Inheritance of Isomorphism Conjectures under colimits

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Abstract. We investigate when Isomorphism Conjectures, such as the ones due to Baum-Connes, Bost and Farrell-Jones, are stable under colimits of groups over directed sets (with not necessarily injective structure maps). We show in particular that both the K-theoretic Farrell-Jones Conjecture and the Bost Conjecture with coefficients hold for those groups for which Higson, Lafforgue and Skandalis have disproved the Baum-Connes Conjecture with coefficients.

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0. Introduction

0.1. Assembly maps.
We want to study the following *assembly maps*:

$$\mathrm{asmb}_n^G: \quad H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G); \mathbf{K}_R) \quad \to \quad H_n^G(\{\bullet\}; \mathbf{K}_R) = K_n(R \rtimes G); \quad (0.1)$$

$$\mathrm{asmb}_n^G: \quad H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{KH}_R) \quad \to \quad H_n^G(\{\bullet\}; \mathbf{KH}_R) = KH_n(R \rtimes G); \quad (0.2)$$

$$\mathrm{asmb}_n^G: \quad H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) \quad \to \quad H_n^G(\{\bullet\}; \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(R \rtimes G); \quad (0.3)$$

$$\mathrm{asmb}_n^G: \quad H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{K}_{A,l^1}^{\mathrm{top}}) \quad \to \quad H_n^G(\{\bullet\}; \mathbf{K}_{A,l^1}^{\mathrm{top}}) = K_n(A \rtimes_{l^1} G); \quad (0.4)$$

$$\mathrm{asmb}_n^G: \quad H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{K}_{A,r}^{\mathrm{top}}) \quad \to \quad H_n^G(\{\bullet\}; \mathbf{K}_{A,r}^{\mathrm{top}}) = K_n(A \rtimes_r G); \quad (0.5)$$

$$\mathrm{asmb}_n^G: \quad H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{K}_{A,m}^{\mathrm{top}}) \quad \to \quad H_n^G(\{\bullet\}; \mathbf{K}_{A,m}^{\mathrm{top}}) = K_n(A \rtimes_m G). \quad (0.6)$$

Some explanations are in order. A *family of subgroups of G* is a collection of subgroups of G which is closed under conjugation and taking subgroups. Examples are the family $\mathcal{F}\mathrm{in}$ of finite subgroups and the family $\mathcal{VC}\mathrm{yc}$ of virtually cyclic subgroups.

Let $E_{\mathcal{F}}(G)$ be the *classifying space associated to $\mathcal{F}$*. It is uniquely characterized up to G-homotopy by the properties that it is a G-CW-complex and that $E_{\mathcal{F}}(G)^H$ is contractible if $H \in \mathcal{F}$ and is empty if $H \notin \mathcal{F}$. For more information about these spaces $E_{\mathcal{F}}(G)$ we refer for instance to the survey article [29].

Given a group G acting on a ring (with involution) by structure preserving maps, let $R \rtimes G$ be the twisted group ring (with involution) and denote by $K_n(R \rtimes G)$, $KH_n(R \rtimes G)$ and $L_n^{\langle -\infty \rangle}(R \rtimes G)$ its *algebraic K-theory* in the non-connective

sense (see Gersten [17] or Pedersen-Weibel [34]), *its homotopy K-theory* in the sense of Weibel [41], and its *L-theory with decoration* $-\infty$ in the sense of Ranicki [37, Chapter 17]. Given a group G acting on a C^*-algebra A by automorphisms of C^*-algebras, let $A \rtimes_{l^1} G$ be the Banach algebra obtained from $A \rtimes G$ by completion with respect to the l^1-norm, let $A \rtimes_r G$ be the reduced crossed product C^*-algebra, and let $A \rtimes_m G$ be the maximal crossed product C^*-algebra and denote by $K_n(A \rtimes_{l^1} G)$, $K_n(A \rtimes_r G)$ and $K_n(A \rtimes_m G)$ their *topological K-theory*.

The source and target of the assembly maps are given by G-homology theories (see Definition 1.1 and Theorem 5.1) with the property that for every subgroup $H \subseteq G$ and $n \in \mathbb{Z}$

$$
\begin{array}{rcl}
H_n^G(G/H; \mathbf{K}_R) & \cong & K_n(R \rtimes H); \\
H_n^G(G/H; \mathbf{KH}_R) & \cong & KH_n(R \rtimes H); \\
H_n^G(G/H; \mathbf{L}_R^{\langle -\infty \rangle}) & \cong & L_n^{\langle -\infty \rangle}(R \rtimes H); \\
H_n^G(G/H; \mathbf{K}_{A,l^1}^{\mathrm{top}}) & \cong & K_n(A \rtimes_{l^1} H); \\
H_n^G(G/H; \mathbf{K}_{A,r}^{\mathrm{top}}) & \cong & K_n(A \rtimes_r H); \\
H_n^G(G/H; \mathbf{K}_{A,r}^{\mathrm{top}}) & \cong & K_n(A \rtimes_m H).
\end{array}
$$

All the assembly maps are induced by the projection from $E_{\mathcal{F}\mathrm{in}}(G)$ or $E_{\mathcal{V}\mathcal{C}\mathrm{yc}}(G)$ respectively to the one-point-space $\{\bullet\}$.

Remark 0.7. It might be surprising to the reader that we restrict to C*-algebra coefficients A in the assembly map (0.4). Indeed, our main results rely heavily on the validity of the conjecture for hyperbolic groups, which, so far, is only known for C*-algebra coefficients. Moreover we also want to study the passage from the l^1-setting to the C^*-setting. Hence we decided to restrict ourselves to the case of C^*-coefficients throughout. We mention that on the other hand the assembly map (0.4) can also be defined for Banach algebra coefficients [33].

0.2. Conventions. Before we go on, let us fix some conventions. A group G is always discrete. Hyperbolic group is to be understood in the sense of Gromov (see for instance [11], [12], [18], [19]). Ring means associative ring with unit and ring homomorphisms preserve units. Homomorphisms of Banach algebras are assumed to be norm decreasing.

0.3. Isomorphism Conjectures. The *Farrell-Jones Conjecture* for algebraic K-theory for a group G and a ring R with G-action by ring automorphisms says that the assembly map (0.1) is bijective for all $n \in \mathbb{Z}$. Its version for homotopy K-theory says that the assembly map (0.2) is bijective for all $n \in \mathbb{Z}$. If R is a ring with involution and G acts on R by automorphism of rings with involutions, the L-theoretic version of the Farrell-Jones Conjecture predicts that the assembly map (0.3) is bijective for all $n \in \mathbb{Z}$. The Farrell-Jones Conjecture for algebraic K- and L-theory was originally formulated in the paper by Farrell-Jones [15, 1.6 on page 257] for the trivial G-action on R. Its homotopy K-theoretic version can be found in [4, Conjecture 7.3], again for trivial G-action on R.

Let G be a group acting on the C^*-algebra A by automorphisms of C^*-algebras. The *Bost Conjecture with coefficients* and the *Baum-Connes Conjecture with coefficients* respectively predict that the assembly map (0.4) and (0.5) respectively are bijective for all $n \in \mathbb{Z}$. The original statement of the Baum-Connes Conjecture with trivial coefficients can be found in [9, Conjecture 3.15 on page 254].

Our formulation of these conjectures follows the homotopy theoretic approach in [13]. The original assembly maps are defined differently. We do not give the proof that our maps agree with the original ones but at least refer to [13, page 239], where the Farrell-Jones Conjecture is treated and to Hambleton-Pedersen [21], where such identification is given for the Baum-Connes Conjecture with trivial coefficients.

0.4. Inheritance under colimits. The main purpose of this paper is to prove that these conjectures are inherited under colimits over directed systems of groups (with not necessarily injective structure maps). We want to show:

Theorem 0.8 (Inheritance under colimits). *Let $\{G_i \mid i \in I\}$ be a directed system of groups with (not necessarily injective) structure maps $\phi_{i,j} \colon G_i \to G_j$. Let $G = \operatorname{colim}_{i \in I} G_i$ be its colimit with structure maps $\psi_i \colon G_i \to G$. Let R be a ring (with involution) and let A be a C^*-algebra with structure preserving G-action. Given $i \in I$ and a subgroup $H \subseteq G_i$, we let H act on R and A by restriction with the group homomorphism $(\psi_i)|_H \colon H \to G$. Fix $n \in \mathbb{Z}$. Then:*

(i) *If the assembly map*

$$\operatorname{asmb}_n^H \colon H_n^H(E_{\mathcal{VC}yc}(H); \mathbf{K}_R) \to H_n^H(\{\bullet\}; \mathbf{K}_R) = K_n(R \rtimes H)$$

of (0.1) is bijective for all $n \in \mathbb{Z}$, all $i \in I$ and all subgroups $H \subseteq G_i$, then for every subgroup $K \subseteq G$ of G the assembly map

$$\operatorname{asmb}_n^K \colon H_n^K(E_{\mathcal{VC}yc}(K); \mathbf{K}_R) \to H_n^K(\{\bullet\}; \mathbf{K}_R) = K_n(R \rtimes K)$$

of (0.1) is bijective for all $n \in \mathbb{Z}$.

The corresponding version is true for the assembly maps given in (0.2), (0.3), (0.4), and (0.6);

(ii) *Suppose that all structure maps $\phi_{i,j}$ are injective and that the assembly map*

$$\operatorname{asmb}_n^{G_i} \colon H_n^{G_i}(E_{\mathcal{VC}yc}(G_i); \mathbf{K}_R) \to H_n^{G_i}(\{\bullet\}; \mathbf{K}_R) = K_n(R \rtimes G_i)$$

of (0.1) is bijective for all $n \in \mathbb{Z}$ and $i \in I$. Then the assembly map

$$\operatorname{asmb}_n^G \colon H_n^G(E_{\mathcal{VC}yc}(G); \mathbf{K}_R) \to H_n^G(\{\bullet\}; \mathbf{K}_R) = K_n(R \rtimes G)$$

of (0.1) is bijective for all $n \in \mathbb{Z}$;

The corresponding statement is true for the assembly maps given in (0.2), (0.3), (0.4), (0.5), and (0.6).

Theorem 0.8 will follow from Theorem 3.5 and Lemma 5.2 as soon as we have proved Theorem 5.1. Notice that the version (0.5) does not appear in assertion (i). A counterexample will be discussed below. The (fibered) version of Theorem 0.8 (i) in the case of algebraic K-theory and L-theory with coefficients in $\mathbb{Z}$ with trivial G-action has been proved by Farrell-Linnell [16, Theorem 7.1].

0.5. Colimits of hyperbolic groups.
In [23, Section 7] Higson, Lafforgue and Skandalis construct counterexamples to the *Baum-Connes-Conjecture with coefficients*, actually with a commutative C^*-algebra as coefficients. They formulate precise properties for a group G which imply that it does *not* satisfy the Baum-Connes Conjecture with coefficients. Gromov [20] describes the construction of such a group G as a colimit over a directed system of groups $\{G_i \mid i \in I\}$, where each G_i is hyperbolic.

This construction did raise the hope that these groups G may also be counterexamples to the Baum-Connes Conjecture with trivial coefficients. But — to the authors' knowledge — this has not been proved and no counterexample to the Baum-Connes Conjecture with trivial coefficients is known.

Of course one may wonder whether such counterexamples to the Baum-Connes Conjecture with coefficients or with trivial coefficients respectively may also be counterexamples to the Farrell-Jones Conjecture or the Bost Conjecture with coefficients or with trivial coefficients respectively. However, this can be excluded by the following result.

Theorem 0.9. *Let G be the colimit of the directed system $\{G_i \mid i \in I\}$ of groups (with not necessarily injective structure maps). Suppose that each G_i is hyperbolic. Let $K \subseteq G$ be a subgroup. Then:*

 (i) *The group K satisfies for every ring R on which K acts by ring automorphisms the Farrell-Jones Conjecture for algebraic K-theory with coefficients in R, i.e., the assembly map (0.1) is bijective for all $n \in \mathbb{Z}$;*

 (ii) *The group K satisfies for every ring R on which K acts by ring automorphisms the Farrell-Jones Conjecture for homotopy K-theory with coefficients in R, i.e., the assembly map (0.2) is bijective for all $n \in \mathbb{Z}$;*

 (iii) *The group K satisfies for every C^*-algebra A on which K acts by C^*-automorphisms the Bost Conjecture with coefficients in A, i.e., the assembly map (0.4) is bijective for all $n \in \mathbb{Z}$.*

Proof. If G is the colimit of the directed system $\{G_i \mid i \in I\}$, then the subgroup $K \subseteq G$ is the colimit of the directed system $\{\psi_i^{-1}(K) \mid i \in I\}$, where $\psi_i \colon G_i \to G$ is the structure map. Hence it suffices to prove Theorem 0.9 in the case $G = K$. This case follows from Theorem 0.8 (i) as soon as one can show that the Farrell-Jones Conjecture for algebraic K-theory, the Farrell-Jones Conjecture for homotopy K-theory, or the Bost Conjecture respectively holds for every subgroup H of a hyperbolic group G with arbitrary coefficients R and A respectively.

