A Survey on the
Baum-Connes Conjecture

Wolfgang Lück*

Fachbereich Mathematik und Informatik

Westfälische Wilhelms-Universität

Münster

Einsteinstr. 62

48149 Münster

Germany

lueck@math.uni-muenster.de

http://www.math.uni-muenster.de/u/lueck

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1. The statement of the Baum-Connes Conjecture

G will always denote a discrete group.

Conjecture 1 (Baum-Connes Conjecture for G) *The assembly map*

$$\text{asmb} : K_p^G(\underline{E}G) \to K_p(C_r^*(G))$$

which sends $[M, P^*]$ to $\text{index}_{C_r^*(G)}(P^*)$ *is bijective for $p \in \mathbb{Z}$.*

Here are some explanations:

- $C_r^*(G)$ is the **reduced C^*-algebra** of G, i.e. the completion of the complex group ring with respect to the operator norm coming from the embedding

  $$\mathbb{C}G \to \mathcal{B}(l^2(G), l^2(G)), \quad g \mapsto r_g,$$

  where r_g is given by right multiplication with $g \in G$. For example

  $$C_r^*(\mathbb{Z}^n) = C(T^n).$$

- $K_p(C_r^*(G))$ is the **topological** K-**theory** of $C_r^*(G)$.

  This is for $p = 0$ the same as the algebraic K-group. So elements in $K_0(C_r^*(G))$ are represented by finitely generated modules over the ring $C_r^*(G)$.

  These groups are two-periodic, i.e.

  $$K_p(C_r^*(G)) = K_{p+2}(C_r^*(G))$$

  for $p \in \mathbb{Z}$.

  In odd dimensions the topology plays a role, namely

  $$K_n(C_r^*(G)) = \pi_{n-1}(GL(C_r^*(G)))$$

  for $n \geq 1$.

- $\underline{E}G$ is the **classifying space for proper** G-**actions**. It is characterized uniquely up to G-homotopy by the property that it is a G-CW-complex whose isotropy groups are all finite and whose H-fixed point sets for $H \subset G$ are contractible.

If G is torsionfree, this coincides with EG.

These spaces are interesting in their own right and have often very nice geometric models. For instance

— Rips complex for word hyperbolic groups;

— Teichmüller space for mapping class groups;

— Outer space for the group of outer automorphisms of free groups;

— L/K for a connected Lie group L, a maximal compact subgroup $K \subseteq L$ and $G \subseteq L$ a discrete subgroup;

— Riemannian manifolds with non-positive sectional curvature and proper isometric G-action;

— Trees with proper G-action.

- $K_p^G(X)$ for a proper G-CW-complex X is the **equivariant K-homology** of X as defined for instance by Kasparov.

  If G acts freely on X, there is a canonical isomorphism

  $$K_0^G(X) \xrightarrow{\cong} K_0(G\backslash X)$$

  to the K-homology of $G\backslash X$.

  For $H \subset G$ finite, $K_0^G(G/H)$ is $\mathrm{Rep}_\mathbb{C}(H)$.

  An element in $K_p^G(\underline{E}G)$ is given by a pair (M, P^*) which consists of a smooth manifold with proper cocompact G-action and an elliptic G-complex P^* of differential operators of order 1;

- $\mathrm{index}_{C_r^*(G)}$ is the $C_r^*(G)$-valued index due to Mishchenko and Fomenko;

Remark 2 The Baum-Connes Conjecture makes also sense for topological groups and is in particular for Lie groups and for p-adic groups closely related to their representation theory.

Remark 3 The **Farrell-Jones Isomorphisms Conjecture** for the algebraic K-theory $K_n(RG)$ and the algebraic L-theory $L_n(RG)$ of group rings is analogous to the the Baum-Connes Conjecture. Both can be viewed as a special case of an assembly principle and their setup and computational questions can be treated simultaneously. However, the proofs of these in special cases have so far been rather different.

The Farrell-Jones Conjecture plays an important role in the classification of manifolds, in particular in connection with the **Borel Conjecture** that two aspherical closed topological manifolds are homeomorphic if and only if their fundamental groups are isomorphic.

2. Some applications of the Baum-Connes Conjecture

Next we explain the relevance of the Baum-Connes Conjecture.

