushes all the Nil-problems appearing in algebraic K- and L-theory into the source of the assembly map so that they do not occur if one tries to prove the Farrell-Jones Conjecture 2.2. Of course they do come up again when one wants to compute the source of the assembly map.

We now take up the promised detailed discussion of some notions like equivariant homology theories and classifying spaces for families we used above. The reader who is familiar with these concepts may of course skip the following subsections.

2.1.1 *G*-CW-Complexes

A *G-CW-complex* X is a G-space X together with a filtration $X_{-1} = \emptyset \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X$ such that $X = \operatorname{colim}_{n \to \infty} X_n$ and for each n there is a G-pushout

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow[\coprod_{i \in I_n} Q_i^n]{} & X_n \end{array} .$$

This definition makes also sense for topological groups. The following alternative definition only applies to discrete groups. A G-CW-complex is a CW-complex with a G-action by cellular maps such that for each open cell e and each $g \in G$ with $ge \cap e \neq \emptyset$ we have $gx = x$ for all $x \in e$. There is an obvious notion of a G-CW-pair.

A G-CW-complex X is called *finite* if it is built out of finitely many G-cells $G/H_i \times D^n$. This is the case if and only if it is *cocompact*, i.e. the quotient space $G \backslash X$ is compact. More information about G-CW-complexes can be found for instance in [197, Sections 1 and 2], [304, Sections II.1 and II.2].

2.1.2 Families of Subgroups

A *family $\mathcal{F}$ of subgroups of G* is a set of subgroups of G closed under conjugation, i.e. $H \in \mathcal{F}, g \in G$ implies $g^{-1}Hg \in \mathcal{F}$, and finite intersections, i.e. $H, K \in \mathcal{F}$ implies $H \cap K \in \mathcal{F}$. Throughout the text we will use the notations

$$\{1\}, \quad \mathcal{FCY}, \quad \mathcal{FIN}, \quad \mathcal{CYC}, \quad \mathcal{VCY}_I, \quad \mathcal{VCY} \quad \text{and} \quad \mathcal{ALL}$$

for the families consisting of the trivial, all finite cyclic, all finite, all (possibly infinite) cyclic, all virtually cyclic of the first kind, all virtually cyclic, respectively all subgroups of a given group G. Recall that a group is called *virtually cyclic* if it is finite or contains an infinite cyclic subgroup of finite index. A group is *virtually cyclic of the first kind* if it admits a surjection onto an infinite cyclic group with finite kernel, compare Lemma 2.15.

2.1.3 Classifying Spaces for Families

Let $\mathcal{F}$ be a family of subgroups of G. A G-CW-complex, all whose isotropy groups belong to $\mathcal{F}$ and whose H-fixed point sets are contractible for all $H \in \mathcal{F}$, is called a *classifying space for the family* $\mathcal{F}$ and will be denoted $E_{\mathcal{F}}(G)$. Such a space is unique up to G-homotopy because it is characterized by the property that for any G-CW-complex X, all whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map from X to $E_{\mathcal{F}}(G)$. These spaces were introduced by tom Dieck [303], [304, I.6].

A functorial "bar-type" construction is given in [82, section 7].

If $\mathcal{F} \subset \mathcal{G}$ are families of subgroups for G, then by the universal property there is up to G-homotopy precisely one G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$.

The space $E_{\{1\}}(G)$ is the same as the space EG which is by definition the total space of the universal G-principal bundle $G \to EG \to BG$, or, equivalently, the universal covering of BG. A model for $E_{\mathcal{ALL}}(G)$ is given by the space $G/G = \mathrm{pt}$ consisting of one point.

The space $E_{\mathcal{FIN}}(G)$ is also known as *the classifying space for proper G-actions* and denoted by $\underline{E}G$ in the literature. Recall that a G-CW-complex X is proper if and only if all its isotropy groups are finite (see for instance [197, Theorem 1.23 on page 18]). There are often nice models for $E_{\mathcal{FIN}}(G)$. If G is word hyperbolic in the sense of Gromov, then the Rips-complex is a finite model [216], [217].

If G is a discrete subgroup of a Lie group L with finitely many path components, then for any maximal compact subgroup $K \subseteq L$ the space L/K with its left G-action is a model for $E_{\mathcal{FIN}}(G)$ [2, Corollary 4.14]. More information about $E_{\mathcal{FIN}}(G)$ can be found for instance in [28, section 2], [180], [199], [206], [207] and [282].

2.1.4 G-Homology Theories

Fix a group G and an associative commutative ring Λ with unit. A *G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules* is a collection of covariant functors $\mathcal{H}_n^G$ from the category of G-CW-pairs to the category of Λ-modules indexed by $n \in \mathbb{Z}$ together with natural transformations

$$\partial_n^G(X,A)\colon \mathcal{H}_n^G(X,A) \to \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A,\emptyset)$$

for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

(i) G-homotopy invariance

If f_0 and f_1 are G-homotopic maps $(X,A) \to (Y,B)$ of G-CW-pairs, then $\mathcal{H}_n^G(f_0) = \mathcal{H}_n^G(f_1)$ for $n \in \mathbb{Z}$.

(ii) Long exact sequence of a pair

Given a pair (X,A) of G-CW-complexes, there is a long exact sequence

$$\ldots \xrightarrow{\mathcal{H}_{n+1}^G(j)} \mathcal{H}_{n+1}^G(X,A) \xrightarrow{\partial_{n+1}^G} \mathcal{H}_n^G(A) \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X) \xrightarrow{\mathcal{H}_n^G(j)} \mathcal{H}_n^G(X,A) \xrightarrow{\partial_n^G} \mathcal{H}_{n-1}^G(A) \xrightarrow{\mathcal{H}_{n-1}^G(i)} \ldots,$$

where $i\colon A \to X$ and $j\colon X \to (X,A)$ are the inclusions.

(iii) Excision

Let (X,A) be a G-CW-pair and let $f\colon A \to B$ be a cellular G-map of G-CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a G-CW-pair. Then the canonical map $(F,f)\colon (X,A) \to (X \cup_f B, B)$ induces for each $n \in \mathbb{Z}$ an isomorphism

$$\mathcal{H}_n^G(F,f)\colon \mathcal{H}_n^G(X,A) \xrightarrow{\cong} \mathcal{H}_n^G(X \cup_f B, B).$$

(iv) Disjoint union axiom

Let $\{X_i \mid i \in I\}$ be a family of G-CW-complexes. Denote by $j_i\colon X_i \to \coprod_{i\in I} X_i$ the canonical inclusion. Then the map

$$\bigoplus_{i\in I} \mathcal{H}_n^G(j_i)\colon \bigoplus_{i\in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G\left(\coprod_{i\in I} X_i\right)$$

is bijective for each $n \in \mathbb{Z}$.

Of course a G-homology theory for the trivial group $G = \{1\}$ is a homology theory (satisfying the disjoint union axiom) in the classical non-equivariant sense.

The disjoint union axiom ensures that we can pass from finite G-CW-complexes to arbitrary ones using the following lemma.

Lemma 2.7. *Let $\mathcal{H}_*^G$ be a G-homology theory. Let X be a G-CW-complex and $\{X_i \mid i \in I\}$ be a directed system of G-CW-subcomplexes directed by inclusion such that $X = \cup_{i\in I} X_i$. Then for all $n \in \mathbb{Z}$ the natural map*

$$\operatorname{colim}_{i\in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

is bijective.

Proof. Compare for example with [301, Proposition 7.53 on page 121], where the non-equivariant case for $I = \mathbb{N}$ is treated. □

Example 2.8 (Bredon Homology). The most basic G-homology theory is *Bredon homology.* The *orbit category* $\mathsf{Or}(G)$ has as objects the homogeneous spaces G/H and as morphisms G-maps. Let X be a G-CW-complex. It defines a contravariant functor from the orbit category $\mathsf{Or}(G)$ to the category of CW-complexes by sending G/H to $\mathrm{map}_G(G/H, X) = X^H$. Composing it with the functor cellular chain complex yields a contravariant functor

$$C_*^c(X)\colon \mathsf{Or}(G) \to \mathbb{Z}\text{-}\mathsf{CHCOM}$$

into the category of $\mathbb{Z}$-chain complexes. Let Λ be a commutative ring and let

$$M\colon \mathsf{Or}(G) \to \Lambda\text{-}\mathsf{MODULES}$$

be a covariant functor. Then one can form the tensor product over the orbit category (see for instance [197, 9.12 on page 166]) and obtains the Λ-chain complex $C_*^c(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} M$. Its homology is the Bredon homology of X with coefficients in M

$$H_*^G(X; M) = H_*(C_*^c(X) \otimes_{\mathbb{Z}\mathsf{Or}(G)} M).$$

Thus we get a G-homology theory H_*^G with values in Λ-modules. For a trivial group G this reduces to the cellular homology of X with coefficients in the Λ-module M.

More information about equivariant homology theories will be given in Section 6.1.

2.2 Varying the Family of Subgroups

Suppose we are given a family of subgroups $\mathcal{F}'$ and a subfamily $\mathcal{F} \subset \mathcal{F}'$. Since all isotropy groups of $E_{\mathcal{F}}(G)$ lie in $\mathcal{F}'$ we know from the universal property of $E_{\mathcal{F}'}(G)$ (compare Subsection 2.1.3) that there is a G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{F}'}(G)$ which is unique up to G-homotopy. For every G-homology theory $\mathcal{H}_*^G$ we hence obtain a *relative assembly map*

$$A_{\mathcal{F}\to\mathcal{F}'}\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{F}'}(G)).$$

If $\mathcal{F}' = \mathcal{ALL}$, then $E_{\mathcal{F}'}(G) = \mathrm{pt}$ and $A_{\mathcal{F}\to\mathcal{F}'}$ specializes to the assembly map $A_{\mathcal{F}}$ we discussed in the previous section. If we now gradually increase the family, we obtain a factorization of the classical assembly $A = A_{\{1\}\to\mathcal{ALL}}$ into several relative assembly maps. We obtain for example from the inclusions

$$\{1\} \subset \mathcal{FCY} \subset \mathcal{FIN} \subset \mathcal{VCY} \subset \mathcal{ALL}$$

for every G-homology theory $\mathcal{H}_n^G(-)$ the following commutative diagram.

$$\begin{tikzcd}
\mathcal{H}_n^G(EG) \arrow[rr, "A"] \arrow[d] & & \mathcal{H}_n^G(\mathrm{pt}) \\
\mathcal{H}_n^G(E_{\mathcal{FCY}}(G)) \arrow[r] & \mathcal{H}_n^G(E_{\mathcal{FIN}}(G)) \arrow[r] \arrow[ur, "A_{\mathcal{FIN}}"] & \mathcal{H}_n^G(E_{\mathcal{VCY}}(G)). \arrow[u, "A_{\mathcal{VCY}}"']
\end{tikzcd} \tag{10}$$

Here A is the "classical" assembly map and $A_{\mathcal{FIN}}$ and $A_{\mathcal{VCY}}$ are the assembly maps that for specific G-homology theories appear in the Baum-Connes and Farrell-Jones Conjectures.

Such a factorization is extremely useful because one can study the relative assembly map $A_{\mathcal{F}\to\mathcal{F}'}$ in terms of absolute assembly maps corresponding to groups in the bigger family. For example the relative assembly map

$$A_{\mathcal{FIN}\to\mathcal{VCY}}\colon \mathcal{H}_n^G(E_{\mathcal{FIN}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{VCY}}(G))$$

is an isomorphism if for all virtually cyclic subgroups V of G the assembly map

$$A_{\mathcal{FIN}} = A_{\mathcal{FIN}\to\mathcal{ALL}}\colon \mathcal{H}_n^V(E_{\mathcal{FIN}}(V)) \to \mathcal{H}_n^V(\mathrm{pt})$$

is an isomorphism. Of course here we need to assume that the G-homology theory $\mathcal{H}_*^G$ and the V-homology theory $\mathcal{H}_*^V$ are somehow related. In fact all the G-homology theories $\mathcal{H}_*^G$ we care about are defined simultaneously for all groups G and for varying G these G-homology theories are related via a so called "induction structure". Induction structures will be discussed in detail in Section 6.1.

For a family $\mathcal{F}$ of subgroups of G and a subgroup $H \subset G$ we define a family of subgroups of H

$$\mathcal{F} \cap H = \{K \cap H \mid K \in \mathcal{F}\}.$$

The general statement about relative assembly maps reads now as follows.

Theorem 2.9 (Transitivity Principle). *Let $\mathcal{H}_*^?(-)$ be an equivariant homology theory in the sense of Section 6.1. Suppose $\mathcal{F} \subset \mathcal{F}'$ are two families of subgroups of G. Suppose that $K \cap H \in \mathcal{F}$ for each $K \in \mathcal{F}$ and $H \in \mathcal{F}'$ (this is automatic if $\mathcal{F}$ is closed under taking subgroups). Let N be an integer. If for every $H \in \mathcal{F}'$ and every $n \leq N$ the assembly map*

$$A_{\mathcal{F}\cap H\to\mathcal{ALL}}\colon \mathcal{H}_n^H(E_{\mathcal{F}\cap H}(H)) \to \mathcal{H}_n^H(\mathrm{pt})$$

is an isomorphism, then for every $n \leq N$ the relative assembly map

$$A_{\mathcal{F}\to\mathcal{F}'}\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{F}'}(G))$$

is an isomorphism.

Proof. If we equip $E_{\mathcal{F}}(G) \times E_{\mathcal{F}'}(G)$ with the diagonal G-action, it is a model for $E_{\mathcal{F}}(G)$. Now apply Lemma 6.4 in the special case $Z = E_{\mathcal{F}'}(G)$. □

This principle can be used in many ways. For example we will derive from it that the general versions of the Baum-Connes and Farrell-Jones Conjectures specialize to the conjectures we discussed in Chapter 1 in the case where the group is torsion free. If we are willing to make compromises, e.g. to invert 2, to rationalize the theories or to restrict ourselves to small dimensions or special classes of groups, then it is often possible to get away with a smaller family, i.e. to conclude from the Baum-Connes or Farrell-Jones Conjectures that an assembly map with respect to a family smaller than the family of finite or virtually cyclic subgroups is an isomorphism. The left hand side becomes more computable and this leads to new corollaries of the Baum-Connes and Farrell-Jones Conjectures.

2.2.1 The General Versions Specialize to the Torsion Free Versions

If G is a torsion free group, then the family $\mathcal{FIN}$ obviously coincides with the trivial family $\{1\}$. Since a nontrivial torsion free virtually cyclic group is infinite cyclic we also know that the family $\mathcal{VCY}$ reduces to the family of all cyclic subgroups, denoted $\mathcal{CYC}$.

Proposition 2.10. *Let G be a torsion free group.*

(i) If R is a regular ring, then the relative assembly map

$$A_{\{1\}\to\mathcal{CYC}}\colon H_n^G(E_{\{1\}}(G);\mathbf{K}_R)\to H_n^G(E_{\mathcal{CYC}}(G);\mathbf{K}_R)$$

is an isomorphism.

(ii) For every ring R the relative assembly map

$$A_{\{1\}\to\mathcal{CYC}}\colon H_n^G(E_{\{1\}}(G);\mathbf{L}_R^{\langle-\infty\rangle})\to H_n^G(E_{\mathcal{CYC}}(G);\mathbf{L}_R^{\langle-\infty\rangle})$$

is an isomorphism.

Proof. Because of the Transitivity Principle 2.9 it suffices in both cases to prove that the classical assembly map $A = A_{\{1\}\to\mathcal{ALL}}$ is an isomorphism in the case where G is an infinite cyclic group. For regular rings in the K-theory case and with the $-\infty$-decoration in the L-theory case this is true as we discussed in Remark 1.15 respectively Remark 1.26. □

As an immediate consequence we obtain.

Corollary 2.11. *(i) For a torsion free group the Baum-Connes Conjecture 2.3 is equivalent to its "torsion free version" Conjecture 1.31.*

(ii) For a torsion free group the Farrell-Jones Conjecture 2.2 for algebraic K- is equivalent to the "torsion free version" Conjecture 1.11, provided R is regular.

(iii) For a torsion free group the Farrell-Jones Conjecture 2.2 for algebraic L-theory is equivalent to the "torsion free version" Conjecture 1.19.

2.2.2 The Baum-Connes Conjecture and the Family $\mathcal{VCY}$

Replacing the family $\mathcal{FIN}$ of finite subgroups by the family $\mathcal{VCY}$ of virtually cyclic subgroups would not make any difference in the Baum-Connes Conjecture 2.3. The Transitivity Principle 2.9 and the fact that the Baum-Connes Conjecture 2.3 is known for virtually cyclic groups implies the following.

Proposition 2.12. *For every group G and every $n \in \mathbb{Z}$ the relative assembly map for topological K-theory*

$$A_{\mathcal{FIN}\to\mathcal{VCY}}\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathrm{top}}) \to H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}^{\mathrm{top}})$$

is an isomorphism.

2.2.3 The Baum-Connes Conjecture and the Family $\mathcal{FCY}$

The following result is proven in [215].

Proposition 2.13. *For every group G and every $n \in \mathbb{Z}$ the relative assembly map for topological K-theory*

$$A_{\mathcal{FCY}\to\mathcal{FIN}}\colon H_n^G(E_{\mathcal{FCY}}(G);\mathbf{K}^{\mathrm{top}}) \to H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathrm{top}})$$

is an isomorphism.

In particular the Baum-Connes Conjecture predicts that the $\mathcal{FCY}$-assembly map

$$A_{\mathcal{FCY}}\colon H_n^G(E_{\mathcal{FCY}}(G);\mathbf{K}^{\mathrm{top}}) \to K_n(C_r^*(G))$$

is always an isomorphism.

2.2.4 Algebraic K-Theory for Special Coefficient Rings

In the algebraic K-theory case we can reduce to the family of finite subgroups if we assume special coefficient rings.

Proposition 2.14. *Suppose R is a regular ring in which the orders of all finite subgroups of G are invertible. Then for every $n \in \mathbb{Z}$ the relative assembly map for algebraic K-theory*

$$A_{\mathcal{FIN}\to\mathcal{VCY}}\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R) \to H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R)$$

is an isomorphism. In particular if R is a regular ring which is a $\mathbb{Q}$-algebra (for example a field of characteristic 0) the above applies to all groups G.

Proof. We first show that RH is regular for a finite group H. Since R is Noetherian and H is finite, RH is Noetherian. It remains to show that every RH-module M has a finite dimensional projective resolution. By assumption M considered as an R-module has a finite dimensional projective resolution.

If one applies $RH \otimes_R -$ this yields a finite dimensional RH-resolution of $RH \otimes_R \operatorname{res} M$. Since $|H|$ is invertible, the RH-module M is a direct summand of $RH \otimes_R \operatorname{res} M$ and hence has a finite dimensional projective resolution.

Because of the Transitivity Principle 2.9 we need to prove that the $\mathcal{FIN}$-assembly map $A_{\mathcal{FIN}}$ is an isomorphism for virtually cyclic groups V. Because of Lemma 2.15 we can assume that either $V \cong H \rtimes \mathbb{Z}$ or $V \cong K_1 *_H K_2$ with finite groups H, K_1 and K_2. From [313] we obtain in both cases long exact sequences involving the algebraic K-theory of the constituents, the algebraic K-theory of V and also additional Nil-terms. However, in both cases the Nil-terms vanish if RH is a regular ring (compare Theorem 4 on page 138 and the Remark on page 216 in [313]). Thus we get long exact sequences

$$\ldots \to K_n(RH) \to K_n(RH) \to K_n(RV) \to K_{n-1}(RH) \to K_{n-1}(RH) \to \ldots$$

and

$$\begin{aligned} \ldots \to K_n(RH) \to K_n(RK_1) \oplus K_n(RK_2) \to K_n(RV) \\ \to K_{n-1}(RH) \to K_{n-1}(RK_1) \oplus K_{n-1}(RK_2) \to \ldots \end{aligned}$$

One obtains analogous exact sequences for the sources of the various assembly maps from the fact that the sources are equivariant homology theories and one can find specific models for $E_{\mathcal{FIN}}(V)$. These sequences are compatible with the assembly maps. The assembly maps for the finite groups H, K_1 and K_2 are bijective. Now a Five-Lemma argument shows that also the one for V is bijective. □

In particular for regular coefficient rings R which are $\mathbb{Q}$-algebras the K-theoretic Farrell-Jones Conjecture specializes to the conjecture that the assembly map

$$A_{\mathcal{FIN}} \colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \to H_n^G(\mathrm{pt}; \mathbf{K}_R) \cong K_n(RG)$$

is an isomorphism.

In the proof above we used the following important fact about virtually cyclic groups.

Lemma 2.15. *If G is an infinite virtually cyclic group then we have the following dichotomy.*

(I) Either G admits a surjection with finite kernel onto the infinite cyclic group $\mathbb{Z}$, or
(II) G admits a surjection with finite kernel onto the infinite dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2$.

Proof. This is not difficult and proven as Lemma 2.5 in [113]. □

2.2.5 Splitting off Nil-Terms and Rationalized Algebraic K-Theory

Recall that the Nil-terms, which prohibit the classical assembly map from being an isomorphism, are direct summands of the K-theory of the infinite cyclic group (see Remark 1.15). Something similar remains true in general [16].

Proposition 2.16. *(i) For every group G, every ring R and every $n \in \mathbb{Z}$ the relative assembly map*

$$A_{\mathcal{FIN} \to \mathcal{VCY}} \colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \to H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R)$$

is split-injective.

(ii) Suppose R is such that $K_{-i}(RV) = 0$ for all virtually cyclic subgroups V of G and for sufficiently large i (for example $R = \mathbb{Z}$ will do, compare Proposition 3.2). Then the relative assembly map

$$A_{\mathcal{FIN} \to \mathcal{VCY}} \colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle})$$

is split-injective.

Combined with the Farrell-Jones Conjectures we obtain that the homology group $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R)$ is a direct summand in $K_n(RG)$. It is much better understood (compare Chapter 8) than the remaining summand which is isomorphic to $H_n^G(E_{\mathcal{VCY}}(G), E_{\mathcal{FIN}}(G); \mathbf{K}_R)$. This remaining summand is the one which plays the role of the Nil-terms for a general group. It is known that for $R = \mathbb{Z}$ the negative dimensional Nil-groups which are responsible for virtually cyclic groups vanish [113]. They vanish rationally, in dimension 0 by [76] and in higher dimensions by [182]. For more information see also [75]. Analogously to the proof of Proposition 2.14 we obtain the following proposition.

Proposition 2.17. *We have*

$$\begin{aligned} H_n^G(E_{\mathcal{VCY}}(G), E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathbb{Z}}) = 0 \quad & \textit{for } n < 0 \quad \textit{and} \\ H_n^G(E_{\mathcal{VCY}}(G), E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0 \quad & \textit{for all } n \in \mathbb{Z}. \end{aligned}$$

In particular the Farrell-Jones Conjecture for the algebraic K-theory of the integral group ring predicts that the map

$$A_{\mathcal{FIN}} \colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is always an isomorphism.

2.2.6 Inverting 2 in L-Theory

Proposition 2.18. *For every group G, every ring R with involution, every decoration j and all $n \in \mathbb{Z}$ the relative assembly map*

$$A_{\mathcal{FIN}\to\mathcal{VCY}}\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle j\rangle})[\frac{1}{2}] \to H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle j\rangle})[\frac{1}{2}]$$

is an isomorphism.

Proof. According to the Transitivity Principle it suffices to prove the claim for a virtually cyclic group. Now argue analogously to the proof of Proposition 2.14 using the exact sequences in [48] and the fact that the UNil-terms appearing there vanish after inverting two [48]. Also recall from Remark 1.22 that after inverting 2 there are no differences between the decorations. □

In particular the L-theoretic Farrell-Jones Conjecture implies that for every decoration j the assembly map

$$A_{\mathcal{FIN}}\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle j\rangle})[\frac{1}{2}] \to L_n^{\langle j\rangle}(RG)[\frac{1}{2}]$$

is an isomorphism.

2.2.7 L-theory and Virtually Cyclic Subgroups of the First Kind

Recall that a group is virtually cyclic of the first kind if it admits a surjection with finite kernel onto the infinite cyclic group. The family of these groups is denoted $\mathcal{VCY}_I$.

Proposition 2.19. *For all groups G, all rings R and all $n \in \mathbb{Z}$ the relative assembly map*

$$A_{\mathcal{FIN}\to\mathcal{VCY}_I}\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \to H_n^G(E_{\mathcal{VCY}_I}(G);\mathbf{L}_R^{\langle -\infty\rangle})$$

is an isomorphism.

Proof. The point is that there are no UNil-terms for infinite virtually cyclic groups of the first kind. This follows essentially from [254] and [255] as carried out in [204]. □

2.2.8 Rationally $\mathcal{FIN}$ Reduces to $\mathcal{FCY}$

We will see later (compare Theorem 8.4, 8.5 and 8.10) that in all three cases, topological K-theory, algebraic K-theory and L-theory, the rationalized left hand side of the $\mathcal{FIN}$-assembly map can be computed very explicitly using the equivariant Chern-Character. As a by-product these computations yield that after rationalizing the family $\mathcal{FIN}$ can be reduced to the family $\mathcal{FCY}$ of finite cyclic groups and that the rationalized relative assembly maps $A_{\{1\}\to\mathcal{FCY}}$ are injective.

Proposition 2.20. *For every ring R, every group G and all $n \in \mathbb{Z}$ the relative assembly maps*

$$A_{\mathcal{FCY}\to\mathcal{FIN}} \colon H_n^G(E_{\mathcal{FCY}}(G);\mathbf{K}_R) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R) \otimes_{\mathbb{Z}} \mathbb{Q}$$
$$A_{\mathcal{FCY}\to\mathcal{FIN}} \colon H_n^G(E_{\mathcal{FCY}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \otimes_{\mathbb{Z}} \mathbb{Q}$$
$$A_{\mathcal{FCY}\to\mathcal{FIN}} \colon H_n^G(E_{\mathcal{FCY}}(G);\mathbf{K}^{\mathrm{top}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathrm{top}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

are isomorphisms and the corresponding relative assembly maps $A_{\{1\}\to\mathcal{FCY}}$ are all rationally injective.

Recall that the statement for topological K-theory is even known integrally, compare Proposition 2.13. Combining the above with Proposition 2.17 and Proposition 2.18 we see that the Farrell-Jones Conjecture predicts in particular that the $\mathcal{FCY}$-assembly maps

$$A_{\mathcal{FCY}} \colon H_n^G(E_{\mathcal{FCY}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \otimes_{\mathbb{Z}} \mathbb{Q} \to L_n^{\langle -\infty\rangle}(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$$
$$A_{\mathcal{FCY}} \colon H_n^G(E_{\mathcal{FCY}}(G);\mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

are always isomorphisms.

3 More Applications

3.1 Applications VI

3.1.1 Low Dimensional Algebraic *K*-Theory

As opposed to topological K-theory and L-theory, which are periodic, the algebraic K-theory groups of coefficient rings such as $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$ are known to be bounded below. Using the spectral sequences for the left hand side of an assembly map that will be discussed in Subsection 8.4.1, this leads to vanishing results in negative dimensions and a concrete description of the groups in the first non-vanishing dimension.

The following conjecture is a consequence of the K-theoretic Farrell-Jones Conjecture in the case $R = \mathbb{Z}$. Note that by the results discussed in Subsection 2.2.5 we know that in negative dimensions we can reduce to the family of finite subgroups. Explanations about the colimit that appears follow below.

Conjecture 3.1 (The Farrell-Jones Conjecture for $K_n(\mathbb{Z}G)$ for $n \leq -1$). For every group G we have

$$K_{-n}(\mathbb{Z}G) = 0 \quad \text{for } n \geq 2,$$

and the map

$$\operatorname{colim}_{H \in \mathsf{Sub}_{\mathcal{FIN}}(G)} K_{-1}(\mathbb{Z}H) \xrightarrow{\cong} K_{-1}(\mathbb{Z}G)$$

is an isomorphism.

We can consider a family $\mathcal{F}$ of subgroups of G as a category $\mathtt{Sub}_{\mathcal{F}}(G)$ as follows. The objects are the subgroups H with $H \in \mathcal{F}$. For $H, K \in \mathcal{F}$ let $\mathrm{conhom}_G(H,K)$ be the set of all group homomorphisms $f\colon H \to K$, for which there exists a group element $g \in G$ such that f is given by conjugation with g. The group of inner automorphism $\mathrm{inn}(K)$ consists of those automorphisms $K \to K$, which are given by conjugation with an element $k \in K$. It acts on $\mathrm{conhom}(H,K)$ from the left by composition. Define the set of morphisms in $\mathtt{Sub}_{\mathcal{F}}(G)$ from H to K to be $\mathrm{inn}(K)\backslash \mathrm{conhom}(H,K)$. Composition of group homomorphisms defines the composition of morphisms in $\mathtt{Sub}_{\mathcal{F}}(G)$. We mention that $\mathtt{Sub}_{\mathcal{F}}(G)$ is a quotient category of the orbit category $\mathtt{Or}_{\mathcal{F}}(G)$ which we will introduce in Section 6.4. Note that there is a morphism from H to K only if H is conjugate to a subgroup of K. Clearly $K_n(R(-))$ yields a functor from $\mathtt{Sub}_{\mathcal{F}}(G)$ to abelian groups since inner automorphisms on a group G induce the identity on $K_n(RG)$. Using the inclusions into G, one obtains a map

$$\mathrm{colim}_{H \in \mathtt{Sub}_{\mathcal{F}}(G)} K_n(RH) \to K_n(RG).$$

The colimit can be interpreted as the 0-th Bredon homology group

$$H_0^G(E_{\mathcal{F}}(G); K_n(R(?)))$$

(compare Example 2.8) and the map is the edge homomorphism in the equivariant Atiyah-Hirzebruch spectral sequence discussed in Subsection 8.4.1. In Conjecture 3.1 we consider the first non-vanishing entry in the lower left hand corner of the E_2-term because of the following vanishing result [113, Theorem 2.1] which generalizes vanishing results for finite groups from [57].

Proposition 3.2. *If V is a virtually cyclic group, then $K_{-n}(\mathbb{Z}V) = 0$ for $n \geq 2$.*

If our coefficient ring R is a regular ring in which the orders of all finite subgroups of G are invertible, then we know already from Subsection 2.2.4 that we can reduce to the family of finite subgroups. In the proof of Proposition 2.14 we have seen that then RH is again regular if $H \subset G$ is finite. Since negative K-groups vanish for regular rings [268, 5.3.30 on page 295], the following is implied by the Farrell-Jones Conjecture 2.2.

Conjecture 3.3 (Farrell-Jones Conjecture for $K_0(\mathbb{Q}G)$). Suppose R is a regular ring in which the orders of all finite subgroups of G are invertible (for example a field of characteristic 0), then

$$K_{-n}(RG) = 0 \quad \text{for } n \geq 1$$

and the map

$$\mathrm{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \xrightarrow{\cong} K_0(RG)$$

is an isomorphism.