Firstly we prove this for the Bost Conjecture. Mineyev and Yu [30, Theorem 17] show that every hyperbolic group G admits a G-invariant metric $\widehat{d}$ which is weakly geodesic and strongly bolic. Since every subgroup H of G clearly acts properly on G with respect to any discrete metric, it follows that H belongs to the class $\mathcal{C}'$ as described by Lafforgue in [27, page 5] (see also the remarks at the top of page 6 in [27]). Now the claim is a direct consequence of [27, Theorem 0.0.2].

The claim for the Farrell-Jones Conjecture is proved for algebraic K-theory and homotopy K-theory in Bartels-Lück-Reich [6] which is based on the results of [5].   □

There are further groups with unusual properties that can be obtained as colimits of hyperbolic groups. This class contains for instance a torsion-free non-cyclic group all whose proper subgroups are cyclic constructed by Ol'shanskii [32]. Further examples are mentioned in [31, p.5] and [38, Section 4].

We mention that if one can prove the L-theoretic version of the Farrell-Jones Conjecture for subgroups of hyperbolic groups with arbitrary coefficients, then it is also true for subgroups of colimits of hyperbolic groups by the argument above.

0.6. Discussion of (potential) counterexamples.
If G is an infinite group which satisfies Kazhdan's property (T), then the assembly map (0.6) for the maximal group C^*-algebra fails to be an isomorphism if the assembly map (0.5) for the reduced group C^*-algebra is injective (which is true for a very large class of groups and in particular for all hyperbolic groups by [25]). The reason is that a group has property (T) if and only if the trivial representation 1_G is isolated in the dual $\widehat{G}$ of G. This implies that $C_m^*(G)$ has a splitting $\mathbb{C} \oplus \ker(1_G)$, where we regard 1_G as a representation of $C_m^*(G)$. If G is infinite, then the first summand is in the kernel of the regular representation $\lambda \colon C_m^*(G) \to C_r^*(G)$ (see for instance [14]), hence the K-theory map $\lambda \colon K_0(C_m^*(G)) \to K_0(C_r^*(G))$ is not injective. For a finite group H we have $A \rtimes_r H = A \rtimes_m H$ and hence we can apply [13, Lemma 4.6] to identify the domains of (0.5) and (0.6). Under this identification the composition of the assembly map (0.6) with λ is the assembly map (0.5) and the claim follows.

Hence the Baum-Connes Conjecture for the maximal group C^*-algebras is not true in general since the Baum-Connes Conjecture for the reduced group C^*-algebras has been proved for some groups with property (T) by Lafforgue [26] (see also [39]). So in the sequel our discussion refers always to the Baum-Connes Conjecture for the reduced group C^*-algebra.

One may speculate that the Baum-Connes Conjecture with trivial coefficients is less likely to be true for a given group G than the Farrell-Jones Conjecture or the Bost Conjecture. Some evidence for this speculation comes from lack of functoriality of the reduced group C^*-algebra. A group homomorphism $\alpha \colon H \to G$ induces in general not a C^*-homomorphism $C_r^*(H) \to C_r^*(G)$, one has to require that its kernel is amenable. Here is a counterexample, namely, if F is a non-abelian free group, then $C_r^*(F)$ is a simple algebra [35] and hence there is no unital algebra homomorphism $C_r^*(F) \to C_r^*(\{1\}) = \mathbb{C}$. On the other hand, any group

homomorphism $\alpha\colon H \to G$ induces a homomorphism

$$H_n^H(E_{\mathcal{F}\mathrm{in}}(H); \mathbf{K}_{\mathbb{C},r}^{\mathrm{top}}) \xrightarrow{\mathrm{ind}_\alpha} H_n^G(\alpha_* E_{\mathcal{F}\mathrm{in}}(H); \mathbf{K}_{\mathbb{C},r}^{\mathrm{top}}) \xrightarrow{H_n^G(f)} H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{K}_{\mathbb{C},r}^{\mathrm{top}})$$

where G acts trivially on $\mathbb{C}$ and $f\colon \alpha_* E_{\mathcal{F}\mathrm{in}}(H) \to E_{\mathcal{F}\mathrm{in}}(G)$ is the up to G-homotopy unique G-map. Notice that the induction map ind_α exists since the isotropy groups of $E_{\mathcal{F}\mathrm{in}}(H)$ are finite. Moreover, this map is compatible under the assembly maps for H and G with the map $K_n(C_r^*(\alpha))\colon K_n(C_r^*(H)) \to K_n(C_r^*(G))$ provided that α has amenable kernel and hence $C_r^*(\alpha)$ is defined. So the Baum-Connes Conjecture implies that every group homomorphism $\alpha\colon H \to G$ induces a group homomorphism $\alpha_*\colon K_n(C_r^*(H)) \to K_n(C_r^*(G))$, although there may be no C^*-homomorphism $C_r^*(H) \to C_r^*(G)$ induced by α. No such direct construction of α_* is known in general.

Here is another failure of the reduced group C^*-algebra. Let G be the colimit of the directed system $\{G_i \mid i \in I\}$ of groups (with not necessarily injective structure maps). Suppose that for every $i \in I$ and preimage H of a finite group under the canonical map $\psi_i\colon G_i \to G$ the Baum-Connes Conjecture for the maximal group C^*-algebra holds (This is for instance true by [22] if $\ker(\psi_i)$ has the Haagerup property). Then

$$\mathrm{colim}_{i \in I} H_n^{G_i}(E_{\mathcal{F}\mathrm{in}}(G_i); \mathbf{K}_{\mathbb{C},m}^{\mathrm{top}}) \xrightarrow{\cong} \mathrm{colim}_{i \in I} H_n^{G_i}(E_{\psi_i^* \mathcal{F}\mathrm{in}}(G_i); \mathbf{K}_{\mathbb{C},m}^{\mathrm{top}})$$
$$\xrightarrow{\cong} H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{K}_{\mathbb{C},m}^{\mathrm{top}})$$

is a composition of two isomorphisms. The first map is bijective by the Transitivity Principle 3.3, the second by Lemma 2.4 and Lemma 5.2. This implies that the following composition is an isomorphism

$$\mathrm{colim}_{i \in I} H_n^{G_i}(E_{\mathcal{F}\mathrm{in}}(G_i); \mathbf{K}_{\mathbb{C},r}^{\mathrm{top}}) \to \mathrm{colim}_{i \in I} H_n^{G_i}(E_{\psi_i^* \mathcal{F}\mathrm{in}}(G_i); \mathbf{K}_{\mathbb{C},r}^{\mathrm{top}})$$
$$\to H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{K}_{\mathbb{C},r}^{\mathrm{top}})$$

Namely, these two compositions are compatible with the passage from the maximal to the reduced setting. This passage induces on the source and on the target isomorphisms since $E_{\mathcal{F}\mathrm{in}}(G_i)$ and $E_{\mathcal{F}\mathrm{in}}(G)$ have finite isotropy groups, for a finite group H we have $C_r^*(H) = C_m^*(H)$ and hence we can apply [13, Lemma 4.6]. Now assume furthermore that the Baum-Connes Conjecture for the reduced group C^*-algebra holds for G_i for each $i \in I$ and for G. Then we obtain an isomorphism

$$\mathrm{colim}_{i \in I} K_n(C_r^*(G_i)) \xrightarrow{\cong} K_n(C_r^*(G)).$$

Again it is in general not at all clear whether there exists such a map in the case, where the structure maps $\psi_i\colon G_i \to G$ do not have amenable kernels and hence do not induce maps $C_r^*(G_i) \to C_r^*(G)$.

These arguments do not apply to the Farrell-Jones Conjecture or the Bost Conjecture. Namely any group homomorphism $\alpha\colon H \to G$ induces maps $R \rtimes H \to R \rtimes G$, $A \rtimes_{l^1} H \to A \rtimes_{l^1} G$, and $A \rtimes_m H \to A \rtimes_m G$ for a ring R or a C^*-algebra

A with structure preserving G-action, where we equip R and A with the H-action coming from α. Moreover we will show for a directed system $\{G_i \mid i \in I\}$ of groups (with not necessarily injective structure maps) and $G = \mathrm{colim}_{i \in I} G_i$ that there are canonical isomorphisms (see Lemma 5.2)

$$\begin{aligned}
\mathrm{colim}_{i \in I} K_n(R \rtimes G_i) &\xrightarrow{\cong} K_n(R \rtimes G); \\
\mathrm{colim}_{i \in I} KH_n(R \rtimes G_i) &\xrightarrow{\cong} KH_n(R \rtimes G); \\
\mathrm{colim}_{i \in I} L_n^{\langle -\infty \rangle}(R \rtimes G_i) &\xrightarrow{\cong} L_n^{\langle -\infty \rangle}(R \rtimes G); \\
\mathrm{colim}_{i \in I} K_n(A \rtimes_{l^1} G_i) &\xrightarrow{\cong} K_n(A \rtimes_{l^1} G); \\
\mathrm{colim}_{i \in I} K_n(A \rtimes_m G_i) &\xrightarrow{\cong} K_n(A \rtimes_m G).
\end{aligned}$$

Let A be a C^*-algebra with G-action by C^*-automorphisms. We can consider A as a ring only. Notice that we get a commutative diagram

$$\begin{array}{ccc}
H_n^G(E_{\mathcal{V}\mathcal{C}\mathrm{yc}}(G); \mathbf{K}_A) & \longrightarrow & KH_n(A \rtimes G) \\
\downarrow & & \downarrow \\
H_n^G(E_{\mathcal{V}\mathcal{C}\mathrm{yc}}(G); \mathbf{KH}_A) & \longrightarrow & KH_n(A \rtimes G) \\
\cong \uparrow & & \uparrow \mathrm{id} \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{KH}_A) & \longrightarrow & KH_n(A \rtimes G) \\
\downarrow & & \downarrow \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{K}_{A,l^1}^{\mathrm{top}}) & \longrightarrow & K_n(A \rtimes_{l^1} G) \\
\cong \downarrow & & \downarrow \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{K}_{A,m}^{\mathrm{top}}) & \longrightarrow & K_n(A \rtimes_m G) \\
\cong \downarrow & & \downarrow \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{K}_{A,r}^{\mathrm{top}}) & \longrightarrow & K_n(A \rtimes_r G)
\end{array}$$

where the horizontal maps are assembly maps and the vertical maps are change of theory and rings maps or induced by the up to G-homotopy unique G-map $E_{\mathcal{F}\mathrm{in}}(G) \to E_{\mathcal{V}\mathcal{C}\mathrm{yc}}(G)$. The second left vertical map, which is marked with $\cong$, is bijective. This is shown in [4, Remark 7.4] in the case, where G acts trivially on R, the proof carries directly over to the general case. The fourth and fifth vertical left arrow, which are marked with $\cong$, are bijective, since for a finite group H we have $A \rtimes H = A \rtimes_{l^1} H = A \rtimes_r H = A \rtimes_m H$ and hence we can apply [13, Lemma 4.6]. In particular the Bost Conjecture and the Baum-Connes Conjecture together imply that the map $K_n(A \rtimes_{l^1} G) \to K_n(A \rtimes_r G)$ is bijective, the map $K_n(A \rtimes_{l^1} G) \to K_n(A \rtimes_m G)$ is split injective and the map $K_n(A \rtimes_m G) \to K_n(A \rtimes_r G)$ is split surjective.

The upshot of this discussions is:

- The counterexamples of Higson, Lafforgue and Skandalis [23, Section 7] to the Baum-Connes Conjecture with coefficients are not counterexamples to the Farrell-Jones Conjecture or the Bost Conjecture;

- The counterexamples of Higson, Lafforgue and Skandalis [23, Section 7] show that the map $K_n(A \rtimes_{l^1} G) \to K_n(A \rtimes_r G)$ is in general not bijective;

- The passage from the topological K-theory of the Banach algebra $l^1(G)$ to the reduced group C^*-algebra is problematic and may cause failures of the Baum-Connes Conjecture;

- The Bost Conjecture and the Farrell-Jones Conjecture are more likely to be true than the Baum-Connes Conjecture;

- There is — to the authors' knowledge — no promising candidate of a group G for which a strategy is in sight to show that the Farrell-Jones Conjecture or the Bost Conjecture are false. (Whether it is reasonable to believe that these conjectures are true for all groups is a different question.)