- **Computations**

  Since $K_p^G(-)$ is an equivariant homology theory for proper G-CW-complexes, it is much easier to compute $K_p^G(\underline{E}G)$ than to compute $K_p(C_r^*(G))$. Techniques like Mayer-Vietoris sequences apply.

- **Novikov-Conjecture for G**

  The Hirzebruch signature formula says

  $$\mathrm{sign}(M) \;=\; \langle \mathcal{L}(M), [M] \rangle.$$

  Given a map $f : M \to BG$ and $x \in H^*(BG)$, define the higher signature by

  $$\mathrm{sign}_x(M, f) \;=\; \langle f^*(x) \cup \mathcal{L}(M), [M] \rangle.$$

  The **Novikov Conjecture** says that these are homotopy invariants, i.e. for $f :$

$M \to BG$, $g : N \to BG$ and a homotopy equivalence $u : M \to N$ with $g \circ u \simeq f$ we have

$$\text{sign}_x(M, f) \;=\; \text{sign}_x(N, g).$$

The Baum-Connes Conjecture for G implies the Novikov Conjecture for G.

- **Stable Gromov-Lawson-Rosenberg Conjecture for** G

  Let M be a closed Spin-manifold with fundamental group G. Let B be the Bott manifold. The **Stable Gromov-Lawson-Rosenberg Conjecture** says that $M \times B^k$ carries a Riemannian metric of positive scalar curvature for some $k \geq 0$ if and only if

  $$\text{index}_{C_r^*(G)}(\widetilde{M}, \widetilde{D}) \;=\; 0.$$

  Here D is the Dirac operator and $\widetilde{D}$ its lift to $\widetilde{M}$.

Stolz has shown that the Baum-Connes Conjecture for G implies the stable Gromov-Lawson-Rosenberg Conjecture for G. The unstable version of the Gromov-Lawson-Rosenberg Conjecture , i.e. $k = 0$, is false in general by a construction of Schick.

- **The Kadison Conjecture**

  The Baum-Connes Conjecture implies

  Conjecture 4 (Kadison Conjecture)
  Let G be torsionfree. Let $p \in C_r^*(G)$ be an idempotent, i.e. $p^2 = p$. Then $p = 0, 1$.

3. Status of the Baum-Connes Conjecture

There is a stronger version of the Baum-Connes conjecture, where certain coefficients are allowed:

Conjecture 5 (Baum-Connes Conjecture with Coefficients)

For every separable C^*-algebra A with an action of a countable group G and every $n \in \mathbb{Z}$ the assembly map

$$K_n(\underline{E}G; A) \to K_n(A \rtimes G)$$

is an isomorphism.

Definition 6 (a-T-menable group) *A group G is **a-T-menable**, or, equivalently, has the **Haagerup property** if G admits a metrically proper isometric action on some affine Hilbert space.*

Metrically proper means that for any bounded subset B the set $\{g \in G \mid gB \cap B \neq \emptyset\}$ is finite. Amenable groups and free groups are a-T-menable.

In the following table we list prominent classes of groups and state whether they are known to satisfy the Baum-Connes Conjecture (with coefficients). The proofs are due to **Connes, Guentner, Kasparov, Higson, Lafforgue, Weinberger, Yu and others**.

type of group	Status
a-T-menable groups	true with coefficients
discrete subgroups of Lie groups with finitely many path components	injectivity true

type of group	Status
linear groups	injectivity is true
arithmetic groups	injectivity is true
Groups with finite BG and finite asymptotic dimension	injectivity is true
G acts properly and isometrically on a complete Riemannian manifold M with non-positive sectional curvature	rational injectivity is true
$\pi_1(M)$ for a closed Riemannian manifold M with negative sectional curvature	true for all subgroups
word hyperbolic groups	true for all subgroups (true with coefficients by an unpublished result of Lafforgue)

type of group	Status
one-relator groups	true with coefficients
Haken 3-manifold groups (in particular knot groups)	true with coefficients
$SL(n,\mathbb{Z}), n \geq 3$	injectivity is true
Artin's braid group B_n	true with coefficients
Thompson's group F	true with coefficients

Remark 7 A group which both is a-T-menable and has Kazhdan's property T is finite. Meanwhile Lafforgue has shown that there are groups which have Kazhdan's property T and do satisfy the Baum-Connes Conjecture.