The conjecture above holds if G is virtually poly-cyclic. Surjectivity is proven in [227] (see also [67] and Chapter 8 in [235]), injectivity in [271]. We will show in Lemma 3.11 (i) that the map appearing in the conjecture is always rationally injective for $R = \mathbb{C}$.

The conjectures above describe the first non-vanishing term in the equivariant Atiyah-Hirzebruch spectral sequence. Already the next step is much harder to analyze in general because there are potentially non-vanishing differentials. We know however that after rationalizing the equivariant Atiyah-Hirzebruch spectral sequence for the left hand side of the $\mathcal{FIN}$-assembly map collapses. As a consequence we obtain that the following conjecture follows from the K-theoretic Farrell-Jones Conjecture 2.2.

Conjecture 3.4. For every group G, every ring R and every $n \in \mathbb{Z}$ the map

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_n(RH) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective.

Note that for $K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$ the conjecture above is always true but not very interesting, because for a finite group H it is known that $\widetilde{K}_0(\mathbb{Z}H) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, compare [298, Proposition 9.1], and hence the left hand side reduces to $K_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$. However, the full answer for $K_0(\mathbb{Z}G)$ should involve the negative K-groups, compare Example 8.8.

Analogously to Conjecture 3.4 the following can be derived from the K-theoretic Farrell-Jones Conjecture 2.2, compare [208].

Conjecture 3.5. The map

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} \operatorname{Wh}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \to \operatorname{Wh}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is always injective.

In general one does not expect this map to be an isomorphism. There should be additional contributions coming from negative K-groups. Conjecture 3.5 is true for groups satisfying a mild homological finiteness condition, see Theorem 5.26.

Remark 3.6 (The Conjectures as Generalized Induction Theorems). The above discussion shows that one may think of the Farrell-Jones Conjectures 2.2 and the Baum-Connes Conjecture 2.3 as “generalized induction theorems”. The prototype of an induction theorem is Artin's theorem about the complex representation ring $R_{\mathbb{C}}(G)$ of a finite group G. Let us recall Artin's theorem.

For finite groups H the complex representation ring $R_{\mathbb{C}}(H)$ coincides with $K_0(\mathbb{C}H)$. Artin's Theorem [283, Theorem 17 in 9.2 on page 70] implies that the obvious induction homomorphism

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{CYC}}(G)} R_{\mathbb{C}}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. Note that this is a very special case of Theorem 8.4 or 8.5, compare Remark 8.9.

Artin's theorem says that rationally one can compute $R_{\mathbb{C}}(G)$ if one knows all the values $R_{\mathbb{C}}(C)$ (including all maps coming from induction with group homomorphisms induced by conjugation with elements in G) for all cyclic subgroups $C \subseteq G$. The idea behind the Farrell-Jones Conjectures 2.2 and the Baum-Connes Conjecture 2.3 is analogous. We want to compute the functors $K_n(RG)$, $L_n(RG)$ and $K_n(C_r^*(G))$ from their values (including their functorial properties under induction) on elements of the family $\mathcal{FIN}$ or $\mathcal{VCY}$.

The situation in the Farrell Jones and Baum-Connes Conjectures is more complicated than in Artin's Theorem, since we have already seen in Remarks 1.15, 1.26 and 1.35 that a computation of $K_n(RG)$, $L_n^{\langle -\infty \rangle}(RG)$ and $K_n(C_r^*(G))$ does involve also the values $K_p(RH)$, $L_p^{\langle -\infty \rangle}(RH)$ and $K_p(C_r^*(H))$ for $p \le n$. A degree mixing occurs.

3.1.2 *G*-Theory

Instead of considering finitely generated projective modules one may apply the standard K-theory machinery to the category of finitely generated modules. This leads to the definition of the groups $G_n(R)$ for $n \ge 0$. For instance $G_0(R)$ is the abelian group whose generators are isomorphism classes $[M]$ of finitely generated R-modules and whose relations are given by $[M_0] - [M_1] + [M_2]$ for any exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ of finitely generated modules. One may ask whether versions of the Farrell-Jones Conjectures for G-theory instead of K-theory might be true. The answer is negative as the following discussion explains.

For a finite group H the ring $\mathbb{C}H$ is semisimple. Hence any finitely generated $\mathbb{C}H$-module is automatically projective and $K_0(\mathbb{C}H) = G_0(\mathbb{C}H)$. Recall that a group G is called *virtually poly-cyclic* if there exists a subgroup of finite index $H \subseteq G$ together with a filtration $\{1\} = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots \subseteq H_r = H$ such that H_{i-1} is normal in H_i and the quotient H_i/H_{i-1} is cyclic. More generally for all $n \in \mathbb{Z}$ the forgetful map

$$f\colon K_n(\mathbb{C}G) \to G_n(\mathbb{C}G)$$

is an isomorphism if G is virtually poly-cyclic, since then $\mathbb{C}G$ is regular [273, Theorem 8.2.2 and Theorem 8.2.20] and the forgetful map f is an isomorphism for regular rings, compare [268, Corollary 53.26 on page 293]. In particular this applies to virtually cyclic groups and so the left hand side of the Farrell-Jones assembly map does not see the difference between K- and G-theory if we work with complex coefficients. We obtain a commutative diagram

$$\begin{tikzcd} \operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(\mathbb{C}H) \arrow[r] \arrow[d, "\cong"'] & K_0(\mathbb{C}G) \arrow[d, "f"] \\ \operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} G_0(\mathbb{C}H) \arrow[r] & G_0(\mathbb{C}G) \end{tikzcd} \qquad (11)$$

where, as indicated, the left hand vertical map is an isomorphism. Conjecture 3.3, which is implied by the Farrell-Jones Conjecture, says that the upper horizontal arrow is an isomorphism. A G-theoretic analogue of the Farrell-Jones Conjecture would say that the lower horizontal map is an isomorphism. There are however cases where the upper horizontal arrow is known to be an isomorphism, but the forgetful map f on the right is not injective or not surjective:

If G contains a non-abelian free subgroup, then the class $[\mathbb{C}G] \in G_0(\mathbb{C}G)$ vanishes [202, Theorem 9.66 on page 364] and hence the map $f\colon K_0(\mathbb{C}G) \to G_0(\mathbb{C}G)$ has an infinite kernel ($[\mathbb{C}G]$ generates an infinite cyclic subgroup in $K_0(\mathbb{C}G)$). The Farrell-Jones Conjecture for $K_0(\mathbb{C}G)$ is known for non-abelian free groups.

The Farrell-Jones Conjecture is also known for $A = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2$ and hence $K_0(\mathbb{C}A)$ is countable, whereas $G_0(\mathbb{C}A)$ is not countable [202, Example 10.13 on page 375]. Hence the map f cannot be surjective.

At the time of writing we do not know a counterexample to the statement that for an amenable group G, for which there is an upper bound on the orders of its finite subgroups, the forgetful map $f\colon K_0(\mathbb{C}G) \to G_0(\mathbb{C}G)$ is an isomorphism. We do not know a counterexample to the statement that for a group G, which is not amenable, $G_0(\mathbb{C}G) = \{0\}$. We also do not know whether $G_0(\mathbb{C}G) = \{0\}$ is true for $G = \mathbb{Z} * \mathbb{Z}$.

For more information about $G_0(\mathbb{C}G)$ we refer for instance to [202, Subsection 9.5.3].

3.1.3 Bass Conjectures

Complex representations of a finite group can be studied using characters. We now want to define the Hattori-Stallings rank of a finitely generated projective $\mathbb{C}G$-module which should be seen as a generalization of characters to infinite groups.

Let $\operatorname{con}(G)$ be the set of conjugacy classes (g) of elements $g \in G$. Denote by $\operatorname{con}(G)_f$ the subset of $\operatorname{con}(G)$ consisting of those conjugacy classes (g) for which each representative g has finite order. Let $\operatorname{class}_0(G)$ and $\operatorname{class}_0(G)_f$ be the $\mathbb{C}$-vector space with the set $\operatorname{con}(G)$ and $\operatorname{con}(G)_f$ as basis. This is the same as the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions on $\operatorname{con}(G)$ and $\operatorname{con}(G)_f$ with finite support. Define the *universal* $\mathbb{C}$-*trace* as

$$\operatorname{tr}^u_{\mathbb{C}G}\colon \mathbb{C}G \to \operatorname{class}_0(G), \quad \sum_{g \in G} \lambda_g \cdot g \;\mapsto\; \sum_{g \in G} \lambda_g \cdot (g). \tag{12}$$

It extends to a function $\operatorname{tr}^u_{\mathbb{C}G}\colon M_n(\mathbb{C}G) \to \operatorname{class}_0(G)$ on (n,n)-matrices over $\mathbb{C}G$ by taking the sum of the traces of the diagonal entries. Let P be a finitely generated projective $\mathbb{C}G$-module. Choose a matrix $A \in M_n(\mathbb{C}G)$ such that $A^2 = A$ and the image of the $\mathbb{C}G$-map $r_A\colon \mathbb{C}G^n \to \mathbb{C}G^n$ given by right multiplication with A is $\mathbb{C}G$-isomorphic to P. Define the *Hattori-Stallings rank* of P as

$$\mathrm{HS}_{\mathbb{C}G}(P) = \mathrm{tr}^u_{\mathbb{C}G}(A) \in \mathrm{class}_0(G). \tag{13}$$

The Hattori-Stallings rank depends only on the isomorphism class of the $\mathbb{C}G$-module P and induces a homomorphism $\mathrm{HS}_{\mathbb{C}G} \colon K_0(\mathbb{C}G) \to \mathrm{class}_0(G)$.

Conjecture 3.7 (Strong Bass Conjecture for $K_0(\mathbb{C}G)$). The $\mathbb{C}$-vector space spanned by the image of the map

$$\mathrm{HS}_{\mathbb{C}G} \colon K_0(\mathbb{C}G) \to \mathrm{class}_0(G)$$

is $\mathrm{class}_0(G)_f$.

This conjecture is implied by the surjectivity of the map

$$\mathrm{colim}_{H \in \mathsf{Sub}_{\mathcal{FIN}}(G)} K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{C} \to K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C}, \tag{14}$$

(compare Conjecture 3.3) and hence by the K-theoretic Farrell-Jones Conjecture for $K_0(\mathbb{C}G)$. We will see below that the surjectivity of the map (14) also implies that the map $K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathrm{class}_0(G)$, which is induced by the Hattori-Stallings rank, is injective. Hence we do expect that the Hattori-Stallings rank induces an isomorphism

$$K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathrm{class}_0(G)_f.$$

There are also versions of the Bass conjecture for other coefficients than $\mathbb{C}$. It follows from results of Linnell [191, Theorem 4.1 on page 96] that the following version is implied by the Strong Bass Conjecture for $K_0(\mathbb{C}G)$.

Conjecture 3.8 (The Strong Bass Conjecture for $K_0(\mathbb{Z}G)$). The image of the composition

$$K_0(\mathbb{Z}G) \to K_0(\mathbb{C}G) \xrightarrow{\mathrm{HS}_{\mathbb{C}G}} \mathrm{class}_0(G)$$

is contained in the $\mathbb{C}$-vector space of those functions $f \colon \mathrm{con}(G) \to \mathbb{C}$ which vanish for $(g) \in \mathrm{con}(g)$ with $g \neq 1$.

The conjecture says that for every finitely generated projective $\mathbb{Z}G$-module P the Hattori-Stallings rank of $\mathbb{C}G \otimes_{\mathbb{Z}G} P$ looks like the Hattori-Stallings rank of a free $\mathbb{C}G$-module. A natural explanation for this behaviour is the following conjecture which clearly implies the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$.

Conjecture 3.9 (Rational $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$-Conjecture). For every group G the map

$$\widetilde{K}_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \to \widetilde{K}_0(\mathbb{Q}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

induced by the change of coefficients is trivial.

Finally we mention the following variant of the Bass Conjecture.

Conjecture 3.10 (The Weak Bass Conjecture). Let P be a finitely generated projective $\mathbb{Z}G$-module. The value of the Hattori-Stallings rank of $\mathbb{C}G \otimes_{\mathbb{Z}G} P$ at the conjugacy class of the identity element is given by

$$\mathrm{HS}_{\mathbb{C}G}(\mathbb{C}G \otimes_{\mathbb{Z}G} P)((1)) \;=\; \dim_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P).$$

Here $\mathbb{Z}$ is considered as a $\mathbb{Z}G$-module via the augmentation.

The K-theoretic Farrell-Jones Conjecture implies all four conjectures above. More precisely we have the following proposition.

Proposition 3.11. *(i) The map*

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is always injective. If the map is also surjective (compare Conjecture 3.3) then the Hattori-Stallings rank induces an isomorphism

$$K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathrm{class}_0(G)_f$$

and in particular the Strong Bass Conjecture for $K_0(\mathbb{C}G)$ and hence also the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$ hold.

(ii) The surjectivity of the map

$$A_{\mathcal{VCY}} \colon H_0^G(E_{\mathcal{VCY}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the Rational $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$ Conjecture and hence also the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$.

(iii) The Strong Bass Conjecture for $K_0(\mathbb{C}G)$ implies the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$. The Strong Bass Conjecture for $K_0(\mathbb{Z}G)$ implies the Weak Bass Conjecture.

Proof. (i) follows from the following commutative diagram, compare [198, Lemma 2.15 on page 220].

$$\begin{tikzcd}
\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[rr] \arrow[d, "\cong"'] & & K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[d] \\
\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} \mathrm{class}_0(H) \arrow[r, "\cong"] & \mathrm{class}_0(G)_f \arrow[r, "i"] & \mathrm{class}_0(G).
\end{tikzcd}$$

Here the vertical maps are induced by the Hattori-Stallings rank, the map i is the natural inclusion and in particular injective and we have the indicated isomorphisms.

(ii) According to Proposition 2.17 the surjectivity of the map $A_{\mathcal{VCY}}$ appearing in (ii) implies the surjectivity of the corresponding assembly map $A_{\mathcal{FIN}}$ (rationalized and with $\mathbb{Z}$ as coefficient ring) for the family of finite subgroups. The map $A_{\mathcal{FIN}}$ is natural with respect to the change of the coefficient ring from $\mathbb{Z}$ to $\mathbb{Q}$. By Theorem 8.5 we know that for every coefficient ring R there is an isomorphism from

$$\bigoplus_{p,q,p+q=0} \bigoplus_{(C)\in(\mathcal{FCY})} H_p(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \Theta_C \cdot K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q}$$

to the 0-dimensional part of the left hand side of the rationalized $\mathcal{FIN}$-assembly map $A_{\mathcal{FIN}}$. The isomorphism is natural with respect to a change of coefficient rings. To see that the Rational $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$ Conjecture follows, it hence suffices to show that the summand corresponding to $C = \{1\}$ and $p = q = 0$ is the only one where the map induced from $\mathbb{Z} \to \mathbb{Q}$ is possibly non-trivial. But $K_q(\mathbb{Q}C) = 0$ if $q < 0$, because $\mathbb{Q}C$ is semisimple and hence regular, and for a finite cyclic group $C \neq \{1\}$ we have by [198, Lemma 7.4]

$$\Theta_C \cdot K_0(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q} = \operatorname{coker}\left(\bigoplus_{D\subsetneq C} K_0(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \;\to\; K_0(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}\right) = 0,$$

since by a result of Swan $K_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(\mathbb{Z}H) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism for a finite group H, see [298, Proposition 9.1].

(iii) As already mentioned the first statement follows from [191, Theorem 4.1 on page 96]. The second statement follows from the formula

$$\sum_{(g)\in\operatorname{con}(G)} \operatorname{HS}_{\mathbb{C}G}(\mathbb{C} \otimes_{\mathbb{Z}} P)(g) = \dim_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P).$$

□

The next result is due to Berrick, Chatterji and Mislin [36, Theorem 5.2]. The Bost Conjecture is a variant of the Baum-Connes Conjecture and is explained in Subsection 4.1.3.

Theorem 3.12. *If the assembly map appearing in the Bost Conjecture 4.2 is rationally surjective, then the Strong Bass Conjecture for $K_0(\mathbb{C}G)$ (compare 3.7) is true.*

We now discuss some further questions and facts that seem to be relevant in the context of the Bass Conjectures.

Remark 3.13 (Integral $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$-Conjecture). We do not know a counterexample to the Integral $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$ Conjecture, i.e. to the statement that the map

$$\widetilde{K}_0(\mathbb{Z}G) \to \widetilde{K}_0(\mathbb{Q}G)$$

itself is trivial. But we also do not know a proof which shows that the K-theoretic Farrell-Jones Conjecture implies this integral version. Note that the Integral $\widetilde{K}_0(\mathbb{Z}G)$-to-$\widetilde{K}_0(\mathbb{Q}G)$ Conjecture would imply that the following diagram commutes.

$$\begin{array}{ccc} K_0(\mathbb{Z}G) & \longrightarrow & K_0(\mathbb{Q}G) \\ {\scriptstyle p_*}\downarrow & & \uparrow{\scriptstyle i} \\ K_0(\mathbb{Z}) & \xrightarrow[\cong]{\dim_{\mathbb{Z}}} & \mathbb{Z}. \end{array}$$

Here p_* is induced by the projection $G \to \{1\}$ and i sends $1 \in \mathbb{Z}$ to the class of $\mathbb{Q}G$.

Remark 3.14 (The passage from $\widetilde{K}_0(\mathbb{Z}G)$ to $\widetilde{K}_0(\mathcal{N}(G))$). Let $\mathcal{N}(G)$ denote the group von Neumann algebra of G. It is known that for every group G the composition

$$\widetilde{K}_0(\mathbb{Z}G) \to \widetilde{K}_0(\mathbb{Q}G) \to \widetilde{K}_0(\mathbb{C}G) \to \widetilde{K}_0(C_r^*(G)) \to \widetilde{K}_0(\mathcal{N}(G))$$

is the zero-map (see for instance [202, Theorem 9.62 on page 362]). Since the group von Neumann algebra $\mathcal{N}(G)$ is not functorial under arbitrary group homomorphisms such as $G \to \{1\}$, this does *not* imply that the diagram

$$\begin{array}{ccc} K_0(\mathbb{Z}G) & \longrightarrow & K_0(\mathcal{N}(G)) \\ {\scriptstyle p_*}\downarrow & & \uparrow{\scriptstyle i} \\ K_0(\mathbb{Z}) & \xrightarrow[\cong]{\dim_{\mathbb{Z}}} & \mathbb{Z} \end{array}$$

commutes. However, commutativity would follow from the Weak Bass Conjecture 3.10. For a discussion of these questions see [93].

More information and further references about the Bass Conjecture can be found for instance in [24], [36, Section 7], [42], [92], [93], [118], [191] [202, Subsection 9.5.2], and [225, page 66ff].

3.2 Applications VII

3.2.1 Novikov Conjectures

In Subsection 1.9.1 we discussed the Novikov Conjectures. Recall that one possible reformulation of the original Novikov Conjecture says that for every group G the rationalized classical assembly map in L-theory

$$A\colon H_n(BG; \mathbf{L}^p(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \to L_n^p(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective. Since A can be identified with $A_{\{1\}\to\mathcal{ALL}}$ and we know from Subsection 2.2.8 that the relative assembly map

$$A_{\{1\}\to\mathcal{FIN}}\colon H_n^G(E_{\{1\}}(G); \mathbf{L}_{\mathbb{Z}}^p) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_{\mathbb{Z}}^p) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective we obtain the following proposition.

Proposition 3.15. *The rational injectivity of the assembly map appearing in L-theoretic Farrell-Jones Conjecture (Conjecture 2.2) implies the L-theoretic Novikov Conjecture (Conjecture 1.51) and hence the original Novikov Conjecture 1.52.*

Similarly the Baum-Connes Conjecture 2.3 implies the injectivity of the rationalized classical assembly map

$$A\colon H_n(BG;\mathbf{K}^{\mathrm{top}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In the next subsection we discuss how one can relate assembly maps for topological K-theory with L-theoretic assembly maps. The results imply in particular the following proposition.

Proposition 3.16. *The rational injectivity of the assembly map appearing in the Baum-Connes Conjecture (Conjecture 2.3) implies the Novikov Conjecture (Conjecture 1.52).*

Finally we would like to mention that by combining the results about the rationalization of $A_{\{1\}\to\mathcal{FIN}}$ from Subsection 2.2.8 with the splitting result about $A_{\mathcal{FIN}\to\mathcal{VCY}}$ from Subsection 2.2.5 we obtain the following result

Proposition 3.17. *The rational injectivity of the assembly map appearing in the Farrell-Jones Conjecture for algebraic K-theory (Conjecture 2.2) implies the K-theoretic Novikov Conjecture, i.e. the injectivity of*

$$A\colon H_n(BG;\mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 3.18 (Integral Injectivity Fails). In general the classical assembly maps $A = A_{\{1\}}$ themselves, i.e. without rationalizing, are not injective. For example one can use the Atiyah-Hirzebruch spectral sequence to see that for $G = \mathbb{Z}/5$

$$H_1(BG;\mathbf{K}^{\mathrm{top}}) \quad \text{and} \quad H_1(BG;\mathbf{L}^{\langle -\infty\rangle}(\mathbb{Z}))$$

contain 5-torsion, whereas for every finite group G the topological K-theory of $\mathbb{C}G$ is torsionfree and the torsion in the L-theory of $\mathbb{Z}G$ is always 2-torsion, compare Proposition 8.1 (i) and Proposition 8.3 (i).

3.2.2 Relating Topological *K*-Theory and *L*-Theory

For every real C^*-algebra A there is an isomorphism $L_n^p(A)[1/2] \xrightarrow{\cong} K_n(A)[1/2]$ [269]. This can be used to compare L-theory to topological K-theory and leads to the following result.

Proposition 3.19. *Let $\mathcal{F} \subseteq \mathcal{FIN}$ be a family of finite subgroups of G. If the topological K-theory assembly map*

$$A_{\mathcal{F}}\colon H_n^G(E_{\mathcal{F}}(G);\mathbf{K}^{\mathrm{top}})[\frac{1}{2}] \to K_n(C_r^*(G))[\frac{1}{2}]$$

is injective, then for an arbitrary decoration j also the map

$$A_{\mathcal{F}} \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{L}_{\mathbb{Z}}^{\langle j \rangle})[\frac{1}{2}] \to L_n^{\langle j \rangle}(\mathbb{Z}G)[\frac{1}{2}]$$

is injective.

Proof. First recall from Remark 1.22 that after inverting 2 there is no difference between the different decorations and we can hence work with the p-decoration. One can construct for any subfamily $\mathcal{F} \subseteq \mathcal{FIN}$ the following commutative diagram [200, Section 7.5]

$$\begin{array}{ccc}
H_n^G(E_{\mathcal{F}}(G); \mathbf{L}^p_{\mathbb{Z}}[1/2]) & \xrightarrow{A^1_{\mathcal{F}}} & L_n^p(\mathbb{Z}G)[1/2] \\
{\scriptstyle i_1}\big\downarrow\cong & & {\scriptstyle j_1}\big\downarrow\cong \\
H_n^G(E_{\mathcal{F}}(G); \mathbf{L}^p_{\mathbb{Q}}[1/2]) & \xrightarrow{A^2_{\mathcal{F}}} & L_n^p(\mathbb{Q}G)[1/2] \\
{\scriptstyle i_2}\big\downarrow\cong & & {\scriptstyle j_2}\big\downarrow \\
H_n^G(E_{\mathcal{F}}(G); \mathbf{L}^p_{\mathbb{R}}[1/2]) & \xrightarrow{A^3_{\mathcal{F}}} & L_n^p(\mathbb{R}G)[1/2] \\
{\scriptstyle i_3}\big\downarrow\cong & & {\scriptstyle j_3}\big\downarrow \\
H_n^G(E_{\mathcal{F}}(G); \mathbf{L}^p_{C^*_r(?;\mathbb{R})}[1/2]) & \xrightarrow{A^4_{\mathcal{F}}} & L_n^p(C^*_r(G;\mathbb{R}))[1/2] \\
{\scriptstyle i_4}\big\downarrow\cong & & {\scriptstyle j_4}\big\downarrow\cong \\
H_n^G(E_{\mathcal{F}}(G); \mathbf{K}^{\mathrm{top}}_{\mathbb{R}}[1/2]) & \xrightarrow{A^5_{\mathcal{F}}} & K_n(C^*_r(G;\mathbb{R}))[1/2] \\
{\scriptstyle i_5}\big\downarrow & & {\scriptstyle j_5}\big\downarrow \\
H_n^G(E_{\mathcal{F}}(G); \mathbf{K}^{\mathrm{top}}_{\mathbb{C}}[1/2]) & \xrightarrow{A^6_{\mathcal{F}}} & K_n(C^*_r(G))[1/2]
\end{array}$$

Here

$$\mathbf{L}^p_{\mathbb{Z}}[1/2], \quad \mathbf{L}^p_{\mathbb{Q}}[1/2], \quad \mathbf{L}^p_{\mathbb{R}}[1/2], \quad \mathbf{L}_{C^*_r(?;\mathbb{R})}[1/2],$$
$$\mathbf{K}^{\mathrm{top}}_{\mathbb{R}}[1/2] \quad \text{and} \quad \mathbf{K}^{\mathrm{top}}_{\mathbb{C}}[1/2]$$

are covariant $\mathsf{Or}(G)$-spectra (compare Section 6.2 and in particular Proposition 6.7) such that the n-th homotopy group of their evaluations at G/H are given by

$$L_n^p(\mathbb{Z}H)[1/2], \quad L_n^p(\mathbb{Q}H)[1/2], \quad L_n^p(\mathbb{R}H)[1/2], \quad L_n^p(C^*_r(H;\mathbb{R}))[1/2],$$
$$K_n(C^*_r(H;\mathbb{R}))[1/2] \quad \text{respectively} \quad K_n(C^*_r(H)[1/2].$$

All horizontal maps are assembly maps induced by the projection $\mathrm{pr} \colon E_{\mathcal{F}}(G) \to \mathrm{pt}$. The maps i_k and j_k for $k = 1, 2, 3$ are induced from a change of rings. The isomorphisms i_4 and j_4 come from the general isomorphism for any real C^*-algebra A

$$L_n^p(A)[1/2] \xrightarrow{\cong} K_n(A)[1/2]$$

and its spectrum version [269, Theorem 1.11 on page 350]. The maps i_1, j_1, i_2 are isomorphisms by [256, page 376] and [258, Proposition 22.34 on page 252]. The map i_3 is bijective since for a finite group H we have $\mathbb{R}H = C_r^*(H;\mathbb{R})$. The maps i_5 and j_5 are given by extending the scalars from $\mathbb{R}$ to $\mathbb{C}$ by induction. For every real C^*-algebra A the composition

$$K_n(A)[1/2] \to K_n(A \otimes_{\mathbb{R}} \mathbb{C})[1/2] \to K_n(M_2(A))[1/2]$$

is an isomorphism and hence j_5 is split injective. An $\mathsf{Or}(G)$-spectrum version of this argument yields that also i_5 is split injective. □

Remark 3.20. One may conjecture that the right vertical maps j_2 and j_3 are isomorphisms and try to prove this directly. Then if we invert 2 everywhere the Baum-Connes Conjecture 2.3 for the real reduced group C^*-algebra, would be equivalent to the Farrell-Jones Isomorphism Conjecture for $L_*(\mathbb{Z}G)[1/2]$.

3.3 Applications VIII

3.3.1 The Modified Trace Conjecture

Denote by Λ^G the subring of $\mathbb{Q}$ which is obtained from $\mathbb{Z}$ by inverting all orders $|H|$ of finite subgroups H of G, i.e.

$$\Lambda^G = \mathbb{Z}\left[|H|^{-1} \mid H \subset G,\ |H| < \infty\right]. \tag{15}$$

The following conjecture generalizes Conjecture 1.37 to the case where the group need no longer be torsionfree. For the standard trace compare (7).

Conjecture 3.21 (Modified Trace Conjecture for a group G). Let G be a group. Then the image of the homomorphism induced by the standard trace

$$\mathrm{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R} \tag{16}$$

is contained in Λ^G.

The following result is proved in [203, Theorem 0.3].

Theorem 3.22. *Let G be a group. Then the image of the composition*

$$K_0^G(E_{\mathcal{FIN}}(G)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{A_{\mathcal{FIN}} \otimes_{\mathbb{Z}} \mathrm{id}} K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\mathrm{tr}_{C_r^*(G)}} \mathbb{R}$$

is Λ^G. Here $A_{\mathcal{FIN}}$ is the map appearing in the Baum-Connes Conjecture 2.3. In particular the Baum-Connes Conjecture 2.3 implies the Modified Trace Conjecture.

The original version of the Trace Conjecture due to Baum and Connes [27, page 21] makes the stronger statement that the image of $\mathrm{tr}_{C_r^*(G)} \colon K_0(C_r^*(G)) \to \mathbb{R}$ is the additive subgroup of $\mathbb{Q}$ generated by all numbers $\frac{1}{|H|}$, where $H \subset G$ runs though all finite subgroups of G. Roy has constructed a counterexample to this version in [274] based on her article [275]. The examples of Roy do *not* contradict the Modified Trace Conjecture 3.21 or the Baum-Connes Conjecture 2.3.

3.3.2 The Stable Gromov-Lawson-Rosenberg Conjecture

The Stable Gromov-Lawson-Rosenberg Conjecture is a typical conjecture relating Riemannian geometry to topology. It is concerned with the question when a given manifold admits a metric of positive scalar curvature. To discuss its relation with the Baum-Connes Conjecture we will need the real version of the Baum-Connes Conjecture, compare Subsection 4.1.1.

Let $\Omega_n^{\mathrm{Spin}}(BG)$ be the bordism group of closed Spin-manifolds M of dimension n with a reference map to BG. Let $C_r^*(G;\mathbb{R})$ be the real reduced group C^*-algebra and let $KO_n(C_r^*(G;\mathbb{R})) = K_n(C_r^*(G;\mathbb{R}))$ be its topological K-theory. We use KO instead of K as a reminder that we here use the real reduced group C^*-algebra. Given an element $[u\colon M \to BG] \in \Omega_n^{\mathrm{Spin}}(BG)$, we can take the $C_r^*(G;\mathbb{R})$-valued index of the equivariant Dirac operator associated to the G-covering $\overline{M} \to M$ determined by u. Thus we get a homomorphism

$$\mathrm{ind}_{C_r^*(G;\mathbb{R})} \colon \Omega_n^{\mathrm{Spin}}(BG) \to KO_n(C_r^*(G;\mathbb{R})). \tag{17}$$

A *Bott manifold* is any simply connected closed Spin-manifold B of dimension 8 whose $\widehat{A}$-genus $\widehat{A}(B)$ is 8. We fix such a choice, the particular choice does not matter for the sequel. Notice that $\mathrm{ind}_{C_r^*(\{1\};\mathbb{R})}(B) \in KO_8(\mathbb{R}) \cong \mathbb{Z}$ is a generator and the product with this element induces the Bott periodicity isomorphisms $KO_n(C_r^*(G;\mathbb{R})) \xrightarrow{\cong} KO_{n+8}(C_r^*(G;\mathbb{R}))$. In particular

$$\mathrm{ind}_{C_r^*(G;\mathbb{R})}(M) = \mathrm{ind}_{C_r^*(G;\mathbb{R})}(M \times B), \tag{18}$$

if we identify $KO_n(C_r^*(G;\mathbb{R})) = KO_{n+8}(C_r^*(G;\mathbb{R}))$ via Bott periodicity.