0.7. Homology theories and spectra. The general strategy of this paper is to present most of the arguments in terms of equivariant homology theories. Many of the arguments for the Farrell-Jones Conjecture, the Bost Conjecture or the Baum-Connes Conjecture become the same, the only difference lies in the homology theory we apply them to. This is convenient for a reader who is not so familiar with spectra and prefers to think of K-groups and not of K-spectra.

The construction of these equivariant homology theories is a second step and done in terms of spectra. Spectra cannot be avoided in algebraic K-theory by definition and since we want to compare also algebraic and topological K-theory, we need spectra descriptions here as well. Another nice feature of the approach to equivariant topological K-theory via spectra is that it yields a theory which can be applied to all G-CW-complexes. This will allow us to consider in the case $G = \mathrm{colim}_{i \in I} G_i$ the equivariant K-homology of the G_i-CW-complex $\psi_i^* E_{\mathcal{F}\mathrm{in}}(G) = E_{\psi^* \mathcal{F}\mathrm{in}}(G_i)$ although $\psi_i^* E_{\mathcal{F}\mathrm{in}}(G)$ has infinite isotropy groups if the structure map $\psi_i \colon G_i \to G$ has infinite kernel.

Details of the constructions of the relevant spectra, namely, the proof of Theorem 7.1, will be deferred to [2]. We will use the existence of these spectra as a black box. These constructions require some work and technical skills, but their details are not at all relevant for the results and ideas of this paper and their existence is not at all surprising.

0.8. Twisting by cocycles. In the L-theory case one encounters also non-orientable manifolds. In this case twisting with the first Stiefel-Whitney class is required. In a more general setup one is given a group G, a ring R with involution and a group homomorphism $w \colon G \to \mathrm{cent}(R^\times)$ to the center of the multiplicative group of units in R. So far we have used the standard involution on the group ring RG, which is given by $\overline{r \cdot g} = \overline{r} \cdot g^{-1}$. One may also consider the w-twisted involution given by $\overline{r \cdot g} = \overline{r} w(g) \cdot g^{-1}$. All the results in this paper generalize

directly to this case since one can construct a modified L-theory spectrum functor (over G) using the w-twisted involution and then the homology arguments are just applied to the equivariant homology theory associated to this w-twisted L-theory spectrum.

1. Equivariant homology theories

In this section we briefly explain basic axioms, notions and facts about equivariant homology theories as needed for the purposes of this article. The main examples which will play a role in connection with the Bost, the Baum-Connes and the Farrell-Jones Conjecture will be presented later in Theorem 5.1.

Fix a group G and a ring Λ. In most cases Λ will be $\mathbb{Z}$. The following definition is taken from [28, Section 1].

Definition 1.1 (G-homology theory). A G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules is a collection of covariant functors $\mathcal{H}_n^G$ from the category of G-CW-pairs to the category of Λ-modules indexed by $n \in \mathbb{Z}$ together with natural transformations $\partial_n^G(X, A) : \mathcal{H}_n^G(X, A) \to \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A, \emptyset)$ for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

- G-homotopy invariance

  If f_0 and f_1 are G-homotopic maps $(X, A) \to (Y, B)$ of G-CW-pairs, then $\mathcal{H}_n^G(f_0) = \mathcal{H}_n^G(f_1)$ for $n \in \mathbb{Z}$;

- Long exact sequence of a pair

  Given a pair (X, A) of G-CW-complexes, there is a long exact sequence

  $$\ldots \xrightarrow{\mathcal{H}_{n+1}^G(j)} \mathcal{H}_{n+1}^G(X, A) \xrightarrow{\partial_{n+1}^G} \mathcal{H}_n^G(A) \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X)$$
  $$\xrightarrow{\mathcal{H}_n^G(j)} \mathcal{H}_n^G(X, A) \xrightarrow{\partial_n^G} \ldots,$$

  where $i : A \to X$ and $j : X \to (X, A)$ are the inclusions;

- Excision

  Let (X, A) be a G-CW-pair and let $f : A \to B$ be a cellular G-map of G-CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a G-CW-pair. Then the canonical map $(F, f) : (X, A) \to (X \cup_f B, B)$ induces an isomorphism

  $$\mathcal{H}_n^G(F, f) : \mathcal{H}_n^G(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(X \cup_f B, B);$$

- Disjoint union axiom

  Let $\{X_i \mid i \in I\}$ be a family of G-CW-complexes. Denote by $j_i \colon X_i \to \coprod_{i \in I} X_i$ the canonical inclusion. Then the map

  $$\bigoplus_{i \in I} \mathcal{H}_n^G(j_i) \colon \bigoplus_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G\left(\coprod_{i \in I} X_i\right)$$

  is bijective.

Let $\mathcal{H}_*^G$ and $\mathcal{K}_*^G$ be G-homology theories. A *natural transformation* $T_* \colon \mathcal{H}_*^G \to \mathcal{K}_*^G$ *of G-homology theories* is a sequence of natural transformations $T_n \colon \mathcal{H}_n^G \to \mathcal{K}_n^G$ of functors from the category of G-CW-pairs to the category of Λ-modules which are compatible with the boundary homomorphisms.

Lemma 1.2. *Let $T_* \colon \mathcal{H}_*^G \to \mathcal{K}_*^G$ be a natural transformation of G-homology theories. Suppose that $T_n(G/H)$ is bijective for every homogeneous space G/H and $n \in \mathbb{Z}$.*

Then $T_n(X, A)$ is bijective for every G-CW-pair (X, A) and $n \in \mathbb{Z}$.

Proof. The disjoint union axiom implies that both G-homology theories are compatible with colimits over directed systems indexed by the natural numbers (such as the system given by the skeletal filtration $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq \cup_{n \geq 0} X_n = X$). The argument for this claim is analogous to the one in [40, 7.53]. Hence it suffices to prove the bijectivity for finite-dimensional pairs. Using the axioms of a G-homology theory, the five lemma and induction over the dimension one reduces the proof to the special case $(X, A) = (G/H, \emptyset)$. □

Next we present a slight variation of the notion of an equivariant homology theory introduced in [28, Section 1]. We have to treat this variation since we later want to study coefficients over a fixed group Γ which we will then pullback via group homomorphisms with Γ as target. Namely, fix a group Γ. A group (G, ξ) over Γ is a group G together with a group homomorphism $\xi \colon G \to \Gamma$. A map $\alpha \colon (G_1, \xi_1) \to (G_2, \xi_2)$ of groups over Γ is a group homomorphisms $\alpha \colon G_1 \to G_2$ satisfying $\xi_2 \circ \alpha = \xi_1$.

Let $\alpha \colon H \to G$ be a group homomorphism. Given an H-space X, define the *induction of X with α* to be the G-space denoted by $\alpha_* X$ which is the quotient of $G \times X$ by the right H-action $(g, x) \cdot h := (g\alpha(h), h^{-1}x)$ for $h \in H$ and $(g, x) \in G \times X$. If $\alpha \colon H \to G$ is an inclusion, we also write ind_H^G instead of α_*. If (X, A) is an H-CW-pair, then $\alpha_*(X, A)$ is a G-CW-pair.

Definition 1.3 (Equivariant homology theory over a group Γ). An *equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules over a group Γ* assigns to every group (G, ξ) over Γ a G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules and comes with the following so called *induction structure*: given a homomorphism $\alpha \colon (H, \xi) \to (G, \mu)$ of groups over Γ and an H-CW-pair (X, A), there are for each $n \in \mathbb{Z}$ natural homomorphisms

$$\operatorname{ind}_\alpha \colon \mathcal{H}_n^H(X, A) \to \mathcal{H}_n^G(\alpha_*(X, A)) \tag{1.4}$$

satisfying

- Compatibility with the boundary homomorphisms
$$\partial_n^G \circ \mathrm{ind}_\alpha = \mathrm{ind}_\alpha \circ \partial_n^H;$$

- Functoriality

  Let $\beta \colon (G, \mu) \to (K, \nu)$ be another morphism of groups over Γ. Then we have for $n \in \mathbb{Z}$

$$\mathrm{ind}_{\beta \circ \alpha} = \mathcal{H}_n^K(f_1) \circ \mathrm{ind}_\beta \circ \mathrm{ind}_\alpha \colon \mathcal{H}_n^H(X, A) \to \mathcal{H}_n^K((\beta \circ \alpha)_*(X, A)),$$

  where $f_1 \colon \beta_* \alpha_*(X, A) \xrightarrow{\cong} (\beta \circ \alpha)_*(X, A)$, $(k, g, x) \mapsto (k\beta(g), x)$ is the natural K-homeomorphism;

- Compatibility with conjugation

  Let (G, ξ) be a group over Γ and let $g \in G$ be an element with $\xi(g) = 1$. Then the conjugation homomorphisms $c(g) \colon G \to G$ defines a morphism $c(g) \colon (G, \xi) \to (G, \xi)$ of groups over Γ. Let $f_2 \colon (X, A) \to c(g)_*(X, A)$ be the G-homeomorphism which sends x to $(1, g^{-1}x)$ in $G \times_{c(g)} (X, A)$.

  Then for every $n \in \mathbb{Z}$ and every G-CW-pair (X, A) the homomorphism $\mathrm{ind}_{c(g)} \colon \mathcal{H}_n^G(X, A) \to \mathcal{H}_n^G(c(g)_*(X, A))$ agrees with $\mathcal{H}_n^G(f_2)$.

- Bijectivity

  If $\alpha \colon (H, \xi) \to (G, \mu)$ is a morphism of groups over Γ such that the underlying group homomorphism $\alpha \colon H \to G$ is an inclusion of groups, then $\mathrm{ind}_\alpha \colon \mathcal{H}_n^H(\{\bullet\}) \to \mathcal{H}_n^G(\alpha_*\{\bullet\}) = \mathcal{H}_n^G(G/H)$ is bijective for all $n \in \mathbb{Z}$.

Definition 1.3 reduces to the one of an equivariant homology in [28, Section 1] if one puts $\Gamma = \{1\}$.

Lemma 1.5. *Let $\alpha \colon (H, \xi) \to (G, \mu)$ be a morphism of groups over Γ. Let (X, A) be an H-CW-pair such that $\ker(\alpha)$ acts freely on $X - A$. Then*

$$\mathrm{ind}_\alpha \colon \mathcal{H}_n^H(X, A) \to \mathcal{H}_n^G(\alpha_*(X, A))$$

is bijective for all $n \in \mathbb{Z}$.

Proof. Let $\mathcal{F}$ be the set of all subgroups of H whose intersection with $\ker(\alpha)$ is trivial. Obviously, this is a family, i.e., closed under conjugation and taking subgroups. A H-CW-pair (X, A) is called a $\mathcal{F}$-H-CW-pair if the isotropy group of any point in $X - A$ belongs to $\mathcal{F}$. A H-CW-pair (X, A) is a $\mathcal{F}$-H-CW-pair if and only if $\ker(\alpha)$ acts freely on $X - A$.

The n-skeleton of $\alpha_*(X, A)$ is α_* applied to the n-skeleton of (X, A). Let (X, A) be an H-CW-pair and let $f \colon A \to B$ be a cellular H-map of H-CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a H-CW-pair. Then there is an obvious natural isomorphism of G-CW-pairs

$$\alpha_*(X \cup_f B, B) \xrightarrow{\cong} (\alpha_* X \cup_{\alpha_* f} \alpha_* B, \alpha_* B).$$

Now we proceed as in the proof of Lemma 1.2 but now considering the transformations

$$\mathrm{ind}_\alpha \colon \mathcal{H}_n^H(X, A) \to \mathcal{H}_n^G(\alpha_*(X, A))$$

only for $\mathcal{F}$-H-CW-pairs (X, A). Thus we can reduce the claim to the special case $(X, A) = H/L$ for some subgroup $L \subseteq H$ with $L \cap \ker(\alpha) = \{1\}$. This special case follows from the following commutative diagram whose vertical arrows are bijective by the axioms and whose upper horizontal arrow is bijective since α induces an isomorphism $\alpha|_L \colon L \to \alpha(L)$.

$$
\begin{array}{ccc}
\mathcal{H}_n^L(\{\bullet\}) & \xrightarrow{\mathrm{ind}_{\alpha|_L \colon L \to \alpha(L)}} & \mathcal{H}_n^{\alpha(L)}(\{\bullet\}) \\
\downarrow{\scriptstyle \mathrm{ind}_L^H} & & \downarrow{\scriptstyle \mathrm{ind}_{\alpha(L)}^G} \\
\mathcal{H}_n^H(H/L) & \xrightarrow{\mathrm{ind}_\alpha} & \mathcal{H}_n^G(\alpha_* H/L) = \mathcal{H}_n^G(G/\alpha(L))
\end{array}
$$

$\square$

2. Equivariant homology theories and colimits

Fix a group Γ and an equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules over Γ.