Remark 8 There are counterexamples to the Baum-Connes Conjecture with coefficients. No counterexamples to the Baum-Connes Conjecture are known. Groups with expanders are the only candidates which may yield counterexamples.

4. The Trace Conjecture

Conjecture 9 (Trace Conjecture for G)
The image of the composite

$$K_0(C_r^*(G)) \to K_0(\mathcal{N}(G)) \xrightarrow{\operatorname{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

is the additive subgroup of $\mathbb{R}$ generated by $\{\frac{1}{|H|} \mid H \subset G, |H| < \infty\}$. Here $\mathcal{N}(G)$ is the group von Neumann algebra and $\operatorname{tr}_{\mathcal{N}(G)}$ the von Neumann trace.

Notice that $\mathbb{C}[G] \subset C_r^*(G) \subset \mathcal{N}(G)$ and equality holds if and only if G is finite.

Lemma 10 *The Trace Conjecture for G implies the Kadison Conjecture for torsion-free G.*

Proof:

$$0 \leq p \leq 1 \;\Rightarrow\; 0 = \operatorname{tr}(0) \leq \operatorname{tr}(p) \leq \operatorname{tr}(1) = 1$$
$$\Rightarrow\; \operatorname{tr}(p) \in \mathbb{Z} \cap [0,1] \;\Rightarrow\; \operatorname{tr}(p) = 0, 1$$
$$\Rightarrow\; \operatorname{tr}(p) = \operatorname{tr}(0), \operatorname{tr}(1) \;\Rightarrow\; p = 0, 1.$$

Lemma 11 *Let G be torsionfree. Then the Baum-Connes Conjecture for G implies the Trace Conjecture for G.*

Proof: The following diagram commutes

$$
\begin{array}{ccccccc}
K_0^G(EG) & \longrightarrow & K_0(C_*^r(G)) & \longrightarrow & K_0(\mathcal{N}(G)) & \longrightarrow & \mathbb{R} \\
\downarrow{\scriptstyle\cong} & & & & & & \uparrow \\
K_0(BG) & \longrightarrow & K_0(*) & \xrightarrow{\quad\cong\quad} & & & \mathbb{Z}
\end{array}
$$

This follows from the Atiyah index theorem. Namely, the upper horizontal composite sends $[M, P^*] \in K_0^G(EG)$ to the L^2-index in the sense of Atiyah

$$L^2 - \mathrm{index}(M, P^*) \in \mathbb{R},$$

the right vertical arrow sends $[M, P^*]$ to $[G\backslash M, G\backslash P^*]$ and the lower horizontal composite sends $[G\backslash M, G\backslash P^*]$ to the ordinary index

$$\mathrm{index}(G\backslash M, G\backslash P^*) \in \mathbb{Z}.$$

The L^2-index theorem of Atiyah says

$$L^2 - \mathrm{index}(M, P^*) \;=\; \mathrm{index}(G\backslash M, G\backslash P^*).$$

5. Roy's counterexample to the trace conjecture

Theorem 12 (Roy 99) *The Trace Conjecture is false in general.*

Proof: Define an algebraic smooth variety

$$M = \{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 \mid$$
$$z_0^{15} + z_1^{15} + z_2^{15} + z_3^{15} = 0\}.$$

The group $G = \mathbb{Z}/3 \times \mathbb{Z}/3$ acts on it by

$$[z_0, z_1, z_2, z_3] \mapsto [\exp(2\pi i/3) \cdot z_0, z_1, z_2, z_3]$$
$$[z_0, z_1, z_2, z_3] \mapsto [z_0, z_3, z_1, z_2]$$

One obtains

$$M^G = \emptyset;$$
$$\mathrm{sign}(M) = -1105;$$
$$\pi_1(M) = \{1\}.$$

An equivariant version of a construction due to Davis and Januszkiewicz yields

- A closed oriented aspherical manifold N with G-action;

- A G-map $f : N \to M$ of degree one;

- An isomorphism $f^*TM \cong TN$.

