Topological rigidity for non-aspherical manifolds

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- This is joint work with Matthias Kreck (Bonn)

Conjecture (Borel Conjecture)

Let M and N be closed aspherical topological manifolds. Then every homotopy equivalence $M \to N$ is homotopic to a homeomorphism.

Conjecture (n-dimensional Poincaré Conjecture)

Let M be a closed topological manifold. Then every homotopy equivalence $M \to S^n$ is homotopic to a homeomorphism.

- Manifold will always mean connected oriented closed topological manifold.

Definition (Borel-manifold)

A manifold M is called a Borel manifold if for any orientation preserving homotopy equivalence $f\colon N \to M$ of manifolds there exists an orientation preserving homeomorphism $h\colon N \to M$ such that f and h induce the same map on the fundamental groups up to conjugation.
It is called a strong Borel manifold if every orientation preserving homotopy equivalence $f\colon N \to M$ of manifolds is homotopic to a homeomorphism $h\colon N \to M$.

- The Borel Conjecture is equivalent to the statement that every aspherical manifold is strongly Borel.
- If M is aspherical, then: Borel $\Leftrightarrow$ Strongly Borel.
- The n-dimensional Poincaré Conjecture is equivalent to the statement that S^n is strongly Borel.
- Both conjectures become false in the smooth category.
- Question: Which manifolds are (strongly) Borel?
- Slogan: Interpolation between the Borel and the Poincaré Conjecture.

- If $\dim(M) \leq 2$, then M is strongly Borel.
- The Lens space $L(7,1,1)$ is not Borel.
- The following assertions are equivalent:
  - $S^1 \times S^2$ is strongly Borel;
  - $S^1 \times S^2$ is Borel;
  - The 3-dimensional Poincaré Conjecture is true.

  Idea of proof: Suppose $S^1 \times S^2$ is Borel. Then:

$$M \simeq S^3 \Rightarrow M\#(S^1 \times S^2) \simeq S^3\#(S^1 \times S^2) \cong S^1 \times S^3$$
$$\Rightarrow M\#(S^1 \times S^2) \cong S^3\#(S^1 \times S^2) \Rightarrow M \cong S^3.$$

  The other direction uses the prime decomposition and the characterization of $S^1 \times S^2$ as the only non-irreducible prime 3-manifold with infinite π.

Theorem (Dimension 3)

Suppose that Thurston's Geometrization Conjecture for irreducible 3-manifolds with infinite fundamental group and the 3-dimensional Poincaré Conjecture are true.
Then every 3-manifold with torsionfree fundamental group is a strong Borel manifold.

- The main input in the proof are Waldhausen's rigidity results for Haken manifolds.
- Conclusion: If $\pi_1(M)$ is torsionfree, then $\pi_1(M)$ determines the homeomorphism type.

Example (Examples in dimension 4)

- Let M be a 4-manifold with Spin structure such that its fundamental group is finite cyclic.
  Then M is Borel. This follows from a classification result of Hambleton-Kreck.
- If M is simply connected and Borel, then it has a Spin structure. This follows results from the star operation $M \mapsto *M$.
- T^4 and $S^1 \times S^3$ are strongly Borel.
- $S^2 \times S^2$ is Borel but not strongly Borel.

Theorem (Connected sums)

Let M and N be manifolds of the same dimension $n \geq 5$ such that neither $\pi_1(M)$ nor $\pi_1(N)$ contains elements of order 2 or that $n = 0, 3 \mod 4$.
If both M and N are (strongly) Borel, then the same is true for their connected sum $M \# N$.

- The proof is based on Cappell's work on splitting obstructions and of UNIL-groups and recent improvements by Banagl, Connolly, Davis, Ranicki.

Theorem (Products of two spheres)

- *Suppose that $k + d \neq 3$. Then $S^k \times S^d$ is a strong Borel manifold if and only if both k and d are odd;*
- *Suppose $k, d > 1$ and $k + d \geq 4$. Then the manifold $S^k \times S^d$ is Borel if and only if the following conditions are satisfied:*
  1. *Neither k nor d is divisible by 4;*
  2. *If $k = 2 \mod 4$, then there is a map $g_k \colon S^k \times S^d \to S^d$ such that its Arf invariant $\operatorname{Arf}(g_k)$ is non-trivial and its restriction to $pt \times S^d$ is an orientation preserving homotopy equivalence $pt \times S^d \to S^d$.*
  3. *The same condition with the role of k and d interchanged.*

- The condition (2) appearing in the last Theorem implies that the Arf invariant homomorphism

$$\mathrm{Arf}_k \colon \Omega_k^{\mathrm{fr}} \to \mathbb{Z}/2$$

is surjective. This is the famous *Arf-invariant-one-problem*.

Definition (structure set)

The *structure set* $S^{top}(M)$ of a manifold M consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold N as source.
Two such homotopy equivalences $f_0 \colon N_0 \to M$ and $f_1 \colon N_1 \to M$ are equivalent if there exists a homeomorphism $g \colon N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

- Let $\text{ho-aut}_\pi(M)$ be the group of homotopy classes of self equivalences inducing the identity on π_1 up to conjugation.
- It acts on the structure set by composition.

Theorem (Surgery criterion for Borel manifolds)

- *A manifold M is a strong Borel manifold if and only if $\mathcal{S}^{\text{top}}(M)$ consists of one element;*
- *A manifold M is a Borel manifold if and only if $\mathcal{S}^{\text{top}}(M)/\operatorname{ho-aut}_{\pi}(M)$ consists of one element.*

Theorem (Ranicki)

There is an exact sequence of abelian groups called algebraic surgery exact sequence

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}}$$
$$\mathcal{S}^{\text{top}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

It can be identified with the classical geometric surgery sequence due to Sullivan and Wall in high dimensions.