Let X be a G-CW-complex. Let $\alpha \colon H \to G$ be a group homomorphism. Denote by $\alpha^* X$ the H-CW-complex obtained from X by *restriction with* α. We have already introduced the induction $\alpha_* Y$ of an H-CW-complex Y. The functors α_* and α^* are adjoint to one another. In particular the adjoint of the identity on $\alpha^* X$ is a natural G-map

$$f(X, \alpha) \colon \alpha_* \alpha^* X \to X. \tag{2.1}$$

It sends an element in $G \times_\alpha \alpha^* X$ given by (g, x) to gx.

Consider a map $\alpha \colon (H, \xi) \to (G, \mu)$ of groups over Γ. Define the Λ-map

$$a_n = a_n(X, \alpha) \colon \mathcal{H}_n^H(\alpha^* X) \xrightarrow{\mathrm{ind}_\alpha} \mathcal{H}_n^G(\alpha_* \alpha^* X) \xrightarrow{\mathcal{H}_n^G(f(X,\alpha))} \mathcal{H}_n^G(X).$$

If $\beta \colon (G, \mu) \to (K, \nu)$ is another morphism of groups over Γ, then by the axioms of an induction structure the composite $\mathcal{H}_n^H(\alpha^* \beta^* X) \xrightarrow{a_n(\beta^* X, \alpha)} \mathcal{H}_n^G(\beta^* X) \xrightarrow{a_n(X, \beta)} \mathcal{H}_n^K(X)$ agrees with $a_n(X, \beta \circ \alpha) \colon \mathcal{H}_n^H(\alpha^* \beta^* X) = \mathcal{H}_n^H((\beta \circ \alpha)^* X) \to \mathcal{H}_n^G(X)$ for a K-CW-complex X.

Consider a directed system of groups $\{G_i \mid i \in I\}$ with $G = \mathrm{colim}_{i \in I} G_i$ and structure maps $\psi_i \colon G_i \to G$ for $i \in I$ and $\phi_{i,j} \colon G_i \to G_j$ for $i, j \in I, i \leq j$. We obtain for every G-CW-complex X a system of Λ-modules $\{\mathcal{H}^{G_i}(\psi_i^* X) \mid i \in I\}$ with structure maps $a_n(\psi_j^* X, \phi_{i,j}) \colon \mathcal{H}^{G_i}(\psi_i^* X) \to \mathcal{H}^{G_j}(\psi_j^* X)$. We get a map of Λ-modules

$$t_n^G(X, A) \;:=\; \mathrm{colim}_{i \in I} \, a_n(X, \psi_i) \colon \; \mathrm{colim}_{i \in I} \, \mathcal{H}_n^{G_i}(\psi_i^*(X, A)) \;\to\; \mathcal{H}_n^G(X, A) \tag{2.2}$$

The next definition is an extension of [4, Definition 3.1].

Definition 2.3 ((Strongly) continuous equivariant homology theory). An equivariant homology theory $\mathcal{H}_*^?$ over Γ is called *continuous* if for every group (G, ξ) over Γ and every directed system of subgroups $\{G_i \mid i \in I\}$ of G with $G = \bigcup_{i \in I} G_i$ the Λ-map (see (2.2))

$$t_n^G(\{\bullet\})\colon \operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(\{\bullet\}) \to \mathcal{H}_n^G(\{\bullet\})$$

is an isomorphism for every $n \in \mathbb{Z}$.

An equivariant homology theory $\mathcal{H}_*^?$ over Γ is called *strongly continuous* if for every group (G, ξ) over Γ and every directed system of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i\colon G_i \to G$ for $i \in I$ the Λ-map

$$t_n^G(\{\bullet\})\colon \operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(\{\bullet\}) \to \mathcal{H}_n^G(\{\bullet\})$$

is an isomorphism for every $n \in \mathbb{Z}$.

Here and in the sequel we view G_i as a group over Γ by $\xi \circ \psi_i\colon G_i \to \Gamma$ and $\psi_i\colon G_i \to G$ as a morphism of groups over Γ.

Lemma 2.4. *Let (G, ξ) be a group over Γ. Consider a directed system of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i\colon G_i \to G$ for $i \in I$. Let (X, A) be a G-CW-pair. Suppose that $\mathcal{H}_*^?$ is strongly continuous.*

Then the Λ-homomorphism (see (2.2))

$$t_n^G(X, A)\colon \operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(\psi_i^*(X, A)) \ \xrightarrow{\cong} \ \mathcal{H}_n^G(X, A)$$

is bijective for every $n \in \mathbb{Z}$.

Proof. The functor sending a directed systems of Λ-modules to its colimit is an exact functor and compatible with direct sums over arbitrary index maps. If (X, A) is a pair of G-CW-complexes, then $(\psi_i^* X, \psi_i^* A)$ is a pair of G_i-CW-complexes. Hence the collection of maps $\{t_n^G(X, A) \mid n \in \mathbb{Z}\}$ is a natural transformation of G-homology theories of pairs of G-CW-complexes which satisfy the disjoint union axiom. Hence in order to show that $t_n^G(X, A)$ is bijective for all $n \in \mathbb{Z}$ and all pairs of G-CW-complexes (X, A), it suffices by Lemma 1.2 to prove this in the special case $(X, A) = (G/H, \emptyset)$.

For $i \in I$ let $k_i\colon G_i/\psi_i^{-1}(H) \to \psi_i^*(G/H)$ be the G_i-map sending $g_i \psi_i^{-1}(H)$ to $\psi_i(g_i)H$. Consider a directed system of Λ-modules $\{\mathcal{H}_n^{G_i}(G_i/\psi_i^{-1}(H)) \mid i \in I\}$ whose structure maps for $i, j \in I, i \leq j$ are given by the composite

$$\mathcal{H}_n^{G_i}(G_i/\psi_i^{-1}(H)) \xrightarrow{\operatorname{ind}_{\phi_{i,j}}} \mathcal{H}_n^{G_j}(G_j \times_{\phi_{i,j}} G_i/\psi_i^{-1}(H)) \xrightarrow{\mathcal{H}_n^{G_j}(f_{i,j})} \mathcal{H}_n^{G_j}(G_j/\psi_j^{-1}(H))$$

for the G_j-map $f_{i,j}\colon G_j \times_{\phi_{i,j}} G_i/\psi_i^{-1}(H) \to G_j/\psi_j^{-1}(H)$ sending $(g_j, g_i \psi_i^{-1}(H))$

to $g_j\phi_{i,j}(g_i)\psi_j^{-1}(H)$. Then the following diagram commutes

$$
\begin{array}{ccc}
\operatorname{colim}_{i\in I}\mathcal{H}_n^{\psi_i^{-1}(H)}(\{\bullet\}) & \xrightarrow[\cong]{\operatorname{colim}_{i\in I}\operatorname{ind}_{\psi_i^{-1}(H)}^{G_i}} & \operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(G_i/\psi_i^{-1}(H)) \\
\Big\downarrow{\scriptstyle t_n^H(\{\bullet\})\,\cong} & & \Big\downarrow{\scriptstyle\operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(k_i)} \\
& & \operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(\psi_i^*(G/H)) \\
& & \Big\downarrow{\scriptstyle t_n^G(G/H)} \\
\mathcal{H}_n^H(\{\bullet\}) & \xrightarrow[\cong]{\operatorname{ind}_H^G} & \mathcal{H}_n^G(G/H)
\end{array}
$$

where the horizontal maps are the isomorphism given by induction. For the directed system $\{\psi_i^{-1}(H)\mid i\in I\}$ with structure maps $\phi_{i,j}|_{\psi_i^{-1}(H)}\colon\psi_i^{-1}(H)\to\psi_j^{-1}(H)$, the group homomorphism $\operatorname{colim}_{i\in I}\psi_i|_{\psi_i^{-1}(H)}\colon\operatorname{colim}_{i\in I}\psi_i^{-1}(H)\to H$ is an isomorphism. This follows by inspecting the standard model for the colimit over a directed system of groups. Hence the left vertical arrow is bijective since $\mathcal{H}_*^?$ is strongly continuous by assumption. Therefore it remains to show that the map

$$\operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(k_i)\colon\operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(G_i/\psi_i^{-1}(H))\to\operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(\psi_i^*G/H)\quad(2.5)$$

is surjective.

Notice that the map given by the direct sum of the structure maps

$$\bigoplus_{i\in I}\mathcal{H}_n^{G_i}(\psi_i^*G/H)\to\operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(\psi_i^*G/H)$$

is surjective. Hence it remains to show for a fixed $i\in I$ that the image of the structure map

$$\mathcal{H}_n^{G_i}(\psi_i^*G/H)\to\operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(\psi_i^*G/H)$$

is contained in the image of the map (2.5).

We have the decomposition of the G_i-set ψ_i^*G/H into its G_i-orbits

$$\coprod_{G_i(gH)\in G_i\backslash(\psi_i^*G/H)}G_i/\psi_i^{-1}(gHg^{-1})\xrightarrow{\cong}\psi_i^*G/H,\quad g_i\psi_i^{-1}(gHg^{-1})\mapsto\psi_i(g_i)gH.$$

It induces an identification of Λ-modules

$$\bigoplus_{G_i(gH)\in G_i\backslash(\psi_i^*G/H)}\mathcal{H}_n^{G_i}\big(G_i/\psi_i^{-1}(gHg^{-1})\big)\;=\;\mathcal{H}_n^{G_i}(\psi_i^*G/H).$$

Hence it remains to show for fixed elements $i\in I$ and $G_i(gH)\in G_i\backslash(\psi_i^*G/H)$ that the obvious composition

$$\mathcal{H}_n^{G_i}\big(G_i/\psi_i^{-1}(gHg^{-1})\big)\subseteq\mathcal{H}_n^{G_i}(\psi_i^*G/H)\to\operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(\psi_i^*G/H)$$

is contained in the image of the map (2.5).

Choose an index j with $j \geq i$ and $g \in \operatorname{im}(\psi_j)$. Then the structure map for $i \leq j$ is a map $\mathcal{H}_n^{G_i}(\psi_i^* G/H) \to \mathcal{H}_n^{G_j}(\psi_j^* G/H)$ which sends the summand corresponding to $G_i(gH) \in G_i \backslash (\psi_i^* G/H)$ to the summand corresponding to $G_j(1H) \in G_j \backslash (\psi_j^* G/H)$ which is by definition the image of

$$\mathcal{H}_n^{G_j}(k_j) \colon \mathcal{H}_n^{G_j}(G_j/\psi_j^{-1}(H)) \to \mathcal{H}_n^{G_j}(\psi_j^* G/H).$$

Obviously the image of composite of the last map with the structure map

$$\mathcal{H}_n^{G_j}(\psi_j^* G/H) \to \operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(\psi_i^* G/H)$$

is contained in the image of the map (2.5). Hence the map (2.5) is surjective. This finishes the proof of Lemma 2.4. $\qquad\square$

3. Isomorphism Conjectures and colimits

A *family $\mathcal{F}$ of subgroups of G* is a collection of subgroups of G which is closed under conjugation and taking subgroups. Let $E_{\mathcal{F}}(G)$ be the *classifying space associated to $\mathcal{F}$*. It is uniquely characterized up to G-homotopy by the properties that it is a G-CW-complex and that $E_{\mathcal{F}}(G)^H$ is contractible if $H \in \mathcal{F}$ and is empty if $H \notin \mathcal{F}$. For more information about these spaces $E_{\mathcal{F}}(G)$ we refer for instance to the survey article [29]. Given a group homomorphism $\phi \colon K \to G$ and a family $\mathcal{F}$ of subgroups of G, define the family $\phi^* \mathcal{F}$ of subgroups of K by

$$\phi^* \mathcal{F} \quad = \quad \{H \subseteq K \mid \phi(H) \in \mathcal{F}\}. \tag{3.1}$$

If ϕ is an inclusion of subgroups, we also write $\mathcal{F}|_K$ instead of $\phi^* \mathcal{F}$.

Definition 3.2 (Isomorphism Conjecture for $\mathcal{H}_*^?$). Fix a group Γ and an equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules over Γ.

A group (G, ξ) over Γ together with a family of subgroups $\mathcal{F}$ of G satisfies the *Isomorphism Conjecture (for $\mathcal{H}_*^?$)* if the projection $\operatorname{pr} \colon E_{\mathcal{F}}(G) \to \{\bullet\}$ to the one-point-space $\{\bullet\}$ induces an isomorphism

$$\mathcal{H}_n^G(\operatorname{pr}) \colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \xrightarrow{\cong} \mathcal{H}_n^G(\{\bullet\})$$

for all $n \in \mathbb{Z}$.