There is an extension of groups

$$1 \to \pi = \pi_1(N) \to \Gamma \xrightarrow{\cong} G \to 1$$

and a Γ-action on $\widetilde{N}$ extending the π-action on $\widetilde{N}$ and covering the G-action on N. We compute using the Hirzebruch signature formula

$$\mathrm{sign}(N) = \langle \mathcal{L}(N), [N] \rangle = \langle f^*\mathcal{L}(M), [N] \rangle$$
$$= \langle \mathcal{L}(M), f_*([N]) \rangle = \langle \mathcal{L}(M), [M]) \rangle = \mathrm{sign}(M).$$

Next we prove that any finite subgroup $H \subset \Gamma$ satisfies

$$|H| \in \{1, 3\}.$$

Since $\widetilde{N}$ turns out to be a CAT(0)-space, any finite subgroup $H \subset \Gamma$ has a fixed point by a result of Bruhat and Tits. This implies

$$\widetilde{N}^H \neq \emptyset \Rightarrow N^{p(H)} \neq \emptyset \Rightarrow M^{p(H)} \neq \emptyset \Rightarrow p(H) \neq G.$$

Since $\pi_1(N)$ is torsionfree, $p|_H : H \rightarrow p(H)$ is bijective.

On $\widetilde{N}$ we have the signature operator $\widetilde{S}$. We claim that the composite

$$K_0^\Gamma(\underline{E}\Gamma) \xrightarrow{\text{asmb}} K_0(C_r^*(\Gamma)) \rightarrow K_0(\mathcal{N}(\Gamma)) \xrightarrow{\text{tr}_{\mathcal{N}(\Gamma)}} \mathbb{R}$$

sends $[\widetilde{N}, \widetilde{S}]$ to

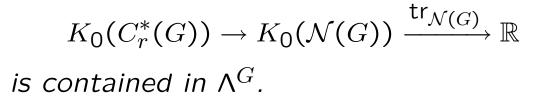$$\frac{1}{[\Gamma : \pi]} \cdot \text{sign}(N) \; = \; \frac{-1105}{9}.$$

The Trace Conjecture for Γ says

$$\frac{-1105}{9} \in \{r \in \mathbb{R} \mid 3 \cdot r \in \mathbb{Z}\}.$$

This is not true (by some very deep number theoretic considerations).

6. The Modified Trace Conjecture

Conjecture 13 Modified Trace Conjecture) *Let $\Lambda^G \subset \mathbb{Q}$ be the subring of $\mathbb{Q}$ obtained from $\mathbb{Z}$ by inverting the orders of finite subgroups of G. Then the image of composite*

$$K_0(C_r^*(G)) \to K_0(\mathcal{N}(G)) \xrightarrow{\mathrm{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

is contained in Λ^G.

Theorem 14 (The Baum-Connes Conjecture implies the Modified Trace Conjecture (L.)) *The image of the composite*

$$K_0^G(\underline{E}G) \xrightarrow{\mathrm{asmb}} K_0(C_r^*(G))$$
$$\to K_0(\mathcal{N}(G)) \xrightarrow{\mathrm{tr}_{\mathcal{N}(G)}} \mathbb{R}$$

is contained in Λ^G.

In particular the Baum-Connes Conjecture for G implies the Modified Trace Conjecture for G.

7. The equivariant Chern character

Theorem 15 (Artin's Theorem) *Let G be finite. Then the map*

$$\bigoplus_{C \subset G} \mathrm{ind}_C^G : \bigoplus_{C \subset G} \mathrm{Rep}_{\mathbb{C}}(C) \to \mathrm{Rep}_{\mathbb{C}}(G)$$

is surjective after inverting $|C|$, where $C \subset G$ runs through the cyclic subgroups of G.

