“$\mathbf{L^2}$-Betti numbers of mapping tori and groups”
by
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Abstract: We prove the following two conjectures of Gromov. Firstly, all L^2-Betti numbers of a manifold fibered over S^1 are trivial. Secondly, the first L^2-Betti number of a finitely presented group Γ vanishes provided that Γ is an extension $\{1\} \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \pi \longrightarrow \{1\}$ of finitely presented groups such that Δ is infinite and π contains $\mathbb{Z}$ as a subgroup. We conclude for such a group Γ that its deficiency is less than or equal to one and that any closed 4-manifold with Γ as fundamental group satisfies $\chi(M) \geq |\sigma(M)|$.

0. Introduction

In his preprint [12, page 152 and page 156] Gromov states the following two conjectures:

Let a compact aspherical manifold M be fibered over the circle S^1. Then all L^2-Betti numbers $b_p(M)$ are trivial.

Let $\{1\} \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \pi \longrightarrow \{1\}$ be an extension of infinite groups which are fundamental groups of finite aspherical CW-complexes. Then the first L^2-Betti number $b_1(\Gamma)$ is trivial.

We will give affirmative answers to these conjectures. The first conjecture follows from Theorem 2.1 which states that all L^2-Betti numbers $b_p(T_f)$ of a mapping torus T_f of an endomorphism f of a finite CW-complex F vanish. We prove in Theorem 4.1 for an extension $\{1\} \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \pi \longrightarrow \{1\}$ of finitely presented groups that the first L^2-Betti number $b_1(\Gamma)$ vanishes provided that Δ is infinite and π contains $\mathbb{Z}$ as a subgroup. This implies the second conjecture above. Let Γ be an infinite finitely presented group with trivial first L^2-Betti number $b_1(\Gamma)$. As applications we show in Theorem 5.1 that a closed 4-manifold with Γ as fundamental group satisfies $\chi(M) \geq |\sigma(M)|$ for $\chi(M)$ the Euler characteristic and $\sigma(M)$ the signature. This generalizes a result of Johnson and Kotschick [16]. We prove in Theorem 6.1 that the deficiency of Γ satisfies $\mathrm{def}(\Gamma) \leq 1$.

L^2-Betti numbers were introduced by Atiyah [1]. In Section 1 we recall their definitions and basic properties from the topological point of view. They also have an analytic meaning, namely, the p-th L^2-Betti number of a closed Riemannian manifold measures the size of the space of harmonic L^2-integrable smooth p-forms of the universal covering [7]. For general information and applications of L^2-Betti numbers, and in particular of conditions that determine when they vanish, the reader may refer for example to [1], [4], [5], [6],[7],[8], [11],[12], [18], [19], [21] and [22].

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The paper is organized as follows:

1. Preliminaries concerning L^2-Betti numbers
2. The vanishing of the L^2-Betti numbers of a mapping torus
3. The first L^2-Betti number of a total space of a fibration
4. Groups with vanishing first L^2-Betti number
5. 4-manifolds satisfying $\chi(M) \geq |\sigma(M)|$
6. Deficiency of groups

1. Preliminaries concerning L^2-Betti numbers

In this section we give the basic definitions and properties of L^2-Betti numbers.

Let Γ be a countable group and $l^2(\Gamma)$ be the Hilbert space of square integrable formal sums $\sum_{\gamma\in\Gamma} \lambda_\gamma \gamma$ with coefficients $\lambda_\gamma \in \mathbb{C}$. The *von Neumann algebra* $\mathcal{N}(\Gamma)$ is the algebra $B(l^2(\Gamma), l^2(\Gamma))^\Gamma$ of bounded operators from $l^2(\Gamma) \longrightarrow l^2(\Gamma)$ which commute with the left Γ-action on $l^2(\Gamma)$. The *von Neumann trace* $tr(f)$ of an element $f \in \mathcal{N}(\Gamma)$ is the complex number $\langle f(e), e\rangle$ where $e \in \Gamma$ is the unit element. This extends to square matrices over $\mathcal{N}(\Gamma)$ by taking the sum of the traces of the diagonal entries. A *Hilbert* $\mathcal{N}(\Gamma)$*-module* is a Hilbert space M together with a left Γ-action by unitary operators such that there exists an isometric Γ-equivariant embedding into $H \otimes l^2(\Gamma)$ for a separable Hilbert space H (which is not part of the structure). We call M finitely generated if H can be chosen to be $\mathbb{C}^n$ for some positive integer n. The von Neumann dimension $dim(M)$ of a finitely generated Hilbert $\mathcal{N}(\Gamma)$-module M is the non-negative real number $tr(pr)$ for any projection pr in $M(n, n, \mathcal{N}(\Gamma)) = B(\oplus_{i=1}^n l^2(\Gamma), \oplus_{i=1}^n l^2(\Gamma))^\Gamma$ whose image is isometrically Γ-isomorphic to M. A *weakly exact sequence* $0 \longrightarrow M \xrightarrow{i} N \xrightarrow{p} P \longrightarrow 0$ of Hilbert $\mathcal{N}(\Gamma)$-modules is a sequence of bounded operators such that i is injective, the closure of the image of i is the kernel of p and the closure of the image of p is P. Given such a sequence of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules, the relation $dim(M) - dim(N) + dim(P) = 0$ holds. We have $dim(M) = 0$ precisely if $M = \{0\}$. A (finitely generated) Hilbert $\mathcal{N}(\Gamma)$-chain complex C is a chain complex of (finitely generated) Hilbert $\mathcal{N}(\Gamma)$-modules with bounded Γ-equivariant operators as differential. Its L^2*-homology* is defined to be $H_p(C) = ker(c_p)/\overline{im(c_{p+1})}$. Notice that one divides by the closure of the image and not just by the image so that the L^2-homology is not ordinary homology. Now one can define the *p-th* L^2*-Betti number* as $b_p(C) = dim(H_p(C))$ provided that $H_p(C)$ is finitely generated.