From now on fix a group Γ and an equivariant homology theory $\mathcal{H}_*^?$ over Γ.

Theorem 3.3 (Transitivity Principle). *Let (G, ξ) be a group over Γ. Let $\mathcal{F} \subseteq \mathcal{G}$ be families of subgroups of G. Assume that for every element $H \in \mathcal{G}$ the group $(H, \xi|_H)$ over Γ satisfies the Isomorphism Conjecture for $\mathcal{F}|_H$.*

Then the up to G-homotopy unique map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$ induces an isomorphism $\mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{G}}(G))$ for all $n \in \mathbb{Z}$. In particular, (G, ξ) satisfies the Isomorphism Conjecture for $\mathcal{G}$ if and only if (G, ξ) satisfies the Isomorphism Conjecture for $\mathcal{F}$.

Proof. The proof is completely analogous to the one in [4, Theorem 2.4, Lemma 2.2], where only the case $\Gamma = \{1\}$ is treated. $\square$

Theorem 3.4. *Let (G, ξ) be a group over Γ. Let $\mathcal{F}$ be a family of subgroups of G.*

(i) *Let G be the directed union of subgroups $\{G_i \mid i \in I\}$. Suppose that $\mathcal{H}_*^?$ is continuous and for every $i \in I$ the Isomorphism Conjecture holds for $(G_i, \xi|_{G_i})$ and $\mathcal{F}|_{G_i}$.*

Then the Isomorphism Conjecture holds for (G, ξ) and $\mathcal{F}$;

(ii) *Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i \colon G_i \to G$. Suppose that $\mathcal{H}_*^?$ is strongly continuous and for every $i \in I$ the Isomorphism Conjecture holds for $(G_i, \xi \circ \psi_i)$ and $\psi_i^* \mathcal{F}$.*

Then the Isomorphism Conjecture holds for (G, ξ) and $\mathcal{F}$.

Proof. (i) The proof is analogous to the one in [4, Proposition 3.4].

(ii) This follows from the following commutative square whose horizontal arrows are bijective because of Lemma 2.4 and the identification $\psi_i^* E_{\mathcal{F}}(G) = E_{\psi_i^* \mathcal{F}}(G_i)$

$$
\begin{array}{ccc}
\operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(E_{\psi_i^* \mathcal{F}}(G_i)) & \xrightarrow[\cong]{t_n^G(E_{\mathcal{F}}(G))} & \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(\{\bullet\}) & \xrightarrow[\cong]{t_n^G(\{\bullet\})} & \mathcal{H}_n^G(\{\bullet\})
\end{array}
$$

$\square$

Fix a class of groups $\mathcal{C}$ closed under isomorphisms, taking subgroups and taking quotients, e.g., the class of finite groups or the class of virtually cyclic groups. For a group G let $\mathcal{C}(G)$ be the family of subgroups of G which belong to $\mathcal{C}$.

Theorem 3.5. *Let (G, ξ) be a group over Γ.*

(i) *Let G be the directed union $G = \bigcup_{i \in I} G_i$ of subgroups G_i Suppose that $\mathcal{H}_*^?$ is continuous and that the Isomorphism Conjecture is true for $(G_i, \xi|_{G_i})$ and $\mathcal{C}(G_i)$ for all $i \in I$.*

Then the Isomorphism Conjecture is true for (G, ξ) and $\mathcal{C}(G)$;

(ii) *Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i \colon G_i \to G$. Suppose that $\mathcal{H}_*^?$ is strongly continuous and that the Isomorphism Conjecture is true for $(H, \mathcal{C}(H))$ for every $i \in I$ and every subgroup $H \subseteq G_i$.*

Then for every subgroup $K \subseteq G$ the Isomorphism Conjecture is true for $(K, \xi|_K)$ and $\mathcal{C}(K)$.

Proof. (i) This follows from Theorem 3.4 (i) since $\mathcal{C}(G_i) = \mathcal{C}(G)|_{G_i}$ holds for $i \in I$.
(ii) If G is the colimit of the directed system $\{G_i \mid i \in I\}$, then the subgroup $K \subseteq G$ is the colimit of the directed system $\{\psi_i^{-1}(K) \mid i \in I\}$. Hence we can assume $G = K$ without loss of generality.

Since $\mathcal{C}$ is closed under quotients by assumption, we have $\mathcal{C}(G_i) \subseteq \psi_i^* \mathcal{C}(G)$ for every $i \in I$. Hence we can consider for any $i \in I$ the composition

$$H_n^{G_i}(E_{\mathcal{C}(G_i)}(G_i)) \to H_n^{G_i}(E_{\psi_i^* \mathcal{C}(G)}(G_i)) \to H_n^{G_i}(\{\bullet\}).$$

Because of Theorem 3.4 (ii) it suffices to show that the second map is bijective. By assumption the composition of the two maps is bijective. Hence it remains to show that the first map is bijective. By Theorem 3.3 this follows from the assumption that the Isomorphism Conjecture holds for every subgroup $H \subseteq G_i$ and in particular for any $H \in \psi_i^* \mathcal{C}(G)$ for $\mathcal{C}(G_i)|_H = \mathcal{C}(H)$. $\qquad\square$

4. Fibered Isomorphism Conjectures and colimits

In this section we also deal with the Fibered version of the Isomorphism Conjectures. (This is not directly needed for the purpose of this paper and the reader may skip this section.) This is a stronger version of the Farrell-Jones Conjecture. The Fibered Farrell-Jones Conjecture does imply the Farrell-Jones Conjecture and has better inheritance properties than the Farrell-Jones Conjecture.

We generalize (and shorten the proof of) the result of Farrell-Linnell [16, Theorem 7.1] to a more general setting about equivariant homology theories as developed in Bartels-Lück [3].

Definition 4.1 (Fibered Isomorphism Conjecture for $\mathcal{H}_*^?$). Fix a group Γ and an equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules over Γ. A group (G, ξ) over Γ together with a family of subgroups $\mathcal{F}$ of G satisfies the *Fibered Isomorphism Conjecture (for $\mathcal{H}_*^?$)* if for each group homomorphism $\phi \colon K \to G$ the group $(K, \xi \circ \phi)$ over Γ satisfies the Isomorphism Conjecture with respect to the family $\phi^* \mathcal{F}$.

Theorem 4.2. *Let (G, ξ) be a group over Γ. Let $\mathcal{F}$ be a family of subgroups of G. Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i \colon G_i \to G$. Suppose that $\mathcal{H}_*^?$ is strongly continuous and for every $i \in I$ the Fibered Isomorphism Conjecture holds for $(G_i, \xi \circ \psi_i)$ and $\psi_i^* \mathcal{F}$.*

Then the Fibered Isomorphism Conjecture holds for (G, ξ) and $\mathcal{F}$.

Proof. Let $\mu \colon K \to G$ be a group homomorphism. Consider the pullback of groups

$$
\begin{array}{ccc}
K_i & \xrightarrow{\ \mu_i\ } & G_i \\
\overline{\psi_i} \downarrow & & \psi_i \downarrow \\
K & \xrightarrow{\ \mu\ } & G
\end{array}
$$

Explicitly $K_i = \{(k, g_i) \in K \times G_i \mid \mu(k) = \psi_i(g_i)\}$. Let $\overline{\phi}_{i,j} \colon K_i \to K_j$ be the map induced by $\phi_{i,j} \colon G_i \to G_j$, id_K and id_G and the pullback property. One easily checks by inspecting the standard model for the colimit over a directed set that we obtain a directed system $\overline{\phi}_{i,j} \colon K_i \to K_j$ of groups indexed by the directed set I and the system of maps $\overline{\psi}_i \colon K_i \to K$ yields an isomorphism $\mathrm{colim}_{i \in I} K_i \xrightarrow{\cong} K$. The following diagram commutes

$$
\begin{array}{ccc}
\mathrm{colim}_{i \in I} \, \mathcal{H}_n^{K_i}\left(\overline{\psi}_i^{\,*} \mu^* E_{\mathcal{F}}(G)\right) & \xrightarrow[\cong]{t_n^K(\mu^* E_{\mathcal{F}}(G))} & \mathcal{H}_n^K\left(\mu^* E_{\mathcal{F}}(G)\right) \\
\downarrow & & \downarrow \\
\mathrm{colim}_{i \in I} \, \mathcal{H}_n^{K_i}(\{\bullet\}) & \xrightarrow[\cong]{t_n^K(\{\bullet\})} & \mathcal{H}_n^K(\{\bullet\})
\end{array}
$$

where the vertical arrows are induced by the obvious projections onto $\{\bullet\}$ and the horizontal maps are the isomorphisms from Lemma 2.4. Notice that $\overline{\psi}_i^{\,*} \mu^* E_{\mathcal{F}}(G)$ is a model for $E_{\overline{\psi}_i^{\,*} \mu^* \mathcal{F}}(K_i) = E_{\mu_i^* \psi_i^* \mathcal{F}}(K_i)$. Hence each map $\mathcal{H}_n^{K_i}\left(\overline{\psi}_i^{\,*} \mu^* E_{\mathcal{F}}(G)\right) \to \mathcal{H}_n^{K_i}(\{\bullet\})$ is bijective since $(G_i, \xi \circ \psi_i)$ satisfies the Fibered Isomorphism Conjecture for $\psi_i^* \mathcal{F}$ and hence $(K_i, \xi \circ \psi_i \circ \mu_i)$ satisfies the Isomorphism Conjecture for $\mu_i^* \psi_i^* \mathcal{F}$. This implies that the left vertical arrow is bijective. Hence the right vertical arrow is an isomorphism. Since $\mu^* E_{\mathcal{F}}(G)$ is a model for $E_{\mu^* \mathcal{F}}(K)$, this means that $(K, \xi \circ \mu)$ satisfies the Isomorphism Conjecture for $\mu^* \mathcal{F}$. Since $\mu \colon K \to G$ is any group homomorphism, (G, ξ) satisfies the Fibered Isomorphism Conjecture for $\mathcal{F}$. $\square$

The proof of the following results are analogous to the one in [3, Lemma 1.6] and [4, Lemma 1.2], where only the case $\Gamma = \{1\}$ is treated.

Lemma 4.3. *Let (G, ξ) be a group over Γ and let $\mathcal{F} \subset \mathcal{G}$ be families of subgroups of G. Suppose that (G, ξ) satisfies the Fibered Isomorphism Conjecture for the family $\mathcal{F}$.*

Then (G, ξ) satisfies the Fibered Isomorphism Conjecture for the family $\mathcal{G}$.

Lemma 4.4. *Let (G, ξ) be a group over Γ. Let $\phi \colon K \to G$ be a group homomorphism and let $\mathcal{F}$ be a family of subgroups of G. If (G, ξ) satisfies the Fibered Isomorphism Conjecture for the family $\mathcal{F}$, then $(K, \xi \circ \phi)$ satisfies the Fibered Isomorphism Conjecture for the family $\phi^* \mathcal{F}$.*

For the remainder of this section fix a class of groups $\mathcal{C}$ closed under isomorphisms, taking subgroups and taking quotients, e.g., the families $\mathcal{F}\mathrm{in}$ or $\mathcal{V}\mathcal{C}\mathrm{yc}$.

Lemma 4.5. *Let (G, ξ) be a group over Γ. Suppose that the Fibered Isomorphism Conjecture holds for (G, ξ) and $\mathcal{C}(G)$. Let $H \subseteq G$ be a subgroup.*

Then the Fibered Isomorphism Conjecture holds for $(H, \xi|_H)$ and $\mathcal{C}(H)$.

Proof. This follows from Lemma 4.4 applied to the inclusion $H \to G$ since $\mathcal{C}(H) = \mathcal{C}(G)|_H$. $\square$

Theorem 4.6. *Let (G, ξ) be a group over Γ.*

(i) *Let G be the directed union $G = \bigcup_{i \in I} G_i$ of subgroups G_i. Suppose that $\mathcal{H}_*^?$ is continuous and that the Fibered Isomorphism Conjecture is true for $(G_i, \xi|_{G_i})$ and $\mathcal{C}(G_i)$ for all $i \in I$.*

   Then the Fibered Isomorphism Conjecture is true for (G, ξ) and $\mathcal{C}(G)$;

(ii) *Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i \colon G_i \to G$. Suppose that $\mathcal{H}_*^?$ is strongly continuous and that the Fibered Isomorphism Conjecture is true for $(G_i, \xi \circ \psi_i)$ and $\mathcal{C}(G_i))$ for all $i \in I$.*

   Then the Fibered Isomorphism Conjecture is true for (G, ξ) and $\mathcal{C}(G)$.