Conjecture 3.23 (Stable Gromov-Lawson-Rosenberg Conjecture). Let M be a closed connected Spin-manifold of dimension $n \geq 5$. Let $u_M \colon M \to B\pi_1(M)$ be the classifying map of its universal covering. Then $M \times B^k$ carries for some integer $k \geq 0$ a Riemannian metric with positive scalar curvature if and only if

$$\mathrm{ind}_{C_r^*(\pi_1(M);\mathbb{R})}([M, u_M]) \;=\; 0 \qquad \in KO_n(C_r^*(\pi_1(M);\mathbb{R})).$$

If M carries a Riemannian metric with positive scalar curvature, then the index of the Dirac operator must vanish by the Bochner-Lichnerowicz formula [267]. The converse statement that the vanishing of the index implies the existence of a Riemannian metric with positive scalar curvature is the hard part of the conjecture. The following result is due to Stolz. A sketch of the proof can be found in [297, Section 3], details are announced to appear in a different paper.

Theorem 3.24. *If the assembly map for the real version of the Baum-Connes Conjecture (compare Subsection 4.1.1) is injective for the group G, then the Stable Gromov-Lawson-Rosenberg Conjecture 3.23 is true for all closed* Spin-*manifolds of dimension ≥ 5 with $\pi_1(M) \cong G$.*

The requirement $\dim(M) \geq 5$ is essential in the Stable Gromov-Lawson-Rosenberg Conjecture, since in dimension four new obstructions, the Seiberg-Witten invariants, occur. The unstable version of this conjecture says that M carries a Riemannian metric with positive scalar curvature if and only if $\mathrm{ind}_{C_r^*(\pi_1(M);\mathbb{R})}([M, u_M]) = 0$. Schick [278] constructs counterexamples to the unstable version using minimal hypersurface methods due to Schoen and Yau (see also [91]). It is not known at the time of writing whether the unstable version is true for finite fundamental groups. Since the Baum-Connes Conjecture 2.3 is true for finite groups (for the trivial reason that $E_{\mathcal{FIN}}(G) = \mathrm{pt}$ for finite groups G), Theorem 3.24 implies that the Stable Gromov-Lawson Conjecture 3.23 holds for finite fundamental groups (see also [270]).

The index map appearing in (17) can be factorized as a composition

$$\mathrm{ind}_{C_r^*(G;\mathbb{R})} \colon \Omega_n^{\mathrm{Spin}}(BG) \xrightarrow{D} KO_n(BG) \xrightarrow{A} KO_n(C_r^*(G;\mathbb{R})), \tag{19}$$

where D sends $[M, u]$ to the class of the G-equivariant Dirac operator of the G-manifold $\overline{M}$ given by u and $A = A_{\{1\}}$ is the real version of the classical assembly map. The homological Chern character defines an isomorphism

$$KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \in \mathbb{Z}} H_{n+4p}(BG; \mathbb{Q}).$$

Recall that associated to M there is the $\widehat{A}$-*class*

$$\widehat{\mathcal{A}}(M) \in \prod_{p \geq 0} H^p(M; \mathbb{Q}) \tag{20}$$

which is a certain polynomial in the Pontrjagin classes. The map D appearing in (19) sends the class of $u \colon M \to BG$ to $u_*(\widehat{\mathcal{A}}(M) \cap [M])$, i.e. the image of the Poincaré dual of $\widehat{\mathcal{A}}(M)$ under the map induced by u in rational homology. Hence $D([M, u]) = 0$ if and only if $u_*(\widehat{\mathcal{A}}(M) \cap [M])$ vanishes. For $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$ define the *higher* $\widehat{A}$-*genus of* (M, u) *associated to* x to be

$$\widehat{A}_x(M, u) = \langle \widehat{\mathcal{A}}(M) \cup u^* x, [M] \rangle = \langle x, u_*(\widehat{\mathcal{A}}(M) \cap [M]) \rangle \ \in \mathbb{Q}. \tag{21}$$

The vanishing of $\widehat{\mathcal{A}}(M)$ is equivalent to the vanishing of all higher $\widehat{A}$-genera $\widehat{A}_x(M,u)$. The following conjecture is a weak version of the Stable Gromov-Lawson-Rosenberg Conjecture.

Conjecture 3.25 (Homological Gromov-Lawson-Rosenberg Conjecture). Let G be a group. Then for any closed Spin-manifold M, which admits a Riemannian metric with positive scalar curvature, the $\widehat{A}$-genus $\widehat{A}_x(M,u)$ vanishes for all maps $u\colon M \to BG$ and elements $x \in \prod_{k\geq 0} H^k(BG;\mathbb{Q})$.

From the discussion above we obtain the following result.

Proposition 3.26. *If the assembly map*

$$KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \to KO_n(C_r^*(G;\mathbb{R})) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective for all $n \in \mathbb{Z}$, then the Homological Gromov-Lawson-Rosenberg Conjecture holds for G.

4 Generalizations and Related Conjectures

4.1 Variants of the Baum-Connes Conjecture

4.1.1 The Real Version

There is an obvious real version of the Baum-Connes Conjecture, which predicts that for all $n \in \mathbb{Z}$ and groups G the assembly map

$$A^{\mathbb{R}}_{\mathcal{FIN}}\colon H_n^G(E_{\mathcal{F}}(G);\mathbf{K}^{\mathrm{top}}_{\mathbb{R}}) \to KO_n(C_r^*(G;\mathbb{R}))$$

is an isomorphism. Here $H_n^G(-;\mathbf{K}^{\mathrm{top}}_{\mathbb{R}})$ is an equivariant homology theory whose distinctive feature is that $H_n^G(G/H;\mathbf{K}^{\mathrm{top}}_{\mathbb{R}}) \cong KO_n(C_r^*(H;\mathbb{R}))$. Recall that we write $KO_n(-)$ only to remind ourselves that the C^*-algebra we apply it to is a real C^*-algebra, like for example the real reduced group C^*-algebra $C_r^*(G;\mathbb{R})$. The following result appears in [31].

Proposition 4.1. *The Baum-Connes Conjecture 2.3 implies the real version of the Baum-Connes Conjecture.*

In the proof of Proposition 3.19 we have already seen that after inverting 2 the "real assembly map" is a retract of the complex assembly map. In particular with 2-inverted or after rationalizing also injectivity results or surjectivity results about the complex Baum-Connes assembly map yield the corresponding results for the real Baum-Connes assembly map.

4.1.2 The Version for Maximal Group C^*-Algebras

For a group G let $C^*_{\max}(G)$ be its *maximal group C^*-algebra*, compare [242, 7.1.5 on page 229]. The maximal group C^*-algebra has the advantage that every homomorphism of groups $\phi\colon G \to H$ induces a homomorphism $C^*_{\max}(G) \to C^*_{\max}(H)$ of C^*-algebras. This is not true for the reduced group C^*-algebra $C^*_r(G)$. Here is a counterexample: since $C^*_r(F)$ is a simple algebra if F is a non-abelian free group [245], there is no unital algebra homomorphism $C^*_r(F) \to C^*_r(\{1\}) = \mathbb{C}$.

One can construct a version of the Baum-Connes assembly map using an equivariant homology theory $H_n^G(-;\mathbf{K}^{\mathrm{top}}_{\max})$ which evaluated on G/H yields the K-theory of $C^*_{\max}(H)$ (use Proposition 6.7 and a suitable modification of $\mathbf{K}^{\mathrm{top}}$, compare Section 6.3).

Since on the left hand side of a $\mathcal{FIN}$-assembly map only the maximal group C^*-algebras for finite groups H matter, and clearly $C^*_{\max}(H) = \mathbb{C}H = C^*_r(H)$ for such H, this left hand side coincides with the left hand side of the usual Baum-Connes Conjecture. There is always a C^*-homomorphism $p\colon C^*_{\max}(G) \to C^*_r(G)$ (it is an isomorphism if and only if G is amenable [242, Theorem 7.3.9 on page 243]) and hence we obtain the following factorization of the usual Baum-Connes assembly map

$$\begin{tikzcd} & K_n(C^*_{\max}(G)) \arrow[d, "K_n(p)"] \\ H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathrm{top}}) \arrow[ur, "A^{\max}_{\mathcal{FIN}}"] \arrow[r, "A_{\mathcal{FIN}}"'] & K_n(C^*_r(G)) \end{tikzcd} \tag{22}$$

It is known that the map $A^{\max}_{\mathcal{FIN}}$ is in general not surjective. The Baum-Connes Conjecture would imply that the map is $A^{\max}_{\mathcal{FIN}}$ is always injective, and that it is surjective if and only if the vertical map $K_n(p)$ is injective.

A countable group G is called *K-amenable* if the map $p\colon C^*_{\max}(G) \to C^*_r(G)$ induces a KK-equivalence (compare [78]). This implies in particular that the vertical map $K_n(p)$ is an isomorphism for all $n \in \mathbb{Z}$. Note that for K-amenable groups the Baum-Connes Conjecture holds if and only if the "maximal" version of the assembly map $A^{\max}_{\mathcal{FIN}}$ is an isomorphism for all $n \in \mathbb{Z}$. A-T-menable groups are K-amenable, compare Theorem 5.1. But $K_0(p)$ is not injective for every infinite group which has property (T) such as for example $SL_n(\mathbb{Z})$ for $n \geq 3$, compare for instance the discussion in [163]. There are groups with property (T) for which the Baum-Connes Conjecture is known (compare Subsection 5.1.2) and hence there are counterexamples to the conjecture that $A^{\max}_{\mathcal{FIN}}$ is an isomorphism.

In Theorem 1.48 and Remark 1.49 we discussed applications of the maximal C^*-algebra version of the Baum-Connes Conjecture.

4.1.3 The Bost Conjecture

Some of the strongest results about the Baum-Connes Conjecture are proven using the so called Bost Conjecture (see [186]). The Bost Conjecture is the version of the Baum-Connes Conjecture, where one replaces the reduced group C^*-algebra $C_r^*(G)$ by the Banach algebra $l^1(G)$ of absolutely summable functions on G. Again one can use the spectra approach (compare Subsection 6.2 and 6.3 and in particular Proposition 6.7) to produce a variant of equivariant K-homology denoted $H_n^G(-;\mathbf{K}_{l^1}^{\mathrm{top}})$ which this time evaluated on G/H yields $K_n(l^1(H))$, the topological K-theory of the Banach algebra $l^1(H)$.

As explained in the beginning of Chapter 2, we obtain an associated assembly map and we believe that it coincides with the one defined using a Banach-algebra version of KK-theory in [186].

Conjecture 4.2 (Bost Conjecture). Let G be a countable group. Then the assembly map

$$A_{\mathcal{FIN}}^{l^1}\colon H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_{l^1}^{\mathrm{top}}) \to K_n(l^1(G))$$

is an isomorphism.

Again the left hand side coincides with the left hand side of the Baum-Connes assembly map because for finite groups H we have $l^1(H) = \mathbb{C}H = C_r^*(H)$. There is always a homomorphism of Banach algebras $q\colon l^1(G) \to C_r^*(G)$ and one obtains a factorization of the usual Baum-Connes assembly map

$$\begin{tikzcd} & K_n(l^1(G)) \arrow[d, "K_n(q)"] \\ H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathrm{top}}) \arrow[ur, "A_{\mathcal{FIN}}^{l^1}"] \arrow[r, "A_{\mathcal{FIN}}"] & K_n(C_r^*(G)). \end{tikzcd}$$

Every group homomorphism $G \to H$ induces a homomorphism of Banach algebras $l^1(G) \to l^1(H)$. So similar as in the maximal group C^*-algebra case this approach repairs the lack of functoriality for the reduced group C^*-algebra.

The disadvantage of $l^1(G)$ is however that indices of operators tend to take values in the topological K-theory of the group C^*-algebras, not in $K_n(l^1(G))$. Moreover the representation theory of G is closely related to the group C^*-algebra, whereas the relation to $l^1(G)$ is not well understood.

For more information about the Bost Conjecture 4.2 see [186], [288].

4.1.4 The Baum-Connes Conjecture with Coefficients

The Baum-Connes Conjecture 2.3 can be generalized to the Baum-Connes Conjecture with Coefficients. Let A be a separable C^*-algebra with an action of the countable group G. Then there is an assembly map

$$KK_n^G(E_{\mathcal{FIN}}(G);A) \to K_n(A \rtimes G) \tag{23}$$

defined in terms of equivariant KK-theory, compare Sections 7.3 and 7.4.

Conjecture 4.3 (Baum-Connes Conjecture with Coefficients). For every separable C^*-algebra A with an action of a countable group G and every $n \in \mathbb{Z}$ the assembly map (23) is an isomorphism.

There are counterexamples to the Baum-Connes Conjecture with Coefficients, compare Remark 5.3. If we take $A = \mathbb{C}$ with the trivial action, the map (23) can be identified with the assembly map appearing in the ordinary Baum-Connes Conjecture 2.3.

Remark 4.4 (A Spectrum Level Description). There is a formulation of the Baum-Connes Conjecture with Coefficients in the framework explained in Section 6.2. Namely, construct an appropriate covariant functor $\mathbf{K}^{\mathrm{top}}(A \rtimes \mathcal{G}^G(-))\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ such that

$$\pi_n(\mathbf{K}^{\mathrm{top}}(A \rtimes \mathcal{G}^G(G/H)) \cong K_n(A \rtimes H)$$

holds for all subgroups $H \subseteq G$ and all $n \in \mathbb{Z}$, and consider the associated G-homology theory $H_*^G(-;\mathbf{K}^{\mathrm{top}}(A \rtimes \mathcal{G}^G(-)))$. Then the map (23) can be identified with the map which the projection $\mathrm{pr}\colon E_{\mathcal{FIN}}(G) \to \mathrm{pt}$ induces for this homology theory.

Remark 4.5 (Farrell-Jones Conjectures with Coefficients). One can also formulate a "Farrell-Jones Conjecture with Coefficients". (This should not be confused with the Fibered Farrell-Jones Conjecture discussed in Subsection 4.2.2.) Fix a ring S and an action of G on it by isomorphisms of rings. Construct an appropriate covariant functor $\mathbf{K}(S \rtimes \mathcal{G}^G(-))\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ such that

$$\pi_n(\mathbf{K}(S \rtimes \mathcal{G}^G(G/H))) \cong K_n(S \rtimes H)$$

holds for all subgroups $H \subseteq G$ and $n \in \mathbb{Z}$, where $S \rtimes H$ is the associated twisted group ring. Now consider the associated G-homology theory $H_*^G(-;\mathbf{K}(S \rtimes \mathcal{G}^G(-)))$. There is an analogous construction for L-theory. A *Farrell-Jones Conjecture with Coefficients* would say that the map induced on these homology theories by the projection $\mathrm{pr}\colon E_{\mathcal{VCY}}(G) \to \mathrm{pt}$ is always an isomorphism. We do not know whether there are counterexamples to the Farrell-Jones Conjectures with Coefficients, compare Remark 5.3.

4.1.5 The Coarse Baum Connes Conjecture

We briefly explain the Coarse Baum-Connes Conjecture, a variant of the Baum-Connes Conjecture, which applies to metric spaces. Its importance lies in the fact that isomorphism results about the Coarse Baum-Connes Conjecture can be used to prove injectivity results about the classical assembly map for topological K-theory. Compare also Section 7.10.

Let X be a proper (closed balls are compact) metric space and H_X a separable Hilbert space with a faithful nondegenerate $*$-representation of $C_0(X)$, the algebra of complex valued continuous functions which vanish at infinity.

A bounded linear operator T has a support $\operatorname{supp} T \subset X \times X$, which is defined as the complement of the set of all pairs (x, x'), for which there exist functions ϕ and $\phi' \in C_0(X)$ such that $\phi(x) \neq 0$, $\phi'(x') \neq 0$ and $\phi' T \phi = 0$. The operator T is said to be a finite propagation operator if there exists a constant α such that $d(x, x') \leq \alpha$ for all pairs in the support of T. The operator is said to be *locally compact* if ϕT and $T\phi$ are compact for every $\phi \in C_0(X)$. An operator is called *pseudolocal* if $\phi T \psi$ is a compact operator for all pairs of continuous functions ϕ and ψ with compact and disjoint supports.

The Roe-algebra $C^*(X) = C(X, H_X)$ is the operator-norm closure of the $*$-algebra of all locally compact finite propagation operators on H_X. The algebra $D^*(X) = D^*(X, H_X)$ is the operator-norm closure of the pseudolocal finite propagation operators. One can show that the topological K-theory of the quotient algebra $D^*(X)/C^*(X)$ coincides up to an index shift with the analytically defined (non-equivariant) K-homology $K_*(X)$, compare Section 7.1. For a uniformly contractible proper metric space the coarse assembly map $K_n(X) \to K_n(C^*(X))$ is the boundary map in the long exact sequence associated to the short exact sequence of C^*-algebras

$$0 \to C^*(X) \to D^*(X) \to D^*(X)/C^*(X) \to 0.$$

For general metric spaces one first approximates the metric space by spaces with nice local behaviour, compare [263]. For simplicity we only explain the case, where X is a discrete metric space. Let $P_d(X)$ the Rips complex for a fixed distance d, i.e. the simplicial complex with vertex set X, where a simplex is spanned by every collection of points in which every two points are a distance less than d apart. Equip $P_d(X)$ with the spherical metric, compare [335].

A discrete metric space has *bounded geometry* if for each $r > 0$ there exists a $N(r)$ such that for all x the ball of radius r centered at $x \in X$ contains at most $N(r)$ elements.

Conjecture 4.6 (Coarse Baum-Connes Conjecture). Let X be a proper discrete metric space of bounded geometry. Then for $n = 0, 1$ the coarse assembly map

$$\operatorname{colim}_d K_n(P_d(X)) \to \operatorname{colim}_d K_n(C^*(P_d(X))) \cong K_n(C^*(X))$$

is an isomorphism.

The conjecture is false if one drops the bounded geometry hypothesis. A counterexample can be found in [336, Section 8]. Our interest in the conjecture stems from the following fact, compare [263, Chapter 8].

Proposition 4.7. *Suppose the finitely generated group G admits a classifying space BG of finite type. If G considered as a metric space via a word length metric satisfies the Coarse Baum-Connes Conjecture 4.6 then the classical assembly map $A\colon K_*(BG) \to K_*(C_r^*G)$ which appears in Conjecture 1.31 is injective.*

The Coarse Baum-Connes Conjecture for a discrete group G (considered as a metric space) can be interpreted as a case of the Baum-Connes Conjecture with Coefficients 4.3 for the group G with a certain specific choice of coefficients, compare [339].

Further information about the coarse Baum-Connes Conjecture can be found for instance in [151], [152], [154], [263], [334], [340], [335], [337], and [338].

4.1.6 The Baum-Connes Conjecture for Non-Discrete Groups

Throughout this subsection let T be a locally compact second countable topological Hausdorff group. There is a notion of a classifying space for proper T-actions $\underline{E}T$ (see [28, Section 1 and 2] [304, Section I.6], [207, Section 1]) and one can define its equivariant topological K-theory $K_n^T(\underline{E}T)$. The definition of a reduced C^*-algebra $C_r^*(T)$ and its topological K-theory $K_n(C_r^*(T))$ makes sense also for T. There is an assembly map defined in terms of equivariant index theory

$$A_{\mathcal{K}}\colon K_n^T(\underline{E}T) \to K_n(C_r^*(T)). \tag{24}$$

The Baum-Connes Conjecture for T says that this map is bijective for all $n \in \mathbb{Z}$ [28, Conjecture 3.15 on page 254].

Now consider the special case where T is a connected Lie group. Let $\mathcal{K}$ be the family of compact subgroups of T. There is a notion of a T-CW-complex and of a classifying space $E_{\mathcal{K}}(T)$ defined as in Subsection 2.1.1 and 2.1.3. The classifying space $E_{\mathcal{K}}(T)$ yields a model for $\underline{E}T$. Let $K \subset T$ be a maximal compact subgroup. It is unique up to conjugation. The space T/K is contractible and in fact a model for $\underline{E}T$ (see [1, Appendix, Theorem A.5], [2, Corollary 4.14], [207, Section 1]). One knows (see [28, Proposition 4.22], [170])

$$K_n^T(\underline{E}T) = K_n^T(T/K) = \begin{cases} R_{\mathbb{C}}(K) & n = \dim(T/K) \mod 2, \\ 0 & n = 1 + \dim(T/K) \mod 2, \end{cases}$$

where $R_{\mathbb{C}}(K)$ is the complex representation ring of K.

Next we consider the special case where T is a totally disconnected group. Let $\mathcal{KO}$ be the family of compact-open subgroups of T. A T-CW-complex and a classifying space $E_{\mathcal{KO}}(T)$ for T and $\mathcal{KO}$ are defined similar as in Subsection 2.1.1 and 2.1.3. Then $E_{\mathcal{KO}}(T)$ is a model for $\underline{E}T$ since any compact subgroup is contained in a compact-open subgroup, and the Baum-Connes Conjecture says that the assembly map yields for $n \in \mathbb{Z}$ an isomorphism

$$A_{\mathcal{KO}}\colon K_n^T(E_{\mathcal{KO}}(T)) \to K_n(C_r^*(T)). \tag{25}$$

For more information see [30].

4.2 Variants of the Farrell-Jones Conjecture

4.2.1 Pseudoisotopy Theory

An important variant of the Farrell-Jones Conjecture deals with the pseudoisotopy spectrum functor $\mathbf{P}$, which we already discussed briefly in Section 1.4.2. In fact it is this variant of the Farrell-Jones Conjecture (and its fibered version which will be explained in the next subsection) for which the strongest results are known at the time of writing.

In Proposition 6.8 we will explain that every functor $\mathbf{E}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$, which sends equivalences of groupoids to stable weak equivalences of spectra, yields a corresponding equivariant homology theory $H_n^G(-;\mathbf{E})$. Now whenever we have a functor $\mathbf{F}\colon \mathsf{SPACES} \to \mathsf{SPECTRA}$, we can precompose it with the functor "classifying space" which sends a groupoid $\mathcal{G}$ to its classifying space $B\mathcal{G}$. (Here $B\mathcal{G}$ is simply the realization of the nerve of $\mathcal{G}$ considered as a category.) In particular this applies to the pseudoisotopy functor $\mathbf{P}$. Thus we obtain a homology theory $H_n^G(-;\mathbf{P}\circ B)$ whose essential feature is that

$$H_n^G(G/H;\mathbf{P}\circ B) \cong \pi_n(\mathbf{P}(BH)),$$

i.e. evaluated at G/H one obtains the homotopy groups of the pseudoisotopy spectrum of the classifying space BH of the group H. As the reader may guess there is the following conjecture.

Conjecture 4.8 (Farrell-Jones Conjecture for Pseudoisotopies of Aspherical Spaces). For every group G and all $n \in \mathbb{Z}$ the assembly map

$$H_n^G(E_{\mathcal{VCY}}(G);\mathbf{P}\circ B) \to H_n^G(\mathrm{pt};\mathbf{P}\circ B) \cong \pi_n(\mathbf{P}(BG))$$

is an isomorphism. Similarly for $\mathbf{P}^{\mathrm{diff}}$, the pseudoisotopy functor which is defined using differentiable pseudoisotopies.

A formulation of a conjecture for spaces which are not necessarily aspherical will be given in the next subsection, see in particular Remark 4.13.

Remark 4.9 (Relating K-Theory and Pseudoisotopy Theory). We already outlined in Subsection 1.4.1 the relationship between K-theory and pseudoisotopies. The comparison in positive dimensions described there can be extended to all dimensions. Vogell constructs in [309] a version of A-theory using retractive spaces that are bounded over $\mathbb{R}^k$ (compare Subsection 1.2.3 and 1.4.2). This leads to a functor $\mathbf{A}^{-\infty}$ from spaces to non-connective spectra. Compare also [56], [310], [311] and [326]. We define $\mathbf{Wh}_{PL}^{-\infty}$ via the fibration sequence

$$X_+ \wedge \mathbf{A}^{-\infty}(\mathrm{pt}) \to \mathbf{A}^{-\infty}(X) \to \mathbf{Wh}_{PL}^{-\infty}(X),$$

where the first map is the assembly map. The natural equivalence

$$\Omega^2\mathbf{Wh}_{PL}^{-\infty}(X) \simeq \mathbf{P}(X)$$

seems to be hard to trace down in the literature but should be true. We will assume it in the following discussion.

Precompose the functors above with the classifying space functor B to obtain functors from groupoids to spectra. The pseudoisotopy assembly map which appears in Conjecture 4.8 is an isomorphism if and only if the A-theory assembly map

$$H_{n+2}^G(E_{\mathcal{VCY}}(G); \mathbf{A}^{-\infty} \circ B) \to H_{n+2}^G(\mathrm{pt}; \mathbf{A}^{-\infty} \circ B) \cong \pi_{n+2}(\mathbf{A}^{-\infty}(BG))$$

is an isomorphism. This uses a 5-lemma argument and the fact that for a fixed spectrum $\mathbf{E}$ the assembly map

$$H_n^G(E_{\mathcal{F}}(G); B\mathcal{G}^G(-)_+ \wedge \mathbf{E}) \to H_n^G(\mathrm{pt}; B\mathcal{G}^G(-)_+ \wedge \mathbf{E})$$

is always bijective. There is a linearization map $\mathbf{A}^{-\infty}(X) \to \mathbf{K}(\mathbb{Z}\Pi(X)_\oplus)$ (see the next subsection for the notation) which is always 2-connected and a rational equivalence if X is aspherical (recall that $\mathbf{K}$ denotes the non-connective K-theory spectrum). For finer statements about the linearization map, compare also [230].

The above discussion yields in particular the following, compare [111, 1.6.7 on page 261].

Proposition 4.10. *The rational version of the K-theoretic Farrell-Jones Conjecture 2.2 is equivalent to the rational version of the Farrell-Jones Conjecture for Pseudoisotopies of Aspherical Spaces 4.8. If the assembly map in the conjecture for pseudoisotopies is (integrally) an isomorphism for $n \leq -1$, then so is the assembly map in the K-theoretic Farrell-Jones Conjecture for $n \leq 1$.*

4.2.2 Fibered Versions

Next we present the more general fibered versions of the Farrell-Jones Conjectures. These fibered versions have better inheritance properties, compare Section 5.4.

In the previous section we considered functors $\mathbf{F} \colon \mathsf{SPACES} \to \mathsf{SPECTRA}$, like $\mathbf{P}$, $\mathbf{P}^{\mathrm{diff}}$ and $\mathbf{A}^{-\infty}$, and the associated equivariant homology theories $H_n^G(-; \mathbf{F} \circ B)$ (compare Proposition 6.8). Here B denotes the classifying space functor, which sends a groupoid $\mathcal{G}$ to its classifying space $B\mathcal{G}$. In fact all equivariant homology theories we considered so far can be obtained in this fashion for special choices of $\mathbf{F}$. Namely, let $\mathbf{F}$ be one of the functors

$$\mathbf{K}(R\Pi(-)_\oplus), \quad \mathbf{L}^{\langle -\infty \rangle}(R\Pi(-)_\oplus) \quad \text{or} \quad \mathbf{K}^{\mathrm{top}}(C_r^*\Pi(-)_\oplus),$$

where $\Pi(X)$ denotes the fundamental groupoid of a space, $R\mathcal{G}_\oplus$ respectively $C_r^*\mathcal{G}_\oplus$ is the R-linear respectively the C^*-category associated to a groupoid $\mathcal{G}$ and $\mathbf{K}$, $\mathbf{L}^{\langle -\infty \rangle}$ and $\mathbf{K}^{\mathrm{top}}$ are suitable functors which send additive respectively C^*-categories to spectra, compare the proof of Theorem 6.9. There is a natural equivalence $\mathcal{G} \to \Pi B\mathcal{G}$. Hence, if we precompose the functors above with the classifying space functor B we obtain functors which are equivalent to the functors we have so far been calling

$$\mathbf{K}_R, \quad \mathbf{L}_R^{\langle -\infty \rangle} \quad \text{and} \quad \mathbf{K}^{\text{top}},$$

compare Theorem 6.9. Note that in contrast to these three cases the pseudoisotopy functor $\mathbf{P}$ depends on more than just the fundamental groupoid. However Conjecture 4.8 above only deals with aspherical spaces.

Given a G-CW-complex Z and a functor $\mathbf{F}$ from spaces to spectra we obtain a functor $X \mapsto \mathbf{F}(Z \times_G X)$ which digests G-CW-complexes. In particular we can restrict it to the orbit category to obtain a functor

$$\mathbf{F}(Z \times_G -) \colon \mathsf{Or}(G) \to \mathsf{SPECTRA}.$$

According to Proposition 6.7 we obtain a corresponding G-homology theory

$$H_n^G(-; \mathbf{F}(Z \times_G -))$$

and associated assembly maps. Note that restricted to the orbit category the functor $EG \times_G -$ is equivalent to the classifying space functor B and so $H_n^G(-; \mathbf{F} \circ B)$ can be considered as a special case of this construction.

Conjecture 4.11 (Fibered Farrell-Jones Conjectures). Let R be a ring (with involution). Let $\mathbf{F} \colon \mathsf{SPACES} \to \mathsf{SPECTRA}$ be one of the functors

$$\mathbf{K}(R\Pi(-)_\oplus), \quad \mathbf{L}^{\langle -\infty \rangle}(R\Pi(-)_\oplus), \quad \mathbf{P}(-), \quad \mathbf{P}^{\text{diff}}(-) \quad \text{or} \quad \mathbf{A}^{-\infty}(-).$$

Then for every free G-CW-complex Z and all $n \in \mathbb{Z}$ the associated assembly map

$$H_n^G(E_{\mathcal{VCY}}(G); \mathbf{F}(Z \times_G -)) \to H_n^G(\text{pt}; \mathbf{F}(Z \times_G -)) \cong \pi_n(\mathbf{F}(Z/G))$$

is an isomorphism.

