# The stable Cannon Conjecture

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# Outline

- We state the main conjectures and results
- We briefly recall the notion of a hyperbolic group
- The existence of a normal map
- The total surgery obstruction
- ANR-homology manifolds and Quinn's obstruction

# The main conjectures

## Definition (Finite Poincaré complex)

A (connected) finite $n$-dimensional $CW$-complex $X$ is a finite $n$-dimensional Poincaré complex if there is $[X] \in H_n(X;\mathbb{Z}^w)$ such that the induced $\mathbb{Z}\pi$-chain map

$$- \cap [X] \colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$$

is a $\mathbb{Z}\pi$-chain homotopy equivalence.

## Theorem (Closed manifolds are Poincaré complexes)

*A closed $n$-dimensional manifold $M$ is a finite $n$-dimensional Poincaré complex with $w = w_1(X)$.*

## Definition (Poincaré duality group)

A Poincaré duality group $G$ of dimension $n$ is a finitely presented group satisfying:

- $G$ is of type FP;
- $H^i(G;\mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

## Theorem (Wall)

*If $G$ is a $d$-dimensional Poincaré duality group for $d \geq 3$ and $\widetilde{K}_0(\mathbb{Z}G) = 0$, then there is a model for $BG$ which is a finite Poincaré complex of dimension $d$.*

### Corollary

*If $M$ is a closed aspherical manifold of dimension $d$, then $\pi_1(X)$ is a $d$-dimensional Poincaré duality group.*

### Theorem (Hadamard)

*If $M$ is a closed smooth Riemannian manifold whose section curvature is negative, then $\pi_1(M)$ is a torsionfree hyperbolic group with $\partial G = S^{n-1}$.*

### Theorem (Bieri-Eckmann, Linnell)

*Every 2-dimensional Poincaré duality group is the fundamental group of a closed surface.*

## Conjecture (Gromov)

*Let $G$ be a torsionfree hyperbolic group whose boundary is a sphere $S^{n-1}$. Then there is a closed aspherical manifold $M$ with $\pi_1(M) \cong G$.*

## Theorem (Bartels-Lück-Weinberger)

*Gromov's Conjecture is true for $n \geq 6$.*

## Conjecture (Wall)

*Every Poincaré duality group is the fundamental group of an aspherical closed manifold.*

## Conjecture (Cannon's Conjecture in the torsionfree case)

*A torsionfree hyperbolic group $G$ has $S^2$ as boundary if and only if it is the fundamental group of a closed hyperbolic $3$-manifold.*

**Theorem (Cannon-Cooper, Eskin-Fisher-Whyte, Kapovich-Leeb)**

*A Poincaré duality group $G$ of dimension $3$ is the fundamental group of an aspherical closed 3-manifold if and only if it is quasiisometric to the fundamental group of an aspherical closed 3-manifold.*

**Theorem (Bowditch)**

*If a Poincaré duality group of dimension $3$ contains an infinite normal cyclic subgroup, then it is the fundamental group of a closed Seifert $3$-manifold.*

**Theorem (Bestvina)**

*Let $G$ be a hyperbolic $3$-dimensional Poincaré duality group. Then its boundary is homeomorphic to $S^2$.*

**Theorem (Bestvina-Mess)**

*Let $G$ be an infinite torsionfree hyperbolic group which is prime, not infinite cyclic, and the fundamental group of a closed $3$-manifold $M$. Then $M$ is hyperbolic and $G$ satisfies the Cannon Conjecture.*

- In order to prove the Cannon Conjecture, it suffices to show for a hyperbolic group $G$, whose boundary is $S^2$, that it is quasiisometric to the fundamental group of some aspherical closed 3-manifold.

## Theorem

*Let $G$ be the fundamental group of an aspherical oriented closed 3-manifold. Then $G$ satisfies:*

- *$G$ is residually finite and Hopfian.*
- *All its $L^2$-Betti numbers $b_n^{(2)}(G)$ vanish;*
- *Its deficiency is $0$. In particular it possesses a presentation with the same number of generators and relations.*
- *Suppose that $M$ is hyperbolic. Then $G$ is virtually compact special and linear over $\mathbb{Z}$. It contains a subgroup of finite index $G'$ which can be written as an extension $1 \to \pi_1(S) \to G \to \mathbb{Z} \to 1$ for some closed orientable surface $S$.*

- Recall that any finitely presented groups occurs as the fundamental group of a closed $d$-dimensional smooth manifold for every $d \geq 4$.