Let C be a finite cyclic group. The **Artin defect** is the cokernel of the map

$$\bigoplus_{D \subset C, D \neq C} \mathrm{ind}_D^C : \bigoplus_{D \subset C, D \neq C} \mathrm{Rep}_{\mathbb{C}}(D) \to \mathrm{Rep}_{\mathbb{C}}(C).$$

For an appropriate idempotent

$$\theta_C \in \mathrm{Rep}_{\mathbb{Q}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right]$$

the Artin defect becomes after inverting the order of $|C|$ canonically isomorphic to

$$\theta_c \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right].$$

Theorem 16 (Equivariant Chern character (L.)) *Let X be a proper G-CW-complex. For a finite cyclic subgroup $C \subset G$ let (C) be its conjugacy class, $N_G C$ ist normalizer, $C_G C$ its centralizer and $W_G C = N_G C / C_G C$. Then there is a natural isomorphism called* equivariant Chern character

$$\bigoplus_{(C)} K_p(C_G C \backslash X^C) \otimes_{\mathbb{Z}[W_G C]} \theta_c \cdot \mathsf{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \wedge^G$$

$$\mathsf{ch}^G \downarrow \cong$$

$$K_p^G(X) \otimes_{\mathbb{Z}} \wedge^G$$

Example 17 Suppose that G is torsion-free. Then the trivial subgroup $\{1\}$ is the only finite cyclic subgroup of C. We have $C_G\{1\} = N_G\{1\} = G$ and $W_G\{1\} = \{1\}$. We get an isomorphism

$$\bigoplus_{(C)} K_p(C_G C \backslash X^C) \otimes_{\mathbb{Z}[W_G C]} \theta_c \cdot \mathsf{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \wedge^G$$

$$\downarrow \cong$$

$$K_0(G \backslash X) \otimes_{\mathbb{Z}} \mathbb{Z}$$

$$\downarrow \cong$$

$$K_p(G \backslash X)$$

Under this identification the inverse of ch^G becomes the canonical isomorphism

$$K_p^G(X) \xrightarrow{\cong} K_p(G\backslash X).$$

Example 18 Let G be finite and $X = \{*\}$. Then we get an improvement of Artin's theorem, namely, the equivariant Chern character induces an isomorphism

$$\bigoplus_{(C)} \mathbb{Z} \otimes_{\mathbb{Z}[W_G C]} \theta_c \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right]$$

$$\mathrm{ch}^G \Big\downarrow \cong$$

$$\mathrm{Rep}_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right]$$

Example 19 Take G to be any (discrete) group and $X = \underline{E}G$. There is a natural isomorphism

$$K_p(BC_G C)\otimes_{\mathbb{Z}}\Lambda^G \xrightarrow{\cong} K_p(C_G C\backslash(\underline{E}G)^C)\otimes_{\mathbb{Z}}\Lambda^G.$$

The equivariant Chern character induces an isomorphism

$$\bigoplus_{(C)} K_p(BC_G C) \otimes_{\mathbb{Z}[W_G C]} \theta_c \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G$$

$$\mathrm{ch}^G \Big\downarrow \cong$$

$$K_0^G(\underline{E}G) \otimes_{\mathbb{Z}} \Lambda^G$$

Corollary 20 *The ordinary Chern charac-
ter induces for a CW-complex Y an iso-
morphism*

$$\oplus_k H_{2k+p}(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_p(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

*If the Baum-Connes Conjecture holds for
G, then we obtain an isomorphism*

$$\bigoplus_{(C)} \bigoplus_k H_{p+2k}(BC_G C) \otimes_{\mathbb{Z}[W_G C]} \theta_c \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$\mathsf{ch}^G \Big\downarrow \cong$$

$$K_p^G(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

8. A generalization of Atiyah's L^2-index theorem

Theorem 21 (Generalized L^2-index theorem (L 01)) *The following diagram commutes*

$$\begin{array}{ccccccc}
K_0^G(EG) & \longrightarrow & K_0^G(\underline{E}G) & \longrightarrow & K_0(C_*^r(G)) & \longrightarrow & K_0(\mathcal{N}(G) \\
\downarrow {\scriptstyle \cong} & & & & & & \uparrow \\
K_0(BG) & \longrightarrow & K_0(*) & \xrightarrow{\hspace{1cm} \cong \hspace{1cm}} & & & K_0(\mathcal{N}(1))
\end{array}$$

or, equivalently, we get for a free cocompact G-manifold M with elliptic G- complex P^* of differential operators of order 1 in $K_0(\mathcal{N}(G))$

$$\operatorname{index}_{\mathcal{N}(G)}(M, P^*) = \operatorname{index}(G\backslash M, G\backslash P^*) \cdot [\mathcal{N}(G)].$$