Let X be a CW-complex with finite d-skeleton and fundamental group π. Consider a group homomorphism $\phi : \pi \longrightarrow \Gamma$. Let $\widetilde{X}$ be the universal covering. For dimensions less than or equal to d, define the finitely generated Hilbert Γ-chain complex $C(X;\phi^* l^2(\Gamma))$ by $l^2(\Gamma) \otimes_{\mathbb{Z}\pi} C(\widetilde{X})$. Here the right π-action on $l^2(\Gamma)$ is induced by $\phi : \pi \longrightarrow \Gamma$ and the Hilbert $\mathcal{N}(\Gamma)$-structure comes from the identification of $l^2(\Gamma) \otimes_{\mathbb{Z}\pi} C_p(\widetilde{X})$ with $\oplus_{i=1}^n l^2(\Gamma)$ given by a cellular $\mathbb{Z}\pi$-basis. The cellular basis is not unique and a different choice of cellular basis will give a different identification. Since the two different identifications differ only by a unitary Γ-equivariant operator, the Hilbert $\mathcal{N}(\Gamma)$-module structure is independent of the choice of cellular basis. The differentials in dimension less than or equal to d are bounded Γ-equivariant operators. These considerations prompt the following

Definition 1.1 *Let X be a CW-complex with finite d-skeleton and $\phi : \pi_1(X) \longrightarrow \Gamma$ be a homomorphism. Define for $p < d$ the* p-th L^2-Betti number *of X with coefficients in $\phi^* l^2(\Gamma)$ by*

$$b_p(X;\phi^* l^2(\Gamma)) = b_p(C(X;\phi^* l^2(\Gamma))).$$

In case $\Gamma = \pi_1(X)$ and $\phi = id$, we abbreviate this to read

$$b_p(X) = b_p(X; id^* l^2(\pi_1(X))).$$

If $B\pi$ has finite d-skeleton we define for $p < d$

$$b_p(\pi) = b_p(B\pi). \qquad \blacksquare$$

The next lemma shows in particular that the definition of $b_p(\pi)$ for $p < d$ is independent of the choice of $B\pi$. Notice that a group π is finitely presented if and only if $B\pi$ has finite 2-skeleton. Most of the claims of the next lemma are already in the literature provided X and Y are finite and $\Gamma = \pi$ and $\phi = id$. We require this more general setup for Theorem 2.1 which is needed in its present form to prove Theorem 3.1 and Theorem 4.1.

Lemma 1.2 *Let X and Y be CW-complexes having finite d-skeletons. Let $\phi : \pi_1(Y) \longrightarrow \Gamma$ be a group homomorphism.*

1. *Suppose $f : X \longrightarrow Y$ is s-connected for $s \geq 2$. Then for $p < \min\{s, d\}$*

   $$b_p(X;(\phi \circ f_*)^* l^2(\Gamma)) = b_p(Y;\phi^* l^2(\Gamma)).$$

   If $s < d$, then

   $$b_s(X;(\phi \circ f_*)^* l^2(\Gamma)) \geq b_s(Y;\phi^* l^2(\Gamma)).$$

2. *If Y has finite 2-skeleton, then for $p = 0, 1$*

   $$b_p(\pi_1(Y)) = b_p(Y).$$

3. *If $i : \Gamma \longrightarrow \Gamma'$ is injective, then for $p < d$*

$$b_p(Y; (i \circ \phi)^* l^2(\Gamma')) = b_p(Y; \phi^* l^2(\Gamma)).$$

4. *Let $p : \overline{Y} \longrightarrow Y$ be a n-sheeted finite covering. Denote by $\Gamma_n \subset \Gamma$ the image of $\phi \circ p_*$ and by $\phi_n : \pi_1(\overline{Y}) \longrightarrow \Gamma_n$ the induced map. If Γ_n has index n in Γ, then for $p < d$*

$$b_p(\overline{Y}; \phi_n^* l^2(\Gamma_n)) = n \cdot b_p(Y; \phi^* l^2(\Gamma)).$$

In particular for $p < d$

$$b_p(\overline{Y}) = n \cdot b_p(Y).$$

5. *Assume $d \geq 1$. If the image of $\phi : \pi_1(Y) \longrightarrow \Gamma$ is finite of cardinality n, then*

$$b_0(Y; \phi^* l^2(\Gamma)) = \frac{1}{n}.$$

Otherwise

$$b_0(Y; \phi^* l^2(\Gamma)) = 0.$$

and in particular

$$b_0(\pi) = \frac{1}{|\pi|}.$$

6. *If Y is a finite CW-complex, then*

$$\chi(Y) = \sum_{p \geq 0} (-1)^p \cdot b_p(Y; \phi^* l^2(\Gamma)).$$

Proof : 1.) In the sequel we write $\pi = \pi_1(Y)$. Let $\widetilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$ be a lift of f to the universal coverings. The induced $\mathbb{Z}\pi$-chain map $\mathbb{Z}\pi \otimes_{\mathbb{Z}\pi_1(X)} C(\widetilde{X}) \longrightarrow C(\widetilde{Y})$ is s-connected. Hence it suffices to show the following chain complex analogue (which we will use later): Let $f : C \longrightarrow D$ be a s-connected $\mathbb{Z}\pi$-chain map of free $\mathbb{Z}\pi$-chain complexes such that the d-dimensional chain complexes obtained by truncating $C|_d$ and $D|_d$ are finitely generated. Then we have $b_p(l^2(\Gamma) \otimes_{\mathbb{Z}\pi} C) = b_p(l^2(\Gamma) \otimes_{\mathbb{Z}\pi} D)$ if $p < \min\{s, d\}$ and $b_s(C) \geq b_s(D)$ if $s < d$.

The strategy of the proof is precisely the same as in [19, Lemma 2.4, Theorem 2.5 and Lemma 4.3] which we describe briefly. One extends f to a $\mathbb{Z}\pi$-chain homotopy equivalence $f' : C' \longrightarrow D$ such that $C|_s = C'|_s$ and C'_{s+1} is finitely generated free if $s < d$. Obviously $b_p(l^2(\Gamma) \otimes_{\mathbb{Z}\pi} C) = b_p(l^2(\Gamma) \otimes_{\mathbb{Z}\pi} C')$ for $p < \min\{s, d\}$ and $b_s(l^2(\Gamma) \otimes_{\mathbb{Z}\pi} C) \geq b_s(l^2(\Gamma) \otimes_{\mathbb{Z}\pi} C')$ for $s < d$. Hence it suffices to show $b_p(l^2(\Gamma) \otimes_{\mathbb{Z}\pi} C) = b_p(l^2(\Gamma) \otimes_{\mathbb{Z}\pi} D)$ for $p < d$ provided $f : C \longrightarrow D$ is a homotopy equivalence. We may assume that f is an inclusion with a free contractible quotient D/C, otherwise substitute D by the mapping cylinder. The exact sequence $0 \longrightarrow C \longrightarrow D \longrightarrow D/C \longrightarrow 0$ splits yielding an isomorphism between D and $C \oplus D/C$. This reduces the claim to the assertion that $b_p(l^2(\Gamma) \otimes_{\mathbb{Z}\pi} C) = 0$ for $p < d$ provided that C is contractible. This follows from the fact that C is a direct sum $\oplus_{p\geq 1}E(p)$ of free $\mathbb{Z}\pi$-chain complexes $E(p)$ such that $E(p)$ is concentrated in dimensions p and $p+1$ and the non-trivial differential is a $\mathbb{Z}\pi$-isomorphism.