Remark 4.12 (A Fibered Baum-Connes Conjecture). With the family $\mathcal{FIN}$ instead of $\mathcal{VCY}$ and the functor $\mathbf{F} = \mathbf{K}^{\text{top}}(C_r^* \Pi(-)_\oplus)$ one obtains a *Fibered Baum-Connes Conjecture.*

Remark 4.13 (The Special Case $Z = \widetilde{X}$). Suppose $Z = \widetilde{X}$ is the universal covering of a space X equipped with the action of its fundamental group $G = \pi_1(X)$. Then in the algebraic K- and L-theory case the conjecture above specializes to the "ordinary" Farrell-Jones Conjecture 2.2. In the pseudoisotopy and A-theory case one obtains a formulation of an (unfibered) conjecture about $\pi_n(\mathbf{P}(X))$ or $\pi_n(\mathbf{A}^{-\infty}(X))$ for spaces X which are not necessarily aspherical.

Remark 4.14 (Relation to the Original Formulation). In [111] Farrell and Jones formulate a fibered version of their conjectures for every (Serre) fibration $Y \to X$ over a connected CW-complex X. In our set-up this corresponds to choosing Z to be the total space of the fibration obtained from $Y \to X$ by pulling back along the universal covering projection $\widetilde{X} \to X$. This space is a free G-space for $G = \pi_1(X)$. Note that an arbitrary free G-CW-complex Z can always be obtained in this fashion from a map $Z/G \to BG$, compare [111, Corollary 2.2.1 on page 264].

Remark 4.15 (Relating K-Theory and Pseudoisotopy Theory in the Fibered Case). The linearization map $\pi_n(\mathbf{A}^{-\infty}(X)) \to K_n(\mathbb{Z}\Pi(X))$ is always 2-connected, but for spaces which are not aspherical it need not be a rational equivalence. Hence the comparison results discussed in Remark 4.9 apply for the fibered versions only in dimensions $n \leq 1$.

4.2.3 The Isomorphism Conjecture for *NK*-groups

In Remark 1.15 we defined the groups $NK_n(R)$ for a ring R. They are the simplest kind of Nil-groups responsible for the infinite cyclic group. Since the functor $\mathbf{K}_R$ is natural with respect to ring homomorphism we can define $\mathbf{NK}_R$ as the (objectwise) homotopy cofiber of $\mathbf{K}_R \to \mathbf{K}_{R[t]}$. There is an associated assembly map.

Conjecture 4.16 (Isomorphism Conjecture for NK-groups). The assembly map

$$H_n^G(E_{\mathcal{VCY}}(G); \mathbf{NK}_R) \to H_n^G(\mathrm{pt}; \mathbf{NK}_R) \cong NK_n(RG)$$

is always an isomorphism.

There is a weak equivalence $\mathbf{K}_{R[t]} \simeq \mathbf{K}_R \vee \mathbf{NK}_R$ of functors from $\mathsf{GROUPOIDS}$ to $\mathsf{SPECTRA}$. This implies for a fixed family $\mathcal{F}$ of subgroups of G and $n \in \mathbb{Z}$ that whenever two of the three assembly maps

$$\begin{aligned} A_{\mathcal{F}} \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{R[t]}) &\to K_n(R[t][G]), \\ A_{\mathcal{F}} \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_R) &\to K_n(R[G]), \\ A_{\mathcal{F}} \colon H_n^G(E_{\mathcal{F}}(G); \mathbf{NK}_R) &\to NK_n(RG) \end{aligned}$$

are bijective, then so is the third (compare [19, Section 7]). Similarly one can define a functor $\mathbf{E}_R$ from the category $\mathsf{GROUPOIDS}$ to $\mathsf{SPECTRA}$ and weak equivalences

$$\mathbf{K}_{R[t,t^{-1}]} \to \mathbf{E}_R \leftarrow \mathbf{K}_R \vee \Sigma\mathbf{K}_R \vee \mathbf{NK}_R \vee \mathbf{NK}_R,$$

which on homotopy groups corresponds to the Bass-Heller-Swan decomposition (see Remark 1.15). One obtains a two-out-of-three statement as above with the $\mathbf{K}_{R[t]}$-assembly map replaced by the $\mathbf{K}_{R[t,t^{-1}]}$-assembly map.

4.2.4 Algebraic K-Theory of the Hecke Algebra

In Subsection 4.1.6 we mentioned the classifying space $E_{\mathcal{KO}}(G)$ for the family of compact-open subgroups and the Baum-Connes Conjecture for a totally disconnected group T. There is an analogous conjecture dealing with the algebraic K-theory of the Hecke algebra.

Let $\mathcal{H}(T)$ denote the *Hecke algebra* of T which consists of locally constant functions $G \to \mathbb{C}$ with compact support and inherits its multiplicative structure from the convolution product. The Hecke algebra $\mathcal{H}(T)$ plays the same role for T as the complex group ring $\mathbb{C}G$ for a discrete group G and reduces to this notion if T happens to be discrete. There is a T-homology theory $\mathcal{H}_*^T$ with the property that for any open and closed subgroup $H \subseteq T$ and all $n \in \mathbb{Z}$ we have $\mathcal{H}_n^T(T/H) = K_n(\mathcal{H}(H))$, where $K_n(\mathcal{H}(H))$ is the algebraic K-group of the Hecke algebra $\mathcal{H}(H)$.

Conjecture 4.17 (Isomorphism Conjecture for the Hecke-Algebra). For a totally disconnected group T the assembly map

$$A_{\mathcal{KO}} \colon \mathcal{H}_n^T(E_{\mathcal{KO}}(T)) \to \mathcal{H}^T(\mathrm{pt}) = K_n(\mathcal{H}(T)) \tag{26}$$

induced by the projection $\mathrm{pr} \colon E_{\mathcal{KO}}(T) \to \mathrm{pt}$ is an isomorphism for all $n \in \mathbb{Z}$.

In the case $n = 0$ this reduces to the statement that

$$\operatorname{colim}_{T/H \in \mathsf{Or}_{\mathcal{KO}}(T)} K_0(\mathcal{H}(H)) \to K_0(\mathcal{H}(T)). \tag{27}$$

is an isomorphism. For $n \leq -1$ one obtains the statement that $K_n(\mathcal{H}(G)) = 0$. The group $K_0(\mathcal{H}(T))$ has an interpretation in terms of the smooth representations of T. The G-homology theory can be constructed using an appropriate functor $\mathbf{E} \colon \mathsf{Or}_{\mathcal{KO}}(T) \to \mathsf{SPECTRA}$ and the recipe explained in Section 6.2. The desired functor $\mathbf{E}$ is given in [276].

5 Status of the Conjectures

In this section, we give the status, at the time of writing, of some of the conjectures mentioned earlier. We begin with the Baum-Connes Conjecture.

5.1 Status of the Baum-Connes-Conjecture

5.1.1 The Baum-Connes Conjecture with Coefficients

We begin with the Baum-Connes Conjecture with Coefficients 4.3. It has better inheritance properties than the Baum-Connes Conjecture 2.3 itself and contains it as a special case.

Theorem 5.1. (Baum-Connes Conjecture with Coefficients and a-T-menable Groups). *The discrete group G satisfies the Baum-Connes Conjecture with Coefficients 4.3 and is K-amenable provided that G is a-T-menable.*

This theorem is proved in Higson-Kasparov [149, Theorem 1.1], where more generally second countable locally compact topological groups are treated (see also [164]).

A group G is *a-T-menable*, or, equivalently, has the *Haagerup property* if G admits a metrically proper isometric action on some affine Hilbert space. *Metrically proper* means that for any bounded subset B the set $\{g \in G \mid gB \cap B \neq \emptyset\}$ is finite. An extensive treatment of such groups is presented in [66]. Any a-T-menable group is countable. The class of a-T-menable groups is closed under taking subgroups, under extensions with finite quotients and under finite products. It is not closed under semi-direct products. Examples of a-T-menable groups are countable amenable groups, countable free groups, discrete subgroups of $SO(n,1)$ and $SU(n,1)$, Coxeter groups, countable groups acting properly on trees, products of trees, or simply connected CAT(0) cubical complexes. A group G has Kazhdan's *property (T)* if, whenever it acts isometrically on some affine Hilbert space, it has a fixed point. An infinite a-T-menable group does not have property (T). Since $SL(n,\mathbb{Z})$ for $n \geq 3$ has property (T), it cannot be a-T-menable.

Using the Higson-Kasparov result Theorem 5.1 and known inheritance properties of the Baum-Connes Conjecture with Coefficients (compare Section 5.4 and [233],[234]) Mislin describes an even larger class of groups for which the conjecture is known [225, Theorem 5.23].

Theorem 5.2 (The Baum-Connes Conjecture with Coefficients and the Class of Groups LH$\mathcal{ETH}$). *The discrete group G satisfies the Baum-Connes Conjecture with Coefficients 4.3 provided that G belongs to the class* **LH**$\mathcal{ETH}$.

The class **LH**$\mathcal{ETH}$ is defined as follows. Let **H**$\mathcal{TH}$ be the smallest class of groups which contains all a-T-menable groups and contains a group G if there is a 1-dimensional contractible G-CW-complex whose stabilizers belong already to **H**$\mathcal{TH}$. Let **H**$\mathcal{ETH}$ be the smallest class of groups containing **H**$\mathcal{TH}$ and containing a group G if either G is countable and admits a surjective map $p\colon G \to Q$ with Q and $p^{-1}(F)$ in **H**$\mathcal{ETH}$ for every finite subgroup $F \subseteq Q$ or if G admits a 1-dimensional contractible G-CW-complex whose stabilizers belong already to **H**$\mathcal{ETH}$. Let **LH**$\mathcal{ETH}$ be the class of groups G whose finitely generated subgroups belong to **H**$\mathcal{ETH}$.

The class **LH**$\mathcal{ETH}$ is closed under passing to subgroups, under extensions with torsion free quotients and under finite products. It contains in particular one-relator groups and Haken 3-manifold groups (and hence all knot groups). All these facts of the class **LH**$\mathcal{ETH}$ and more information can be found in [225].

Vincent Lafforgue has an unpublished proof of the Baum-Connes Conjecture with Coefficients 4.3 for word-hyperbolic groups.

Remark 5.3. There are counterexamples to the Baum-Connes Conjecture with (commutative) Coefficients 4.3 as soon as the existence of finitely generated groups containing arbitrary large expanders in their Cayley graph is shown [150, Section 7]. The existence of such groups has been claimed by Gromov [138], [139]. Details of the construction are described by Ghys in [134]. At the time of writing no counterexample to the Baum-Connes Conjecture 2.3 (without coefficients) is known to the authors.

5.1.2 The Baum-Connes Conjecture

Next we deal with the Baum-Connes Conjecture 2.3 itself. Recall that all groups which satisfy the Baum-Connes Conjecture with Coefficients 4.3 do in particular satisfy the Baum-Connes Conjecture 2.3.

Theorem 5.4 (Status of the Baum-Connes Conjecture). *A group G satisfies the Baum-Connes Conjecture 2.3 if it satisfies one of the following conditions.*

(i) It is a discrete subgroup of a connected Lie groups L, whose Levi-Malcev decomposition $L = RS$ into the radical R and semisimple part S is such that S is locally of the form

$$S = K \times SO(n_1, 1) \times \ldots \times SO(n_k, 1) \times SU(m_1, 1) \times \ldots \times SU(m_l, 1)$$

for a compact group K.

(ii) The group G has property (RD) and admits a proper isometric action on a strongly bolic weakly geodesic uniformly locally finite metric space.

(iii) G is a subgroup of a word hyperbolic group.

(iv) G is a discrete subgroup of $Sp(n, 1)$.

Proof. The proof under condition (i) is due to Julg-Kasparov [166]. The proof under condition (ii) is due to Lafforgue [183] (see also [288]). Word hyperbolic groups have property (RD) [84]. Any subgroup of a word hyperbolic group satisfies the conditions appearing in the result of Lafforgue and hence satisfies the Baum-Connes Conjecture 2.3 [222, Theorem 20]. The proof under condition (iv) is due to Julg [165]. □

Lafforgue's result about groups satisfying condition (ii) yielded the first examples of infinite groups which have Kazhdan's property (T) and satisfy the Baum-Connes Conjecture 2.3. Here are some explanations about condition (ii).

A *length function* on G is a function $L\colon G \to \mathbb{R}_{\geq 0}$ such that $L(1) = 0$, $L(g) = L(g^{-1})$ for $g \in G$ and $L(g_1g_2) \leq L(g_1) + L(g_2)$ for $g_1, g_2 \in G$ holds. The word length metric L_S associated to a finite set S of generators is an example. A length function L on G has *property (RD)* ("rapid decay") if there exist $C, s > 0$ such that for any $u = \sum_{g \in G} \lambda_g \cdot g \in \mathbb{C}G$ we have

$$||\rho_G(u)||_\infty \ \leq \ C \cdot \left(\sum_{g \in G} |\lambda_g|^2 \cdot (1 + L(g))^{2s} \right)^{1/2},$$

where $||\rho_G(u)||_\infty$ is the operator norm of the bounded G-equivariant operator $l^2(G) \to l^2(G)$ coming from right multiplication with u. A group G has *property (RD)* if there is a length function which has property (RD). More information about property (RD) can be found for instance in [63], [184] and [307, Chapter 8]. Bolicity generalizes Gromov's notion of hyperbolicity for metric spaces. We refer to [169] for a precise definition.

Remark 5.5. We do not know whether all groups appearing in Theorem 5.4 satisfy also the Baum-Connes Conjecture with Coefficients 4.3.

Remark 5.6. It is not known at the time of writing whether the Baum-Connes Conjecture is true for $SL(n, \mathbb{Z})$ for $n \geq 3$.

Remark 5.7 (The Status for Topological Groups). We only dealt with the Baum-Connes Conjecture for discrete groups. We already mentioned that Higson-Kasparov [149] treat second countable locally compact topological groups. The Baum-Connes Conjecture for second countable almost connected groups G has been proven by Chabert-Echterhoff-Nest [60] based on the work of Higson-Kasparov [149] and Lafforgue [186]. The Baum-Connes Conjecture with Coefficients 4.3 has been proved for the connected Lie groups L appearing in Theorem 5.4 (i) by [166] and for $Sp(n, 1)$ by Julg [165].

5.1.3 The Injectivity Part of the Baum-Connes Conjecture

In this subsection we deal with injectivity results about the assembly map appearing in the Baum-Connes Conjecture 2.3. Recall that rational injectivity already implies the Novikov Conjecture 1.52 (see Proposition 3.16) and the Homological Stable Gromov-Lawson-Rosenberg Conjecture 3.25 (see Proposition 3.26 and 2.20).

Theorem 5.8 (Rational Injectivity of the Baum-Connes Assembly Map). *The assembly map appearing in the Baum-Connes Conjecture 2.3 is rationally injective if G belongs to one of the classes of groups below.*

(i) Groups acting properly isometrically on complete manifolds with non-positive sectional curvature.
(ii) Discrete subgroups of Lie groups with finitely many path components.
(iii) Discrete subgroups of p-adic groups.

Proof. The proof of assertions (i) and (ii) is due to Kasparov [171], the one of assertion (iii) to Kasparov-Skandalis [172]. □

A metric space (X, d) admits a *uniform embedding into Hilbert space* if there exist a separable Hilbert space H, a map $f\colon X \to H$ and non-decreasing functions ρ_1 and ρ_2 from $[0, \infty) \to \mathbb{R}$ such that $\rho_1(d(x, y)) \leq ||f(x) - f(y)|| \leq \rho_2(d(x, y))$ for $x, y \in X$ and $\lim_{r\to\infty} \rho_i(r) = \infty$ for $i = 1, 2$. A metric is *proper* if for each $r > 0$ and $x \in X$ the closed ball of radius r centered at x is compact. The question whether a discrete group G equipped with a proper left G-invariant metric d admits a uniform embedding into Hilbert space is independent of the choice of d, since the induced coarse structure does not depend on d [289, page 808]. For more information about groups admitting a uniform embedding into Hilbert space we refer to [87], [140].

The class of finitely generated groups, which embed uniformly into Hilbert space, contains a subclass A, which contains all word hyperbolic groups, finitely generated discrete subgroups of connected Lie groups and finitely generated amenable groups and is closed under semi-direct products [338, Definition 2.1, Theorem 2.2 and Proposition 2.6]. Gromov [138], [139] has announced examples of finitely generated groups which do not admit a uniform embedding into Hilbert space. Details of the construction are described in Ghys [134].

The next theorem is proved by Skandalis-Tu-Yu [289, Theorem 6.1] using ideas of Higson [148].

Theorem 5.9 (Injectivity of the Baum-Connes Assembly Map). *Let G be a countable group. Suppose that G admits a G-invariant metric for which G admits a uniform embedding into Hilbert space. Then the assembly map appearing in the Baum-Connes Conjecture with Coefficients 4.3 is injective.*

We now discuss conditions which can be used to verify the assumption in Theorem 5.9.

Remark 5.10 (Linear Groups). A group G is called *linear* if it is a subgroup of $GL_n(F)$ for some n and some field F. Guentner-Higson-Weinberger [140] show that every countable linear group admits a uniform embedding into Hilbert space and hence Theorem 5.9 applies.

Remark 5.11 (Groups Acting Amenably on a Compact Space). A continuous action of a discrete group G on a compact space X is called *amenable* if there exists a sequence

$$p_n \colon X \to M^1(G) = \{f \colon G \to [0,1] \mid \sum_{g \in G} f(g) = 1\}$$

of weak-$*$-continuous maps such that for each $g \in G$ one has

$$\lim_{n \to \infty} \sup_{x \in X} ||g * (p_n(x) - p_n(g \cdot x))||_1 \;=\; 0.$$

Note that a group G is amenable if and only if its action on the one-point-space is amenable. More information about this notion can be found for instance in [5], [6].

Higson-Roe [153, Theorem 1.1 and Proposition 2.3] show that a finitely generated group equipped with its word length metric admits an amenable action on a compact metric space, if and only if it belongs to the class A defined in [338, Definition 2.1], and hence admits a uniform embedding into Hilbert space. Hence Theorem 5.9 implies the result of Higson [148, Theorem 1.1] that the assembly map appearing in the Baum-Connes Conjecture with Coefficients 4.3 is injective if G admits an amenable action on some compact space.

Word hyperbolic groups and the class of groups mentioned in Theorem 5.8 (ii) fall under the class of groups admitting an amenable action on some compact space [153, Section 4].

Remark 5.12. Higson [148, Theorem 5.2] shows that the assembly map appearing in the Baum-Connes Conjecture with Coefficients 4.3 is injective if $\underline{E}G$ admits an appropriate compactification. This is a C^*-version of the result for K-and L-theory due to Carlsson-Pedersen [55], compare Theorem 5.27.

Remark 5.13. We do not know whether the groups appearing in Theorem 5.8 and 5.9 satisfy the Baum-Connes Conjecture 2.3.

Next we discuss injectivity results about the classical assembly map for topological K-theory.

The *asymptotic dimension* of a proper metric space X is the infimum over all integers n such that for any $R > 0$ there exists a cover $\mathcal{U}$ of X with the property that the diameter of the members of $\mathcal{U}$ is uniformly bounded and every ball of radius R intersects at most $(n+1)$ elements of $\mathcal{U}$ (see [137, page 28]).

The next result is due to Yu [337].

Theorem 5.14 (The C^*-Theoretic Novikov Conjecture and Groups of Finite Asymptotic Dimension). *Let G be a group which possesses a finite model for BG and has finite asymptotic dimension. Then the assembly map in the Baum-Connes Conjecture 1.31*

$$K_n(BG) \to K_n(C_r^*(G))$$

is injective for all $n \in \mathbb{Z}$.

5.1.4 The Coarse Baum-Connes Conjecture

The coarse Baum-Connes Conjecture was explained in Section 4.1.5. Recall the descent principle (Proposition 4.7): if a countable group can be equipped with a G-invariant metric such that the resulting metric space satisfies the Coarse Baum-Connes Conjecture, then the classical assembly map for topological K-theory is injective.

Recall that a discrete metric space has bounded geometry if for each $r > 0$ there exists a $N(r)$ such that for all x the ball of radius $N(r)$ centered at $x \in X$ contains at most $N(r)$ elements.

The next result is due to Yu [338, Theorem 2.2 and Proposition 2.6].

Theorem 5.15 (Status of the Coarse Baum-Connes Conjecture). *The Coarse Baum-Connes Conjecture 4.6 is true for a discrete metric space X of* *bounded geometry if X admits a uniform embedding into Hilbert space. In particular a countable group G satisfies the Coarse Baum-Connes Conjecture 4.6 if G equipped with a proper left G-invariant metric admits a uniform embedding into Hilbert space.*

Also Yu's Theorem 5.14 is proven via a corresponding result about the Coarse Baum-Connes Conjecture.

5.2 Status of the Farrell-Jones Conjecture

Next we deal with the Farrell-Jones Conjecture.

5.2.1 The Fibered Farrell-Jones Conjecture

The Fibered Farrell-Jones Conjecture 4.11 was discussed in Subsection 4.2.2. Recall that it has better inheritance properties (compare Section 5.4) and contains the ordinary Farrell-Jones Conjecture 2.2 as a special case.

Theorem 5.16 (Status of the Fibered Farrell-Jones Conjecture).

(i) Let G be a discrete group which satisfies one of the following conditions.
   - *(a) There is a Lie group L with finitely many path components and G is a cocompact discrete subgroup of L.*
   - *(b) The group G is virtually torsionfree and acts properly discontinuously, cocompactly and via isometries on a simply connected complete nonpositively curved Riemannian manifold.*

   Then
   - *(1) The version of the Fibered Farrell-Jones Conjecture 4.11 for the topological and the differentiable pseudoisotopy functor is true for G.*
   - *(2) The version of the Fibered Farrell-Jones Conjecture 4.11 for K-theory and $R = \mathbb{Z}$ is true for G in the range $n \leq 1$, i.e. the assembly map is bijective for $n \leq 1$.*

Moreover we have the following statements.

(ii) The version of the Fibered Farrell-Jones Conjecture 4.11 for K-theory and $R = \mathbb{Z}$ is true in the range $n \leq 1$ for braid groups.

(iii) The L-theoretic version of the Fibered Farrell-Jones Conjecture 4.11 with $R = \mathbb{Z}$ holds after inverting 2 for elementary amenable groups.

Proof. (i) For assertion (1) see [111, Theorem 2.1 on page 263], [111, Proposition 2.3] and [119, Theorem A]. Assertion (2) follows from (1) by Remark 4.15.

(ii) See [119].

(iii) is proven in [117, Theorem 5.2]. For crystallographic groups see also [333]. □

A surjectivity result about the Fibered Farrell-Jones Conjecture for Pseudoisotopies appears as the last statement in Theorem 5.20.

The rational comparison result between the K-theory and the pseudoisotopy version (see Proposition 4.10) does not work in the fibered case, compare Remark 4.15. However, in order to exploit the good inheritance properties one can first use the pseudoisotopy functor in the fibered set-up, then specialize to the unfibered situation and finally do the rational comparison to K-theory.

Remark 5.17. The version of the Fibered Farrell-Jones Conjecture 4.11 for L-theory and $R = \mathbb{Z}$ seems to be true if G satisfies the condition (a) appearing in Theorem 5.16. Farrell and Jones [111, Remark 2.1.3 on page 263] say that they can also prove this version without giving the details.

Remark 5.18. Let G be a virtually poly-cyclic group. Then it contains a maximal normal finite subgroup N such that the quotient G/N is a discrete cocompact subgroup of a Lie group with finitely many path components [331, Theorem 3, Remark 4 on page 200]. Hence by Subsection 5.4.3 and Theorem 5.16 the version of the Fibered Farrell-Jones Conjecture 4.11 for the topological and the differentiable pseudoisotopy functor, and for K-theory and $R = \mathbb{Z}$ in the range $n \leq 1$, is true for G. Earlier results of this type were treated for example in [100], [105].

5.2.2 The Farrell-Jones Conjecture

Here is a sample of some results one can deduce from Theorem 5.16.

Theorem 5.19 (The Farrell-Jones Conjecture and Subgroups of Lie groups). *Suppose H is a subgroup of G, where G is a discrete cocompact subgroup of a Lie group L with finitely many path components. Then*

(i) The version of the Farrell-Jones Conjecture for K-theory and $R = \mathbb{Z}$ is true for H rationally, i.e. the assembly map appearing in Conjecture 2.2 is an isomorphism after applying $- \otimes_{\mathbb{Z}} \mathbb{Q}$.
(ii) The version of the Farrell-Jones Conjecture for K-theory and $R = \mathbb{Z}$ is true for H in the range $n \leq 1$, i.e. the assembly map appearing in Conjecture 2.2 is an isomorphism for $n \leq 1$.

Proof. The results follow from Theorem 5.16, since the Fibered Farrell-Jones Conjecture 4.11 passes to subgroups [111, Theorem A.8 on page 289] (compare Section 5.4.2) and implies the Farrell-Jones Conjecture 2.2. ☐

We now discuss results for torsion free groups. Recall that for $R = \mathbb{Z}$ the K-theoretic Farrell-Jones Conjecture in dimensions ≤ 1 together with the L-theoretic version implies already the Borel Conjecture 1.27 in dimension ≥ 5 (see Theorem 1.28).

A complete Riemannian manifold M is called *A-regular* if there exists a sequence of positive real numbers $A_0, A_1, A_2, \ldots$ such that $||\nabla^n K|| \leq A_n$, where $||\nabla^n K||$ is the supremum-norm of the n-th covariant derivative of the curvature tensor K. Every locally symmetric space is A-regular since ∇K is identically zero. Obviously every closed Riemannian manifold is A-regular.

Theorem 5.20 (Status of the Farrell-Jones Conjecture for Torsion-free Groups). *Consider the following conditions for the group G.*

(i) $G = \pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature which is A-regular.
(ii) $G = \pi_1(M)$ for a closed Riemannian manifold M with non-positive sectional curvature.
(iii) $G = \pi_1(M)$ for a complete Riemannian manifold with negatively pinched sectional curvature.
(iv) G is a torsion free discrete subgroup of $GL(n, \mathbb{R})$.
(v) G is a torsion free solvable discrete subgroup of $GL(n, \mathbb{C})$.
(vi) $G = \pi_1(X)$ for a non-positively curved finite simplicial complex X.
(vii) G is a strongly poly-free group in the sense of Aravinda-Farrell-Roushon [10, Definition 1.1]. The pure braid group satisfies this hypothesis.

Then

(1) Suppose that G satisfies one of the conditions (i) to (vii). Then the K-theoretic Farrell-Jones Conjecture is true for $R = \mathbb{Z}$ in dimensions $n \leq 1$. In particular Conjecture 1.3 holds for G.
(2) Suppose that G satisfies one of the conditions (i), (ii), (iii) or (iv). Then G satisfies the Farrell-Jones Conjecture for Torsion Free Groups and L-Theory 1.19 for $R = \mathbb{Z}$.
(3) Suppose that G satisfies (ii). Then the Farrell-Jones Conjecture for Pseudoisotopies of Aspherical Spaces 4.8 holds for G.
(4) Suppose that G satisfies one of the conditions (i), (iii) or (iv). Then the assembly map appearing in the version of the Fibered Farrell-Jones Conjecture for Pseudoisotopies 4.11 is surjective, provided that the G-space Z appearing in Conjecture 4.11 is connected.

Proof. Note that (ii) is a special case of (i) because every closed Riemannian manifold is A-regular. If M is a pinched negatively curved complete Riemannian manifold, then there is another Riemannian metric for which M is negatively curved complete and A-regular. This fact is mentioned in [115, page 216] and attributed there to Abresch [3] and Shi [285]. Hence also (iii) can be considered as a special case of (i). The manifold $M = G\backslash GL(n, \mathbb{R})/O(n)$ is a non-positively curved complete locally symmetric space and hence in particular A-regular. So (iv) is a special case of (i).

Assertion (1) under the assumption (i) is proved by Farrell-Jones in [115, Proposition 0.10 and Lemma 0.12]. The earlier work [110] treated the case (ii). Under assumption (v) assertion (1) is proven by Farrell-Linnell [117, Theorem 1.1]. The result under assumption (vi) is proved by Hu [156], under assumption (vii) it is proved by Aravinda-Farrell-Roushon [10, Theorem 1.3].

Assertion (2) under assumption (i) is proven by Farrell-Jones in [115]. The case (ii) was treated earlier in [112].

Assertion (3) is proven by Farrell-Jones in [110] and assertion (4) by Jones in [161]. □

Remark 5.21. As soon as certain collapsing results (compare [114], [116]) are extended to orbifolds, the results under (4) above would also apply to groups with torsion and in particular to $SL_n(\mathbb{Z})$ for arbitrary n.

5.2.3 The Farrell-Jones Conjecture for Arbitrary Coefficients

The following result due to Bartels-Reich [21] deals with algebraic K-theory for arbitrary coefficient rings R. It extends Bartels-Farrell-Jones-Reich [19].

Theorem 5.22. *Suppose that G is the fundamental group of a closed Riemannian manifold with negative sectional curvature. Then the K-theoretic part of the Farrell-Jones Conjecture 2.2 is true for any ring R, i.e. the assembly map*

$$A_{\mathcal{VCY}} \colon H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) \ \to \ K_n(RG)$$

is an isomorphism for all $n \in \mathbf{Z}$.

Note that the assumption implies that G is torsion free and hence the family $\mathcal{VCY}$ reduces to the family $\mathcal{CYC}$ of cyclic subgroups. Recall that for a regular ring R the theorem above implies that the classical assembly

$$A \colon H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$

is an isomorphism, compare Proposition 2.10 (i).

5.2.4 Injectivity Part of the Farrell-Jones Conjecture

The next result about the classical K-theoretic assembly map is due to Bökstedt-Hsiang-Madsen [38].

Theorem 5.23 (Rational Injectivity of the Classical K-Theoretic Assembly Map). *Let G be a group such that the integral homology $H_j(BG;\mathbb{Z})$ is finitely generated for each $j \in \mathbb{Z}$. Then the rationalized assembly map*

$$A \colon H_n(BG; \mathbf{K}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n^G(E_{\{1\}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective for all $n \in \mathbf{Z}$.

Because of the homological Chern character (see Remark 1.12) we obtain for the groups treated in Theorem 5.23 an injection

$$\bigoplus_{s+t=n} H_s(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} (K_t(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}) \to K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{28}$$

Next we describe a generalization of Theorem 5.23 above from the trivial family $\{1\}$ to the family $\mathcal{FIN}$ of finite subgroups due to Lück-Reich-Rognes-Varisco [208]. Let $\mathbf{K}_{\mathbb{Z}}^{\mathrm{con}}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ be the connective version of the functor $\mathbf{K}_{\mathbb{Z}}$ of (37). In particular $H_n(G/H;\mathbf{K}_{\mathbb{Z}}^{\mathrm{con}})$ is isomorphic to $K_n(\mathbb{Z}H)$ for $n \geq 0$ and vanishes in negative dimensions. For a prime p we denote by $\mathbb{Z}_p$ the p-adic integers. Let $K_n(R;\mathbb{Z}_p)$ denote the homotopy groups $\pi_n(\mathbf{K}^{\mathrm{con}}(R)\widehat{_p})$ of the p-completion of the connective K-theory spectrum of the ring R.