- M is strongly Borel $\Leftrightarrow A_{n+1}$ is surjective and A_n is injective.
- The Farrell-Jones Conjecture predicts for torsionfree π that

$$H_n(B\pi;\mathbf{L}) \xrightarrow{A_n} L_n(\mathbb{Z}\pi)$$

is bijective for $n \in \mathbb{Z}$.

Example

- Consider $M = S^k \times S^d$ for $k + d \geq 4$.
- Then $\pi_1(M)$ is trivial and the assembly map can be identified with

$$H_m(S^k \times S^d; \mathbf{L}\langle 1 \rangle) \to H_m(\mathrm{pt}; \mathbf{L}\langle 1 \rangle).$$

- $\mathcal{S}^{\mathrm{top}}(S^k \times S^d) \cong H_{k+d}(S^k \times S^d, \mathrm{pt}; \mathbf{L}\langle 1 \rangle) \cong L_d(\mathbb{Z}) \oplus L_k(\mathbb{Z})$.
- Hence $S^k \times S^d$ is strongly Borel if and only if k and d are odd.
- To prove that $S^k \times S^d$ is Borel, one has to construct enough selfhomotopy equivalences of $S^k \times S^d$.

Theorem (A necessary homological criterion for being Borel)

Let M be a Borel manifold and let $c\colon M \to B\pi$ be the classifying map. Then for every $i \geq 1$ with $\mathcal{L}(M)_i = 0$ the map

$$c_* \colon H_{n-4i}(M;\mathbb{Q}) \to H_{n-4i}(B\pi;\mathbb{Q})$$

is injective.

- This criterion is obviously empty for aspherical manifolds.
- Input in the proof: The image of $[f\colon N \to M]$ under the map

$$\mathcal{S}^{\mathrm{top}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}\langle 1\rangle) \to \bigoplus_{i\geq 1} H_{4i+n}(M;\mathbb{Q})$$

is

$$f_*\left(\mathcal{L}(N) \cap [N]\right) - \mathcal{L}(M) \cap [M].$$

Theorem (Sphere bundles over surfaces)

Let K be S^1 or a 2-dimensional manifold different from S^2. Let $S^d \to E \to K$ be a fiber bundle over K for $d \geq 3$.
Then E is a Borel manifold. It is a strong Borel manifold if and only if $K = S^1$.

Theorem (Sphere bundles over 3-manifolds)

Let K be an aspherical 3-dimensional manifold. Suppose that the Farrell-Jones Conjecture holds for $\pi_1(K)$. Let $S^d \to E \xrightarrow{p} K$ be a fiber bundle over K with orientable E such that $d \geq 4$ or such that $d = 2, 3$ and there is a map $i \colon K \to E$ with $p \circ i \simeq \mathrm{id}_K$. Then

- *E is strongly Borel if and only if $H_1(K; \mathbb{Z}/2) = 0$;*
- *If $d = 3 \mod 4$ and $d \geq 7$, then $K \times S^d$ is Borel;*
- *If $d = 0 \mod 4$ and $d \geq 8$ and $H_1(K; \mathbb{Z}/2) \neq 0$, then $K \times S^d$ is not Borel.*

Theorem (Chang-Weinberger)

Let M^{4k+3} be a manifold for $k \geq 1$ whose fundamental group has torsion.
Then there are infinitely many pairwise not homeomorphic smooth manifolds which are homotopy equivalent to M but not homeomorphic to M. In particular M is not Borel.

Theorem (Homology spheres)

Let M be a manifold of dimension $n \geq 5$ with fundamental group $\pi = \pi_1(M)$.

- *Let M be an integral homology sphere. Then M is a strong simple Borel manifold if and only if*

$$L^s_{n+1}(\mathbb{Z}) \xrightarrow{\cong} L^s_{n+1}(\mathbb{Z}\pi).$$

- *Suppose that M is a rational homology sphere and Borel. Suppose that π satisfies the Novikov Conjecture. Then*

$$H_{n+1-4i}(B\pi;\mathbb{Q}) = 0$$

for $i \geq 1$ and $n+1-4i \neq 0$.

Theorem (Another construction of strongly Borel manifolds)

Start with a strongly Borel manifold M of dimension $n \geq 5$. Choose an emdedding $S^1 \times D^{n-1} \to M$ which induces an injection on π_1. Choose a high dimensional knot $K \subseteq S^n$ with complement X such that the inclusion $\partial X \cong S^1 \times S^{n-2} \to X$ induces an isomorphism on π_1. Put

$$M' = M - (S^1 \times D^{n-1}) \cup_{S^1 \times S^{n-2}} X.$$

Then M' is strongly Borel.

- If M is aspherical, then M' is in general not aspherical.

Problem (Classification of certain low-dimensional manifolds)

Classify up to orientation preserving homotopy equivalence, homeomorphism (or diffeomorphism in the smooth case) all manifolds in dimension $1 \leq k < n \leq 6$ satisfying:

- $\pi = \pi_1(M)$ *is isomorphic to* $\pi_1(K)$ *for a manifold* K *of dimension* $k \leq 2$.
- $\pi_2(M)$ *vanishes.*

- The case $\pi = \{1\}$ was already solved by Wall;
- In dimension ≤ 5 we give a complete answer in terms of the second Stiefel Whitney class.
- In dimension 6 we give in the Spin case a complete answer in terms of the equivariant intersection pairing of the universal covering.
- Such a manifold is never strongly Borel but always Borel.

- In nearly all examples of Borel manifolds we have constructed — what we call — a *generalized topological space form*, i.e., manifolds M, whose universal covering $\widetilde{M}$ is contractibel or homotopy-equivalent to a wedge of k-spheres S^k for some $2 \leq k < \infty$.