2.) follows from 1.) applied to the classifying map $Y \longrightarrow B\pi$.

3.) follows from the elementary proof of [19, Lemma 4.6].

4.) In the sequel res denotes restriction for the subgroup $\pi_1(\overline{Y}) \subset \pi_1(Y)$ respectively $\Gamma_n \subset \Gamma$. Define a bounded Γ_n-equivariant operator for $p \leq d$

$$I_p : l^2(\Gamma_n) \otimes_{\mathbb{Z}\pi_1(\overline{Y})} res(C_p(\widetilde{Y})) \longrightarrow res\left(l^2(\Gamma) \otimes_{\mathbb{Z}\pi_1(Y)} C_p(\widetilde{Y})\right) \qquad u \otimes v \mapsto u \otimes v.$$

This map is a well-defined Γ_n-equivariant isometry since $C_p(\widetilde{Y})$ is finitely generated free and $\pi_1(\overline{Y}) \subset \pi_1(Y)$ and $\Gamma_n \subset \Gamma$ have the same finite index, namely n. As the collection I_p is compatible with the differentials, $C(\overline{Y};\phi_n^* l^2(\Gamma_n))$ and $res\left(C(Y;\phi^* l^2(\Gamma))\right)$ have the same L^2-Betti numbers over $\mathcal{N}(\Gamma_n)$ for $p < d$. Given a finitely generated Hilbert $\mathcal{N}(\Gamma)$-module M, we have $dim_{\mathcal{N}(\Gamma_n)}(res(M)) = n \cdot dim_{\mathcal{N}(\Gamma)}(M)$ since $tr_{\mathcal{N}(\Gamma_n)}(res(k)) = n \cdot tr_{\mathcal{N}(\Gamma)}(k)$ holds for any bounded Γ-equivariant endomorphism k of $\oplus_{i=1}^l l^2(\Gamma)$. This establishes assertion 4.).

5.) We can assume by assertion 3.) that ϕ is surjective. Choose a set of generators $s_1, s_2, \ldots s_g$ for π. Then $\phi(s_1), \phi(s_2), \ldots \phi(s_g)$ is a set of generators for Γ. Moreover, $C(Y;\phi^* l^2(\Gamma))$ is given in dimension 1 and 0 by

$$\oplus_{i=1}^g l^2(\Gamma) \xrightarrow{\oplus_{i=1}^g r(\phi(s_i)-1)} l^2(\Gamma)$$

where $r(\phi(s_i)-1)$ is right multiplication with $\phi(s_i)-1$. Hence we can assume $\pi = \Gamma$ and $\phi = \mathrm{id}$. It remains to show $b_0(\pi) = 0$ if π is infinite and $b_0(\pi) = 1/|\pi|$ if π is finite. This follows from the observation that $l^2(\pi)^\pi$ is zero for infinite π and $\mathbb{C}$ with the trivial π-action for finite π.

6.) follows from the additivity of the von Neumann dimension under weakly exact sequences. ■

Finally we mention the following combinatorial way of computing $b_1(\pi)$ for a finitely presented group π proved in [21]. Let $\langle s_1, \ldots, s_g | R_1, \ldots, R_r\rangle$ be any finite presentation of π. Let A be the $(r, g-1)$-matrix over $\mathbb{Z}\pi$ given by the the Fox derivatives $A_{i,j} = \frac{\partial R_i}{\partial s_j}$ for $1 \leq i \leq r$ and $1 \leq j \leq g-1$.(The index j does not take the value g.) Define for $u = \sum_{w\in\pi} \lambda_w \cdot w \in \mathbb{R}\pi$ its $\mathbb{R}\Gamma$-*trace* $tr_{\mathbb{R}\pi}(u) = \lambda_e \in \mathbb{R}$ if e is the unit element in π. This extends to a square (n,n)-matrix B with entries in $\mathbb{R}\pi$ by putting $tr_{\mathbb{R}\pi}(B) = \sum_{k=1}^n tr_{\mathbb{R}\pi}(b_{k,k})$. Let K be any real number satisfying $K \geq \| A \|$ where $\| A \|$ is the operator norm of the bounded operator $\oplus_{i=1}^r l^2(\pi) \longrightarrow \oplus_{j=1}^{g-1} l^2(\pi)$ induced by A. A possible choice is the product of $\sqrt{g-1}$ and the maximum of the word length of the relations R_i in terms of the s_j. Denote by A^* the matrix obtained from A by transposing and applying to each entry the involution on $\mathbb{R}\pi$ sending $\sum_{w\in\pi}\lambda_w\cdot w$ to $\sum_{w\in\pi}\lambda_w\cdot w^{-1}$. Denote by $(I_{g-1}-K^{-2}\cdot A^*A)^n$ the n-fold product of the square $(g-1,g-1)$-matrix $(I_{g-1}-K^{-2}\cdot A^*A)$ for I_{g-1} the unit matrix. Then the sequence of non-negative real numbers $tr_{\mathbb{R}\pi}(1-K^{-2}\cdot A^*A)^n$ is monotone decreasing and converges for $n\to\infty$ to $b_1(\pi)$. In this context we mention Conjectures 9.1 and 9.2 in [19] which imply for torsion-free π that $b_1(\pi)$ is an integer.