Theorem 5.24 (Rational Injectivity of the Farrell-Jones Assembly Map for Connective K-Theory). *Suppose that the group G satisfies the following two conditions:*

(H) For each finite cyclic subgroup $C \subseteq G$ and all $j \geq 0$ the integral homology group $H_j(BZ_GC;\mathbb{Z})$ of the centralizer Z_GC of C in G is finitely generated.
(K) There exists a prime p such that for each finite cyclic subgroup $C \subseteq G$ and each $j \geq 1$ the map induced by the change of coefficients homomorphism

$$K_j(\mathbb{Z}C;\mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_j(\mathbb{Z}_pC;\mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective.

Then the rationalized assembly map

$$A_{\mathcal{VCY}}\colon H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathbb{Z}}^{\mathrm{con}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an injection for all $n \in \mathbb{Z}$.

Remark 5.25. The methods of Chapter 8 apply also to $\mathbf{K}_{\mathbb{Z}}^{\mathrm{con}}$ and yield under assumption (H) and (K) an injection

$$\bigoplus_{s+t=n,\ t\geq 0} \quad \bigoplus_{(C)\in(\mathcal{FCY})} H_s(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot K_t(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \to \ K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Notice that in the index set for the direct sum appearing in the source we require $t \geq 0$. This reflects the fact that the result deals only with the connective K-theory spectrum. If one drops the restriction $t \geq 0$ the Farrell-Jones Conjecture 2.2 predicts that the map is an isomorphism, compare Subsection 2.2.5 and Theorem 8.5. If we restrict the injection to the direct sum given by $C = 1$, we rediscover the map (28) whose injectivity follows already from Theorem 5.23.

The condition (K) appearing in Theorem 5.24 is conjectured to be true for all primes p (compare [280], [290] and [291]) but no proof is known. The weaker version of condition (K), where C is the trivial group is also needed in Theorem 5.23. But that case is known to be true and hence does not appear in its formulation. The special case of condition (K), where $j = 1$ is implied by the Leopoldt Conjecture for abelian fields (compare [229, IX, § 3]), which is known to be true [229, Theorem 10.3.16]. This leads to the following result.

Theorem 5.26 (Rational Contribution of Finite Subgroups to $\mathrm{Wh}(G)$). *Let G be a group. Suppose that for each finite cyclic subgroup $C \subseteq G$ and each $j \leq 4$ the integral homology group $H_j(BZ_GC)$ of the centralizer Z_GC of C in G is finitely generated. Then the map*

$$\operatorname{colim}_{H \in \mathtt{Sub}_{\mathcal{FIN}}(G)} \mathrm{Wh}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \;\to\; \mathrm{Wh}(G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

is injective, compare Conjecture 3.5.

The result above should be compared to the result which is proven using Fuglede-Kadison determinants in [209, Section 5], [202, Theorem 9.38 on page 354]: for *every* (discrete) group G and every finite normal subgroup $H \subseteq G$ the map $\mathrm{Wh}(H) \otimes_{\mathbb{Z}G} \mathbb{Z} \to \mathrm{Wh}(G)$ induced by the inclusion $H \to G$ is rationally injective.

The next result is taken from Rosenthal [271], where the techniques and results of Carlsson-Pedersen [55] are extended from the trivial family $\{1\}$ to the family of finite subgroups $\mathcal{FIN}$.

Theorem 5.27. *Suppose there exists a model E for the classifying space $E_{\mathcal{FIN}}(G)$ which admits a metrizable compactification $\overline{E}$ to which the group action extends. Suppose $\overline{E}^H$ is contractible and E^H is dense in $\overline{E}^H$ for every finite subgroup $H \subset G$. Suppose compact subsets of E become small near $\overline{E} - E$. Then for every ring R the assembly map*

$$A_{\mathcal{FIN}} \colon H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \;\to\; K_n(RG)$$

is split injective.

A compact subset $K \subset E$ is said to become *small near* $\overline{E} - E$ if for every neighbourhood $U \subset \overline{E}$ of a point $x \in \overline{E} - E$ there exists a neighbourhood $V \subset \overline{E}$ such that $g \in G$ and $gK \cap V \neq \emptyset$ implies $gK \subset U$. Presumably there is an analogous result for $L^{\langle -\infty \rangle}$-theory under the assumption that $K_{-n}(RH)$ vanishes for finite subgroups H of G and n large enough. This would extend the corresponding result for the family $\mathcal{F} = \{1\}$ which appears in Carlsson-Pedersen [55].

We finally discuss injectivity results about assembly maps for the trivial family. The following result is due to Ferry-Weinberger [129, Corollary 2.3] extending earlier work of Farrell-Hsiang [99].

Theorem 5.28. *Suppose $G = \pi_1(M)$ for a complete Riemannian manifold of non-positive sectional curvature. Then the L-theory assembly map*

$$A \colon H_n(BG; \mathbf{L}_{\mathbb{Z}}^{\epsilon}) \to L_n^{\epsilon}(\mathbb{Z}G)$$

is injective for $\epsilon = h, s$.

In fact Ferry-Weinberger also prove a corresponding splitting result for the classical A-theory assembly map. In [155] Hu shows that a finite complex of non-positive curvature is a retract of a non-positively curved PL-manifold and concludes split injectivity of the classical L-theoretic assembly map for $R = \mathbb{Z}$.

The next result due to Bartels [17] is the algebraic K- and L-theory analogue of Theorem 5.14.

Theorem 5.29 (The K-and L-Theoretic Novikov Conjecture and Groups of Finite Asymptotic Dimension). *Let G be a group which admits a finite model for BG. Suppose that G has finite asymptotic dimension. Then*

(i) The assembly maps appearing in the Farrell-Jones Conjecture 1.11

$$A\colon H_n(BG;\mathbf{K}(R)) \to K_n(RG)$$

is injective for all $n \in \mathbb{Z}$.

(ii) If furthermore R carries an involution and $K_{-j}(R)$ vanishes for sufficiently large j, then the assembly maps appearing in the Farrell-Jones Conjecture 1.19

$$A\colon H_n(BG;\mathbf{L}^{\langle -\infty \rangle}(R)) \to L_n^{\langle -\infty \rangle}(RG)$$

is injective for all $n \in \mathbb{Z}$.

Further results related to the Farrell-Jones Conjecture 2.2 can be found for instance in [9], [33].

5.3 List of Groups Satisfying the Conjecture

In the following table we list prominent classes of groups and state whether they are known to satisfy the Baum-Connes Conjecture 2.3 (with coefficients 4.3) or the Farrell-Jones Conjecture 2.2 (fibered 4.11). Some of the classes are redundant. A question mark means that the authors do not know about a corresponding result. The reader should keep in mind that there may exist results of which the authors are not aware.

type of group	Baum-Connes Conjecture 2.3 (with coefficients 4.3)	Farrell-Jones Conjecture 2.2 for K-theory for $R = \mathbb{Z}$ (fibered 4.11)	Farrell-Jones Conjecture 2.2 for L-theory for $R = \mathbb{Z}$ (fibered 4.11)
a-T-menable groups	true with coefficients (see Theorem 5.1)	?	injectivity is true after inverting 2 (see Propositions 2.18 and 3.19)
amenable groups	true with coefficients (see Theorem 5.1)	?	injectivity is true after inverting 2 (see Propositions 2.18 and 3.19)
elementary amenable groups	true with coefficients (see Theorem 5.1)	?	true fibered after inverting 2 (see Theorem 5.16)
virtually poly-cyclic	true with coefficients (see Theorem 5.1)	true rationally, true fibered in the range $n \leq 1$ (compare Remark 5.18)	true fibered after inverting 2 (see Theorem 5.16)
torsion free virtually solvable subgroups of $GL(n,\mathbb{C})$	true with coefficients (see Theorem 5.1)	true in the range ≤ 1 [117, Theorem 1.1]	true fibered after inverting 2 [117, Corollary 5.3]

type of group	Baum-Connes Conjecture 2.3 (with coefficients 4.3)	Farrell-Jones Conjecture 2.2 for K-theory for $R = \mathbb{Z}$ (fibered 4.11)	Farrell-Jones Conjecture 2.2 for L-theory for $R = \mathbb{Z}$ (fibered 4.11)
discrete subgroups of Lie groups with finitely many path components	injectivity true (see Theorem 5.9 and Remark 5.11)	?	injectivity is true after inverting 2 (see Propositions 2.18 and 3.19)
subgroups of groups which are discrete cocompact subgroups of Lie groups with finitely many path components	injectivity true (see Theorem 5.9 and Remark 5.11)	true rationally, true fibered in the range $n \leq 1$ (see Theorem 5.16)	probably true fibered (see Remark 5.17). Injectivity is true after inverting 2 (see Propositions 2.18 and 3.19)
linear groups	injectivity is true (see Theorem 5.9 and Remark 5.10)	?	injectivity is true after inverting 2 (see Propositions 2.18 and 3.19)
arithmetic groups	injectivity is true (see Theorem 5.9 and Remark 5.10)	?	injectivity is true after inverting 2 (see Propositions 2.18 and 3.19)
torsion free discrete subgroups of $GL(n,\mathbb{R})$	injectivity is true (see Theorem 5.9 and Remark 5.11)	true in the range $n \leq 1$ (see Theorem 5.20)	true (see Theorem 5.20)

type of group	Baum-Connes Conjecture 2.3 (with coefficients 4.3)	Farrell-Jones Conjecture 2.2 for K-theory for $R = \mathbb{Z}$ (fibered 4.11)	Farrell-Jones Conjecture 2.2 for L-theory for $R = \mathbb{Z}$ (fibered 4.11)
Groups with finite BG and finite asymptotic dimension	injectivity is true (see Theorem 5.14)	injectivity is true for arbitrary coefficients R (see Theorem 5.29)	injectivity is true for regular R as coefficients (see Theorem 5.29)
G acts properly and isometrically on a complete Riemannian manifold M with non-positive sectional curvature	rational injectivity is true (see Theorem 5.8)	?	rational injectivity is true (see Propositions 2.18 and 3.19)
$\pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature	rationally injective (see Theorem 5.8)	?	injectivity true (see Theorem 5.28)
$\pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature which is A-regular	rationally injective (see Theorem 5.8)	true in the range $n \leq 1$, rationally surjective (see Theorem 5.20)	true (see Theorem 5.20)
$\pi_1(M)$ for a complete Riemannian manifold M with pinched negative sectional curvature	rational injectivity is true (see Theorem 5.9)	true in the range $n \leq 1$, rationally surjective (see Theorem 5.20)	true (see Theorem 5.20)
$\pi_1(M)$ for a closed Riemannian manifold M with non-positive sectional curvature	rationally injective (see Theorem 5.8)	true fibered in the range $n \leq 1$, true rationally (see Theorem 5.20)	true (see Theorem 5.20)
$\pi_1(M)$ for a closed Riemannian manifold M with negative sectional curvature	true for all subgroups (see Theorem 5.4)	true for all coefficients R (see Theorem 5.22)	true (see Theorem 5.20)

type of group	Baum-Connes Conjecture 2.3 (with coefficients 4.3)	Farrell-Jones Conjecture 2.2 for K-theory and $R = \mathbb{Z}$ (fibered 4.11)	Farrell-Jones Conjecture 2.2 for L-theory for $R = \mathbb{Z}$ (fibered 4.11)
word hyperbolic groups	true for all subgroups (see Theorem 5.4). Unpublished proof with coefficients by V. Lafforgue	?	injectivity is true after inverting 2 (see Propositions 2.18 and 3.19)
one-relator groups	true with coefficients (see Theorem 5.2)	rational injectivity is true for the fibered version (see [20])	injectivity is true after inverting 2 (see Propositions 2.18 and 3.19)
torsion free one-relator groups	true with coefficients (see Theorem 5.2)	true for R regular [313, Theorem 19.4 on page 249 and Theorem 19.5 on page 250]	true after inverting 2 [47, Corollary 8]
Haken 3-manifold groups (in particular knot groups)	true with coefficients (see Theorem 5.2)	true in the range $n \leq 1$ for R regular [313, Theorem 19.4 on page 249 and Theorem 19.5 on page 250]	true after inverting 2 [47, Corollary 8]
$SL(n,\mathbb{Z}), n \geq 3$	injectivity is true	compare Remark 5.21	injectivity is true after inverting 2 (see Propositions 2.18 and 3.19)
Artin's braid group B_n	true with coefficients [225, Theorem 5.25], [277]	true fibered in the range $n \leq 1$, true rationally [119]	injectivity is true after inverting 2 (see Propositions 2.18 and 3.19)
pure braid group C_n	true with coefficients	true in the range $n \leq 1$ (see Theorem 5.20)	injectivity is true after inverting 2 (see Propositions 2.18 and 3.19)
Thompson's group F	true with coefficients [94]	?	injectivity is true after inverting 2 (see Propositions 2.18 and 3.19)

Remark 5.30. The authors have no information about the status of these conjectures for mapping class groups of higher genus or the group of outer automorphisms of free groups. Since all of these spaces have finite models for $E_{\mathcal{FIN}}(G)$ Theorem 5.24 applies in these cases.

5.4 Inheritance Properties

In this Subsection we list some inheritance properties of the various conjectures.

5.4.1 Directed Colimits

Let $\{G_i \mid i \in I\}$ be a directed system of groups. Let $G = \operatorname{colim}_{i \in I} G_i$ be the colimit. We do not require that the structure maps are injective. If the Fibered Farrell-Jones Conjecture 4.11 is true for each G_i, then it is true for G [117, Theorem 6.1].

Suppose that $\{G_i \mid i \in I\}$ is a system of subgroups of G directed by inclusion such that $G = \operatorname{colim}_{i \in I} G_i$. If each G_i satisfies the Farrell-Jones Conjecture 2.2, the Baum-Connes Conjecture 2.3 or the Baum-Connes Conjecture with Coefficients 4.3, then the same is true for G [32, Theorem 1.1], [225, Lemma 5.3]. We do not know a reference in Farrell-Jones case. An argument in that case uses Lemma 2.7, the fact that $K_n(RG) = \operatorname{colim}_{i \in I} K_n(RG_i)$ and that for suitable models we have $E_{\mathcal{F}}(G) = \bigcup_{i \in I} G \times_{G_i} E_{\mathcal{F} \cap G_i}(G_i)$.

5.4.2 Passing to Subgroups

The Baum-Connes Conjecture with Coefficients 4.3 and the Fibered Farrell-Jones Conjecture 4.11 pass to subgroups, i.e. if they hold for G, then also for any subgroup $H \subseteq G$. This claim for the Baum-Connes Conjecture with Coefficients 4.3 has been stated in [28], a proof can be found for instance in [59, Theorem 2.5]. For the Fibered Farrell-Jones Conjecture this is proved in [111, Theorem A.8 on page 289] for the special case $R = \mathbb{Z}$, but the proof also works for arbitrary rings R.

It is not known whether the Baum-Connes Conjecture 2.3 or the Farrell-Jones Conjecture 2.2 itself passes to subgroups.

5.4.3 Extensions of Groups

Let $p\colon G \to K$ be a surjective group homomorphism. Suppose that the Baum-Connes Conjecture with Coefficients 4.3 or the Fibered Farrell-Jones Conjecture 4.11 respectively holds for K and for $p^{-1}(H)$ for any subgroup $H \subset K$ which is finite or virtually cyclic respectively. Then the Baum-Connes Conjecture with Coefficients 4.3 or the Fibered Farrell-Jones Conjecture 4.11 respectively holds for G. This is proved in [233, Theorem 3.1] for the Baum-Connes

Conjecture with Coefficients 4.3, and in [111, Proposition 2.2 on page 263] for the Fibered Farrell-Jones Conjecture 4.11 in the case $R = \mathbb{Z}$. The same proof works for arbitrary coefficient rings.

It is not known whether the corresponding statement holds for the Baum-Connes Conjecture 2.3 or the Farrell-Jones Conjecture 2.2 itself.

Let $H \subseteq G$ be a normal subgroup of G. Suppose that H is a-T-menable. Then G satisfies the Baum-Connes Conjecture with Coefficients 4.3 if and only if G/H does [59, Corollary 3.14]. The corresponding statement is not known for the Baum-Connes Conjecture 2.3.

5.4.4 Products of Groups

The group $G_1 \times G_2$ satisfies the Baum-Connes Conjecture with Coefficients 4.3 if and only if both G_1 and G_2 do [59, Theorem 3.17], [233, Corollary 7.12]. The corresponding statement is not known for the Baum-Connes Conjecture 2.3.

Let $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$ denote the infinite dihedral group. Whenever a version of the Fibered Farrell-Jones Conjecture 4.11 is known for $G = \mathbb{Z} \times \mathbb{Z}$, $G = \mathbb{Z} \times D_\infty$ and $D_\infty \times D_\infty$, then that version of the Fibered Farrell-Jones Conjecture is true for $G_1 \times G_2$ if and only if it is true for G_1 and G_2.

5.4.5 Subgroups of Finite Index

It is not known whether the Baum-Connes Conjecture 2.3, the Baum-Connes Conjecture with Coefficients 4.3, the Farrell-Jones Conjecture 2.2 or the Fibered Farrell-Jones Conjecture 4.11 is true for a group G if it is true for a subgroup $H \subseteq G$ of finite index.

5.4.6 Groups Acting on Trees

Let G be a countable discrete group acting without inversion on a tree T. Then the Baum-Connes Conjecture with Coefficients 4.3 is true for G if and only if it holds for all stabilizers of the vertices of T. This is proved by Oyono-Oyono [234, Theorem 1.1]. This implies that Baum-Connes Conjecture with Coefficients 4.3 is stable under amalgamated products and HNN-extensions. Actions on trees in the context the Farrell-Jones Conjecture 2.2 will be treated in [20].

6 Equivariant Homology Theories

We already defined the notion of a G-homology theory in Subsection 2.1.4. If G-homology theories for different G are linked via a so called induction structure one obtains the notion of an equivariant homology theory. In this section we give a precise definition and we explain how a functor from the orbit category $\mathsf{Or}(G)$ to the category of spectra leads to a G-homology theory (see Proposition 6.7) and how more generally a functor from the category of groupoids leads to an equivariant homology theory (see Proposition 6.8). We then describe the main examples of such spectra valued functors which were already used in order to formulate the Farrell-Jones and the Baum-Connes Conjectures in Chapter 2.

6.1 The Definition of an Equivariant Homology Theory

The notion of a G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules for a commutative ring Λ was defined in Subsection 2.1.4. We now recall the axioms of an equivariant homology theory from [201, Section 1]. We will see in Section 6.3 that the G-homology theories we used in the formulation of the Baum-Connes and the Farrell-Jones Conjectures in Chapter 2 are in fact the values at G of suitable equivariant homology theories.

Let $\alpha\colon H \to G$ be a group homomorphism. Given a H-space X, define the *induction of X with α* to be the G-space $\operatorname{ind}_\alpha X$ which is the quotient of $G \times X$ by the right H-action $(g,x)\cdot h := (g\alpha(h), h^{-1}x)$ for $h \in H$ and $(g,x) \in G \times X$. If $\alpha\colon H \to G$ is an inclusion, we also write ind_H^G instead of $\operatorname{ind}_\alpha$.

An *equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules* consists of a G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules for each group G together with the following so called *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a H-CW-pair (X,A) such that $\ker(\alpha)$ acts freely on X, there are for each $n \in \mathbb{Z}$ natural isomorphisms

$$\operatorname{ind}_\alpha\colon \mathcal{H}_n^H(X,A) \xrightarrow{\cong} \mathcal{H}_n^G(\operatorname{ind}_\alpha(X,A))$$

satisfying the following conditions.

(i) Compatibility with the boundary homomorphisms
$\partial_n^G \circ \operatorname{ind}_\alpha = \operatorname{ind}_\alpha \circ \partial_n^H$.

(ii) Functoriality
Let $\beta\colon G \to K$ be another group homomorphism such that $\ker(\beta\circ\alpha)$ acts freely on X. Then we have for $n \in \mathbb{Z}$

$$\operatorname{ind}_{\beta\circ\alpha} \;=\; \mathcal{H}_n^K(f_1)\circ \operatorname{ind}_\beta \circ \operatorname{ind}_\alpha\colon \mathcal{H}_n^H(X,A) \to \mathcal{H}_n^K(\operatorname{ind}_{\beta\circ\alpha}(X,A)),$$

where $f_1\colon \operatorname{ind}_\beta \operatorname{ind}_\alpha(X,A) \xrightarrow{\cong} \operatorname{ind}_{\beta\circ\alpha}(X,A)$, $(k,g,x) \mapsto (k\beta(g),x)$ is the natural K-homeomorphism.

(iii) Compatibility with conjugation
For $n \in \mathbb{Z}$, $g \in G$ and a G-CW-pair (X,A) the homomorphism

$$\operatorname{ind}_{c(g)\colon G\to G}\colon \mathcal{H}_n^G(X,A) \to \mathcal{H}_n^G(\operatorname{ind}_{c(g)\colon G\to G}(X,A))$$

agrees with $\mathcal{H}_n^G(f_2)$, where the G-homeomorphism

$$f_2\colon (X,A) \to \mathrm{ind}_{c(g)\colon G \to G}(X,A)$$

sends x to $(1, g^{-1}x)$ and $c(g)\colon G \to G$ sends g' to $gg'g^{-1}$.

This induction structure links the various homology theories for different groups G.

If the G-homology theory $\mathcal{H}_*^G$ is defined or considered only for proper G-CW-pairs (X,A), we call it a *proper G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules.*

Example 6.1. Let $\mathcal{K}_*$ be a homology theory for (non-equivariant) CW-pairs with values in Λ-modules. Examples are singular homology, oriented bordism theory or topological K-homology. Then we obtain two equivariant homology theories with values in Λ-modules, whose underlying G-homology theories for a group G are given by the following constructions

$$\begin{aligned} \mathcal{H}_n^G(X,A) &= \mathcal{K}_n(G\backslash X, G\backslash A); \\ \mathcal{H}_n^G(X,A) &= \mathcal{K}_n(EG \times_G (X,A)). \end{aligned}$$

Example 6.2. Given a proper G-CW-pair (X,A), one can define the G-bordism group $\Omega_n^G(X,A)$ as the abelian group of G-bordism classes of maps $f\colon (M, \partial M) \to (X,A)$ whose sources are oriented smooth manifolds with cocompact orientation preserving proper smooth G-actions. The definition is analogous to the one in the non-equivariant case. This is also true for the proof that this defines a proper G-homology theory. There is an obvious induction structure coming from induction of equivariant spaces. Thus we obtain an equivariant proper homology theory $\Omega_*^?$.

Example 6.3. Let Λ be a commutative ring and let

$$M\colon \mathsf{GROUPOIDS} \to \Lambda\text{-}\mathsf{MODULES}$$

be a contravariant functor. For a group G we obtain a covariant functor

$$M^G\colon \mathsf{Or}(G) \to \Lambda\text{-}\mathsf{MODULES}$$

by its composition with the transport groupoid functor $\mathcal{G}^G$ defined in (30). Let $H_*^G(-;M)$ be the G-homology theory given by the Bredon homology with coefficients in M^G as defined in Example 2.8. There is an induction structure such that the collection of the $H^G(-;M)$ defines an equivariant homology theory $H_*^?(-;M)$. This can be interpreted as the special case of Proposition 6.8, where the covariant functor $\mathsf{GROUPOIDS} \to \Omega\text{-}\mathsf{SPECTRA}$ is the composition of M with the functor sending a Λ-module to the associated Eilenberg-MacLane spectrum. But there is also a purely algebraic construction.

The next lemma was used in the proof of the Transitivity Principle 2.9.

Lemma 6.4. *Let $\mathcal{H}_*^?$ be an equivariant homology theory with values in Λ-modules. Let G be a group and let $\mathcal{F}$ a family of subgroups of G. Let Z be a G-CW-complex. Consider $N \in \mathbb{Z} \cup \{\infty\}$. For $H \subseteq G$ let $\mathcal{F} \cap H$ be the family of subgroups of H given by $\{K \cap H \mid K \in \mathcal{F}\}$. Suppose for each $H \subset G$, which occurs as isotropy group in Z, that the map induced by the projection* $\mathrm{pr}\colon E_{\mathcal{F}\cap H}(H) \to \mathrm{pt}$

$$\mathcal{H}_n^H(\mathrm{pr})\colon \mathcal{H}_n^H(E_{\mathcal{F}\cap H}(H)) \to \mathcal{H}_n^H(\mathrm{pt})$$

is bijective for all $n \in \mathbb{Z}, n \leq N$.

Then the map induced by the projection $\mathrm{pr}_2\colon E_{\mathcal{F}}(G) \times Z \to Z$

$$\mathcal{H}_n^G(\mathrm{pr}_2)\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z) \to \mathcal{H}_n^G(Z)$$

is bijective for $n \in \mathbb{Z}, n \leq N$.

Proof. We first prove the claim for finite-dimensional G-CW-complexes by induction over $d = \dim(Z)$. The induction beginning $\dim(Z) = -1$, i.e. $Z = \emptyset$, is trivial. In the induction step from $(d-1)$ to d we choose a G-pushout

$$\begin{array}{ccc} \coprod_{i\in I_d} G/H_i \times S^{d-1} & \longrightarrow & Z_{d-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{i\in I_d} G/H_i \times D^d & \longrightarrow & Z_d \end{array}$$

If we cross it with $E_{\mathcal{F}}(G)$, we obtain another G-pushout of G-CW-complexes. The various projections induce a map from the Mayer-Vietoris sequence of the latter G-pushout to the Mayer-Vietoris sequence of the first G-pushout. By the Five-Lemma it suffices to prove that the following maps

$$\mathcal{H}_n^G(\mathrm{pr}_2)\colon \mathcal{H}_n^G\left(E_{\mathcal{F}}(G) \times \coprod_{i\in I_d} G/H_i \times S^{d-1}\right) \to \mathcal{H}_n^G\left(\coprod_{i\in I_d} G/H_i \times S^{d-1}\right);$$
$$\mathcal{H}_n^G(\mathrm{pr}_2)\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z_{d-1}) \to \mathcal{H}_n^G(Z_{d-1});$$
$$\mathcal{H}_n^G(\mathrm{pr}_2)\colon \mathcal{H}_n^G\left(E_{\mathcal{F}}(G) \times \coprod_{i\in I_d} G/H_i \times D^{d}\right) \to \mathcal{H}_n^G\left(\coprod_{i\in I_d} G/H_i \times D^{d}\right)$$

are bijective for $n \in \mathbb{Z}, n \leq N$. This follows from the induction hypothesis for the first two maps. Because of the disjoint union axiom and G-homotopy invariance of $\mathcal{H}_*^?$ the claim follows for the third map if we can show for any $H \subseteq G$ which occurs as isotropy group in Z that the map

$$\mathcal{H}_n^G(\mathrm{pr}_2)\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times G/H) \to \mathcal{H}^G(G/H) \tag{29}$$

is bijective for $n \in \mathbb{Z}, n \leq N$. The G-map

$$G \times_H \mathrm{res}_G^H E_{\mathcal{F}}(G) \to G/H \times E_{\mathcal{F}}(G) \qquad (g,x) \;\mapsto\; (gH, gx)$$

is a G-homeomorphism where res_G^H denotes the restriction of the G-action to an H-action. Obviously $\operatorname{res}_G^H E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F}\cap H}(H)$. We conclude from the induction structure that the map (29) can be identified with the map

$$\mathcal{H}_n^G(\mathrm{pr})\colon \mathcal{H}_n^H(E_{\mathcal{F}\cap H}(H)) \to \mathcal{H}^H(\mathrm{pt})$$

which is bijective for all $n \in \mathbb{Z}, n \leq N$ by assumption. This finishes the proof in the case that Z is finite-dimensional. The general case follows by a colimit argument using Lemma 2.7. □

6.2 Constructing Equivariant Homology Theories

Recall that a (non-equivariant) spectrum yields an associated (non-equivariant) homology theory. In this section we explain how a spectrum over the orbit category of a group G defines a G-homology theory. We would like to stress that our approach using spectra over the orbit category should be distinguished from approaches to equivariant homology theories using spectra with G-action or the more complicated notion of G-equivariant spectra in the sense of [190], see for example [53] for a survey. The latter approach leads to a much richer structure but only works for compact Lie groups.

We briefly fix some conventions concerning spectra. We always work in the very convenient *category* SPACES *of compactly generated spaces* (see [295], [330, I.4]). In that category the adjunction homeomorphism $\operatorname{map}(X\times Y, Z) \xrightarrow{\cong} \operatorname{map}(X, \operatorname{map}(Y,Z))$ holds without any further assumptions such as local compactness and the product of two CW-complexes is again a CW-complex. Let SPACES^+ be the category of pointed compactly generated spaces. Here the objects are (compactly generated) spaces X with base points for which the inclusion of the base point is a cofibration. Morphisms are pointed maps. If X is a space, denote by X_+ the pointed space obtained from X by adding a disjoint base point. For two pointed spaces $X = (X, x)$ and $Y = (Y, y)$ define their *smash product* as the pointed space

$$X \wedge Y = X \times Y/(\{x\} \times Y \cup X \times \{y\}),$$

and the *reduced cone* as

$$\operatorname{cone}(X) = X \times [0,1]/(X \times \{1\} \cup \{x\} \times [0,1]).$$

A *spectrum* $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps* $\sigma(n)\colon E(n) \wedge S^1 \longrightarrow E(n+1)$. A *map* of spectra $\mathbf{f}\colon \mathbf{E} \to \mathbf{E}'$ is a sequence of maps $f(n)\colon E(n) \to E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e. we have $f(n+1)\circ\sigma(n) = \sigma'(n)\circ(f(n)\wedge \mathrm{id}_{S^1})$ for all $n \in \mathbb{Z}$. Maps of spectra are sometimes called functions in the literature, they should not be confused with the notion of a map of spectra in the stable category (see [4, III.2.]). The category of spectra and maps will be denoted $\mathsf{SPECTRA}$. Recall that the homotopy groups of a spectrum are defined by

$$\pi_i(\mathbf{E}) \;=\; \operatorname{colim}_{k\to\infty} \pi_{i+k}(E(k)),$$

where the system $\pi_{i+k}(E(k))$ is given by the composition

$$\pi_{i+k}(E(k)) \xrightarrow{S} \pi_{i+k+1}(E(k)\wedge S^1) \xrightarrow{\sigma(k)_*} \pi_{i+k+1}(E(k+1))$$

of the suspension homomorphism S and the homomorphism induced by the structure map. A *weak equivalence* of spectra is a map $\mathbf{f}\colon \mathbf{E}\to\mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups.