### Theorem (Bestvina-Mess)

*A torsionfree hyperbolic G is a Poincaré duality group of dimension n if and only if its boundary $\partial G$ and $S^{n-1}$ have the same Čech cohomology.*

### Theorem

*If the boundary of a hyperbolic group contains an open subset homeomorphic to $\mathbb{R}^n$, then the boundary is homeomorphic to $S^n$.*

# The main results

**Theorem (Ferry-Lück-Weinberger, (preprint, 2018), Vanishing of the surgery obstruction)**

*Let $G$ be a hyperbolic 3-dimensional Poincaré duality group. Then there is a normal map of degree one (in the sense of surgery theory)*

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\overline{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \end{array}$$

*satisfying*

1. *The space $BG$ is a finite 3-dimensional $CW$-complex;*
2. *The map $H_n(f,\mathbb{Z})\colon H_n(M;\mathbb{Z}) \xrightarrow{\cong} H_n(BG;\mathbb{Z})$ is bijective for all $n \geq 0$;*
3. *The simple algebraic surgery obstruction $\sigma(f,\overline{f}) \in L_3^s(\mathbb{Z}G)$ vanishes.*

## Theorem (Ferry-Lück-Weinberger, (preprint, 2018), Stable Cannon Conjecture)

*Let $G$ be a hyperbolic 3-dimensional Poincaré duality group. Let $N$ be any smooth, PL or topological manifold respectively which is closed and whose dimension is $\geq 2$.*

*Then there is a closed smooth, PL or topological manifold $M$ and a normal map of degree one*

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\underline{f}} & \xi \times TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \times N \end{array}$$

*such that the map $f$ is a simple homotopy equivalence.*

- Obviously the last two theorems follow from the Cannon Conjecture.
- By the product formula for surgery theory the second last theorem implies the last theorem.
- The manifold $M$ appearing in the last theorem is unique up to homeomorphism by the Borel Conjecture, provided that $\pi_1(N)$ satisfies the Farrell-Jones Conjecture.
- If we take $N = T^k$ for some $k \geq 2$, then the Cannon Conjecture is equivalent to the statement that this $M$ is homeomorphic to $M' \times T^k$ for some closed 3-manifold $M'$.

## Definition (Hyperbolic space)

A $\delta$-hyperbolic space $X$ is a geodesic space whose geodesic triangles are all $\delta$-thin.
A geodesic space is called hyperbolic if it is $\delta$-hyperbolic for some $\delta > 0$.

- A geodesic space with bounded diameter is hyperbolic.
- A tree is 0-hyperbolic.
- A simply connected complete Riemannian manifold $M$ with $\sec(M) \leq \kappa$ for some $\kappa < 0$ is hyperbolic as a metric space.
- $\mathbb{R}^n$ is hyperbolic if and only if $n \leq 1$.

## Definition (Boundary of a hyperbolic space)

Let $X$ be a hyperbolic space. Define its **boundary** $\partial X$ to be the set of equivalence classes of geodesic rays. Put

$$\overline{X} := X \amalg \partial X.$$

- Two geodesic rays $c_1, c_2 \colon [0,\infty) \to X$ are called **equivalent** if there exists $C > 0$ satisfying $d_X\big(c_1(t), c_2(t)\big) \le C$ for $t \in [0,\infty)$.

## Lemma

*There is a topology on $\overline{X}$ with the properties:*

- *$\overline{X}$ is compact and metrizable;*
- *The subspace topology $X \subseteq \overline{X}$ is the given one;*
- *$X$ is open and dense in $\overline{X}$.*

- Let $M$ be a simply connected complete Riemannian manifold $M$ with $\sec(M) \leq \kappa$ for some $\kappa < 0$. Then $M$ is hyperbolic as a metric space and $\partial M = S^{\dim(M)-1}$.

## Definition (Quasi-isometry)

A map $f\colon X \to Y$ of metric spaces is called a quasi-isometry if there exist real numbers $\lambda, C > 0$ satisfying:

- The inequality

$$\lambda^{-1} \cdot d_X\big(x_1, x_2\big) - C \leq d_Y\big(f(x_1), f(x_2)\big) \leq \lambda \cdot d_X(x_1, x_2) + C$$

  holds for all $x_1, x_2 \in X$;
- For every $y$ in $Y$ there exists $x \in X$ with $d_Y\big(f(x), y\big) < C$.

## Lemma (Švarc-Milnor Lemma)

*Let $X$ be a geodesic space. Suppose that the finitely generated group $G$ acts properly, cocompactly and isometrically on $X$. Choose a base point $x \in X$. Then the map*

$$f\colon G \to X, \quad g \mapsto gx$$

*is a quasi-isometry.*

## Lemma (Quasi-isometry invariance of the Cayley graph)

*The quasi-isometry type of the Cayley graph of a finitely generated group is independent of the choice of a finite set of generators.*

## Lemma (Quasi-isometry invariance of being hyperbolic)

*The property "hyperbolic" is a quasi-isometry invariant of geodesic spaces.*

## Lemma (Quasi-isometry invariance of the boundary)

*A quasi-isometry $f \colon X_1 \to X_2$ of hyperbolic spaces induces a homeomorphism*

$$\partial X_1 \xrightarrow{\cong} \partial X_2.$$

## Definition (Hyperbolic group)

A finitely generated group is called hyperbolic if its Cayley graph is hyperbolic.