2. The vanishing of the L^2-Betti numbers of a mapping torus

Given a self map $f: F\longrightarrow F$, its *mapping cylinder* M_f is obtained by gluing the bottom of the cylinder $F\times[0,1]$ to F by the identification $(x,0)=f(x)$. Its *mapping torus* T_f is obtained from the mapping cylinder by identifying the top and the bottom by the identity. If f is a homotopy equivalence T_f is homotopy equivalent to the total space of a fibration over S^1 with fiber F. Conversely, the total space of such a fibration is homotopy equivalent to the mapping torus of the self homotopy equivalence of F given by the fiber transport with a generator of $\pi_1(S^1)$. The homotopy type of T_f depends only on the homotopy class of f.

Let L denote the colimit of the following system of groups indexed by the integers

$$\ldots\xrightarrow{\pi_1(f)}\pi_1(F)\xrightarrow{\pi_1(f)}\pi_1(F)\xrightarrow{\pi_1(f)}\ldots$$

Denote by $i:\pi_1(F)\longrightarrow L$ the map at the group indexed by zero. The map i is bijective if and only if $\pi_1(f)$ is an isomorphism. Let $\mathbb{Z}$ operate on L by shifting the sequence. Then $\pi_1(T_f)$ is the semidirect product of L and $\mathbb{Z}$ with respect to the operation above. Consider any factorization of the canonical epimorphism $\pi_1(T_f)\longrightarrow\mathbb{Z}$ into a composition of epimorphisms $\pi_1(T_f)\xrightarrow{\phi}\Gamma\xrightarrow{\psi}\mathbb{Z}$. Denote by $\overline{F}$ the covering of F associated to the homomorphism $\phi\circ i:\pi_1(F)\longrightarrow ker(\psi)$ and by $\overline{T_f}$ be the covering of T_f associated to the epimorphism ϕ. Let $\overline{f}:\overline{F}\longrightarrow\overline{F}$ be a lift of f. Then $\overline{T_f}$ is the mapping telescope of $\overline{F}$ infinite to both sides, i.e., the identification space

$$\overline{T_f}=\coprod_{n\in\mathbb{Z}}\overline{F}\times[n,n+1]/\sim$$

where the identification $\sim$ is given by $(x,n+1)\sim(\overline{f}(x),n)$. The group of deck transformations Γ is a semidirect product of $\ker(\psi)$ and $\mathbb{Z}$ and acts in the obvious way. One easily checks that the cellular $\mathbb{Z}\Gamma$-chain complex of $\overline{T_f}$ is the mapping cone of the following $\mathbb{Z}\Gamma$-chain map

$$\mathbb{Z}\Gamma\otimes_{\mathbb{Z}[ker(\psi)]}C(\overline{F})\longrightarrow\mathbb{Z}\Gamma\otimes_{\mathbb{Z}[ker(\psi)]}C(\overline{F})\qquad\gamma\otimes u\mapsto\gamma\otimes u-\gamma t\otimes C(\overline{f})(u)$$

where t is a lift of the generator of $\mathbb{Z}$ to Γ.

The next theorem implies a conjecture by Gromov in [12, page 152]. The case of a manifold fibered over S^1 has already been dealt with in [19, Theorem 4.10].

Theorem 2.1 *Let $f : F \longrightarrow F$ be a self map of a connected CW-complex F with finite d-skeleton for $d \geq 2$. Let $\pi_1(T_f) \xrightarrow{\phi} \Gamma \xrightarrow{\psi} \mathbb{Z}$ be a factorization of the canonical map $\pi_1(T_f) \longrightarrow \mathbb{Z}$ into epimorphisms. Then the mapping torus T_f has a CW-structure with finite d-skeleton and for $p < d$*

$$b_p(T_f; \phi^* l^2(\Gamma)) = 0.$$

Proof : Let T_f^n be obtained from the n-fold mapping telescope of f by identifying the bottom and top by the identity. In this notation T_f^1 is just T_f. There is an obvious n-fold covering $p : T_f^n \longrightarrow T_f$. Let Γ_n be the image of $\phi \circ p_*$ and denote by $\phi_n : \pi_1(T_f^n) \longrightarrow \Gamma_n$ the induced map. Then Γ_n has index n in Γ. Lemma 1.2.4 implies for all integers $p < d$

$$b_p(T_f; \phi^* l^2(\Gamma)) = \frac{b_p(T_f^n; \phi_n^* l^2(\Gamma_n))}{n}.$$

There is a homotopy equivalence $g : T_{f^n} \longrightarrow T_f^n$. Hence we get

$$b_p(T_{f^n}; (\phi_n \circ g_*)^* l^2(\Gamma_n)) = b_p(T_f^n; \phi_n^* l^2(\Gamma_n)).$$

The mapping torus T_{f^n} has a CW-structure such that its d-skeleton is finite and the number of p-cells is $c_{p-1} + c_p$ where c_p is the number of p-cells in F. Thus

$$b_p(T_{f^n}; (\phi_n \circ g_*)^* l^2(\Gamma_n)) \leq c_{p-1} + c_p.$$

We conclude

$$0 \leq b_p(T_f; \phi^* l^2(\Gamma)) \leq \frac{c_{p-1} + c_p}{n}.$$

Taking the limit $n \to \infty$ proves the claim. ■

3. The first L^2-Betti number of a total space of a fibration

In this section we prove

Theorem 3.1 *Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of connected CW-complexes such that F and B have finite 2-skeletons. Then E has finite 2-skeleton up to homotopy. If the image of $\pi_1(F) \longrightarrow \pi_1(E)$ is infinite and $\pi_1(B)$ contains $\mathbb{Z}$ as subgroup, then the first L^2-Betti number of E*

$$b_1(E) = 0. \quad ■$$

The proof of Theorem 3.1 requires some preparations. Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of connected CW-complexes. Denote $\pi = \pi_1(B)$, $\Gamma = \pi_1(E)$ and let Δ be the image of $\pi_1(F) \longrightarrow \pi_1(E)$. We obtain the group extension $\{1\} \longrightarrow \Delta \longrightarrow \Gamma \xrightarrow{p_*} \pi \longrightarrow \{1\}$. The *pointed fiber transport* is a homomorphism of monoids into the monoid of pointed homotopy classes of pointed self maps of the fiber

$$\sigma : \Gamma \longrightarrow [F, F]^+.$$

We recall its definition. Let $w : I \longrightarrow E$ be a loop at the base point $e \in E$. Put $b = p(e)$ and $F = p^{-1}(b)$. Choose a solution h of the following lifting problem

$$\begin{array}{ccc} F_b \times \{0\} \cup \{e\} \times I & \xrightarrow{\quad i \cup w \quad} & E \\ j \Big\downarrow & \overset{\text{h}}{\nearrow} & \Big\downarrow p \\ F \times I & \xrightarrow{\quad p \circ w \circ \mathrm{pr}_I \quad} & B \end{array}$$

where i and j are the obvious inclusions. Then σ assigns to the class $w \in \Gamma$ the pointed homotopy class of the map $(F, e) \longrightarrow (F, e)$ sending x to $h(x, 1)$.