Given a spectrum $\mathbf{E}$ and a pointed space X, we can define their smash product $X\wedge\mathbf{E}$ by $(X\wedge\mathbf{E})(n) := X\wedge E(n)$ with the obvious structure maps. It is a classical result that a spectrum $\mathbf{E}$ defines a homology theory by setting

$$H_n(X,A;\mathbf{E}) = \pi_n\left((X_+\cup_{A_+}\operatorname{cone}(A_+))\wedge\mathbf{E}\right).$$

We want to extend this to G-homology theories. This requires the consideration of spaces and spectra over the orbit category. Our presentation follows [82], where more details can be found.

In the sequel $\mathcal{C}$ is a small category. Our main example is the orbit category $\mathsf{Or}(G)$, whose objects are homogeneous G-spaces G/H and whose morphisms are G-maps.

Definition 6.5. A *covariant (contravariant) $\mathcal{C}$-space* X is a covariant (contravariant) functor

$$X\colon \mathcal{C} \;\to\; \mathsf{SPACES}.$$

A map between $\mathcal{C}$-spaces is a natural transformation of such functors. Analogously a *pointed $\mathcal{C}$-space* is a functor from $\mathcal{C}$ to SPACES^+ and a $\mathcal{C}$-spectrum a functor to $\mathsf{SPECTRA}$.

Example 6.6. *Let Y be a left G-space. Define the associated* contravariant $\mathsf{Or}(G)$-space $\operatorname{map}_G(-,Y)$ *by*

$$\operatorname{map}_G(-,Y)\colon\; \mathsf{Or}(G)\to\mathsf{SPACES}, \qquad G/H \;\mapsto \operatorname{map}_G(G/H,Y) = Y^H.$$

If Y is pointed then $\operatorname{map}_G(-,Y)$ *takes values in pointed spaces.*

Let X be a contravariant and Y be a covariant $\mathcal{C}$-space. Define their *balanced product* to be the space

$$X\times_{\mathcal{C}} Y \;:=\; \coprod_{c\in\operatorname{ob}(\mathcal{C})} X(c)\times Y(c)/\sim$$

where $\sim$ is the equivalence relation generated by $(x\phi,y)\sim(x,\phi y)$ for all morphisms $\phi\colon c\to d$ in $\mathcal{C}$ and points $x\in X(d)$ and $y\in Y(c)$. Here $x\phi$ stands for $X(\phi)(x)$ and ϕy for $Y(\phi)(y)$. If X and Y are pointed, then one defines analogously their *balanced smash product* to be the pointed space

$$X \wedge_{\mathcal{C}} Y \;=\; \bigvee_{c \in \mathrm{ob}(\mathcal{C})} X(c) \wedge Y(c)/\sim .$$

In [82] the notation $X \otimes_{\mathcal{C}} Y$ was used for this space. Doing the same construction level-wise one defines the *balanced smash product* $X \wedge_{\mathcal{C}} \mathbf{E}$ of a contravariant pointed $\mathcal{C}$-space and a covariant $\mathcal{C}$-spectrum $\mathbf{E}$.

The proof of the next result is analogous to the non-equivariant case. Details can be found in [82, Lemma 4.4], where also cohomology theories are treated.

Proposition 6.7 (Constructing G-Homology Theories). *Let* $\mathbf{E}$ *be a covariant* $\mathsf{Or}(G)$*-spectrum. It defines a* G*-homology theory* $H_*^G(-;\mathbf{E})$ *by*

$$H_n^G(X,A;\mathbf{E}) \;=\; \pi_n\left(\mathrm{map}_G\left(-,(X_+ \cup_{A_+} \mathrm{cone}(A_+))\right) \wedge_{\mathsf{Or}(G)} \mathbf{E}\right).$$

In particular we have

$$H_n^G(G/H;\mathbf{E}) \;=\; \pi_n(\mathbf{E}(G/H)).$$

Recall that we seek an equivariant homology theory and not only a G-homology theory. If the $\mathsf{Or}(G)$-spectrum in Proposition 6.7 is obtained from a $\mathsf{GROUPOIDS}$-spectrum in a way we will now describe, then automatically we obtain the desired induction structure.

Let $\mathsf{GROUPOIDS}$ be the category of small groupoids with covariant functors as morphisms. Recall that a groupoid is a category in which all morphisms are isomorphisms. A covariant functor $f\colon \mathcal{G}_0 \to \mathcal{G}_1$ of groupoids is called *injective*, if for any two objects x, y in $\mathcal{G}_0$ the induced map $\mathrm{mor}_{\mathcal{G}_0}(x,y) \to \mathrm{mor}_{\mathcal{G}_1}(f(x),f(y))$ is injective. Let $\mathsf{GROUPOIDS}^{\mathrm{inj}}$ be the subcategory of $\mathsf{GROUPOIDS}$ with the same objects and injective functors as morphisms. For a G-set S we denote by $\mathcal{G}^G(S)$ its associated *transport groupoid*. Its objects are the elements of S. The set of morphisms from s_0 to s_1 consists of those elements $g \in G$ which satisfy $gs_0 = s_1$. Composition in $\mathcal{G}^G(S)$ comes from the multiplication in G. Thus we obtain for a group G a covariant functor

$$\mathcal{G}^G\colon \mathsf{Or}(G) \to \mathsf{GROUPOIDS}^{\mathrm{inj}}, \qquad G/H \mapsto \mathcal{G}^G(G/H). \tag{30}$$

A functor of small categories $F\colon \mathcal{C} \to \mathcal{D}$ is called an *equivalence* if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that both $F \circ G$ and $G \circ F$ are naturally equivalent to the identity functor. This is equivalent to the condition that F induces a bijection on the set of isomorphisms classes of objects and for any objects $x, y \in \mathcal{C}$ the map $\mathrm{mor}_{\mathcal{C}}(x,y) \to \mathrm{mor}_{\mathcal{D}}(F(x),F(y))$ induced by F is bijective.

Proposition 6.8 (Constructing Equivariant Homology Theories). *Consider a covariant* $\mathsf{GROUPOIDS}^{\mathrm{inj}}$*-spectrum*

$$\mathbf{E}\colon \mathsf{GROUPOIDS}^{\mathrm{inj}} \to \mathsf{SPECTRA}.$$

Suppose that $\mathbf{E}$ *respects equivalences, i.e. it sends an equivalence of groupoids to a weak equivalence of spectra. Then* $\mathbf{E}$ *defines an equivariant homology theory* $H_*^?(-;\mathbf{E})$*, whose underlying* G*-homology theory for a group* G *is the* G*-homology theory associated to the covariant* $\mathsf{Or}(G)$*-spectrum* $\mathbf{E}\circ\mathcal{G}^G\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ *in the previous Proposition 6.7, i.e.*

$$H_*^G(X,A;\mathbf{E}) \;=\; H_*^G(X,A;\mathbf{E}\circ\mathcal{G}^G).$$

In particular we have

$$H_n^G(G/H;\mathbf{E}) \;\cong\; H_n^H(\mathrm{pt};\mathbf{E}) \;\cong\; \pi_n(\mathbf{E}(I(H))),$$

where $I(H)$ *denotes* H *considered as a groupoid with one object. The whole construction is natural in* $\mathbf{E}$*.*

Proof. We have to specify the induction structure for a homomorphism $\alpha\colon H \to G$. We only sketch the construction in the special case where α is injective and $A = \emptyset$. The details of the full proof can be found in [276, Theorem 2.10 on page 21].

The functor induced by α on the orbit categories is denoted in the same way

$$\alpha\colon \mathsf{Or}(H) \to \mathsf{Or}(G), \qquad H/L \mapsto \mathrm{ind}_\alpha(H/L) = G/\alpha(L).$$

There is an obvious natural equivalence of functors $\mathsf{Or}(H) \to \mathsf{GROUPOIDS}^{\mathrm{inj}}$

$$T\colon \mathcal{G}^H \to \mathcal{G}^G \circ \alpha.$$

Its evaluation at H/L is the equivalence of groupoids $\mathcal{G}^H(H/L) \to \mathcal{G}^G(G/\alpha(L))$ which sends an object hL to the object $\alpha(h)\alpha(L)$ and a morphism given by $h \in H$ to the morphism $\alpha(h) \in G$. The desired isomorphism

$$\mathrm{ind}_\alpha\colon H_n^H(X;\mathbf{E}\circ\mathcal{G}^H) \to H_n^G(\mathrm{ind}_\alpha X;\mathbf{E}\circ\mathcal{G}^G)$$

is induced by the following map of spectra

$$\begin{aligned} &\mathrm{map}_H(-,X_+) \wedge_{\mathsf{Or}(H)} \mathbf{E}\circ\mathcal{G}^H \xrightarrow{\mathrm{id}\wedge\mathbf{E}(T)} \mathrm{map}_H(-,X_+) \wedge_{\mathsf{Or}(H)} \mathbf{E}\circ\mathcal{G}^G\circ\alpha \\ &\xleftarrow{\simeq} (\alpha_*\,\mathrm{map}_H(-,X_+)) \wedge_{\mathsf{Or}(G)} \mathbf{E}\circ\mathcal{G}^G \xleftarrow{\simeq} \mathrm{map}_G(-,\mathrm{ind}_\alpha X_+) \wedge_{\mathsf{Or}(G)} \mathbf{E}\circ\mathcal{G}^G. \end{aligned}$$

Here $\alpha_*\,\mathrm{map}_H(-,X_+)$ is the pointed $\mathsf{Or}(G)$-space which is obtained from the pointed $\mathsf{Or}(H)$-space $\mathrm{map}_H(-,X_+)$ by induction, i.e. by taking the balanced product over $\mathsf{Or}(H)$ with the $\mathsf{Or}(H)$-$\mathsf{Or}(G)$ bimodule $\mathrm{mor}_{\mathsf{Or}(G)}(??,\alpha(?))$ [82, Definition 1.8]. Notice that $\mathbf{E}\circ\mathcal{G}^G\circ\alpha$ is the same as the restriction of the $\mathsf{Or}(G)$-spectrum $\mathbf{E}\circ\mathcal{G}^G$ along α which is often denoted by $\alpha^*(\mathbf{E}\circ\mathcal{G}^G)$ in the literature [82, Definition 1.8]. The second map is given by the adjunction homeomorphism of induction α_* and restriction α^* (see [82, Lemma 1.9]). The third map is the homeomorphism of $\mathsf{Or}(G)$-spaces which is the adjoint of the obvious map of $\mathsf{Or}(H)$-spaces $\mathrm{map}_H(-,X_+) \to \alpha^*\,\mathrm{map}_G(-,\mathrm{ind}_\alpha X_+)$ whose evaluation at H/L is given by ind_α. $\square$

6.3 *K*- and *L*-Theory Spectra over Groupoids

Let RINGS be the category of associative rings with unit. An *involution* on a R is a map $R \to R$, $r \mapsto \overline{r}$ satisfying $\overline{1} = 1$, $\overline{x+y} = \overline{x} + \overline{y}$ and $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$ for all x, $y \in R$. Let $\mathsf{RINGS}^{\mathrm{inv}}$ be the category of rings with involution. Let C^*-$\mathsf{ALGEBRAS}$ be the category of C^*-algebras. There are classical functors for $j \in -\infty \amalg \{j \in \mathbb{Z} \mid j \leq 2\}$

$$\mathbf{K}\colon \mathsf{RINGS} \to \mathsf{SPECTRA}; \tag{31}$$
$$\mathbf{L}^{\langle j \rangle}\colon \mathsf{RINGS}^{\mathrm{inv}} \to \mathsf{SPECTRA}; \tag{32}$$
$$\mathbf{K}^{\mathrm{top}}\colon C^*\text{-}\mathsf{ALGEBRAS} \to \mathsf{SPECTRA}. \tag{33}$$

The construction of such a non-connective algebraic K-theory functor goes back to Gersten [133] and Wagoner [312]. The spectrum for quadratic algebraic L-theory is constructed by Ranicki in [258]. In a more geometric formulation it goes back to Quinn [250]. In the topological K-theory case a construction using Bott periodicity for C^*-algebras can easily be derived from the Kuiper-Mingo Theorem (see [281, Section 2.2]). The homotopy groups of these spectra give the algebraic K-groups of Quillen (in high dimensions) and of Bass (in negative dimensions), the decorated quadratic L-theory groups, and the topological K-groups of C^*-algebras.

We emphasize again that in all three cases we need the non-connective versions of the spectra, i.e. the homotopy groups in negative dimensions are non-trivial in general. For example the version of the Farrell-Jones Conjecture where one uses connective K-theory spectra is definitely false in general, compare Remark 1.15.

Now let us fix a coefficient ring R (with involution). Then sending a group G to the group ring RG yields functors $R(-)\colon \mathsf{GROUPS} \to \mathsf{RINGS}$, respectively $R(-)\colon \mathsf{GROUPS} \to \mathsf{RINGS}^{\mathrm{inv}}$, where GROUPS denotes the category of groups. Let $\mathsf{GROUPS}^{\mathrm{inj}}$ be the category of groups with injective group homomorphisms as morphisms. Taking the reduced group C^*-algebra defines a functor $C^*_r\colon \mathsf{GROUPS}^{\mathrm{inj}} \to C^*$-$\mathsf{ALGEBRAS}$. The composition of these functors with the functors (31), (32) and (33) above yields functors

$$\mathbf{K}R(-)\colon \mathsf{GROUPS} \to \mathsf{SPECTRA}; \tag{34}$$
$$\mathbf{L}^{\langle j \rangle}R(-)\colon \mathsf{GROUPS} \to \mathsf{SPECTRA}; \tag{35}$$
$$\mathbf{K}^{\mathrm{top}}C^*_r(-)\colon \mathsf{GROUPS}^{\mathrm{inj}} \to \mathsf{SPECTRA}. \tag{36}$$

They satisfy

$$\begin{aligned}
\pi_n(\mathbf{K}R(G)) &= K_n(RG);\\
\pi_n(\mathbf{L}^{\langle j \rangle}R(G)) &= L_n^{\langle j \rangle}(RG);\\
\pi_n(\mathbf{K}^{\mathrm{top}}C^*_r(G)) &= K_n(C^*_r(G)),
\end{aligned}$$

for all groups G and $n \in \mathbb{Z}$. The next result essentially says that these functors can be extended to groupoids.

Theorem 6.9 (K- and L-Theory Spectra over Groupoids). *Let R be a ring (with involution). There exist covariant functors*

$$\mathbf{K}_R \colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}; \tag{37}$$

$$\mathbf{L}_R^{\langle j \rangle} \colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}; \tag{38}$$

$$\mathbf{K}^{\mathrm{top}} \colon \mathsf{GROUPOIDS}^{\mathrm{inj}} \to \mathsf{SPECTRA} \tag{39}$$

with the following properties:

(i) If $F \colon \mathcal{G}_0 \to \mathcal{G}_1$ is an equivalence of (small) groupoids, then the induced maps $\mathbf{K}_R(F)$, $\mathbf{L}_R^{\langle j \rangle}(F)$ and $\mathbf{K}^{\mathrm{top}}(F)$ are weak equivalences of spectra.

(ii) Let $I \colon \mathsf{GROUPS} \to \mathsf{GROUPOIDS}$ be the functor sending G to G considered as a groupoid, i.e. to $\mathcal{G}^G(G/G)$. This functor restricts to a functor $\mathsf{GROUPS}^{\mathrm{inj}} \to \mathsf{GROUPOIDS}^{\mathrm{inj}}$.
There are natural transformations from $\mathbf{K}R(-)$ to $\mathbf{K}_R \circ I$, from $\mathbf{L}^{\langle j \rangle}R(-)$ to $\mathbf{L}_R^{\langle j \rangle} \circ I$ and from $\mathbf{K}C_r^*(-)$ to $\mathbf{K}^{\mathrm{top}} \circ I$ such that the evaluation of each of these natural transformations at a given group is an equivalence of spectra.

(iii) For every group G and all $n \in \mathbb{Z}$ we have

$$\begin{aligned} \pi_n(\mathbf{K}_R \circ I(G)) &\cong K_n(RG); \\ \pi_n(\mathbf{L}_R^{\langle j \rangle} \circ I^{\mathrm{inv}}(G)) &\cong L_n^{\langle j \rangle}(RG); \\ \pi_n(\mathbf{K}^{\mathrm{top}} \circ I(G)) &\cong K_n(C_r^*(G)). \end{aligned}$$

Proof. We only sketch the strategy of the proof. More details can be found in [82, Section 2].

Let $\mathcal{G}$ be a groupoid. Similar to the group ring RG one can define an R-linear category $R\mathcal{G}$ by taking the free R-modules over the morphism sets of $\mathcal{G}$. Composition of morphisms is extended R-linearly. By formally adding finite direct sums one obtains an additive category $R\mathcal{G}_{\oplus}$. Pedersen-Weibel [237] (compare also [51]) define a non-connective algebraic K-theory functor which digests additive categories and can hence be applied to $R\mathcal{G}_{\oplus}$. For the comparison result one uses that for every ring R (in particular for RG) the Pedersen-Weibel functor applied to $R_{\oplus}$ (a small model for the category of finitely generated free R-modules) yields the non-connective K-theory of the ring R and that it sends equivalences of additive categories to equivalences of spectra. In the L-theory case $R\mathcal{G}_{\oplus}$ inherits an involution and one applies the construction of [258, Example 13.6 on page 139] to obtain the $L^{\langle 1 \rangle} = L^h$-version. The versions for $j \leq 1$ can be obtained by a construction which is analogous to the Pedersen-Weibel construction for K-theory, compare [55, Section 4]. In the C^*-case one obtains from $\mathcal{G}$ a C^*-category $C_r^*(\mathcal{G})$ and assigns to it its topological K-theory spectrum. There is a construction of the topological K-theory spectrum of a C^*-category in [82, Section 2]. However, the construction given there depends on two statements, which appeared in

[130, Proposition 1 and Proposition 3], and those statements are incorrect, as already pointed out by Thomason in [302]. The construction in [82, Section 2] can easily be fixed but instead we recommend the reader to look at the more recent construction of Joachim [159]. □

6.4 Assembly Maps in Terms of Homotopy Colimits

In this section we describe a homotopy-theoretic formulation of the Baum-Connes and Farrell-Jones Conjectures. For the classical assembly maps which in our set-up correspond to the trivial family such formulations were described in [328].

For a group G and a family $\mathcal{F}$ of subgroups we denote by $\mathsf{Or}_{\mathcal{F}}(G)$ the *restricted orbit category.* Its objects are homogeneous spaces G/H with $H \in \mathcal{F}$. Morphisms are G-maps. If $\mathcal{F} = \mathcal{ALL}$ we get back the (full) orbit category, i.e. $\mathsf{Or}(G) = \mathsf{Or}_{\mathcal{ALL}}(G)$.

Meta-Conjecture 6.10 (Homotopy-Theoretic Isomorphism Conjecture). *Let G be a group and $\mathcal{F}$ a family of subgroups. Let $\mathbf{E}\colon \mathsf{Or}(G) \to$ $\mathsf{SPECTRA}$ be a covariant functor. Then*

$$A_{\mathcal{F}}\colon \operatorname{hocolim}_{\mathsf{Or}_{\mathcal{F}}(G)} \mathbf{E}|_{\mathsf{Or}_{\mathcal{F}}(G)} \;\to\; \operatorname{hocolim}_{\mathsf{Or}(G)} \mathbf{E} \simeq \mathbf{E}(G/G)$$

is a weak equivalence of spectra.

Here hocolim is the homotopy colimit of a covariant functor to spectra, which is itself a spectrum. The map $A_{\mathcal{F}}$ is induced by the obvious functor $\mathsf{Or}_{\mathcal{F}}(G) \to \mathsf{Or}(G)$. The equivalence $\operatorname{hocolim}_{\mathsf{Or}(G)} \mathbf{E} \simeq \mathbf{E}(G/G)$ comes from the fact that G/G is a final object in $\mathsf{Or}(G)$. For information about homotopy-colimits we refer to [40], [82, Section 3] and [88].

Remark 6.11. If we consider the map on homotopy groups that is induced by the map $A_{\mathcal{F}}$ which appears in the Homotopy-Theoretic Isomorphism Conjecture above, then we obtain precisely the map with the same name in Meta-Conjecture 2.1 for the homology theory $H_*^G(-;\mathbf{E})$ associated with $\mathbf{E}$ in Proposition 6.7, compare [82, Section 5]. In particular the Baum-Connes Conjecture 2.3 and the Farrell-Jones Conjecture 2.2 can be seen as special cases of Meta-Conjecture 6.10.

Remark 6.12 (Universal Property of the Homotopy-Theoretic Assembly Map). The Homotopy-Theoretic Isomorphism Conjecture 6.10 is in some sense the most conceptual formulation of an Isomorphism Conjecture because it has a universal property as the universal approximation from the left by a (weakly) excisive $\mathcal{F}$-homotopy invariant functor. This is explained in detail in [82, Section 6]. This universal property is important if one wants to identify different models for the assembly map, compare e.g. [19, Section 6] and [142].

6.5 Naturality under Induction

Consider a covariant functor $\mathbf{E}\colon \mathsf{GROUPOIDS} \to \mathsf{SPECTRA}$ which respects equivalences. Let $H_*^?(-;\mathbf{E})$ be the associated equivariant homology theory (see Proposition 6.8). Then for a group homomorphism $\alpha\colon H \to G$ and H-CW-pair (X,A) we obtain a homomorphism

$$\operatorname{ind}_\alpha\colon H_n^H(X,A;\mathbf{E}) \to H_n^G(\operatorname{ind}_\alpha(X,A);\mathbf{E})$$

which is natural in (X,A). Note that we did not assume that $\ker(\alpha)$ acts freely on X. In fact the construction sketched in the proof of Proposition 6.8 still works even though $\operatorname{ind}_\alpha$ may not be an isomorphism as it is the case if $\ker(\alpha)$ acts freely. We still have functoriality as described in (ii) towards the beginning of Section 6.1.

Now suppose that $\mathcal{H}$ and $\mathcal{G}$ are families of subgroups for H and G such that $\alpha(K) \in \mathcal{G}$ holds for all $K \in \mathcal{H}$. Then we obtain a G-map $f\colon \operatorname{ind}_\alpha E_{\mathcal{H}}(H) \to E_{\mathcal{G}}(G)$ from the universal property of $E_{\mathcal{G}}(G)$. Let $p\colon \operatorname{ind}_\alpha \mathrm{pt} = G/\alpha(H) \to \mathrm{pt}$ be the projection. Let $I\colon \mathsf{GROUPS} \to \mathsf{GROUPOIDS}$ be the functor sending G to $\mathcal{G}^G(G/G)$. Then the following diagram, where the horizontal arrows are induced from the projections to the one point space, commutes for all $n \in \mathbb{Z}$.

$$\begin{array}{ccc}
H_n^H(E_{\mathcal{H}}(H);\mathbf{E}) & \xrightarrow{A_{\mathcal{H}}} & H_n^H(\mathrm{pt};\mathbf{E}) = \pi_n(\mathbf{E}(I(H))) \\
{\scriptstyle H_n^G(f)\circ\operatorname{ind}_\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle H_n^G(p)\circ\operatorname{ind}_\alpha=\pi_n(\mathbf{E}(I(\alpha)))} \\
H_n^G(E_{\mathcal{G}}(G);\mathbf{E}) & \xrightarrow[A_{\mathcal{G}}]{} & H_n^G(\mathrm{pt};\mathbf{E}) = \pi_n(\mathbf{E}(I(G))).
\end{array}$$

If we take the special case $\mathbf{E} = \mathbf{K}_R$ and $\mathcal{H} = \mathcal{G} = \mathcal{VCY}$, we get the following commutative diagram, where the horizontal maps are the assembly maps appearing in the Farrell-Jones Conjecture 2.2 and α_* is the change of rings homomorphism (induction) associated to α.

$$\begin{array}{ccc}
H_n^H(E_{\mathcal{VCY}}(H);\mathbf{K}_R) & \xrightarrow{A_{\mathcal{VCY}}} & K_n(RH) \\
{\scriptstyle H_n^G(f)\circ\operatorname{ind}_\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha_*} \\
H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) & \xrightarrow[A_{\mathcal{VCY}}]{} & K_n(RG).
\end{array}$$

We see that we can define a kind of induction homomorphism on the source of the assembly maps which is compatible with the induction structure given on their target. We get analogous diagrams for the L-theoretic version of the Farrell-Jones-Isomorphism Conjecture 2.2, for the Bost Conjecture 4.2 and for the Baum-Connes Conjecture for maximal group C^*-algebras (see (22) in Subsection 4.1.2).

Remark 6.13. The situation for the Baum-Connes Conjecture 2.3 itself, where one has to work with reduced C^*-algebras, is more complicated. Recall that not every group homomorphism $\alpha\colon H \to G$ induces a homomorphisms of C^*-algebras $C_r^*(H) \to C_r^*(G)$. (It does if $\ker(\alpha)$ is finite.) But it turns out that the source $H_n^H(E_{\mathcal{FIN}}(H); \mathbf{K}^{\mathrm{top}})$ always admits such a homomorphism. The point is that the isotropy groups of $E_{\mathcal{FIN}}(H)$ are all finite and the spectra-valued functor $\mathbf{K}^{\mathrm{top}}$ extends from $\mathsf{GROUPOIDS}^{\mathrm{inj}}$ to the category $\mathsf{GROUPOIDS}^{\mathrm{finker}}$, which has small groupoids as objects but as morphisms only those functors $f\colon \mathcal{G}_0 \to \mathcal{G}_1$ with finite kernels (in the sense that for each object $x \in \mathcal{G}_0$ the group homomorphism $\mathrm{aut}_{\mathcal{G}_0}(x) \to \mathrm{aut}_{\mathcal{G}_1}(f(x))$ has finite kernel). This is enough to get for any group homomorphism $\alpha\colon H \to G$ an induced map $\mathrm{ind}_\alpha\colon H_n^H(X, A; \mathbf{K}^{\mathrm{top}}) \to H_n^G(\mathrm{ind}_\alpha(X, A); \mathbf{K}^{\mathrm{top}})$ provided that X is proper. Hence one can define an induction homomorphism for the source of the assembly map as above.

In particular the Baum-Connes Conjecture 2.3 predicts that for any group homomorphism $\alpha\colon H \to G$ there is an induced induction homomorphism $\alpha_*\colon K_n(C_r^*(H)) \to K_n(C_r^*(G))$ on the targets of the assembly maps although there is no induced homomorphism of C^*-algebras $C_r^*(H) \to C_r^*(G)$ in general.

7 Methods of Proof

In Chapter 2, we formulated the Baum-Connes Conjecture 2.2 and the Farrell-Jones Conjecture 2.3 in abstract homological terms. We have seen that this formulation was very useful in order to understand formal properties of assembly maps. But in order to actually prove cases of the conjectures one needs to interpret the assembly maps in a way that is more directly related to geometry or analysis. In this chapter we wish to explain such approaches to the assembly maps. We briefly survey some of the methods of proof that are used to attack the Baum-Connes Conjecture 2.3 and the Farrell-Jones Conjecture 2.2.

7.1 Analytic Equivariant *K*-Homology

Recall that the covariant functor $\mathbf{K}^{\mathrm{top}}\colon \mathsf{GROUPOIDS}^{\mathrm{inj}} \to \mathsf{SPECTRA}$ introduced in (39) defines an equivariant homology theory $H_*^?(-; \mathbf{K}^{\mathrm{top}})$ in the sense of Section 6.1 such that

$$H_n^G(G/H; \mathbf{K}^{\mathrm{top}}) = H_n^H(\mathrm{pt}; \mathbf{K}^{\mathrm{top}}) = \begin{cases} R(H) & \text{for even n;} \\ 0 & \text{for odd n,} \end{cases}$$

holds for all groups G and subgroups $H \subseteq G$ (see Proposition 6.8). Next we want to give for a proper G-CW-complex X an analytic definition of $H_n^G(X; \mathbf{K}^{\mathrm{top}})$.

Consider a locally compact proper G-space X. Recall that a G-space X is called *proper* if for each pair of points x and y in X there are open neighborhoods V_x of x and W_y of y in X such that the subset $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$ of G is finite. A G-CW-complex X is proper if and only if all its isotropy groups are finite [197, Theorem 1.23]. Let $C_0(X)$ be the C^*-algebra of continuous functions $f\colon X \to \mathbb{C}$ which vanish at infinity. The C^*-norm is the supremum norm. A *generalized elliptic G-operator* is a triple (U, ρ, F), which consists of a unitary representation $U\colon G \to \mathcal{B}(H)$ of G on a Hilbert space H, a $*$-representation $\rho\colon C_0(X) \to \mathcal{B}(H)$ such that $\rho(f \circ l_{g^{-1}}) = U(g) \circ \rho(f) \circ U(g)^{-1}$ holds for $g \in G$, and a bounded selfadjoint G-operator $F\colon H \to H$ such that the operators $\rho(f)(F^2 - 1)$ and $[\rho(f), F]$ are compact for all $f \in C_0(X)$. Here $\mathcal{B}(H)$ is the C^*-algebra of bounded operators $H \to H$, $l_{g^{-1}}\colon H \to H$ is given by multiplication with g^{-1}, and $[\rho(f), F] = \rho(f) \circ F - F \circ \rho(f)$. We also call such a triple (U, ρ, F) an *odd cycle*. If we additionally assume that H comes with a $\mathbb{Z}/2$-grading such that ρ preserves the grading if we equip $C_0(X)$ with the trivial grading, and F reverses it, then we call (U, ρ, F) an *even cycle*. This means that we have an orthogonal decomposition $H = H_0 \oplus H_1$ such that U, ρ and F look like

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix} \qquad \rho = \begin{pmatrix} \rho_0 & 0 \\ 0 & \rho_1 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}. \tag{40}$$

An important example of an even cocycle is described in Section 7.5. A cycle (U, ρ, f) is called *degenerate*, if for each $f \in C_0(X)$ we have $[\rho(f), F] = \rho(f)(F^2 - 1) = 0$. Two cycles (U_0, ρ_0, F_0) and (U_1, ρ_1, F_1) of the same parity are called *homotopic*, if $U_0 = U_1$, $\rho_0 = \rho_1$ and there exists a norm continuous path $F_t, t \in [0, 1]$ in $\mathcal{B}(H)$ such that for each $t \in [0, 1]$ the triple (U_0, ρ_0, F_t) is again a cycle of the same parity. Two cycles (U_0, ρ_0, F_0) and (U_1, ρ_1, F_1) are called *equivalent*, if they become homotopic after taking the direct sum with degenerate cycles of the same parity. Let $K_n^G(C_0(X))$ for even n be the set of equivalence classes of even cycles and $K_n^G(C_0(X))$ for odd n be the set of equivalence classes of odd cycles. These become abelian groups by the direct sum. The neutral element is represented by any degenerate cycle. The inverse of an even cycle is represented by the cycle obtained by reversing the grading of H. The inverse of an odd cycle (U, ρ, F) is represented by $(U, \rho, -F)$.