Proof. (i) The proof is analogous to the one in [4, Proposition 3.4], where the case $\Gamma = \{1\}$ is considered.

(ii) Because $\mathcal{C}$ is closed under taking quotients we conclude $\mathcal{C}(G_i) \subseteq \psi_i^* \mathcal{C}(G)$. Now the claim follows from Theorem 4.2 and Lemma 4.3. $\qquad\square$

Corollary 4.7. (i) *Suppose that $\mathcal{H}_*^?$ is continuous. Then the (Fibered) Isomorphism Conjecture for (G, ξ) and $\mathcal{C}(G)$ is true for all groups (G, ξ) over Γ if and only if it is true for all such groups where G is a finitely generated group;*

(ii) *Suppose that $\mathcal{H}_*^?$ is strongly continuous. Then the Fibered Isomorphism Conjecture for (G, ξ) and $\mathcal{C}(G)$ is true for all groups (G, ξ) over Γ if and only if it is true for all such groups where G is finitely presented.*

Proof. Let (G, ξ) be a group over Γ where G is finitely generated. Choose a finitely generated free group F together with an epimorphism $\psi \colon F \to G$. Let K be the kernel of ψ. Consider the directed system of finitely generated subgroups $\{K_i \mid i \in I\}$ of K. Let $\overline{K_i}$ be the smallest normal subgroup of K containing K_i. Explicitly $\overline{K_i}$ is given by elements which can be written as finite products of elements of the shape $f k_i f^{-1}$ for $f \in F$ and $k \in K_i$. We obtain a directed system of groups $\{F/\overline{K_i} \mid i \in I\}$, where for $i \leq j$ the structure map $\phi_{i,j} \colon F/\overline{K_i} \to F/\overline{K_j}$ is the canonical projection. If $\psi_i \colon F/\overline{K_i} \to F/K = G$ is the canonical projection, then the collection of maps $\{\psi_i \mid i \in I\}$ induces an isomorphism $\operatorname{colim}_{i \in I} F/\overline{K_i} \xrightarrow{\cong} G$. By construction for each $i \in I$ the group $F/\overline{K_i}$ is finitely presented and the Fibered Isomorphism Conjecture holds for $(F/\overline{K_i}, \xi \circ \psi_i)$ and $\mathcal{C}(F/\overline{K_i})$ by assumption. Theorem 4.6 (ii) implies that the Fibered Farrell-Jones Conjecture for (G, ξ) and $\mathcal{C}(G)$ is true. $\qquad\square$

5. Some equivariant homology theories

In this section we will describe the relevant homology theories over a group Γ and show that they are (strongly) continuous. (We have defined the notion of an equivariant homology theory over a group in Definition 1.3.)

5.1. Desired equivariant homology theories. We will need the following

Theorem 5.1 (Construction of equivariant homology theories). *Suppose that we are given a group Γ and a ring R (with involution) or a C^*-algebra A respectively on which Γ acts by structure preserving automorphisms. Then:*

(i) *Associated to these data there are equivariant homology theories with values in $\mathbb{Z}$-modules over the group Γ*

$$H_*^?(-;\mathbf{K}_R)$$
$$H_*^?(-;\mathbf{KH}_R)$$
$$H_*^?(-;\mathbf{L}_R^{\langle -\infty \rangle}),$$
$$H_*^?(-;\mathbf{K}_{A,l^1}^{\mathrm{top}}),$$
$$H_*^?(-;\mathbf{K}_{A,r}^{\mathrm{top}}),$$
$$H_*^?(-;\mathbf{K}_{A,m}^{\mathrm{top}}),$$

where in the case $H_*^?(-;\mathbf{K}_{A,r}^{\mathrm{top}})$ we will have to impose the restriction to the induction structure that a homomorphisms $\alpha\colon (H,\xi) \to (G,\mu)$ over Γ induces a transformation $\mathrm{ind}_\alpha\colon \mathcal{H}_n^H(X,X_0) \to \mathcal{H}_n^G(\alpha_*(X,X_0))$ only if the kernel of the underlying group homomorphism $\alpha\colon H \to G$ acts with amenable isotropy on $X \setminus X_0$;

(ii) *If (G,μ) is a group over Γ and $H \subseteq G$ is a subgroup, then there are for every $n \in \mathbb{Z}$ identifications*

$$
\begin{array}{ccccc}
H_n^H(\{\bullet\};\mathbf{K}_R) & \cong & H_n^G(G/H;\mathbf{K}_R) & \cong & K_n(R \rtimes H); \\
H_n^H(\{\bullet\};\mathbf{KH}_R) & \cong & H_n^G(G/H;\mathbf{KH}_R) & \cong & KH_n(R \rtimes H); \\
H_n^H(\{\bullet\};\mathbf{L}_R^{\langle -\infty \rangle}) & \cong & H_n^G(G/H;\mathbf{L}_R^{\langle -\infty \rangle}) & \cong & L_n^{\langle -\infty \rangle}(R \rtimes H); \\
H_n^H(\{\bullet\};\mathbf{K}_{A,l^1}^{\mathrm{top}}) & \cong & H_n^G(G/H;\mathbf{K}_{A,l^1}^{\mathrm{top}}) & \cong & K_n(A \rtimes_{l^1} H); \\
H_n^H(\{\bullet\};\mathbf{K}_{A,r}^{\mathrm{top}}) & \cong & H_n^G(G/H;\mathbf{K}_{A,r}^{\mathrm{top}}) & \cong & K_n(A \rtimes_r H); \\
H_n^H(\{\bullet\};\mathbf{K}_{A,m}^{\mathrm{top}}) & \cong & H_n^G(G/H;\mathbf{K}_{A,r}^{\mathrm{top}}) & \cong & K_n(A \rtimes_m H).
\end{array}
$$

Here H and G act on R and A respectively via the given Γ-action, $\mu\colon G \to \Gamma$ and the inclusion $H \subseteq G$, $K_n(R \rtimes H)$ is the algebraic K-theory of the twisted group ring $R \rtimes H$, $KH_n(R \rtimes H)$ is the homotopy K-theory of the twisted group ring $R \rtimes H$, $L_n^{\langle -\infty \rangle}(R \rtimes H)$ is the algebraic L-theory with decoration $\langle -\infty \rangle$ of the twisted group ring with involution $R \rtimes H$, $K_n(A \rtimes_{l^1} H)$ is the topological K-theory of the crossed product Banach algebra $A \rtimes_{l^1} H$, $K_n(A \rtimes_r H)$ is the topological K-theory of the reduced crossed product C^*-algebra $A \rtimes_r H$, and $K_n(A \rtimes_m H)$ is the topological K-theory of the maximal crossed product C^*-algebra $A \rtimes_m H$;

(iii) *Let $\zeta\colon \Gamma_0 \to \Gamma_1$ be a group homomorphism. Let R be a ring (with involution) and A be a C^*-algebra on which Γ_1 acts by structure preserving automorphisms. Let (G,μ) be a group over Γ_0. Then in all cases the evaluation*

at (G, μ) of the equivariant homology theory over Γ_0 associated to $\zeta^* R$ or $\zeta^* A$ respectively agrees with the evaluation at $(G, \zeta \circ \mu)$ of the equivariant homology theory over Γ_1 associated to R or A respectively.

(iv) Suppose the group Γ acts on the rings (with involution) R and S or on the C^*-algebras A and B respectively by structure preserving automorphisms. Let $\xi \colon R \to S$ or $\xi \colon A \to B$ be a Γ-equivariant homomorphism of rings (with involution) or C^*-algebras respectively. Then ξ induces natural transformations of homology theories over Γ

$$
\begin{array}{llll}
\xi_*^? \colon & H_*^?(-; \mathbf{K}_R) & \to & H_*^?(-; \mathbf{K}_S); \\
\xi_*^? \colon & H_*^?(-; \mathbf{KH}_R) & \to & H_*^?(-; \mathbf{KH}_S); \\
\xi_*^? \colon & H_*^?(-; \mathbf{L}_R^{\langle -\infty \rangle}) & \to & H_*^?(-; \mathbf{L}_S^{\langle -\infty \rangle}); \\
\xi_*^? \colon & H_*^?(-; \mathbf{K}_{A,l^1}^{\mathrm{top}}) & \to & H_*^?(-; \mathbf{K}_{B,l^1}^{\mathrm{top}}); \\
\xi_*^? \colon & H_*^?(-; \mathbf{K}_{A,r}^{\mathrm{top}}) & \to & H_*^?(-; \mathbf{K}_{B,r}^{\mathrm{top}}); \\
\xi_*^? \colon & H_*^?(-; \mathbf{K}_{A,m}^{\mathrm{top}}) & \to & H_*^?(-; \mathbf{K}_{B,m}^{\mathrm{top}}).
\end{array}
$$

They are compatible with the identifications appearing in assertion (ii);

(v) Let Γ act on the C^*-algebra A by structure preserving automorphisms. We can consider A also as a ring with structure preserving G-action. Then there are natural transformations of equivariant homology theories with values in $\mathbb{Z}$-modules over Γ

$$
H_*^?(-; \mathbf{K}_A) \to H_*^?(-; \mathbf{KH}_A) \to H_*^?(-; \mathbf{K}_{A,l^1}^{\mathrm{top}})
$$
$$
\to H_*^?(-; \mathbf{K}_{A,m}^{\mathrm{top}}) \to H_*^?(-; \mathbf{K}_{A,r}^{\mathrm{top}}).
$$

They are compatible with the identifications appearing in assertion (ii).

5.2. (Strong) Continuity. Next we want to show

Lemma 5.2. *Suppose that we are given a group Γ and a ring R (with involution) or a C^*-algebra A respectively on which G acts by structure preserving automorphisms.*

Then the homology theories with values in $\mathbb{Z}$-modules over Γ

$$
H_*^?(-; \mathbf{K}_R), H_*^?(-; \mathbf{KH}_R), H_*^?(-; \mathbf{L}_R^{\langle -\infty \rangle}), H_*^?(-; \mathbf{K}_{A,l^1}^{\mathrm{top}}), \text{ and } H_*^?(-; \mathbf{K}_{A,m}^{\mathrm{top}})
$$

(see Theorem 5.1) are strongly continuous in the sense of Definition 2.3, whereas

$$
H_*^?(-; \mathbf{K}_{A,r}^{\mathrm{top}})
$$

is only continuous.

Proof. We begin with $H_*^?(-; \mathbf{K}_R)$ and $H_*^?(-; \mathbf{KH}_R)$. We have to show for every directed systems of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i \in I} G_i$ together with a map $\mu \colon G \to \Gamma$ that the canonical maps

$$
\begin{array}{rcl}
\operatorname{colim}_{i \in I} K_n(R \rtimes G_i) & \to & K_n(R \rtimes G); \\
\operatorname{colim}_{i \in I} KH_n(R \rtimes G_i) & \to & KH_n(R \rtimes G),
\end{array}
$$

are bijective for all $n \in \mathbb{Z}$. Obviously $R \rtimes G$ is the colimit of rings $\operatorname{colim}_{i \in I} R \rtimes G_i$. Now the claim follows for $K_n(R \rtimes G)$ for $n \geq 0$ from [36, (12) on page 20].

Using the Bass-Heller-Swan decomposition one gets the results for $K_n(R \rtimes G)$ for all $n \in \mathbb{Z}$ and that the map

$$\operatorname{colim}_{i \in I} N^p K_n(R \rtimes G_i) \quad \rightarrow \quad N^p K_n(R \rtimes G)$$

is bijective for all $n \in \mathbb{Z}$ and all $p \in \mathbb{Z}, p \geq 1$ for the Nil-groups $N^p K_n(RG)$ defined by Bass [8, XII]. Now the claim for homotopy K-theory follows from the spectral sequence due to Weibel [41, Theorem 1.3].

Next we treat $H_*^?(-; \mathbf{L}_R^{\langle -\infty \rangle})$. We have to show for every directed systems of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i \in I} G_i$ together with a map $\mu \colon G \to \Gamma$ that the canonical map

$$\operatorname{colim}_{i \in I} L_n^{\langle -\infty \rangle}(R \rtimes G_i) \to L_n^{\langle -\infty \rangle}(R \times G)$$

is bijective for all $n \in \mathbb{Z}$. Recall from [37, Definition 17.1 and Definition 17.7] that

$$\begin{aligned}
L_n^{\langle -\infty \rangle}(R \rtimes G) &= \operatorname{colim}_{m \to \infty} L_n^{\langle -m \rangle}(R \rtimes G); \\
L_n^{\langle -m \rangle}(R \rtimes G) &= \operatorname{coker}\left(L_{n+1}^{\langle -m+1 \rangle}(R \rtimes G) \to L_{n+1}^{\langle -m+1 \rangle}(R \rtimes G[\mathbb{Z}]) \right) \quad \text{for } m \geq 0.
\end{aligned}$$

Since $L_n^{\langle 1 \rangle}(R \rtimes G)$ is $L_n^h(R \rtimes G)$, it suffices to show that

$$\omega_n \colon \operatorname{colim}_{i \in I} L_n^h(R \rtimes G_i) \quad \rightarrow \quad L_n^h(R \rtimes G) \tag{5.3}$$

is bijective for all $n \in \mathbb{Z}$. We give the proof of surjectivity for $n = 0$ only, the proofs of injectivity for $n = 0$ and of bijectivity for the other values of n are similar.