Let $\overline{F} \longrightarrow F$ be the connected covering of F with Δ as group of deck transformations. Let $c(\gamma) : \Delta \longrightarrow \Delta$ send δ to $\gamma\delta\gamma^{-1}$ for $\gamma \in \Gamma$. A representative of $\sigma(\gamma)$ lifts to a $c(\gamma)$-equivariant (pointed) self map of $\overline{F}$. Its $c(\gamma)$-equivariant (free) homotopy class denoted by $\overline{\sigma}(\gamma)$ depends only on γ. Given $w \in \pi$, denote by $\overline{w} \in \Gamma$ some element satisfying $p_*(\overline{w}) = w$. Define a $\mathbb{Z}\Gamma$-chain map $U(w) : \mathbb{Z}\Gamma \otimes_\Delta C(\overline{F}) \longrightarrow \mathbb{Z}\Gamma \otimes_\Delta C(\overline{F})$ by sending $\gamma \otimes x$ to $\gamma\overline{w} \otimes C(\overline{\sigma}(\overline{w}^{(-1)}))(x)$. The $\mathbb{Z}\Gamma$-chain homotopy class of $U(w)$ is independent of the choice of $\overline{w}$ and of the representative of $\overline{\sigma}(\overline{w}^{(-1)})$ and hence depends only on w. This follows from the fact that a representative for $\overline{\sigma}(\delta)$ for $\delta \in \Delta$ is given by left multiplication with δ on $\overline{F}$. For $u = \sum_{\gamma\in\pi} \lambda_\gamma \gamma \in \mathbb{Z}\pi$ define $U(u)$ by $\sum_{\gamma\in\pi} \lambda_\gamma U(\gamma)$. Then the $\mathbb{Z}\Gamma$-chain homotopy classes of $U(v) \circ U(u)$ and $U(uv)$ agree and $U(1) = id$. Hence we have constructed an algebra homomorphism

$$U : \mathbb{Z}\pi \longrightarrow [\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C(\overline{F}), \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C(\overline{F})]^{op}_{\mathbb{Z}\Gamma}$$

into the opposite (reverse multiplication) of the algebra of $\mathbb{Z}\Gamma$-chain homotopy classes of $\mathbb{Z}\Gamma$-chain self maps of $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C(\overline{F})$. This is called *chain homotopy presentation* of the given fibration. For more information and (elementary) proofs of some of the claims above we refer to [20, section 1 and 6].

As $\mathbb{Z}\Gamma$ is free over $\mathbb{Z}\Delta$ the natural $\mathbb{Z}\Gamma$-map $S_1 : \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} H_*(C(\overline{F})) \longrightarrow H_*(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C(\overline{F}))$ is bijective. The $\mathbb{Z}\Gamma$-isomorphism $S_2 : \mathbb{Z}\Gamma \otimes_\Delta H_*(C(\overline{F})) \longrightarrow \mathbb{Z}\pi \otimes_\mathbb{Z} H_*(C(\overline{F}))$ sends $\gamma \otimes x$ to $p_*(\gamma) \otimes H_*(C(\overline{\sigma}(\gamma)))(x)$ where $\gamma \in \Gamma$ acts on $\mathbb{Z}\pi$ by left multiplication with $p_*(\gamma) \in \pi$, on $H_*(C(\overline{F}))$ by $H_*(\overline{\sigma}(\gamma))$ and diagonally on $\mathbb{Z}\pi \otimes_\mathbb{Z} H_*(C(\overline{F}))$. Define the $\mathbb{Z}\Gamma$-isomorphism

$$S : H_*(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C(\overline{F})) \longrightarrow \mathbb{Z}\pi \otimes_\mathbb{Z} H_*(C(\overline{F}))$$

by $S = S_2 \circ S_1^{-1}$.

Lemma 3.2 *Given $u \in \mathbb{Z}\pi$, let $r(u) : \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$ be right multiplication with u. The following diagram commutes*

$$\begin{array}{ccc} H_*(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C(\overline{F})) & \xrightarrow{H_*(U(u))} & H_*(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C(\overline{F})) \\ \downarrow S & & \downarrow S \\ \mathbb{Z}\pi \otimes_{\mathbb{Z}} H_*(C(\overline{F})) & \xrightarrow{r(u) \otimes id} & \mathbb{Z}\pi \otimes_{\mathbb{Z}} H_*(C(\overline{F})) \end{array}$$

and has isomorphisms as vertical arrows. ■

Proof of Theorem 3.1: Choose a set of generators $s_1, \ldots, s_g$ of π such that the cyclic subgroup $\langle s_1 \rangle$ generated by s_1 is infinite. The following sequence is exact

$$\oplus_{i=1}^g \mathbb{Z}\pi \xrightarrow{\oplus_{i=1}^g r(1-s_i)} \mathbb{Z}\pi \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow \{0\}$$

where $\mathbb{Z}$ here and in the sequel carries always the trivial Δ-,Γ-, respectively, π-action and the augmentation ϵ sends $\sum_{w\in\pi} \lambda_w w$ to $\sum_{w\in\pi} \lambda_w$. As $H_0(C(\overline{F}))$ is $\mathbb{Z}\Gamma$-isomorphic to $\mathbb{Z}$, Lemma 3.2 proves the exactness of the sequence of $\mathbb{Z}\Gamma$-modules

$$\oplus_{i=1}^g H_0(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C(\overline{F})) \xrightarrow{\oplus_{i=1}^g H_0(U(1-s_i))} H_0(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C(\overline{F})) \longrightarrow \mathbb{Z} \longrightarrow \{0\}.$$