## Definition (Boundary of a hyperbolic group)

Define the boundary $\partial G$ of a hyperbolic group to be the boundary of its Cayley graph.

# Basic properties of hyperbolic groups

- A group $G$ is hyperbolic if and only if it acts properly, cocompactly and isometrically on a hyperbolic space. In this case $\partial G = \partial X$.
- Let $M$ be a closed Riemannian manifold with $\sec(M) < 0$. Then $\pi_1(M)$ is hyperbolic with $S^{\dim(M)-1}$ as boundary.
- If $G$ is virtually torsionfree and hyperbolic, then $\mathrm{vcd}(G) = \dim(\partial G) + 1$.
- If the boundary of a hyperbolic group contains an open subset homeomorphic to $\mathbb{R}^n$, then the boundary is homeomorphic to $S^n$.
- A subgroup of a hyperbolic group is either virtually cyclic or contains $\mathbb{Z} * \mathbb{Z}$ as subgroup. In particular $\mathbb{Z}^2$ is not a subgroup of a hyperbolic group.

- A free product of two hyperbolic groups is again hyperbolic.
- A direct product of two finitely generated groups is hyperbolic if and only if one of the two groups is finite and the other is hyperbolic.
- The Rips complex of a hyperbolic group $G$ is a cocompact model for its classifying space $\underline{E}G$ for proper actions. This implies that there is a model of finite type for $BG$ and hence that $G$ is finitely presented and that there are only finitely many conjugacy classes of finite subgroups.
- A finitely generated torsion group is hyperbolic if and only if it is finite.
- A random finitely presented group is hyperbolic.

# The existence of a normal map

**Theorem (Existence of a normal map)**

*Let $X$ be a connected oriented finite 3-dimensional Poincaré complex. Then there are an integer $a \geq 0$ and a vector bundle $\xi$ over $BG$ and a normal map of degree one*

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\overline{f}} & \underline{\xi} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

## Proof.

- Stable vector bundles over $X$ are classified by the first and second Stiefel-Whitney class $w_1(\xi)$ and $w_2(\xi)$ in $H^*(X;\mathbb{Z}/2)$, since $\dim(X) = 3$.
- The analogous statement holds for $M$.
- Let $\xi$ be a $k$-dimensional vector bundle over $X$ such that $w_1(\xi) = w_1(X)$ and $w_2(\xi) = w_1(X) \cup w_1(X)$ holds.

□

## Proof (continued).

- A spectral sequence argument applied to $\Omega_3(X, w_1(X))$ shows that there is a closed 3-manifold $M$ together with a map $f\colon M \to X$ of degree one such that $f^*w_1(X) = w_1(M)$.

- Then
$$w_1(f^*\xi) = f^*w_1(X) = w_1(M) = w_1(TM).$$

- The Wu formula implies
$$\begin{aligned} w_2(f^*\xi) = f^*w_2(\xi) = f^*(w_1(X) \cup w_1(X)) &= f^*w_1(X) \cup f^*w_1(X)) \\ &= w_1(M) \cup w_1(M) = w_2(M) = w_2(TM). \end{aligned}$$

- Hence $f^*\xi$ is stably isomorphic to the stable tangent bundle of $M$.

□

# The total surgery obstruction

- Consider an aspherical finite $n$-dimensional Poincaré complex $X$ such that $G = \pi_1(X)$ is a Farrell-Jones group, i.e., satisfies both the $K$-theoretic and the $L$-theoretic Farrell-Jones Conjecture with coefficients in additive categories, and $\mathcal{N}(X)$ is non-empty. (For simplicity we assume $w_1(X) = 0$ in the sequel.)

- We have to find one normal map of degree one

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\overline{f}} & \underline{\xi} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

whose simple surgery obstruction $\sigma^s(f, \overline{f}) \in L_3^s(\mathbb{Z}G)$ vanishes.

- Recall that the simple surgery obstruction defines a map

$$\sigma^s\colon \mathcal{N}(X) \to L_n^s(\mathbb{Z}G).$$

- Fix a normal map $(f_0, \overline{f_0})$.
- Then there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{N}(X) & \xrightarrow{\sigma^s(-,-)-\sigma^s(f_0,\overline{f_0})} & L_n^s(\mathbb{Z}G) \\
{\scriptstyle s_0}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\uparrow{\scriptstyle \operatorname{asmb}_n^s(X)} \\
H_n(X;\mathbf{L}_{\mathbb{Z}}^s\langle 1\rangle) & \xrightarrow[H_n^G(\operatorname{id}_X;\mathbf{i})]{} & H_n(X;\mathbf{L}_{\mathbb{Z}}^s)
\end{array}$$

whose vertical arrows are bijections thanks to the Farrell-Jones Conjecture and the upper arrow sends the class of $(f, \overline{f})$ to the difference $\sigma^s(f, \overline{f}) - \sigma^s(f, \overline{f_0})$ of simple surgery obstructions.