A proper G-map $f\colon X \to Y$ induces a map of C^*-algebras $C_0(f)\colon C_0(Y) \to C_0(X)$ by composition and thus in the obvious way a homomorphism of abelian groups $K_0^G(f)\colon K_0^G(C_0(X)) \to K_0^G(C_0(Y))$. It depends only on the proper G-homotopy class of f. One can show that this construction defines a G-homology theory on the category of finite proper G-CW-complexes. It extends to a G-homology theory K_*^G for all proper G-CW-complexes by

$$K_n^G(X) = \operatorname{colim}_{Y \in I(X)} K_n^G(C_0(Y)) \tag{41}$$

where $I(X)$ is the set of proper finite G-CW-subcomplexes $Y \subseteq X$ directed by inclusion. This definition is forced upon us by Lemma 2.7. The groups $K_n^G(X)$ and $K_n^G(C_0(X))$ agree for finite proper G-CW-complexes, in general they are different.

The cycles were introduced by Atiyah [11]. The equivalence relation, the group structure and the homological properties of $K_n^G(X)$ were established by

Kasparov [168]. More information about analytic K-homology can be found in Higson-Roe [154].

7.2 The Analytic Assembly Map

For for every G-CW-complex X the projection $\mathrm{pr}\colon X \to \mathrm{pt}$ induces a map

$$H_n^G(X;\mathbf{K}^{\mathrm{top}}) \to H_n^G(\mathrm{pt};\mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G)). \tag{42}$$

In the case where X is the proper G-space $E_{\mathcal{FIN}}(G)$ we obtain the assembly map appearing in the Baum-Connes Conjecture 2.3. We explain its analytic analogue

$$\mathrm{ind}_G\colon K_n^G(X) \to K_n(C_r^*(G)). \tag{43}$$

Note that we need to assume that X is a proper G-space since $K_n^G(X)$ was only defined for such spaces. It suffices to define the map for a finite proper G-CW-complex X. In this case it assigns to the class in $K_n^G(X) = K_n^G(C_0(X))$ represented by a cycle (U,ρ,F) its G-index in $K_n(C_r^*(G))$ in the sense of Mishencko-Fomenko [223]. At least in the simple case, where G is finite, we can give its precise definition. The odd K-groups vanish in this case and $K_0(C_r^*(G))$ reduces to the complex representation ring $R(G)$. If we write F in matrix form as in (40) then $P\colon H \to H$ is a G-equivariant Fredholm operator. Hence its kernel and cokernel are G-representations and the G-index of F is defined as $[\ker(P)] - [\mathrm{coker}(P)] \in R(G)$. In the case of an infinite group the kernel and cokernel are a priori not finitely generated projective modules over $C_r^*(G)$, but they are after a certain pertubation. Moreover the choice of the pertubation does not affect $[\ker(P)] - [\mathrm{coker}(P)] \in K_0(C_r^*(G))$.

The identification of the two assembly maps (42) and (43) has been carried out in Hambleton-Pedersen [142] using the universal characterization of the assembly map explained in [82, Section 6]. In particular for a proper G-CW-complex X we have an identification $H_n^G(X;\mathbf{K}^{\mathrm{top}}) \cong K_n^G(X)$. Notice that $H_n^G(X;\mathbf{K}^{\mathrm{top}})$ is defined for all G-CW-complexes, whereas $K_n^G(X)$ has only been introduced for proper G-CW-complexes.

Thus the Baum-Connes Conjecture 2.3 gives an index-theoretic interpretations of elements in $K_0(C_r^*(G))$ as generalized elliptic operators or cycles (U,ρ,F). We have explained already in Subsection 1.8.1 an application of this interpretation to the Trace Conjecture for Torsionfree Groups 1.37 and in Subsection 3.3.2 to the Stable Gromov-Lawson-Rosenberg Conjecture 3.23.

7.3 Equivariant KK-theory

Kasparov [170] developed *equivariant KK-theory*, which we will briefly explain next. It is one of the basic tools in the proofs of theorems about the Baum-Connes Conjecture 2.3.

A *G-C^*-algebra* A is a C^*-algebra with a G-action by $*$-automorphisms. To any pair of separable G-C^*-algebras (A,B) Kasparov assigns abelian groups $KK_n^G(A,B)$. If G is trivial, we write briefly $KK_n(A,B)$. We do not give the rather complicated definition but state the main properties.

If we equip $\mathbb{C}$ with the trivial G-action, then $KK_n^G(C_0(X),\mathbb{C})$ reduces to the abelian group $K_n^G(C_0(X))$ introduced in Section 7.1. The topological K-theory $K_n(A)$ of a C^*-algebra coincides with $KK_n(\mathbb{C},A)$. The equivariant KK-groups are covariant in the second and contravariant in the first variable under homomorphism of C^*-algebras. One of the main features is the bilinear Kasparov product

$$KK_i^G(A,B)\times KK_j^G(B,C)\to KK_{i+j}(A,C),\ (\alpha,\beta)\mapsto \alpha\otimes_B\beta. \tag{44}$$

It is associative and natural. A homomorphism $\alpha\colon A\to B$ defines an element in $KK_0(A,B)$. There are natural *descent homomorphisms*

$$j_G\colon KK_n^G(A,B)\to KK_n(A\rtimes_r G, B\rtimes_r G), \tag{45}$$

where $A\rtimes_r G$ and $B\rtimes_r G$ denote the reduced crossed product C^*-algebras.

7.4 The Dirac-Dual Dirac Method

A G-C^*-algebra A is called *proper* if there exists a locally compact proper G-space X and a G-homomorphism $\sigma\colon C_0(X)\to\mathcal{B}(A)$, $f\mapsto\sigma_f$ satisfying $\sigma_f(ab)=a\sigma_f(b)=\sigma_f(a)b$ for $f\in C_0(X)$, $a,b\in A$ and for every net $\{f_i\mid i\in I\}$, which converges to 1 uniformly on compact subsets of X, we have $\lim_{i\in I}\parallel\sigma_{f_i}(a)-a\parallel\ =\ 0$ for all $a\in A$. A locally compact G-space X is proper if and only if $C_0(X)$ is proper as a G-C^*-algebra.

Given a proper G-CW-complex X and a G-C^*-algebra A, we put

$$KK_n^G(X;A)=\operatorname{colim}_{Y\in I(X)}KK_n^G(C_0(Y),A), \tag{46}$$

where $I(Y)$ is the set of proper finite G-CW-subcomplexes $Y\subseteq X$ directed by inclusion. We have $KK_n^G(X;\mathbb{C})=K_n^G(X)$. There is an analytic index map

$$\operatorname{ind}_G^A\colon KK_n^G(X;A)\to K_n(A\rtimes_r G), \tag{47}$$

which can be identified with the assembly map appearing in the Baum-Connes Conjecture with Coefficients 4.3. The following result is proved in Tu [305] extending results of Kasparov-Skandalis [169], [172].

Theorem 7.1. *The Baum-Connes Conjecture with coefficients 4.3 holds for a proper G-C^*-algebra A, i.e.* $\operatorname{ind}_G^A\colon KK_n^G(E_{\mathcal{FIN}}(G);A)\to K_n(A\rtimes G)$ *is bijective.*

Now we are ready to state the *Dirac-dual Dirac method* which is the key strategy in many of the proofs of the Baum-Connes Conjecture 2.3 or the Baum-Connes Conjecture with coefficients 4.3.

Theorem 7.2 (Dirac-Dual Dirac Method). *Let G be a countable (discrete) group. Suppose that there exist a proper G-C^*-algebra A, elements $\alpha \in KK_i^G(A, \mathbb{C})$, called the* Dirac element, *and $\beta \in KK_i^G(\mathbb{C}, A)$, called the* dual Dirac element, *satisfying*

$$\beta \otimes_A \alpha \;=\; 1 \quad \in KK_0^G(\mathbb{C}, \mathbb{C}).$$

Then the Baum-Connes Conjecture 2.3 is true, or, equivalently, the analytic index map

$$\mathrm{ind}_G \colon K_n^G(X) \to K_n(C_r^*(G))$$

of 43 is bijective.

Proof. The index map ind_G is a retract of the bijective index map ind_G^A from Theorem 7.1. This follows from the following commutative diagram

$$\begin{array}{ccccc}
K_n^G(E_{\mathcal{FIN}}(G)) & \xrightarrow{-\otimes_{\mathbb{C}}\beta} & KK_n^G(E_{\mathcal{FIN}}(G); A) & \xrightarrow{-\otimes_A \alpha} & K_n^G(E_{\mathcal{FIN}}(G)) \\
\downarrow{\scriptstyle \mathrm{ind}_G} & & \downarrow{\scriptstyle \mathrm{ind}_G^A} & & \downarrow{\scriptstyle \mathrm{ind}_G} \\
K_n(C_r^*(G)) & \xrightarrow{-\otimes_{C_r^*(G)} j_G(\beta)} & K_n(A \rtimes_r G) & \xrightarrow{-\otimes_{A \rtimes_r} j_G(\alpha)} & K_n(C_r^*(G))
\end{array}$$

and the fact that the composition of both the top upper horizontal arrows and lower upper horizontal arrows are bijective. $\square$

7.5 An Example of a Dirac Element

In order to give a glimpse of the basic ideas from operator theory we briefly describe how to define the Dirac element α in the case where G acts by isometries on a complete Riemannian manifold M. Let $T_{\mathbb{C}}M$ be the complexified tangent bundle and let $\mathrm{Cliff}(T_{\mathbb{C}}M)$ be the associated Clifford bundle. Let A be the proper G-C^*-algebra given by the sections of $\mathrm{Cliff}(T_{\mathbb{C}}M)$ which vanish at infinity. Let H be the Hilbert space $L^2(\wedge T_{\mathbb{C}}^*M)$ of L^2-integrable differential forms on $T_{\mathbb{C}}M$ with the obvious $\mathbb{Z}/2$-grading coming from even and odd forms. Let U be the obvious G-representation on H coming from the G-action on M. For a 1-form ω on M and $u \in H$ define a $*$-homomorphism $\rho\colon A \to \mathcal{B}(H)$ by

$$\rho_\omega(u) \;:=\; \omega \wedge u + i_\omega(u).$$

Now $D = (d + d^*)$ is a symmetric densely defined operator $H \to H$ and defines a bounded selfadjoint operator $F\colon H \to H$ by putting $F = \frac{D}{\sqrt{1+D^2}}$. Then (U, ρ, F) is an even cocycle and defines an element $\alpha \in K_0^G(M) = KK_0^G(C_0(M), \mathbb{C})$. More details of this construction and the construction of the dual Dirac element β under the assumption that M has non-positive curvature and is simply connected, can be found for instance in [307, Chapter 9].

7.6 Banach KK-Theory

Skandalis showed that the Dirac-dual Dirac method cannot work for all groups [287] as long as one works with KK-theory in the unitary setting. The problem is that for a group with property (T) the trivial and the regular unitary representation cannot be connected by a continuous path in the space of unitary representations, compare also the discussion in [163]. This problem can be circumvented if one drops the condition unitary and works with a variant of KK-theory for Banach algebras as worked out by Lafforgue [183], [185], [186].

7.7 Controlled Topology and Algebra

To a topological problem one can often associate a notion of "size". We describe a prototypical example. Let M be a Riemannian manifold. Recall that an h-cobordism W over $M = \partial^- W$ admits retractions $r^\pm \colon W \times I \to W$, $(x,t) \mapsto r_t^\pm(x,t)$ which retract W to $\partial^\pm W$, i.e. which satisfy $r_0^\pm = \mathrm{id}_W$ and $r_1^\pm(W) \subset \partial^\pm W$. Given $\epsilon > 0$ we say that W is ϵ-controlled if the retractions can be chosen in such a way that for every $x \in W$ the paths (called tracks of the h-cobordism) $p_x^\pm \colon I \to M$, $t \mapsto r_1^- \circ r_t^\pm(x)$ both lie within an ϵ-neighbourhood of their starting point. The usefulness of this concept is illustrated by the following theorem [124].

Theorem 7.3. *Let M be a compact Riemannian manifold of dimension ≥ 5. Then there exists an $\epsilon = \epsilon_M > 0$, such that every ϵ-controlled h-cobordism over M is trivial.*

If one studies the s-Cobordism Theorem 1.5 and its proof one is naturally lead to algebraic analogues of the notions above. A (geometric) R-module over the space X is by definition a family $M = (M_x)_{x \in X}$ of free R-modules indexed by points of X with the property that for every compact subset $K \subset X$ the module $\oplus_{x \in K} M_x$ is a finitely generated R-module. A morphism ϕ from M to N is an R-linear map $\phi = (\phi_{y,x}) \colon \oplus_{x \in X} M_x \to \oplus_{y \in X} N_y$. Instead of specifying fundamental group data by paths (analogues of the tracks of the h-cobordism) one can work with modules and morphisms over the universal covering $\widetilde{X}$, which are invariant under the operation of the fundamental group $G = \pi_1(X)$ via deck transformations, i.e. we require that $M_{gx} = M_x$ and $\phi_{gy,gx} = \phi_{y,x}$. Such modules and morphisms form an additive category which we denote by $\mathcal{C}^G(\widetilde{X}; R)$. In particular one can apply to it the non-connective K-theory functor $\mathbf{K}$ (compare [237]). In the case where X is compact the category is equivalent to the category of finitely generated free RG-modules and hence $\pi_* \mathbf{K} \mathcal{C}^G(\widetilde{X}; R) \cong K_*(RG)$. Now suppose $\widetilde{X}$ is equipped with a G-invariant metric, then we will say that a morphism $\phi = (\phi_{y,x})$ is ϵ-controlled if $\phi_{y,x} = 0$, whenever x and y are further than ϵ apart. (Note that ϵ-controlled morphisms do not form a category because the composition of two such morphisms will in general be 2ϵ-controlled.)

Theorem 7.3 has the following algebraic analogue [251] (see also Section 4 in [240]).

Theorem 7.4. *Let M be a compact Riemannian manifold with fundamental group G. There exists an $\epsilon = \epsilon_M > 0$ with the following property. The K_1-class of every G-invariant automorphism of modules over $\widetilde{M}$ which together with its inverse is ϵ-controlled lies in the image of the classical assembly map*

$$H_1(BG; \mathbf{K}R) \to K_1(RG) \cong K_1(\mathcal{C}^G(\widetilde{M}; R)).$$

To understand the relation to Theorem 7.3 note that for $R = \mathbb{Z}$ such an ϵ-controlled automorphism represents the trivial element in the Whitehead group which is in bijection with the h-cobordisms over M, compare Theorem 1.5.

There are many variants to the simple concept of "metric ϵ-control" we used above. In particular it is very useful to not measure size directly in M but instead use a map $p \colon M \to X$ to measure size in some auxiliary space X. (For example we have seen in Subsection 1.2.3 and 1.4.2 that "bounded" control over $\mathbb{R}^k$ may be used in order to define or describe negative K-groups.)

Before we proceed we would like to mention that there are analogous control-notions for pseudoisotopies and homotopy equivalences. The tracks of a pseudoisotopy $f \colon M \times I \to M \times I$ are defined as the paths in M which are given by the composition

$$p_x \colon I = \{x\} \times I \subset M \times I \xrightarrow{f} M \times I \xrightarrow{p} M$$

for each $x \in M$, where the last map is the projection onto the M-factor. Suppose $f \colon N \to M$ is a homotopy equivalence, $g \colon M \to N$ its inverse and h_t and h'_t are homotopies from $f \circ g$ to id_M respectively from $g \circ f$ to id_N then the tracks are defined to be the paths in M that are given by $t \mapsto h_t(x)$ for $x \in M$ and $t \mapsto f \circ h'_t(y)$ for $y \in N$. In both cases, for pseudoisotopies and for homotopy equivalences, the tracks can be used to define ϵ-control.

7.8 Assembly as Forget Control

If instead of a single problem over M one defines a family of problems over $M \times [1, \infty)$ and requires the control to tend to zero for $t \to \infty$ in a suitable sense, then one obtains something which is a homology theory in M. Relaxing the control to just bounded families yields the classical assembly map. This idea appears in [252] in the context of pseudoisotopies and in a more categorical fashion suitable for higher algebraic K-theory in [55] and [241]. We spell out some details in the case of algebraic K-theory, i.e. for geometric modules.

Let M be a Riemannian manifold with fundamental group G and let $\mathcal{S}(1/t)$ be the space of all functions $[1, \infty) \to [0, \infty)$, $t \mapsto \delta_t$ such that $t \mapsto t \cdot \delta_t$ is bounded. Similarly let $\mathcal{S}(1)$ be the space of all functions $t \mapsto \delta_t$ which are bounded. Note that $\mathcal{S}(1/t) \subset \mathcal{S}(1)$. A G-invariant morphism ϕ over $\widetilde{M} \times [1,\infty)$ is $\mathcal{S}$-controlled for $\mathcal{S} = \mathcal{S}(1)$ or $\mathcal{S}(1/t)$ if there exists an $\alpha > 0$ and a $\delta_t \in \mathcal{S}$ (both depending on the morphism) such that $\phi_{(x,t),(x',t')} \neq 0$ implies that $|t - t'| \leq \alpha$ and $d_{\widetilde{M}}(x,x') \leq \delta_{\min\{t,t'\}}$. We denote by $\mathcal{C}^G(\widetilde{M} \times [1,\infty), \mathcal{S}; R)$ the category of all $\mathcal{S}$-controlled morphisms. Furthermore $\mathcal{C}^G(\widetilde{M} \times [1,\infty), \mathcal{S}; R)^\infty$ denotes the quotient category which has the same objects, but where two morphisms are identified, if their difference factorizes over an object which lives over $\widetilde{M} \times [1, N]$ for some large but finite number N. This passage to the quotient category is called "taking germs at infinity". It is a special case of a Karoubi quotient, compare [51].

Theorem 7.5 (Classical Assembly as Forget Control). *Suppose M is aspherical, i.e. M is a model for BG, then for all $n \in \mathbb{Z}$ the map*

$$\pi_n(\mathbf{K}\mathcal{C}^G(\widetilde{M} \times [1,\infty), \mathcal{S}(1/t); R)^\infty) \to \pi_n(\mathbf{K}\mathcal{C}^G(\widetilde{M} \times [1,\infty), \mathcal{S}(1); R)^\infty)$$

can be identified up to an index shift with the classical assembly map that appears in Conjecture 1.11, i.e. with

$$H_{n-1}(BG; \mathbf{K}(R)) \to K_{n-1}(RG).$$

Note that the only difference between the left and the right hand side is that on the left morphism are required to become smaller in a $1/t$-fashion, whereas on the right hand side they are only required to stay bounded in the $[1,\infty)$-direction.

Using so called equivariant continuous control (see [7] and [19, Section 2] for the equivariant version) one can define an equivariant homology theory which applies to arbitrary G-CW-complexes. This leads to a "forget-control description" for the generalized assembly maps that appear in the Farrell-Jones Conjecture 2.2. Alternatively one can use stratified spaces and stratified Riemannian manifolds in order to describe generalized assembly maps in terms of metric control. Compare [111, 3.6 on p.270] and [252, Appendix].

7.9 Methods to Improve Control

From the above description of assembly maps we learn that the problem of proving surjectivity results translates into the problem of improving control. A combination of many different techniques is used in order to achieve such control-improvements. We discuss some prototypical arguments which go back to [98] and [102] and again restrict attention to K-theory. Of course this can only give a very incomplete impression of the whole program which is mainly due to Farrell-Hsiang and Farrell-Jones. The reader should consult [120] and [160] for a more detailed survey.

We restrict to the basic case, where M is a compact Riemannian manifold with negative sectional curvature. In order to explain a contracting property of the geodesic flow $\Phi\colon \mathbb{R} \times S\widetilde{M} \to S\widetilde{M}$ on the unit sphere bundle $S\widetilde{M}$, we introduce the notion of foliated control. We think of $S\widetilde{M}$ as a manifold equipped with the one-dimensional foliation by the flow lines of Φ and equip it with its natural Riemannian metric. Two vectors v and w in $S\widetilde{M}$ are called foliated (α, δ)-controlled if there exists a path of length α inside one flow line such that v lies within distance $\delta/2$ of the starting point of that path and w lies within distance $\delta/2$ of its endpoint.

Two vectors v and $w \in S\widetilde{M}$ are called asymptotic if the distance between their associated geodesic rays is bounded. These rays will then determine the same point on the sphere at infinity which can be introduced to compactify $\widetilde{M}$ to a disk. Recall that the universal covering of a negatively curved manifold is diffeomorphic to $\mathbb{R}^n$. Suppose v and w are α-controlled asymptotic vectors, i.e. their distance is smaller than $\alpha > 0$. As a consequence of negative sectional curvature the vectors $\Phi_t(v)$ and $\Phi_t(w)$ are foliated $(C\alpha, \delta_t)$-controlled, where $C > 1$ is a constant and $\delta_t > 0$ tends to zero when t tends to ∞. So roughly speaking the flow contracts the directions transverse to the flow lines and leaves the flow direction as it is, at least if we only apply it to asymptotic vectors.

This property can be used in order to find foliated (α, δ)-controlled representatives of K-theory classes with arbitrary small δ if one is able to define a suitable transfer from M to $S\widetilde{M}$, which yields representatives whose support is in an asymptotic starting position for the flow. Here one needs to take care of the additional problem that in general such a transfer may not induce an isomorphism in K-theory.

Finally one is left with the problem of improving foliated control to ordinary control. Corresponding statements are called "Foliated Control Theorems". Compare [18], [101], [103], [104] and [108].

If such an improvement were possible without further hypothesis, we could prove that the classical assembly map, i.e. the assembly map with respect to the trivial family is surjective. We know however that this is not true in general. It fails for example in the case of topological pseudoisotopies or for algebraic K-theory with arbitrary coefficients. In fact the geometric arguments that are involved in a "Foliated Control Theorem" need to exclude flow lines in $S\widetilde{M}$ which correspond to "short" closed geodesic loops in SM. But the techniques mentioned above can be used in order to achieve ϵ-control for arbitrary small $\epsilon > 0$ outside of a suitably chosen neighbourhood of "short" closed geodesics. This is the right kind of control for the source of the assembly map which involves the family of cyclic subgroups. (Note that a closed a loop in M determines the conjugacy class of a maximal infinite cyclic subgroup inside $G = \pi_1(M)$.) We see that even in the torsionfree case the class of cyclic subgroups of G naturally appears during the proof of a surjectivity result.

Another source for processes which improve control are expanding self-maps. Think for example of an n-torus $\mathbb{R}^n/\mathbb{Z}^n$ and the self-map f_s which is induced by $m_s \colon \mathbb{R}^n \to \mathbb{R}^n$, $x \to sx$ for a large positive integer s. If one pulls an automorphism back along such a map one can improve control, but unfortunately the new automorphism describes a different K-theory class.

Additional algebraic arguments nevertheless make this technique very successful. Compare for example [98]. Sometimes a clever mixture between flows and expanding self-maps is needed in order to achieve the goal, compare [105]. Recent work of Farrell-Jones (see [114], [115], [116] and [162]) makes use of a variant of the Cheeger-Fukaya-Gromov collapsing theory.

Remark 7.6 (Algebraicizing the Farrell-Jones Approach). In this Subsection we sketched some of the geometric ideas which are used in order to obtain control over an h-cobordism, a pseudisotopy or an automorphism of a geometric module representing a single class in K_1. In Subsection 7.8 we used families over the cone $M \times [1, \infty)$ in order to described the whole algebraic K-theory assembly map at once in categorical terms without ever referring to a single K-theory element. The recent work [21] shows that the geometric ideas can be adapted to this more categorical set-up, at least in the case where the group is the fundamental group of a Riemannian manifold with strictly negative curvature. However serious difficulties had to be overcome in order to achieve this. One needs to formulate and prove a Foliated Control Theorem in this context and also construct a transfer map to the sphere bundle for higher K-theory which is in a suitable sense compatible with the control structures.

7.10 The Descent Principle

In Theorem 7.5 we described the classical assembly map as a forget control map using G-invariant geometric modules over $\widetilde{M} \times [1, \infty)$. If in that context one does not require the modules and morphisms to be invariant under the G-action one nevertheless obtains a forget control functor between additive categories for which we introduce the notation

$$\mathcal{D}(1/t) = \mathcal{C}(\widetilde{M} \times [1, \infty), \mathcal{S}(1/t); R)^{\infty} \to \mathcal{D}(1) = \mathcal{C}(\widetilde{M} \times [1, \infty), \mathcal{S}(1); R)^{\infty}.$$

Applying K-theory yields a version of a "coarse" assembly map which is the algebraic K-theory analogue of the map described in Section 4.1.5. A crucial feature of such a construction is that the left hand side can be interpreted as a locally finite homology theory evaluated on $\widetilde{M}$. It is hence an invariant of the proper homotopy type of $\widetilde{M}$. Compare [7] and [326]. It is usually a lot easier to prove that this coarse assembly map is an equivalence. Suppose for example that M has non-positive curvature, choose a point $x_0 \in M$ (this will destroy the G-invariance) and with increasing $t \in [1, \infty)$ move the modules along geodesics towards x_0. In this way one can show that the coarse assembly map is an isomorphism. Such coarse assembly maps exist also in the context of algebraic L-theory and topological K-theory, compare [151], [263].

Results about these maps (compare e.g. [17], [55], [336], [338]) lead to injectivity results for the classical assembly map by the "descent principle" (compare [52], [55], [263]) which we will now briefly describe in the context of algebraic K-theory. (We already stated an analytic version in Section 4.1.5.) For a spectrum $\mathbf{E}$ with G-action we denote by $\mathbf{E}^{hG}$ the homotopy fixed points.

Since there is a natural map from fixed points to homotopy fixed points we obtain a commutative diagram

$$\begin{tikzcd} \mathbf{K}(\mathcal{D}(1/t))^G \arrow[r] \arrow[d] & \mathbf{K}(\mathcal{D}(1))^G \arrow[d] \\ \mathbf{K}(\mathcal{D}(1/t))^{hG} \arrow[r] & \mathbf{K}(\mathcal{D}(1))^{hG}. \end{tikzcd}$$

If one uses a suitable model K-theory commutes with taking fixed points and hence the upper horizontal map can be identified with the classical assembly map by Theorem 7.5. Using that K-theory commutes with infinite products [54], one can show by an induction over equivariant cells, that the vertical map on the left is an equivalence. Since we assume that the map $\mathbf{K}(\mathcal{D}(1/t)) \to \mathbf{K}(\mathcal{D}(1))$ is an equivalence, a standard property of the homotopy fixed point construction implies that the lower horizontal map is an equivalence. It follows that the upper horizontal map and hence the classical assembly map is split injective. A version of this argument which involves the assembly map for the family of finite subgroups can be found in [271].

7.11 Comparing to Other Theories

Every natural transformation of G-homology theories leads to a comparison between the associated assembly maps. For example one can compare topological K-theory to periodic cyclic homology [72], i.e. for every Banach algebra completion $\mathcal{A}(G)$ of $\mathbb{C}G$ inside $C_r^*(G)$ there exists a commutative diagram

$$\begin{tikzcd} K_*(BG) \arrow[r] \arrow[d] & K_*(\mathcal{A}(G)) \arrow[d] \\ H_*(BG; HP_*(\mathbb{C})) \arrow[r] & HP_*(\mathcal{A}(G)). \end{tikzcd}$$

This is used in [72] to prove injectivity results for word hyperbolic groups. Similar diagrams exist for other cyclic theories (compare for example [246]).

A suitable model for the cyclotomic trace $trc\colon K_n(RG) \to TC_n(RG)$ from (connective) algebraic K-theory to topological cyclic homology [38] leads for every family $\mathcal{F}$ to a commutative diagram

$$\begin{tikzcd} H_n(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{Z}}^{con}) \arrow[r] \arrow[d] & K_n^{con}(\mathbb{Z}G) \arrow[d] \\ H_n(E_{\mathcal{F}}(G); \mathbf{TC}_{\mathbb{Z}}) \arrow[r] & TC_n(\mathbb{Z}G). \end{tikzcd}$$

Injectivity results about the left hand and the lower horizontal map lead to injectivity results about the upper horizontal map. This is the principle behind Theorem 5.23 and 5.24.

8 Computations

Our ultimate goal is to compute K- and L-groups such as $K_n(RG)$, $L_n^{\langle -\infty \rangle}(RG)$ and $K_n(C_r^*(G))$. Assuming that the Baum-Connes Conjecture 2.3 or the Farrell-Jones Conjecture 2.2 is true, this reduces to the computation of the left hand side of the corresponding assembly map, i.e. to $H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}^{\mathrm{top}})$, $H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R)$ and $H_n^G(E_{\mathcal{VCY}}(G); \mathbf{L}_R^{\langle -\infty \rangle})$. This is much easier since here we can use standard methods from algebraic topology such as spectral sequences, Mayer-Vietoris sequences and Chern characters. Nevertheless such computations can be pretty hard. Roughly speaking, one can obtain a general reasonable answer after rationalization, but integral computations have only been done case by case and no general pattern is known.

8.1 *K*- and *L*- Groups for Finite Groups

In all these computations the answer is given in terms of the values of $K_n(RG)$, $L_n^{\langle -\infty \rangle}(RG)$ and $K_n(C_r^*(G))$ for finite groups G. Therefore we briefly recall some of the results known for finite groups focusing on the case $R = \mathbb{Z}$

8.1.1 Topological *K*-Theory for Finite Groups

Let G be a finite group. By $r_F(G)$, we denote the number of isomorphism classes of irreducible representations of G over the field F. By $r_{\mathbb{R}}(G;\mathbb{R})$, $r_{\mathbb{R}}(G;\mathbb{C})$, respectively $r_{\mathbb{R}}(G;\mathbb{H})$ we denote the number of isomorphism classes of irreducible real G-representations V, which are of real, complex respectively of quaternionic type, i.e. $\mathrm{aut}_{\mathbb{R}G}(V)$ is isomorphic to the field of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$. Let $RO(G)$ respectively $R(G)$ be the *real* respectively the *complex representation ring.*

Notice that $\mathbb{C}G = l^1(G) = C_r^*(G) = C_{\max}^*(G)$ holds for a finite group, and analogous for the real versions.