Example 22 Let M be a closed oriented $4k$-dimensional manifold. Suppose that the finite group G acts on M freely and orientation preserving. Define the **equivariant signature**

$$\operatorname{sign}^G(M) \in \operatorname{Rep}_{\mathbb{C}}(G)$$

by

$$\text{sign}^G(M) = \left[H_{2k}(M;\mathbb{C})^+\right] - \left[H_{2k}(M;\mathbb{C})^-\right].$$

Then the theorem above implies the well-known statement that for a free G-action we get

$$\begin{aligned}
\text{sign}^G(M) &= \text{sign}(G\backslash M) \cdot [\mathbb{C}G]; \\
\text{sign}(M) &= |G| \cdot \text{sign}(G\backslash M).
\end{aligned}$$

9. Proof of Theorem 14

Let X be a proper G-CW-complex. Define two homomorphisms

$$\bigoplus_{(C)} K_0(C_G C \backslash X^C) \otimes_{\mathbb{Z}[W_G C]} \theta_c \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \wedge^G$$

$$\xi_i \downarrow$$

$$K_0(\mathcal{N}(G)) \otimes_{\mathbb{Z}} \wedge^G$$

as follows. The first one is the composition of the equivariant Chern character with the assembly map

$$\mathrm{asmb}^G \otimes \mathrm{id} : K_0^G(X) \otimes_{\mathbb{Z}} \wedge^G \to K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \wedge^G$$

and the change of rings homomorphism

$$K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \wedge^G \to K_0(\mathcal{N}(G)) \otimes_{\mathbb{Z}} \wedge^G.$$

This is the homomorphism which we want to understand. In particular we are interested in its image. We want to identify it with the easier to compute homomorphism ξ_2.

The homomorphism ξ_2 is induced by the composition

$$\bigoplus_{(C)} K_0(C_G C \backslash X^C) \otimes_{\mathbb{Z}} \theta_c \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G$$

$$\left\downarrow {\scriptstyle \bigoplus_{(C)} K_0(\mathrm{pr}) \otimes_{\mathbb{Z}} \mathrm{incl}} \right.$$

$$\bigoplus_{(C)} K_0(*) \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G$$

$$\left\downarrow {\scriptstyle \cong} \right.$$

$$\bigoplus_{(C)} \mathbb{Z} \otimes_{\mathbb{Z}} \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G$$

$$\left\downarrow {\scriptstyle \cong} \right.$$

$$\bigoplus_{(C)} \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G$$

$$\left\downarrow {\scriptstyle \bigoplus_{(C)} \mathrm{ind}_C^G} \right.$$

$$K_0(\mathcal{N}(G)) \otimes_{\mathbb{Z}} \Lambda^G$$

The proof of the next result uses the generalized L^2-Atiyah index theorem.

Theorem 23 *Let X be a proper G-CW-complex. Then the maps ξ_1 and ξ_2 agree.*

Theorem 24 *The image of the compos-*

ite

$$K_0(\underline{E}G) \otimes_{\mathbb{Z}} \wedge^G \rightarrow K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \wedge^G$$
$$\rightarrow K_0(\mathcal{N}(G)) \otimes_{\mathbb{Z}} \wedge^G$$

is contained in the image of

$$\bigoplus_{(C)} \mathrm{ind}_C^G : \bigoplus_{(C)} \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \wedge^G \rightarrow K_0(\mathcal{N}(G) \otimes_{\mathbb{Z}} \wedge^G.$$

Remark 25 If we compose the second map above with

$$\mathrm{tr}_{\mathcal{N}(G)} : K_0(\mathcal{N}(G)) \otimes_{\mathbb{Z}} \wedge^G \rightarrow \mathbb{R}$$

it is easy to see that its image is contained in $\wedge^G$. Hence the following composition has $\wedge^G$ as image

$$K_0^G(\underline{E}G) \xrightarrow{\mathrm{asmb}} K_0(C_r^*(G))$$
$$\rightarrow K_0(\mathcal{N}(G)) \xrightarrow{\mathrm{tr}_{\mathcal{N}(G)}} \mathbb{R}.$$