- An easy spectral sequence argument yields a short exact sequence

$$0 \to H_n(X;\mathbf{L}^s_{\mathbb{Z}}\langle 1\rangle) \xrightarrow{H_n(\mathrm{id}_X;\mathbf{i})} H_n(X;\mathbf{L}^s_{\mathbb{Z}}) \xrightarrow{\lambda^s_n(X)} L_0(\mathbb{Z}).$$

- Consider the composite

$$\mu^s_n(X)\colon \mathcal{N}(X) \xrightarrow{\sigma^s} L^s_n(\mathbb{Z}G,w) \xrightarrow{\mathrm{asmb}^s_n(X)^{-1}} H_n(X;\mathbf{L}^s_{\mathbb{Z}}) \xrightarrow{\lambda^s_n(X)} L_0(\mathbb{Z}).$$

- We conclude that there is precisely one element, called the total surgery obstruction,

$$s(X) \in L_0(\mathbb{Z}) \cong \mathbb{Z}$$

such that for any element $[(f, \overline{f})]$ in $\mathcal{N}(X)$ its image under $\mu_n^s(X)$ is $s(X)$.

### Theorem (Total surgery obstruction)

- *There exists a normal map of degree one $(f, \overline{f})$ with target $X$ and vanishing simple surgery obstruction $\sigma^s(f, \overline{f}) \in L_n^s(\mathbb{Z}G)$ if and only if $s(X) \in L_0(\mathbb{Z}) \cong \mathbb{Z}$ vanishes.*
- *The total surgery obstruction is a homotopy invariant of $X$ and hence depends only on $G$.*

# ANR-homology manifolds

## Definition (Homology ANR-manifold)

A homology ANR-manifold X is an ANR satisfying:

- $X$ has a countable basis for its topology;
- The topological dimension of $X$ is finite;
- $X$ is locally compact;
- for every $x \in X$ we have for the singular homology

$$H_i(X, X - \{x\}; \mathbb{Z}) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$$

If $X$ is additionally compact, it is called a closed ANR-homology manifold.

- Every closed topological manifold is a closed ANR-homology manifold.
- Let $M$ be homology sphere with non-trivial fundamental group. Then its suspension $\Sigma M$ is a closed ANR-homology manifold but not a topological manifold.

# Quinn's resolution obstruction

## Theorem (Quinn (1987))

*There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ for homology ANR-manifolds with the following properties:*

- *if $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;*
- *$i(M \times N) = i(M) \cdot i(N)$;*
- *Let $M$ be a homology ANR-manifold of dimension $\geq 5$. Then $M$ is a topological manifold if and only if $\iota(M) = 1$.*
- *The Quinn obstruction and the total surgery obstruction are related for an aspherical closed ANR-homology manifold $M$ of dimension $\geq 5$ by*

$$\iota(M) = 8 \cdot s(X) + 1.$$

# Proof of the Theorem about the vanishing of the surgery obstruction

## Proof.

- We have to show for the aspherical finite 3-dimensional Poincaré complex $X$ that its total surgery obstruction vanishes.
- The total surgery obstruction satisfies a product formula

$$s(X \times Y) = s(X) + s(Y).$$

- This implies

$$s(X \times T^3) = s(X).$$

- Hence it suffices to show that $s(X \times T^3)$ vanishes.

## Proof (continued).

- There exists an aspherical closed ANR-homology manifold $M$ and a homotopy equivalence to $f\colon M \to X \times T^3$.
- There is a $Z$-compactification $\overline{\widetilde{X}}$ of $\widetilde{X}$ by the boundary $\partial G = S^2$.
- One then constructs an appropriate $Z$-compactification $\overline{\widetilde{M}}$ of $\widetilde{M}$ so that we get a ANR-homology manifold $\overline{\widetilde{M}}$ whose boundary is a topological manifold and whose interior is $\widetilde{M}$.
- By adding a collar to $\overline{\widetilde{M}}$ one obtains a ANR-homology manifold $Y$ which contains $\widetilde{M}$ as an open subset and contains an open subset $U$ which is homeomorphic to $\mathbb{R}^6$.

□

## Proof (continued).

- Hence we get

$$\begin{aligned} 8s(X \times T^3) + 1 = 8s(M) + 1 = i(M) = i(\widetilde{M}) \\ = i(Y) = i(U) = i(\mathbb{R}^6) = 1. \end{aligned}$$

- This implies $s(X \times T^3) = 0$ and hence $s(X) = 0$.

□