Proposition 8.1. *Let G be a finite group.*

(i) We have

$$K_n(C_r^*(G)) \;\cong\; \begin{cases} R(G) \cong \mathbb{Z}^{r_{\mathbb{C}}(G)} & \text{for } n \text{ even;} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

(ii) There is an isomorphism of topological K-groups

$$K_n(C_r^*(G;\mathbb{R})) \;\cong\; K_n(\mathbb{R})^{r_{\mathbb{R}}(G;\mathbb{R})} \times K_n(\mathbb{C})^{r_{\mathbb{R}}(G;\mathbb{C})} \times K_n(\mathbb{H})^{r_{\mathbb{R}}(G;\mathbb{H})}.$$

Moreover $K_n(\mathbb{C})$ is 2-periodic with values $\mathbb{Z}$, 0 for $n = 0, 1$, $K_n(\mathbb{R})$ is 8-periodic with values $\mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/2$, 0, $\mathbb{Z}$, 0, 0, 0 for $n = 0, 1, \ldots, 7$ and $K_n(\mathbb{H}) = K_{n+4}(\mathbb{R})$ for $n \in \mathbb{Z}$.

Proof. One gets isomorphisms of C^*-algebras

$$C_r^*(G) \;\cong\; \prod_{j=1}^{r_{\mathbb{C}}(G)} M_{n_i}(\mathbb{C})$$

and

$$C_r^*(G;\mathbb{R}) \;\cong\; \prod_{i=1}^{r_{\mathbb{R}}(G;\mathbb{R})} M_{m_i}(\mathbb{R}) \times \prod_{i=1}^{r_{\mathbb{R}}(G;\mathbb{C})} M_{n_i}(\mathbb{C}) \times \prod_{i=1}^{r_{\mathbb{R}}(G;\mathbb{H})} M_{p_i}(\mathbb{H})$$

from [283, Theorem 7 on page 19, Corollary 2 on page 96, page 102, page 106]. Now the claim follows from Morita invariance and the well-known values for $K_n(\mathbb{R})$, $K_n(\mathbb{C})$ and $K_n(\mathbb{H})$ (see for instance [301, page 216]). □

To summarize, the values of $K_n(C_r^*(G))$ and $K_n(C_r^*(G;\mathbb{R}))$ are explicitly known for finite groups G and are in the complex case in contrast to the real case always torsion free.

8.1.2 Algebraic *K*-Theory for Finite Groups

Here are some facts about the algebraic K-theory of integral group rings of finite groups.

Proposition 8.2. *Let G be a finite group.*

(i) $K_n(\mathbb{Z}G) = 0$ *for* $n \leq -2$.

(ii) We have

$$K_{-1}(\mathbb{Z}G) \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2)^s,$$

where

$$r = 1 - r_{\mathbb{Q}}(G) + \sum_{p \mid |G|} r_{\mathbb{Q}_p}(G) - r_{\mathbb{F}_p}(G)$$

and the sum runs over all primes dividing the order of G. (Recall that $r_F(G)$ denotes the number of isomorphism classes of irreducible representations of G over the field F.) There is an explicit description of the integer s in terms of global and local Schur indices [58]. If G contains a normal abelian subgroup of odd index, then $s = 0$.

(iii) The group $\widetilde{K}_0(\mathbb{Z}G)$ is finite.

(iv) The group $\mathrm{Wh}(G)$ *is a finitely generated abelian group and its rank is* $r_{\mathbb{R}}(G) - r_{\mathbb{Q}}(G)$.

(v) The groups $K_n(\mathbb{Z}G)$ are finitely generated for all $n \in \mathbb{Z}$.

(vi) We have $K_{-1}(\mathbb{Z}G) = 0$, $\widetilde{K}_0(\mathbb{Z}G) = 0$ *and* $\mathrm{Wh}(G) = 0$ *for the following finite groups* $G = \{1\}$, $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/4$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, D_6, D_8, *where D_m is the dihedral group of order m.*

If p is a prime, then $K_{-1}(\mathbb{Z}[\mathbb{Z}/p]) = K_{-1}(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}/p]) = 0$.

We have

$$K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong \mathbb{Z}, \quad \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/6]) = 0, \quad \mathrm{Wh}(\mathbb{Z}/6) = 0$$
$$K_{-1}(\mathbb{Z}[D_{12}]) \cong \mathbb{Z}, \quad \widetilde{K}_0(\mathbb{Z}[D_{12}]) = 0, \quad \mathrm{Wh}(D_{12}) = 0.$$

(vii) Let $\mathrm{Wh}_2(G)$ *denote the cokernel of the assembly map*

$$H_2(BG; \mathbf{K}(\mathbb{Z})) \to K_2(\mathbb{Z}G).$$

We have $\mathrm{Wh}_2(G) = 0$ *for* $G = \{1\}$*,* $\mathbb{Z}/2$*,* $\mathbb{Z}/3$ *and* $\mathbb{Z}/4$*. Moreover* $|\,\mathrm{Wh}_2(\mathbb{Z}/6)| \leq 2$*,* $|\,\mathrm{Wh}_2(\mathbb{Z}/2 \times \mathbb{Z}/2)| \geq 2$ *and* $\mathrm{Wh}_2(D_6) = \mathbb{Z}/2$*.*

Proof. (i) and (ii) are proved in [58].

(iii) is proved in [298, Proposition 9.1 on page 573].

(iv) This is proved for instance in [232].

(v) See [181], [248].

(vi) and (vii) The computation $K_{-1}(\mathbb{Z}G) = 0$ for $G = \mathbb{Z}/p$ or $\mathbb{Z}/p \times \mathbb{Z}/p$ can be found in [22, Theorem 10.6, p. 695] and is a special case of [58].

The vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ is proven for $G = D_6$ in [262, Theorem 8.2] and for $G = D_8$ in [262, Theorem 6.4]. The cases $G = \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/6$, and $(\mathbb{Z}/2)^2$ are treated in [79, Corollary 5.17]. Finally, $\widetilde{K}_0(\mathbb{Z}D_{12}) = 0$ follows from [79, Theorem 50.29 on page 266] and the fact that $\mathbb{Q}D_{12}$ as a $\mathbb{Q}$-algebra splits into copies of $\mathbb{Q}$ and matrix algebras over $\mathbb{Q}$, so its maximal order has vanishing class group by Morita equivalence.

The claims about $\mathrm{Wh}_2(\mathbb{Z}/n)$ for $n = 2, 3, 4, 6$ and for $\mathrm{Wh}_2((\mathbb{Z}/2)^2)$ are taken from [85, Proposition 5], [89, p.482] and [296, p. 218 and 221].

We get $K_2(\mathbb{Z}D_6) \cong (\mathbb{Z}/2)^3$ from [296, Theorem 3.1]. The assembly map $H_2(B\mathbb{Z}/2; \mathbf{K}(\mathbb{Z})) \to K_2(\mathbb{Z}[\mathbb{Z}/2])$ is an isomorphism by [89, Theorem on p. 482]. Now construct a commutative diagram

$$\begin{array}{ccc}
H_2(B\mathbb{Z}/2; \mathbf{K}(\mathbb{Z})) & \xrightarrow{\cong} & H_2(BD_6; \mathbf{K}(\mathbb{Z})) \\
\cong\downarrow & & \downarrow \\
K_2(\mathbb{Z}[\mathbb{Z}/2]) & \longrightarrow & K_2(\mathbb{Z}D_6)
\end{array}$$

whose lower horizontal arrow is split injective and whose upper horizontal arrow is an isomorphism by the Atiyah-Hirzebruch spectral sequence. Hence the right vertical arrow is split injective and $\mathrm{Wh}_2(D_6) = \mathbb{Z}/2$. □

Let us summarize. We already mentioned that a complete computation of $K_n(\mathbb{Z})$ is not known. Also a complete computation of $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$ for arbitrary primes p is out of reach (see [221, page 29,30]). There is a complete formula for $K_{-1}(\mathbb{Z}G)$ and $K_n(\mathbb{Z}G) = 0$ for $n \leq -2$ and one has a good understanding of $\mathrm{Wh}(G)$ (see [232]). We have already mentioned Borel's formula for $K_n(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $n \in \mathbb{Z}$ (see Remark 1.14). For more rational information see also 8.9.

8.1.3 Algebraic *L*-Theory for Finite Groups

Here are some facts about L-groups of finite groups.

Proposition 8.3. *Let G be a finite group. Then*

(i) For each $j \leq 1$ the groups $L_n^{\langle j \rangle}(\mathbb{Z}G)$ are finitely generated as abelian groups and contain no p-torsion for odd primes p. Moreover, they are finite for odd n.
(ii) For every decoration $\langle j \rangle$ we have

$$L_n^{\langle j \rangle}(\mathbb{Z}G)[1/2] \;\cong\; L_n^{\langle j \rangle}(\mathbb{R}G)[1/2] \cong \begin{cases} \mathbb{Z}[1/2]^{r_{\mathbb{R}}(G)} & n \equiv 0 \ (4); \\ \mathbb{Z}[1/2]^{r_{\mathbb{C}}(G)} & n \equiv 2 \ (4); \\ 0 & n \equiv 1,3 \ (4). \end{cases}$$

(iii) If G has odd order and n is odd, then $L_n^{\epsilon}(\mathbb{Z}G) = 0$ for $\epsilon = p, h, s$.

Proof. (i) See for instance [143].
(ii) See [258, Proposition 22.34 on page 253].
(iii) See [13] or [143]. □

The L-groups of $\mathbb{Z}G$ are pretty well understood for finite groups G. More information about them can be found in [143].

8.2 Rational Computations for Infinite Groups

Next we state what is known rationally about the K- and L-groups of an infinite (discrete) group, provided the Baum-Connes Conjecture 2.3 or the relevant version of the Farrell-Jones Conjecture 2.2 is known.

In the sequel let $(\mathcal{FCY})$ be the set of conjugacy classes (C) for finite cyclic subgroups $C \subseteq G$. For $H \subseteq G$ let $N_G H = \{g \in G \mid gHg^{-1} = H\}$ be its *normalizer*, let $Z_G H = \{g \in G \mid ghg^{-1} = h \text{ for } h \in H\}$ be its *centralizer*, and put

$$W_G H := N_G H/(H \cdot Z_G H),$$

where $H \cdot Z_G H$ is the normal subgroup of $N_G H$ consisting of elements of the form hu for $h \in H$ and $u \in Z_G H$. Notice that $W_G H$ is finite if H is finite.

Recall that the *Burnside ring* $A(G)$ of a finite group is the Grothendieck group associated to the abelian monoid of isomorphism classes of finite G-sets with respect to the disjoint union. The ring multiplication comes from the cartesian product. The zero element is represented by the empty set, the unit is represented by $G/G = \text{pt}$. For a finite group G the abelian groups $K_q(C_r^*(G))$, $K_q(RG)$ and $L^{\langle -\infty \rangle}(RG)$ become modules over $A(G)$ because these functors come with a Mackey structure and $[G/H]$ acts by $\text{ind}_H^G \circ \text{res}_G^H$.

We obtain a ring homomorphism

$$\chi^G \colon A(G) \to \prod_{(H) \in \mathcal{FIN}} \mathbb{Z}, \qquad [S] \mapsto (|S^H|)_{(H) \in \mathcal{FIN}}$$

which sends the class of a finite G-set S to the element given by the cardinalities of the H-fixed point sets. This is an injection with finite cokernel. This leads to an isomorphism of $\mathbb{Q}$-algebras

$$\chi^G_{\mathbb{Q}} := \chi^G \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}} \colon A(G) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{(H)\in(\mathcal{FIN})} \mathbb{Q}. \tag{48}$$

For a finite cyclic group C let

$$\theta_C \ \in \ A(C) \otimes_{\mathbb{Z}} \mathbb{Z}[1/|C|] \tag{49}$$

be the element which is sent under the isomorphism $\chi^C_{\mathbb{Q}} \colon A(C) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{(H)\in\mathcal{FIN}} \mathbb{Q}$ of (48) to the element, whose entry is one if $(H) = (C)$ and is zero if $(H) \neq (C)$. Notice that θ_C is an idempotent. In particular we get a direct summand $\theta_C \cdot K_q(C^*_r(C)) \otimes_{\mathbb{Z}} \mathbb{Q}$ in $K_q(C^*_r(C)) \otimes_{\mathbb{Z}} \mathbb{Q}$ and analogously for $K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L^{\langle -\infty \rangle}(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$.

8.2.1 Rationalized Topological *K*-Theory for Infinite Groups

The next result is taken from [203, Theorem 0.4 and page 127]. Recall that Λ^G is the ring $\mathbb{Z} \subseteq \Lambda^G \subseteq \mathbb{Q}$ which is obtained from $\mathbb{Z}$ by inverting the orders of the finite subgroups of G.

Theorem 8.4 (Rational Computation of Topological K-Theory for Infinite Groups). *Suppose that the group G satisfies the Baum-Connes Conjecture 2.3. Then there is an isomorphism*

$$\bigoplus_{p+q=n} \ \bigoplus_{(C)\in(\mathcal{FCY})} K_p(BZ_GC) \otimes_{\mathbb{Z}[W_GC]} \theta_C \cdot K_q(C^*_r(C)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\cong} K_n(C^*_r(G)) \otimes_{\mathbb{Z}} \Lambda^G.$$

If we tensor with $\mathbb{Q}$, we get an isomorphism

$$\bigoplus_{p+q=n} \ \bigoplus_{(C)\in(\mathcal{FCY})} H_p(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot K_q(C^*_r(C)) \otimes_{\mathbb{Z}} \mathbb{Q}. \xrightarrow{\cong} K_n(C^*_r(G)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

8.2.2 Rationalized Algebraic *K*-Theory for Infinite Groups

Recall that for algebraic K-theory of the integral group ring we know because of Proposition 2.17 that in the Farrell-Jones Conjecture we can reduce to the family of finite subgroups. A reduction to the family of finite subgroups also works if the coefficient ring is a regular $\mathbb{Q}$-algebra, compare 2.14. The next result is a variation of [201, Theorem 0.4].

Theorem 8.5 (Rational Computation of Algebraic K-Theory). *Suppose that the group G satisfies the Farrell-Jones Conjecture 2.2 for the algebraic K-theory of RG, where either $R = \mathbb{Z}$ or R is a regular ring with $\mathbb{Q} \subset R$. Then we get an isomorphism*

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in(\mathcal{FCY})} H_p(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 8.6. If in Theorem 8.5 we assume the Farrell-Jones Conjecture for the algebraic K-theory of RG but make no assumption on the coefficient ring R, then we still obtain that the map appearing there is split injective.

Example 8.7 (The Comparison Map from Algebraic to Topological K-theory). If we consider $R = \mathbb{C}$ as coefficient ring and apply $-\otimes_{\mathbb{Z}}\mathbb{C}$ instead of $-\otimes_{\mathbb{Z}} \mathbb{Q}$, the formulas simplify. Suppose that G satisfies the Baum-Connes Conjecture 2.3 and the Farrell-Jones Conjecture 2.2 for algebraic K-theory with $\mathbb{C}$ as coefficient ring. Recall that $\mathrm{con}(G)_f$ is the set of conjugacy classes (g) of elements $g \in G$ of finite order. We denote for $g \in G$ by $\langle g \rangle$ the cyclic subgroup generated by g.

Then we get the following commutative square, whose horizontal maps are isomorphisms and whose vertical maps are induced by the obvious change of theory homorphisms (see [201, Theorem 0.5])

$$\begin{array}{ccc}
\bigoplus_{p+q=n} \bigoplus_{(g)\in\mathrm{con}(G)_f} H_p(Z_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \xrightarrow{\cong} & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\
\downarrow & & \downarrow \\
\bigoplus_{p+q=n} \bigoplus_{(g)\in\mathrm{con}(G)_f} H_p(Z_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\mathrm{top}}(\mathbb{C}) & \xrightarrow[\cong]{} & K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C}
\end{array}$$

The Chern character appearing in the lower row of the commutative square above has already been constructed by different methods in [26]. The construction in [201] works also for $\mathbb{Q}$ (and even smaller rings) and other theories like algebraic K- and L-theory. This is important for the proof of Theorem 3.22 and to get the commutative square above.

Example 8.8 (A Formula for $K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$). Suppose that the Farrell-Jones Conjecture is true rationally for $K_0(\mathbb{Z}G)$, i.e. the assembly map

$$A_{\mathcal{VCY}}\colon H_0^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. Then we obtain

$$K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \\ K_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \bigoplus_{(C)\in(\mathcal{FCY})} H_1(BZ_GC;\mathbb{Q}) \otimes_{\mathbb{Q}[W_GC]} \theta_C \cdot K_{-1}(RC) \otimes_{\mathbb{Z}} \mathbb{Q} \ .$$

Notice that $\widetilde{K}_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains only contributions from $K_{-1}(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}$ for finite cyclic subgroups $C \subseteq G$.

Remark 8.9. Note that these statements are interesting already for finite groups. For instance Theorem 8.4 yields for a finite group G and $R = \mathbb{C}$ an isomorphism

$$\bigoplus_{(C)\in(\mathcal{FCY})} \Lambda_G \otimes_{\Lambda_G[W_G C]} \theta_C \cdot R(C) \otimes_{\mathbb{Z}} \Lambda_G \ \cong\ R(G) \otimes_{\mathbb{Z}} \Lambda_G.$$

which in turn implies Artin's Theorem discussed in Remark 3.6.

8.2.3 Rationalized Algebraic L-Theory for Infinite Groups

Here is the L-theory analogue of the results above. Compare [201, Theorem 0.4].

Theorem 8.10 (Rational Computation of Algebraic L-Theory for Infinite Groups). *Suppose that the group G satisfies the Farrell-Jones Conjecture 2.2 for L-theory. Then we get for all $j \in \mathbb{Z}, j \leq 1$ an isomorphism*

$$\bigoplus_{p+q=n} \ \bigoplus_{(C)\in(\mathcal{FCY})} H_p(BZ_G C;\mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C \cdot L_q^{\langle j \rangle}(RC) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} L_n^{\langle j \rangle}(RG) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 8.11 (Separation of Variables). Notice that in Theorem 8.4, 8.5 and 8.10 we see again the principle we called *separation of variables* in Remark 1.13. There is a group homology part which is independent of the coefficient ring R and the K- or L-theory under consideration and a part depending only on the values of the theory under consideration on RC or $C_r^*(C)$ for all finite cyclic subgroups $C \subseteq G$.

8.3 Integral Computations for Infinite Groups

As mentioned above, no general pattern for integral calculations is known or expected. We mention at least one situation where a certain class of groups can be treated simultaneously. Let $\mathcal{MFI}$ be the subset of $\mathcal{FIN}$ consisting of elements in $\mathcal{FIN}$ which are maximal in $\mathcal{FIN}$. Consider the following assertions on the group G.

(M) $M_1, M_1 \in \mathcal{MFI}, M_1 \cap M_2 \neq 1 \ \Rightarrow\ M_1 = M_2$;
(NM) $M \in \mathcal{MFI} \ \Rightarrow\ N_G M = M$;
(VCL_I) If V is an infinite virtually cyclic subgroup of G, then V is of type I (see Lemma 2.15);
(FJK_N) The Isomorphism Conjecture of Farrell-Jones for algebraic K-theory 2.2 is true for $\mathbb{Z}G$ in the range $n \leq N$ for a fixed element $N \in \mathbb{Z} \amalg \{\infty\}$, i.e. the assembly map $A\colon \mathcal{H}_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) \xrightarrow{\cong} K_n(RG)$ is bijective for $n \in \mathbb{Z}$ with $n \leq N$.

Let $\widetilde{K}_n(C_r^*(H))$ be the cokernel of the map $K_n(C_r^*(\{1\})) \to K_n(C_r^*(H))$ and $\overline{L}_n^{\langle j \rangle}(RG)$ be the cokernel of the map $L_n^{\langle j \rangle}(R) \to L_n^{\langle j \rangle}(RG)$. This coincides with $\widetilde{L}_n^{\langle j \rangle}(R)$, which is the cokernel of the map $L_n^{\langle j \rangle}(\mathbb{Z}) \to L_n^{\langle j \rangle}(R)$ if $R = \mathbb{Z}$ but not in general. Denote by $\mathrm{Wh}_n^R(G)$ the n-th Whitehead group of RG which is the $(n-1)$-th homotopy group of the homotopy fiber of the assembly map $BG_+ \wedge \mathbf{K}(R) \to \mathbf{K}(RG)$. It agrees with the previous defined notions if $R = \mathbb{Z}$. The next result is taken from [83, Theorem 4.1].

Theorem 8.12. *Let $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ be a ring such that the order of any finite subgroup of G is invertible in Λ. Let $(\mathcal{MFI})$ be the set of conjugacy classes (H) of subgroups of G such that H belongs to $\mathcal{MFI}$. Then:*

(i) If G satisfies (M), (NM) and the Baum-Connes Conjecture 2.3, then for $n \in \mathbb{Z}$ there is an exact sequence of topological K-groups

$$0 \to \bigoplus_{(H) \in (\mathcal{MFI})} \widetilde{K}_n(C_r^*(H)) \to K_n(C_r^*(G)) \to K_n(G \backslash E_{\mathcal{FIN}}(G)) \to 0,$$

which splits after applying $- \otimes_{\mathbb{Z}} \Lambda$.

(ii) If G satisfies (M), (NM), (VCL_I) and the L-theory part of the Farrell-Jones Conjecture 2.2, then for all $n \in \mathbb{Z}$ there is an exact sequence

$$\ldots \to H_{n+1}(G \backslash E_{\mathcal{FIN}}(G); \mathbf{L}^{\langle -\infty \rangle}(R)) \to \bigoplus_{(H) \in (\mathcal{MFI})} \overline{L}_n^{\langle -\infty \rangle}(RH)$$
$$\to L_n^{\langle -\infty \rangle}(RG) \to H_n(G \backslash E_{\mathcal{FIN}}(G); \mathbf{L}^{\langle -\infty \rangle}(R)) \to \ldots$$

It splits after applying $- \otimes_{\mathbb{Z}} \Lambda$, more precisely

$$L_n^{\langle -\infty \rangle}(RG) \otimes_{\mathbb{Z}} \Lambda \to H_n(G \backslash E_{\mathcal{FIN}}(G); \mathbf{L}^{\langle -\infty \rangle}(R)) \otimes_{\mathbb{Z}} \Lambda$$

is a split-surjective map of Λ-modules.

(iii) If G satisfies (M), (NM) and the Farrell-Jones Conjecture 2.2 for $L_n(RG)[1/2]$, then the conclusion of assertion (ii) still holds if we invert 2 everywhere. Moreover, in the case $R = \mathbb{Z}$ the sequence reduces to a short exact sequence

$$0 \to \bigoplus_{(H) \in (\mathcal{MFI})} \widetilde{L}_n^{\langle j \rangle}(\mathbb{Z}H)[\frac{1}{2}] \to L_n^{\langle j \rangle}(\mathbb{Z}G)[\frac{1}{2}]$$
$$\to H_n(G \backslash E_{\mathcal{FIN}}(G); \mathbf{L}(\mathbb{Z})[\frac{1}{2}] \to 0,$$

which splits after applying $- \otimes_{\mathbb{Z}[\frac{1}{2}]} \Lambda[\frac{1}{2}]$.

(iv) If G satisfies (M), (NM), and (FJK_N), then there is for $n \in \mathbb{Z}, n \leq N$ an isomorphism

$$H_n(E_{\mathcal{VCY}}(G), E_{\mathcal{FIN}}(G); \mathbf{K}_R) \oplus \bigoplus_{(H) \in (\mathcal{MFI})} \mathrm{Wh}_n^R(H) \xrightarrow{\cong} \mathrm{Wh}_n^R(G),$$

where $\mathrm{Wh}_n^R(H) \to \mathrm{Wh}_n^R(G)$ is induced by the inclusion $H \to G$.

Remark 8.13 (Role of $G\backslash E_{\mathcal{FIN}}(G)$). Theorem 8.12 illustrates that for such computations a good understanding of the geometry of the orbit space $G\backslash E_{\mathcal{FIN}}(G)$ is necessary.

Remark 8.14. In [83] it is explained that the following classes of groups do satisfy the assumption appearing in Theorem 8.12 and what the conclusions are in the case $R = \mathbb{Z}$. Some of these cases have been treated earlier in [34], [212].

- Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$;
- Fuchsian groups F;
- One-relator groups G.

Theorem 8.12 is generalized in [204] in order to treat for instance the semi-direct product of the discrete three-dimensional Heisenberg group by $\mathbb{Z}/4$. For this group $G\backslash E_{\mathcal{FIN}}(G)$ is S^3.

A calculation for 2-dimensional crystallographic groups and more general cocompact NEC-groups is presented in [212] (see also [236]). For these groups the orbit spaces $G\backslash E_{\mathcal{FIN}}(G)$ are compact surfaces possibly with boundary.

Example 8.15. Let F be a cocompact Fuchsian group with presentation

$$F = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_t \mid c_1^{\gamma_1} = \ldots = c_t^{\gamma_t} = c_1^{-1} \cdots c_t^{-1}[a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

for integers $g, t \geq 0$ and $\gamma_i > 1$. Then $G\backslash E_{\mathcal{FIN}}(G)$ is a closed orientable surface of genus g. The following is a consequence of Theorem 8.12 (see [212] for more details).

- There are isomorphisms
$$K_n(C_r^*(F)) \cong \begin{cases} \left(2 + \sum_{i=1}^t (\gamma_i - 1)\right) \cdot \mathbb{Z} & n = 0; \\ (2g) \cdot \mathbb{Z} & n = 1. \end{cases}$$
- The inclusions of the maximal subgroups $\mathbb{Z}/\gamma_i = \langle c_i \rangle$ induce an isomorphism
$$\bigoplus_{i=1}^t \mathrm{Wh}_n(\mathbb{Z}/\gamma_i) \xrightarrow{\cong} \mathrm{Wh}_n(F)$$
for $n \leq 1$.
- There are isomorphisms
$$L_n(\mathbb{Z}F)[1/2] \cong \begin{cases} \left(1 + \sum_{i=1}^t \left[\frac{\gamma_i}{2}\right]\right) \cdot \mathbb{Z}[1/2] & n \equiv 0 \ (4); \\ (2g) \cdot \mathbb{Z}[1/2] & n \equiv 1 \ (4); \\ \left(1 + \sum_{i=1}^t \left[\frac{\gamma_i - 1}{2}\right]\right) \cdot \mathbb{Z}[1/2] & n \equiv 2 \ (4); \\ 0 & n \equiv 3 \ (4), \end{cases}$$

where $[r]$ for $r \in \mathbb{R}$ denotes the largest integer less than or equal to r.
From now on suppose that each γ_i is odd. Then the number m above is odd and we get for for $\epsilon = p$ and s

$$L_n^\epsilon(\mathbb{Z}F) \cong \begin{cases} \mathbb{Z}/2 \bigoplus \left(1 + \sum_{i=1}^t \frac{\gamma_i - 1}{2}\right) \cdot \mathbb{Z} & n \equiv 0 \ \ (4); \\ (2g) \cdot \mathbb{Z} & n \equiv 1 \ \ (4); \\ \mathbb{Z}/2 \bigoplus \left(1 + \sum_{i=1}^t \frac{\gamma_i - 1}{2}\right) \cdot \mathbb{Z} & q \equiv 2 \ \ (4); \\ (2g) \cdot \mathbb{Z}/2 & n \equiv 3 \ \ (4). \end{cases}$$

For $\epsilon = h$ we do not know an explicit formula. The problem is that no general formula is known for the 2-torsion contained in $\widetilde{L}_{2q}^h(\mathbb{Z}[\mathbb{Z}/m])$, for m odd, since it is given by the term $\widehat{H}^2(\mathbb{Z}/2; \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/m]))$, see [14, Theorem 2].

Information about the left hand side of the Farrell-Jones assembly map for algebraic K-theory in the case where G is $SL_3(\mathbb{Z})$ can be found in [306].

8.4 Techniques for Computations

We briefly outline some methods that are fundamental for computations and for the proofs of some of the theorems above.

8.4.1 Equivariant Atiyah-Hirzebruch Spectral Sequence

Let $\mathcal{H}_*^G$ be a G-homology theory with values in Λ-modules. Then there are two spectral sequences which can be used to compute it. The first one is the rather obvious equivariant version of the *Atiyah-Hirzebuch spectral sequence.* It converges to $\mathcal{H}_n^G(X)$ and its E^2-term is given in terms of Bredon homology

$$E_{p,q}^2 \;=\; H_p^G(X; \mathcal{H}_q^G(G/H))$$

of X with respect to the coefficient system, which is given by the covariant functor $\mathsf{Or}(G) \to \Lambda\text{-}\mathsf{MODULES}$, $G/H \mapsto \mathcal{H}_q^G(G/H)$. More details can be found for instance in [82, Theorem 4.7].

8.4.2 *p*-Chain Spectral Sequence

There is another spectral sequence, the *p-chain spectral sequence* [83]. Consider a covariant functor $\mathbf{E}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$. It defines a G-homology theory $\mathcal{H}_*^G(-;\mathbf{E})$ (see Proposition 6.7). The p-chain spectral sequence converges to $\mathcal{H}_n^G(X)$ but has a different setup and in particular a different E^2-term than the equivariant Atiyah-Hirzebruch spectral sequence. We describe the E^1-term for simplicity only for a proper G-CW-complex.

A *p-chain* is a sequence of conjugacy classes of finite subgroups

$$(H_0) < \ldots < (H_p)$$

where $(H_{i-1}) < (H_i)$ means that H_{i-1} is subconjugate, but not conjugate to (H_i). Notice for the sequel that the group of automorphism of G/H in $\mathsf{Or}(G)$ is isomorphic to NH/H. To such a p-chain there is associated the NH_p/H_p-NH_0/H_0-set

$$\begin{aligned} S((H_0) < \ldots < (H_p)) \;=\; & \operatorname{map}(G/H_{p-1}, G/H_p)^G \times_{NH_{p-1}/H_{p-1}} \\ & \ldots \times_{NH_1/H_1} \operatorname{map}(G/H_0, G/H_1)^G. \end{aligned}$$

The E^1-term $E^1_{p,q}$ of the p-chain spectral sequence is

$$\bigoplus_{(H_0)<\ldots<(H_p)} \pi_q \left(\left(X^{H_p} \times_{NH_p/H_p} S((H_0) < \ldots < (H_p)) \right)_+ \wedge_{NH_0/H_0} \mathbf{E}(G/H_0) \right)$$

where Y_+ means the pointed space obtained from Y by adjoining an extra base point. There are many situations where the p-chain spectral sequence is much more useful than the equivariant Atiyah-Hirzebruch spectral sequence. Sometimes a combination of both is necessary to carry through the desired calculation.

8.4.3 Equivariant Chern Characters

Equivariant Chern characters have been studied in [201] and [203] and allow to compute equivariant homology theories for proper G-CW-complexes. The existence of the equivariant Chern character says that under certain conditions the Atiyah-Hirzebruch spectral sequence collapses and, indeed, the source of the equivariant Chern character is canonically isomorphic to $\bigoplus_{p+q} E^2_{p,q}$, where $E^2_{p,q}$ is the E^2-term of the equivariant Atiyah-Hirzebruch spectral sequence.

The results of Section 8.2 are essentially proved by applying the equivariant Chern character to the source of the assembly map for the family of finite subgroups.

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Notation

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