The ring $R \rtimes G$ is the colimit of rings $\operatorname{colim}_{i \in I} R \rtimes G_i$. Let $\psi_i \colon R \rtimes G_i \to R \rtimes G$ and $\phi_{i,j} \colon R \rtimes G_i \to R \rtimes G_j$ for $i, j \in I, i \leq j$ be the structure maps. One can define $R \rtimes G$ as the quotient of $\coprod_{i \in I} R \rtimes G_i / \sim$, where $x \in R \rtimes G_i$ and $y \in R \rtimes G_j$ satisfy $x \sim y$ if and only if $\phi_{i,k}(x) = \phi_{j,k}(y)$ holds for some $k \in I$ with $i, j \leq k$. The addition and multiplication is given by adding and multiplying representatives belonging to the same $R \rtimes G_i$. Let $M(m, n; R \rtimes G)$ be the set of (m, n)-matrices with entries in $R \rtimes G$. Given $A_i \in M(m, n; R \rtimes G_i)$, define $\phi_{i,j}(A_i) \in M(m, n; R \rtimes G_j)$ and $\psi_i(A_i) \in M(m, n; R \rtimes G)$ by applying $\phi_{i,j}$ and ψ_i to each entry of the matrix A_i. We need the following key properties which follow directly from inspecting the model for the colimit above:

(i) Given $A \in M(m, n; R \rtimes G)$, there exists $i \in I$ and $A_i \in M(m, n; R \rtimes G_i)$ with $\psi_i(A_i) = A$;

(ii) Given $A_i \in M(m, n; R \rtimes G_i)$ and $A_j \in M(m, n; R \rtimes G_j)$ with $\psi_i(A_i) = \psi_j(A_j)$, there exists $k \in I$ with $i, j \leq k$ and $\phi_{i,k}(A_i) = \phi_{j,k}(A_j)$.

An element $[A]$ in $L_0^h(R \rtimes G)$ is represented by a quadratic form on a finitely generated free $R \rtimes G$-module, i.e., a matrix $A \in GL_n(R \rtimes G)$ for which there exists a matrix $B \in M(n, n; R \rtimes G)$ with $A = B + B^*$, where B^* is given by transposing the matrix B and applying the involution of R elementwise. Fix such a choice of

a matrix B. Choose $i \in I$ and $B_i \in M(n, n; R \rtimes G_i)$ with $\psi_i(B_i) = B$. Then $\psi_i(B_i + B_i^*) = A$ is invertible. Hence we can find $j \in I$ with $i \leq j$ such that $A_j := \phi_{i,j}(B_i + B_i^*)$ is invertible. Put $B_j = \phi_{i,j}(B_i)$. Then $A_j = B_j + B_j^*$ and $\psi_j(A_j) = A$. Hence A_j defines an element $[A_j] \in L_n^h(R \rtimes G_j)$ which is mapped to $[A]$ under the homomorphism $L_n^h(R \rtimes G_j) \to L_n(R \rtimes G)$ induced by ψ_j. Hence the map ω_0 of (5.3) is surjective.

Next we deal with $H_*^?(-; \mathbf{K}_{A,l^1}^{\text{top}})$. We have to show for every directed systems of groups $\{G_i \mid i \in I\}$ with $G = \text{colim}_{i \in I} G_i$ together with a map $\mu \colon G \to \Gamma$ that the canonical map

$$\text{colim}_{i \in I} K_n(A \rtimes_{l^1} G_i) \to K_n(A \rtimes_{l^1} G)$$

is bijective for all $n \in \mathbb{Z}$. Since topological K-theory is a continuous functor, it suffices to show that the colimit (or sometimes also called inductive limit) of the system of Banach algebras $\{A \rtimes_{l^1} G_i \mid i \in I\}$ in the category of Banach algebras with norm decreasing homomorphisms is $A \rtimes_{l^1} G$. So we have to show that for any Banach algebra B and any system of (norm deceasing) homomorphisms of Banach algebras $\alpha_i \colon A \rtimes_{l^1} G_i \to B$ compatible with the structure maps $A \rtimes_{l^1} \phi_{i,j} \colon A \rtimes_{l^1} G_i \to A \rtimes_{l^1} G_j$ there exists precisely one homomorphism of Banach algebras $\alpha \colon A \rtimes_{l^1} G \to B$ with the property that its composition with the structure map $A \rtimes_{l^1} \psi_i \colon A \rtimes_{l^1} G_i \to A \rtimes_{l^1} G$ is α_i for $i \in I$.

It is easy to see that in the category of $\mathbb{C}$-algebras the colimit of the system $\{A \rtimes G_i \mid i \in I\}$ is $A \rtimes G$ with structure maps $A \rtimes \psi_i \colon A \rtimes G_i \to A \rtimes G$. Hence the restrictions of the homomorphisms α_i to the subalgebras $A \rtimes G_i$ yields a homomorphism of central $\mathbb{C}$-algebras $\alpha' \colon A \rtimes G \to B$ uniquely determined by the property that the composition of α' with the structure map $A \rtimes \psi_i \colon A \rtimes G_i \to A \rtimes G$ is $\alpha_i|_{A \rtimes G_i}$ for $i \in I$. If α exists, its restriction to the dense subalgebra $A \rtimes G$ has to be α'. Hence α is unique if it exists. Of course we want to define α to be the extension of α' to the completion $A \rtimes_{l^1} G$ of $A \rtimes G$ with respect to the l^1-norm. So it remains to show that $\alpha' \colon A \rtimes G \to B$ is norm decreasing. Consider an element $u \in A \rtimes G$ which is given by a finite formal sum $u = \sum_{g \in F} a_g \cdot g$, where $F \subset G$ is some finite subset of G and $a_g \in A$ for $g \in F$. We can choose an index $j \in I$ and a finite set $F' \subset G_j$ such that $\psi_j|_{F'} \colon F' \to F$ is one-to-one. For $g \in F$ let $g' \in F'$ denote the inverse image of g under this map. Consider the element $v = \sum_{g' \in F'} a_g \cdot g'$ in $A \rtimes G_j$. By construction we have $A \rtimes \psi_j(v) = u$ and $||v|| = ||u|| = \sum_{i=1}^n ||a_i||$. We conclude

$$||\alpha'(u)|| = ||\alpha' \circ (A \rtimes \psi_j)(v)|| = ||\alpha_j(v)|| \leq ||v|| = ||u||.$$

The proof for $H_*^?(-; \mathbf{K}_{A,m}^{\text{top}})$ follows similarly, using the fact that by definition of the norm on $A \rtimes_m G$ every $*$-homomorphism of $A \rtimes G$ into a C^*-algebra B extends uiniquely to $A \rtimes_m G$. The proof for the continuity of $H_*^?(-; \mathbf{K}_{A,r}^{\text{top}})$ follows from [10, Theorem 4.1]. □

Notice that we have proved all promised results of the introduction as soon as we have completed the proof of Theorem 5.1 which we have used as a black box so far.

6. From spectra over groupoids to equivariant homology theories

In this section we explain how one can construct equivariant homology theories from spectra over groupoids.

A *spectrum* $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps* $\sigma(n)\colon E(n) \wedge S^1 \longrightarrow E(n+1)$. A *(strong) map* of spectra (sometimes also called function in the literature) $\mathbf{f}\colon \mathbf{E} \to \mathbf{E}'$ is a sequence of maps $f(n)\colon E(n) \to E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e., we have $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \mathrm{id}_{S^1})$ for all $n \in \mathbb{Z}$. This should not be confused with the notion of a map of spectra in the stable category (see [1, III.2.]). Recall that the homotopy groups of a spectrum are defined by

$$\pi_i(\mathbf{E}) := \mathrm{colim}_{k \to \infty} \pi_{i+k}(E(k)),$$

where the system $\pi_{i+k}(E(k))$ is given by the composition

$$\pi_{i+k}(E(k)) \xrightarrow{S} \pi_{i+k+1}(E(k) \wedge S^1) \xrightarrow{\sigma(k)_*} \pi_{i+k+1}(E(k+1))$$

of the suspension homomorphism and the homomorphism induced by the structure map. We denote by $\mathtt{Spectra}$ the category of spectra.

A *weak equivalence* of spectra is a map $\mathbf{f}\colon \mathbf{E} \to \mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups.

Given a small groupoid $\mathcal{G}$, denote by $\mathtt{Groupoids} \downarrow \mathcal{G}$ the category of small groupoids over $\mathcal{G}$, i.e., an object is a functor $F_0\colon \mathcal{G}_0 \to \mathcal{G}$ with a small groupoid as source and a morphism from $F_0\colon \mathcal{G}_0 \to \mathcal{G}$ to $F_1\colon \mathcal{G}_1 \to \mathcal{G}$ is a functor $F\colon \mathcal{G}_0 \to \mathcal{G}_1$ satisfying $F_1 \circ F = F_0$. We will consider a group Γ as a groupoid with one object and Γ as set of morphisms. An *equivalence* $F\colon \mathcal{G}_0 \to \mathcal{G}_1$ of groupoids is a functor of groupoids F for which there exists a functor of groupoids $F'\colon \mathcal{G}_1 \to \mathcal{G}_0$ such that $F' \circ F$ and $F \circ F'$ are naturally equivalent to the identity functor. A functor $F\colon \mathcal{G}_0 \to \mathcal{G}_1$ of small groupoids is an equivalence of groupoids if and only if it induces a bijection between the isomorphism classes of objects and for any object $x \in \mathcal{G}_0$ the map $\mathrm{aut}_{\mathcal{G}_0}(x) \to \mathrm{aut}_{\mathcal{G}_1}(F(x))$ induced by F is an isomorphism of groups.

Lemma 6.1. *Let Γ be a group. Consider a covariant functor*

$$\mathbf{E}\colon \mathtt{Groupoids} \downarrow \Gamma \to \mathtt{Spectra}$$

which sends equivalences of groupoids to weak equivalences of spectra.

Then we can associate to it an equivariant homology theory $\mathcal{H}_*^?(,-;\mathbf{E})$ (with values in $\mathbb{Z}$-modules) over Γ such that for every group (G, μ) over Γ and subgroup $H \subseteq G$ we have a natural identification

$$\mathcal{H}_n^H(\{\bullet\}; \mathbf{E}) = \mathcal{H}_n^G(G/H, \mathbf{E}) = \pi_n(\mathbf{E}(H)).$$

If $\mathbf{T}\colon \mathbf{E} \to \mathbf{F}$ is a natural transformation of such functors $\mathtt{Groupoids} \downarrow \Gamma \to \mathtt{Spectra}$, then it induces a transformation of equivariant homology theories over Γ

$$\mathcal{H}_*^?(-; \mathbf{T})\colon \mathcal{H}_*^?(,-;\mathbf{E}) \to \mathcal{H}_*^?(-;\mathbf{F})$$

such that for every group (G, μ) over Γ and subgroup $H \subseteq G$ the homomorphism $\mathcal{H}_n^H(\{\bullet\}; \mathbf{T}): \mathcal{H}_n^H(\{\bullet\}; \mathbf{E}) \to \mathcal{H}_n^H(\{\bullet\}; \mathbf{F})$ agrees under the identification above with $\pi_n(\mathbf{T}(H)): \pi_n(\mathbf{E}(H)) \to \pi_n(\mathbf{F}(H))$.