Let $cone(f)$ denote the mapping cone of a chain map f. We derive from the long homology sequence that $H_0(cone(\oplus_{i=1}^g U(1-s_i)))$ is $\mathbb{Z}\Gamma$-isomorphic to $\mathbb{Z}$. Let $\widetilde{E}$ be the universal covering of E. For a chain complex C let $C|_2$ be the 2-dimensional chain complex obtained by truncating C. Notice that $C(\widetilde{E})|_2$ is a 2-dimensional $\mathbb{Z}\Gamma$-chain complex such that $H_1(C(\widetilde{E})|_2)$ is trivial and $H_0(C(\widetilde{E})|_2)$ is $\mathbb{Z}$ and that $cone(\oplus_{i=1}^g U(1-s_i))$ is a free $\mathbb{Z}\Gamma$-chain complex. Hence there is a $\mathbb{Z}\Gamma$-chain map $f : cone(\oplus_{i=1}^g U(1-s_i))|_2 \longrightarrow C(\widetilde{E})|_2$ inducing the identity on H_0. Since f is 1-connected, we conclude from the proof of Lemma 1.2.1

$$b_1(l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} cone(\oplus_{i=1}^g U(1-s_i))) \geq b_1(l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} C(\widetilde{E})) = b_1(E).$$

There is an obvious exact sequence of $\mathbb{Z}\Gamma$-chain complexes

$$0 \longrightarrow \quad \mathrm{cone}(U(1-s_1)) \longrightarrow \quad cone(\oplus_{i=1}^{g} U(1-s_i)) \longrightarrow \quad \oplus_{i=2}^{g} \Sigma\left(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C(\overline{F})\right) \longrightarrow 0$$

where Σ denotes the suspension. It induces the weakly exact sequence in L^2-homology [4, Theorem 2.1 on page 10]

$$H_1(l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} cone(U(1-s_1))) \longrightarrow H_1(l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} cone(\oplus_{i=1}^{g} U(1-s_i))) \\ \longrightarrow H_1\left(l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} \oplus_{i=2}^{g} \Sigma\left(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C(\overline{F})\right)\right).$$

Clearly $H_1\left(l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} \Sigma\left(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} C(\overline{F})\right)\right)$ agrees with the L^2-homology $H_0(F,(j \circ l)^* l^2(\Gamma))$ where $j : \Delta \longrightarrow \Gamma$ is the inclusion and $l : \pi_1(F) \longrightarrow \Delta$ is the obvious surjection. This L^2-homology group vanishes by Lemma 1.2.5 as Δ is infinite. This implies

$$b_1(l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} cone(U(1-s_1))) \geq b_1(l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} cone(\oplus_{i=1}^{g} U(1-s_i)))$$

and hence

$$b_1(l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} cone(U(1-s_1))) \geq b_1(E).$$

Therefore it remains to show $b_1(l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} cone(U(1-s_1))) = 0$. Let Γ' be the subgroup of Γ generated by Δ and $\overline{s_1} \in \Gamma$. Define $U'(1-s_1) : \mathbb{Z}\Gamma' \otimes_{\mathbb{Z}\Delta} C(\overline{F}) \longrightarrow \mathbb{Z}\Gamma' \otimes_{\mathbb{Z}\Delta} C(\overline{F})$ by mapping $\gamma' \otimes x$ to $\gamma' \otimes x - \gamma'\overline{s_1} \otimes C(\overline{\sigma}(\overline{s_1}^{(-1)}))(x)$. Then $U(1-s_1)$ is obtained from $U'(1-s_1)$ by induction with the inclusion of groups $i : \Gamma' \longrightarrow \Gamma$. This implies

$$b_1(l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma'} cone(U'(1-s_1))) = b_1(l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} cone(U(1-s_1))).$$

Let $T_{\sigma(\overline{s_1})}$ be the mapping torus of $\sigma(\overline{s_1}) : F \longrightarrow F$. (Notice that the free homotopy class of $\sigma(\overline{s_1})$ and hence the homotopy type of $T_{\sigma(\overline{s_1})}$ depend only on $s_1 \in \pi$.) The obvious map $\pi_1(T_{\sigma(\overline{s_1})}) \longrightarrow \mathbb{Z}$ factorizes into the composition of epimorphisms $\pi_1(T_{\sigma(\overline{s_1})}) \xrightarrow{\phi'} \Gamma' \longrightarrow \mathbb{Z}$. We get from Section 2 that $cone(U'(1-s_1))$ is the $\mathbb{Z}\Gamma'$-chain complex of the covering of $T_{\sigma(\overline{s_1})}$ associated to ϕ'. So we get

$$b_1(l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma'} cone(U'(1-s_1))) = b_1(T_{\sigma(\overline{s_1})}; (i \circ \phi')^* l^2(\Gamma)).$$

This finishes the proof of Theorem 3.1 since

$$b_1(T_{\sigma(\overline{s_1})}; (i \circ \phi')^* l^2(\Gamma)) = b_1(T_{\sigma(\overline{s_1})}; (\phi')^* l^2(\Gamma')) = 0$$

follows from Lemma 1.2.3 and Theorem 2.1. ■

4. Groups with vanishing first L^2-Betti number

In this section we give criteria for the triviality of the first L^2-Betti number of a finitely presented group π. The next theorem answers a query by Gromov [12, page 154].

Theorem 4.1 *Let $\{1\} \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \pi \longrightarrow \{1\}$ be an extension of finitely presented groups. Suppose that Δ is infinite and π contains $\mathbb{Z}$ as a subgroup. Then the first L^2-Betti number of Γ*

$$b_1(\Gamma) = 0.$$

Proof : Apply Theorem 3.1 to the fibration of classifying spaces $B\Delta \longrightarrow B\Gamma \longrightarrow B\pi$. ∎

We mention that there are counterexamples to the so called Burnside problem, namely, infinite finitely generated groups which do not contain $\mathbb{Z}$ as a subgroup. At of this writing the author does not know of an example of a infinite finitely presented group which does not contain $\mathbb{Z}$ as a subgroup.

We call a prime 3-manifold *exceptional* if it is closed and no finite cover of it is homotopy equivalent to a Haken, Seifert or hyperbolic 3-manifold. No exceptional prime 3-manifolds are known, and standard conjectures (Thurston geometrization conjecture, Waldhausen conjecture) imply that there are none. The next theorem follows from [19, Corollary 7.7].