Proof. We begin with explaining how we can associate to a group (G, μ) over Γ a G-homology theory $\mathcal{H}_*^G(-; \mathbf{E})$ with the property that for every subgroup $H \subseteq G$ we have an identification

$$\mathcal{H}_n^G(G/H, \mathbf{E}) = \pi_n(\mathbf{E}(H)).$$

We just follow the construction in [13, Section 4]. Let $\mathsf{Or}(G)$ be the *orbit category* of G, i.e., objects are homogenous spaces G/H and morphisms are G-maps. Given a G-set S, the associated *transport groupoid* $t^G(S)$ has S as set of objects and the set of morphisms from $s_0 \in S$ to $s_1 \in S$ consists of the subset $\{g \in G \mid gs_1 = s_2\}$ of G. Composition is given by the group multiplication. A G-map of sets induces a functor between the associated transport groupoids in the obvious way. In particular the projection $G/H \to G/G$ induces a functor of groupoids $\mathrm{pr}_S: t^G(S) \to t^G(G/G) = G$. Thus $t^G(S)$ becomes an object in $\mathsf{Groupoids} \downarrow \Gamma$ by the composite $\mu \circ \mathrm{pr}_S$. We obtain a covariant functor $t^G: \mathsf{Or}(G) \to \mathsf{Groupoids} \downarrow \Gamma$. Its composition with the given functor $\mathbf{E}$ yields a covariant functor

$$\mathbf{E}^G := \mathbf{E} \circ t^G: \mathsf{Or}(G) \to \mathsf{Spectra}.$$

Now define

$$\mathcal{H}_*^G(X, A; \mathbf{E}) := \mathcal{H}_*^G(-; \mathbf{E}^G),$$

where $\mathcal{H}_*^G(-; \mathbf{E}^G)$ is the G-homology theory which is associated to $\mathbf{E}^G: \mathsf{Or}(G) \to \mathsf{Spectra}$ and defined in [13, Section 4 and 7]. Namely, if X is a G-CW-complex, we can assign to it a contravariant functor $\mathrm{map}_G(G/?, X): \mathsf{Or}(G) \to \mathsf{Spaces}$ sending G/H to $\mathrm{map}_G(G/H, X) = X^H$ and put $\mathcal{H}_n^G(X; \mathbf{E}^G) := \pi_n(\mathrm{map}_G(G/?, X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}^G)$ for the spectrum $\mathrm{map}_G(G/?, X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}^G$ (which is denoted in [13] by $\mathrm{map}_G(G/?, X)_+ \otimes_{\mathsf{Or}(G)} \mathbf{E}^G$).

Next we have to explain the induction structure. Consider a group homomorphism $\alpha: (H, \xi) \to (G, \mu)$ of groups over Γ and an H-CW-complex X. We have to construct a homomorphism

$$\mathcal{H}_n^H(X; \mathbf{E}) \to \mathcal{H}_n^G(\alpha_* X; \mathbf{E}).$$

This will be done by constructing a map of spectra

$$\mathrm{map}_H(H/?, X)_+ \wedge_{\mathsf{Or}(H)} \mathbf{E}^H \to \mathrm{map}_G(G/?, \alpha_* X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}^G.$$

We follow the constructions in [13, Section 1]. The homomorphism α induces a covariant functor $\mathsf{Or}(\alpha): \mathsf{Or}(H) \to \mathsf{Or}(G)$ by sending H/L to $\alpha_*(H/L) = G/\alpha(L)$. Given a contravariant functor $Y: \mathsf{Or}(H) \to \mathsf{Spaces}$, we can assign to it its induction with $\mathsf{Or}(\alpha)$ which is a contravariant functor $\alpha_* Y: \mathsf{Or}(G) \to \mathsf{Spaces}$. Given a contravariant functor $Z: \mathsf{Or}(G) \to \mathsf{Spaces}$, we can assign to it its restriction which is the contravariant functor $\alpha^* Z := Z \circ \mathsf{Or}(\alpha): \mathsf{Or}(G) \to \mathsf{Spaces}$. Induction

α_* and α^* form an adjoint pair. Given an H-CW-complex X, there is a natural identification $\alpha_* \left(\text{map}_H(H/?, X) \right) = \text{map}_G(G/?, \alpha_* X)$. Using [13, Lemma 1.9] we get for an H-CW-complex X a natural map of spectra

$$\text{map}_H(H/?, X)_+ \wedge_{\text{Or}(H)} \alpha^* \mathbf{E}^G \to \text{map}_G(G/?, \alpha_* X)_+ \wedge_{\text{Or}(G)} \mathbf{E}^G.$$

Given an H-set S, we obtain a functor of groupoids $t^H(S) \to t^G(\alpha_* S)$ sending $s \in S$ to $(1, s) \in G \times_\alpha S$ and a morphism in $t^H(S)$ given by a group element h to the one in $t^G(\alpha_* S)$ given by $\alpha(h)$. This yields a natural transformation of covariant functors $\text{Or}(H) \to \text{Groupoids} \downarrow \Gamma$ from $t^H \to t^G \circ \text{Or}(\alpha)$. Composing with the functor $\mathbf{E}$ gives a natural transformation of covariant functors $\text{Or}(H) \to \text{Spectra}$ from $\mathbf{E}^H$ to $\alpha^* \mathbf{E}^G$. It induces a map of spectra

$$\text{map}_H(H/?, X)_+ \wedge_{\text{Or}(H)} \mathbf{E}^H \to \text{map}_H(H/?, X)_+ \wedge_{\text{Or}(H)} \alpha^* \mathbf{E}^G.$$

Its composition with the maps of spectra constructed beforehand yields the desired map of spectra $\text{map}_H(H/?, X)_+ \otimes_{\text{Or}(H)} \mathbf{E}^H \to \text{map}_G(G/?, \alpha_* X)_+ \otimes_{\text{Or}(G)} \mathbf{E}^G$.

We omit the straightforward proof that the axioms of an induction structure are satisfied. This finishes the proof of Theorem 5.1.

The statement about the natural transformation $\mathbf{T} \colon \mathbf{E} \to \mathbf{F}$ is obvious. $\square$

7. Some K-theory spectra associated to groupoids

The last step in completing the proof of Theorem 5.1 is to prove the following Theorem 7.1 because then we can apply it in combination with Lemma 6.1. (Actually we only need the version of Theorem 7.1, where $\mathcal{G}$ is given by a group Γ.) Let $\text{Groupoids}^{\text{finker}} \downarrow \mathcal{G}$ be the subcategory of $\text{Groupoids} \downarrow \mathcal{G}$ which has the same objects and for which a morphism from $F_0 \colon \mathcal{G}_0 \to \mathcal{G}$ to $F_1 \colon \mathcal{G}_1 \to \mathcal{G}$ given by a functor $F \colon \mathcal{G}_0 \to \mathcal{G}_1$ satisfying $F_1 \circ F = F_0$ has the property that for every object $x \in \mathcal{G}_0$ the group homomorphism $\text{aut}_{\mathcal{G}_0}(x) \to \text{aut}_{\mathcal{G}_1}(F(x))$ induced by F has a finite kernel. Denote by Rings, $*\text{Rings}$, and $C^*\text{-Algebras}$ the categories of rings, rings with involution and C^*-algebras.

Theorem 7.1. *Let $\mathcal{G}$ be a fixed groupoid. Let $R \colon \mathcal{G} \to \text{Rings}$, $R \colon \mathcal{G} \to *\text{Rings}$, or $A \colon \mathcal{G} \to C^*\text{-Algebras}$ respectively be a covariant functor. Then there exist covariant functors*

$$
\begin{aligned}
\mathbf{K}_R &\colon & \text{Groupoids} \downarrow \mathcal{G} & \to & \text{Spectra}; \\
\mathbf{KH}_R &\colon & \text{Groupoids} \downarrow \mathcal{G} & \to & \text{Spectra}; \\
\mathbf{L}_R^{\langle -\infty \rangle} &\colon & \text{Groupoids} \downarrow \mathcal{G} & \to & \text{Spectra}; \\
\mathbf{K}_{A,l^1}^{\text{top}} &\colon & \text{Groupoids} \downarrow \mathcal{G} & \to & \text{Spectra}; \\
\mathbf{K}_{A,r}^{\text{top}} &\colon & \text{Groupoids}^{\text{finker}} \downarrow \mathcal{G} & \to & \text{Spectra}; \\
\mathbf{K}_{A,m}^{\text{top}} &\colon & \text{Groupoids} \downarrow \mathcal{G} & \to & \text{Spectra},
\end{aligned}
$$

together with natural transformations

$$\begin{aligned}
\mathbf{I}_1 : \mathbf{K} &\to \mathbf{KH}; \\
\mathbf{I}_2 : \mathbf{KH} &\to \mathbf{K}_{A,l^1}^{\mathrm{top}}; \\
\mathbf{I}_3 : \mathbf{K}_{A,l^1}^{\mathrm{top}} &\to \mathbf{K}_{A,m}^{\mathrm{top}}; \\
\mathbf{I}_4 : \mathbf{K}_{A,m}^{\mathrm{top}} &\to \mathbf{K}_{A,r}^{\mathrm{top}},
\end{aligned}$$

of functors from $\mathsf{Groupoids} \downarrow \mathcal{G}$ *or* $\mathsf{Groupoids}^{\mathrm{finker}} \downarrow \mathcal{G}$ *respectively to* $\mathsf{Spectra}$ *such that the following holds:*

(i) *Let* $F_i : \mathcal{G}_i \to \mathcal{G}$ *be objects for* $i = 0, 1$ *and* $F : F_0 \to F_1$ *be a morphism between them in* $\mathsf{Groupoids} \downarrow \mathcal{G}$ *or* $\mathsf{Groupoids}^{\mathrm{finker}} \downarrow \mathcal{G}$ *respectively such that the underlying functor of groupoids* $F : \mathcal{G}_0 \to \mathcal{G}_1$ *is an equivalence of groupoids. Then the functors send* F *to a weak equivalences of spectra;*

(ii) *Let* $F_0 : \mathcal{G}_0 \to \mathcal{G}$ *be an object in* $\mathsf{Groupoids} \downarrow \mathcal{G}$ *or* $\mathsf{Groupoids}^{\mathrm{finker}} \downarrow \mathcal{G}$ *respectively such that the underlying groupoid* $\mathcal{G}_0$ *has only one object* x. *Let* $G = \mathrm{mor}_{\mathcal{G}_0}(x, x)$ *be its automorphisms group. We obtain a ring* $R(y)$, *a ring* $R(y)$ *with involution, or a* C^*-*algebra* $B(y)$ *with* G-*operation by structure preserving maps from the evaluation of the functor* R *or* A *respectively at* $y = F(x)$. *Then:*

$$\begin{aligned}
\pi_n(\mathbf{K}_R(F)) &= K_n(R(y) \rtimes G); \\
\pi_n(\mathbf{KH}_R(F)) &= KH_n(R(y) \rtimes G); \\
\pi_n(\mathbf{L}_R^{\langle -\infty \rangle}(F)) &= L_n^{\langle -\infty \rangle}(R(y) \rtimes G); \\
\pi_n(\mathbf{K}_{A(y),l^1}^{\mathrm{top}}(F)) &= K_n(A(y) \rtimes_{l^1} G); \\
\pi_n(\mathbf{K}_{A(y),r}^{\mathrm{top}}(F)) &= K_n(A(y) \rtimes_r G); \\
\pi_n(\mathbf{K}_{A(y),m}^{\mathrm{top}}(F)) &= K_n(A(y) \rtimes_m G),
\end{aligned}$$

where $K_n(R(y) \rtimes G)$ *is the algebraic* K-*theory of the twisted group ring* $R(y) \rtimes G$, $KH_n(R(y) \rtimes G)$ *is the homotopy* K-*theory of the twisted group ring* $R(y) \rtimes G$, $L_n^{\langle -\infty \rangle}(R(y) \rtimes G)$ *is the algebraic* L-*theory with decoration* $\langle -\infty \rangle$ *of the twisted group ring with involution* $R(y) \rtimes G$, $K_n(A(y) \rtimes_{l^1} G)$ *is the topological* K-*theory of the crossed product Banach algebra* $A(y) \rtimes_{l^1} G$, $K_n(A(y) \rtimes_r G)$ *is the topological* K-*theory of the reduced crossed product* C^*-*algebra* $A(y) \rtimes_r G$, *and* $K_n(A(y) \rtimes_m G)$ *is the topological* K-*theory of the maximal crossed product* C^*-*algebra* $A(y) \rtimes_m G$.

The natural transformations $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ *and* $\mathbf{I}_4$ *become under this identifications the obvious change of rings and theory homomorphisms*

(iii) *These constructions are in the obvious sense natural in* R *and* A *respectively and in* $\mathcal{G}$.

We defer the details of the proof of Theorem 7.1 in [2]. Its proof requires some work but there are many special cases which have already been taken care of. If we

would not insist on groupoids but only on groups as input, these are the standard algebraic K- and L-theory spectra or topological K-theory spectra associated to group rings, group Banach algebras and group C^*-algebras. The construction for the algebraic K- and L-theory and the topological K-theory in the case, where G acts trivially on a ring R or a C^*-algebra are already carried out or can easily be derived from [4], [13], and [24] except for the case of a Banach algebra. The case of the K-theory spectrum associated to an additive category with G-action has already been carried out in [7]. The main work which remains to do is to treat the Banach case and to construct the relevant natural transformation from $\mathbf{KH}$ to $\mathbf{K}^{\mathrm{top}}_{A,l^1}$

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