Theorem 4.2 *Let π be the fundamental group of a prime 3-manifold M. Suppose either that ∂M is a union of tori or that M is closed and not exceptional. Then*

$$b_1(\pi) = 0. \qquad \blacksquare$$

Next we investigate closed 4-manifolds with a geometric structure in the sense of Thurston and compute their L^2-Betti numbers. Note that $b_p(M) = b_{4-p}(M)$ holds for such a manifold by Poincaré duality [19, Proposition 4.2]. The descriptions of the geometries and the computations of the Euler characteristics and signatures can be found in [24, Table 1 on page 122 and Theorem 6.1 on page 135] with a correction in [17].

Theorem 4.3 *Let M be a closed orientable 4-manifold with a geometry in the sense of Thurston and fundamental group π. Then the following are the values of the L^2-Betti numbers b_p, the Euler characteristics χ and the signatures σ of M*

geometry	$b_0 = b_4$	$b_1 = b_3$	b_2	χ	σ
S^4	$\frac{1}{\lvert\pi\rvert}$	0	0	$\frac{2}{\lvert\pi\rvert}$	0
$S^2 \times S^2$	$\frac{1}{\lvert\pi\rvert}$	0	$\frac{2}{\lvert\pi\rvert}$	$\frac{4}{\lvert\pi\rvert}$	0
$P^2(\mathbb{C})$	1	0	1	3	1
$S^2 \times \mathbb{R}^2$	0	0	0	0	0
$S^2 \times \mathbb{H}^2$	0	$-\frac{\chi(M)}{2}$	0	< 0	0
$\mathbb{R}^4$	0	0	0	0	0
$\mathbb{R}^2 \times \mathbb{H}^2$	0	0	0	0	0
$\mathbb{H}^2 \times \mathbb{H}^2$	0	0	$\chi(M)$	> 0	0
$S^3 \times \mathbb{R}^1$	0	0	0	0	0
$\mathbb{H}^3 \times \mathbb{R}$	0	0	0	0	0
$\widetilde{SL_2} \times \mathbb{R}$	0	0	0	0	0
$Nil^3 \times \mathbb{R}$	0	0	0	0	0
Nil^4	0	0	0	0	0
$Sol^4_{m,n}$	0	0	0	0	0
Sol^4_0	0	0	0	0	0
Sol^4_1	0	0	0	0	0
$\mathbb{H}^2(\mathbb{C})$	0	0	$\chi(M)$	$\chi = 3\sigma$	> 0
$\mathbb{H}^4$	0	0	$\chi(M)$	> 0	0

In particular $b_1(M) = b_1(\pi) = 0$ *in all cases except* $S^2 \times \mathbb{H}^2$*.*

Proof : We first give the proofs of the statements of the L^2-Betti numbers. Recall from Lemma 1.2.6 that $\chi(M) = 2b_0(M) - b_1(M) + b_2(M)$ holds. The fundamental group π is finite if and only if the underlying manifold of the geometry is compact. If $\widetilde{M}$ is the universal covering and π is finite, then $b_p(M) = \frac{b_p(\widetilde{M})}{|\pi|}$ from Lemma 1.2.4. Now the claim follows for S^4, $S^2 \times S^2$ and $P^2(\mathbb{C})$. If M and N have the same geometry X , then $b_p(M) = 0$ is equivalent to $b_p(N) = 0$ since $b_p(M) = 0$ is equivalent to the fact that X has no harmonic smooth L^2-integrable p-forms [7]. Hence it suffices to check $b_p(M) = 0$ for one example of a manifold with a given geometry. Suppose the geometry is a product of two lower dimensional geometries. Then such an example is given by a product of two surfaces or of S^1 with a 3-manifold. The L^2-Betti numbers for a closed orientable surfaces F_g of genus $g \geq 1$ are $b_0(F_g) = b_2(F_g) = 0$ and $b_1(F_g) = 2g - 2$ and the L^2-Betti numbers of S^1 are zero [19, Example 4.11]. Using the Künneth formula for L^2-cohomology [22, Theorem 3.16] one obtains the claim for all these cases. Suppose the geometry is Nil^4, Sol^4_0, Sol^4_1 or $Sol^4_{m,n}$. Then an example for M can be constructed which fibers over S^1. For such a space all L^2-Betti numbers are trivial by Theorem 2.1. One can also argue that the fundamental group of a manifold carrying such a geometry contains a normal infinite amenable subgroup and refer to [6, Theorem 0.2]. In the case $X = \mathbb{H}^4$, we get $b_1(M) = b_0(M) = 0$ and $b_2(M) = \chi(M) > 0$ from [8]. If M carries a $\mathbb{H}^2(\mathbb{C})$-geometry, then M is Kähler hyperbolic in the sense of [11] and we derive $b_p(M) = 0$ for $p = 0, 1$ from [11].

The statements about the Euler characteristic and the signature follow in the cases where all L^2-Betti numbers are trivial from the L^2-signature theorem [1] and in the cases where the geometry is compact from the multiplicativity of Euler characteristic and signature under finite coverings. The remaining cases are $S^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^4$ and $\mathbb{H}^2(\mathbb{C})$. Because of the Hirzebruch proportionality principle [15] it suffices to check the examples $S^2 \times F_g$ and $F_g \times F_g$ for some $g \geq 2$. In the cases $S^2 \times \mathbb{H}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$ and one gets $\chi(M) = t \cdot \chi(S^4)$ and $\sigma(M) = t \cdot \sigma(S^4)$ if M carries a $\mathbb{H}^4$-structure and $\chi(M) = s \cdot \chi(P^2(\mathbb{C}))$ and $\sigma(M) = s \cdot \sigma(P^2(\mathbb{C}))$ if M carries a $\mathbb{H}^2(\mathbb{C})$-structure for some non-zero constants t and s. From this the claim follows. ■

5. 4-manifolds satisfying $\chi(M) \geq |\sigma(M)|$

Theorem 5.1 *Let M be a closed oriented 4-manifold with fundamental group π. If $b_1(\pi) = 0$, then the following inequality for the Euler characteristic and the signature holds*

$$\chi(M) \geq |\sigma(M)|.$$

Proof : According to the L^2-signature theorem (see [1]), the signature $\sigma(M)$ is the difference of the von Neumann dimensions of two complementary subspaces of the second L^2-cohomology $H^2(M; l^2(\pi))$ of M. This implies

$$|\sigma(M)| \leq dim_{\mathcal{N}(\pi)}(H^2(M; l^2(\pi))) = dim_{\mathcal{N}(\pi)}(H_2(M; l^2(\pi))) = b_2(M).$$

Since $b_4(M) = b_0(M)$ and $b_3(M) = b_1(M)$ by Poincaré duality [19, Proposition 4.2] and $b_1(M) = 0$ by assumption holds, we conclude $\chi(M) = b_2(M) + 2 \cdot b_0(M)$ from Lemma 1.2.6 and the claim follows. ■

This Theorem 5.1 together with Theorems 4.1, 4.2 and 4.3 generalizes respectively reproves results of [16] (see also [25]). In [16] it is shown among other things that the inequality in Theorem 5.1 holds if π is an extension of a finitely generated group Δ and the integers $\mathbb{Z}$. (We have to assume that Δ is finitely presented). If a connected closed oriented 4-manifold has a geometric structure in the sense of Thurston, then it is pointed out in [16] that inspecting the 18 different cases using the results in [24] and the correction in [17] shows that the inequality in Theorem 5.1 holds for M provided the geometry is not $S^2 \times \mathbb{H}^2$. A simple proof can be found in [16] that the inequality above holds if π satisfies 3-dimensional oriented Poincaré duality. If π is infinite amenable, $b_1(\pi)$ vanishes by [6, Theorem 0.2] and Theorem 5.1 applies (see also [9]). For further information on lower bounds on the Euler characteristic of a closed 4-manifold we refer to [13].

6. Deficiency of groups

The *deficiency* of a finitely presented group π is the maximum over all differences $g - r$ where g respectively r is the number of generators respectively relations of a presentation of π.

Theorem 6.1 *Let π be a finitely presented group.*

1. *If $B\pi$ has finite 3-skeleton, then*
$$def(\pi) \leq 1 - b_0(\pi) + b_1(\pi) - b_2(\pi).$$
2. $def(\pi) \leq 1 - b_0(\pi) + b_1(\pi)$.
3. *If $b_1(\pi) = 0$, then*
$$def(\pi) \leq 1.$$

Proof : Given a presentation with r relations and g generators, let X be the corresponding connected 2-dimensional CW-complex with fundamental group isomorphic to π which has precisely one cell of dimension 0, g cells of dimension 1 and r cells of dimension 2. Since the classifying map $X \longrightarrow B\pi$ is 2-connected, we conclude from Lemma 1.2 that

$$1 - g + r = \chi(X) = b_0(X) - b_1(X) + b_2(X) \geq b_0(\pi) - b_1(\pi) + b_2(\pi)$$

from which assertion 1.) follows. Assertion 2.) is proven similiarly and does imply assertion 3.). ■

From Theorem 4.1 and Theorem 6.1.3. we obtain

Corollary 6.2 *Let $\{1\} \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \pi \longrightarrow \{1\}$ be an extension of finitely presented groups. Suppose that Δ is infinite and π contains $\mathbb{Z}$ as a subgroup. Then*

$$def(\Gamma) \leq 1. \qquad ■$$

If π is finite, we rediscover from Theorem 6.1.1 the well-known fact that $\mathrm{def}(\pi) \leq 0$. The inequality in Theorem 6.1.1 is obviously sharp and $\mathrm{def}(\pi) = 1 - \chi(B\pi)$ if $B\pi$ is a finite 2-dimensional CW-complex. If π is a torsion-free one-relator group, the 2-dimensional CW-complex associated with the presentation is aspherical and hence $B\pi$ is 2-dimensional [23, chapter III §§9 -11]. We conjecture for a torsion-free group having a presentation with

$g \geq 2$ generators and one non-trivial relation that $b_2(\pi) = 0$ and $b_1(\pi) = \operatorname{def}(\pi) - 1 = g - 2$ holds (compare [12, page 156]). This would follow from [19, Conjecture 9.2] saying that the L^2-Betti numbers of a finite CW-complex with torsion-free fundamental groups are integers. Namely, the kernel of the second differential of the L^2-chain complex of $B\pi$ is a proper submodule of $l^2(\pi)$ so that its dimension $b_2(\pi)$ is less than one.

The L^2-homological test for the deficiency described in Theorem 6.1 is useful in the situation considered in Corollary 6.2 where the corresponding tests using homology with $\mathbb{Z}/p$-coefficients appears insufficient. However, in other situations homology with $\mathbb{Z}/p$-coefficients seems to be more useful than L^2-homology as illustrated by the following result which is a direct consequence of [10, Theorem 2.5]

Theorem 6.3 *Let M be a compact 3-manifold with fundamental group π and prime decomposition*

$$M = M_1 \sharp M_2 \sharp \ldots \sharp M_r.$$

Let $s(M)$ be the number of prime factors M_i with non-empty boundary and $t(M)$ be the number of prime factors which are S^2-bundles over S^1. Denote by $\chi(M)$ the Euler characteristic. Then

$$\mathit{def}(\pi_1(M)) = \dim_{\mathbf{Z}/2}(H_1(\pi;\mathbf{Z}/2)) - \dim_{\mathbf{Z}/2}(H_2(\pi;\mathbf{Z}/2)) = s(M) + t(M) - \chi(M). \qquad \blacksquare$$

Let M be a compact irreducible 3-manifold with infinite fundamental group. Any such manifold is aspherical by the Sphere Theorem [14, page 40]. It follows from Theorem 6.3 that $\operatorname{def}(\pi_1(M))$ is $1 - \chi(M)$ if ∂M is not empty and is zero if ∂M is empty. Thus $1 - b_0(M) + b_1(M) - b_2(M) = 1 - \chi(M)$ since $b_3(M) = 0$ holds by [19, Lemma 4.5]. Hence we rediscover Theorem 6.3 in the case $\partial M \neq \emptyset$ from our L^2-homological test in Theorem 6.1. However, if M is closed, the inequality in Theorem 6.1.1 is not sharp.

Finally we mention the in a certain sense complementary result [3, Theorem 2]. If π is a finitely presented group with $\operatorname{def}(\pi) \geq 2$, then π can be written as an amalgamated product $\pi = A *_C B$ where A, B and C are finitely generated, C is proper subgroup of both A and B and has index greater than two in A or B. In particular π contains a free subgroup of rank 2 and is not amenable. This implies that an amenable finitely presented group has deficiency less or equal to one (see also [2] and [9]). This also follows from Theorem 6.1.3 and the fact that $b_1(\pi) = 0$ for a finitely presented amenable group [6, Theorem 0.2].

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