Introduction to surgery theory

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Outline

- State the existence problem and uniqueness problem in surgery theory.
- Explain the notion of Poincaré complex and of Spivak normal fibration.
- Introduce the surgery problem, the surgery step and the surgery obstruction.
- Explain the surgery exact sequence and its applications to topological rigidity.

The goal of surgery theory

Problem (Existence)

Let X be a space. When is X homotopy equivalent to a closed manifold?

Problem (Uniqueness)

Let M and N be two closed manifolds. Are they isomorphic?

- For simplicity we will mostly work with orientable connected closed manifolds.
- We can consider topological manifolds, PL-manifolds or smooth manifolds and then isomorphic means homeomorphic, PL-homeomorphic or diffeomorphic.
- We will begin with the existence problem. We will later see that the uniqueness problem can be interpreted as a relative existence problem thanks to the s-Cobordism Theorem.

Poincaré complexes

- A closed manifold carries the structure of a finite CW-complex. Hence we assume in the sequel in the existence problem that X itself is already a CW-complex.

- Fix a natural number $n \geq 4$. Then every finitely presented group occurs as fundamental group of a closed n-dimensional manifold. Since the fundamental group of a finite CW-complex is finitely presented, we get no constraints on the fundamental group.

- We have already explained that not all finitely presented groups can occur as fundamental groups of closed 3-manifolds. For instance, the fundamental group π of a closed 3-manifold satisfies

$$\dim_{\mathbb{Q}}(H_2(\pi;\mathbb{Q})) \leq \dim_{\mathbb{Q}}(H_1(\pi;\mathbb{Q}))$$

- Let M be a (connected orientable) closed n-dimensional manifold. Then $H_n(M;\mathbb{Z})$ is infinite cyclic. If $[M] \in H_n(M;\mathbb{Z})$ is a generator, then the cap product with $[M]$ yields for $k \in \mathbb{Z}$ isomorphisms

$$- \cap [M] \colon H^{n-k}(M;\mathbb{Z}) \xrightarrow{\cong} H_k(M;\mathbb{Z}).$$

  Obviously X has to satisfy the same property if it is homotopy equivalent to M.

- There is a much more sophisticated Poincaré duality behind the result above which we will explain next.

- Recall that a (not necessarily commutative) ring with involution R is ring R with an involution of rings

$$\bar{\ } \colon R \to R, \quad r \mapsto \bar{r},$$

  i.e., a map satisfying $\bar{\bar{r}} = r$, $\overline{r+s} = \bar{r} + \bar{s}$, $\overline{r \cdot s} = \bar{s} \cdot \bar{r}$ and $\bar{1} = 1$ for $r, s \in R$.

- Our main example is the involution on the group ring $\mathbb{Z}G$ for a group G defined by sending $\sum_{g \in G} a_g \cdot g$ to $\sum_{g \in G} a_g \cdot g^{-1}$.

- Let M be a left R-module. Then $M^* := \hom_R(M, R)$ carries a canonical right R-module structure given by $(fr)(m) = f(m) \cdot r$ for a homomorphism of left R-modules $f \colon M \to R$ and $m \in M$. The involution allows us to view $M^* = \hom_R(M; R)$ as a left R-module, namely, define rf for $r \in R$ and $f \in M^*$ by $(rf)(m) := f(m) \cdot \overline{r}$ for $m \in M$.

- Given an R-chain complex of left R-modules C_* and $n \in \mathbb{Z}$, we define its dual chain complex C^{n-*} to be the chain complex of left R-modules whose p-th chain module is $\hom_R(C_{n-p}, R)$ and whose p-th differential is given by

$$\begin{aligned}(-1)^{n-p+1} \cdot \hom_R(c_{n-p+1}, \mathrm{id}) \colon (C^{n-*})_p &= \hom_R(C_{n-p}, R) \\ \to (C^{n-*})_{p-1} &= \hom_R(C_{n-p+1}, R).\end{aligned}$$

Definition (Finite Poincaré complex)

A (connected) finite n-dimensional CW-complex X is a finite n-dimensional Poincaré complex if there is $[X] \in H_n(X;\mathbb{Z})$ such that the induced $\mathbb{Z}\pi$-chain map

$$- \cap [X] \colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$$

is a $\mathbb{Z}\pi$-chain homotopy equivalence.

- If we apply $\mathrm{id}_{\mathbb{Z}} \otimes_{\mathbb{Z}\pi} -$, we obtain a $\mathbb{Z}$-chain homotopy equivalence

  $$C^{n-*}(X) \to C_*(X)$$

  which induces after taking homology the Poincaré duality isomorphism $- \cap [X] \colon H^{n-k}(M;\mathbb{Z}) \xrightarrow{\cong} H_k(M;\mathbb{Z})$ from above.

Theorem (Closed manifolds are Poincaré complexes)

A closed n-dimensional manifold M is a finite n-dimensional Poincaré complex.

- We conclude that a finite n-dimensional CW-complex X is homotopy equivalent to a closed n-dimensional manifold only if it is up to homotopy a finite n-dimensional Poincaré complex.

The Spivak normal fibration

- We briefly recall the Pontryagin-Thom construction for a closed n-dimensional manifold M.
- Choose an embedding $i\colon M \to S^{n+k}$ normal bundle $\nu(M)$.
- Choose a tubular neighborhood $N \subseteq S^{n+k}$ of M. It comes with a diffeomorphism

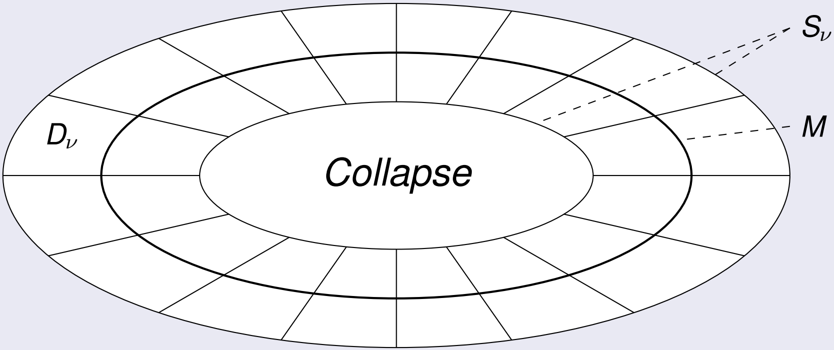
$$f\colon (D\nu(M), S\nu(M)) \xrightarrow{\cong} (N, \partial N)$$

  which is the identity on the zero section.
- Let

$$c\colon S^{n+k} \to \mathrm{Th}(\nu(M)) := D\nu(M)/S\nu(M)$$

  be the collaps map onto the Thom space, which is given by f^{-1} on $\mathrm{int}(N)$ and sends any point outside $\mathrm{int}(N)$ to the base point.

Figure (Pontrjagin-Thom construction)



- New: The Hurewicz homomorphism

$$\pi_{n+k}(Th(M)) \to H_{n+k}(\mathrm{Th}(M))$$

sends $[c]$ to a generator of the infinite cyclic group $H_{n+k}(\mathrm{Th}(M))$.

- Since DE is a compact $(n+k)$-dimensional manifold with boundary SE, the group $H_{n+k}(\mathrm{Th}(M)) \cong H_{n+k}(DE, SE)$ is infinite cyclic since it is isomorphic by Poincaré duality to $H^0(DE)$.

- The bijectivity of the Hurewicz homomorphisms above follows from the fact that c is a diffeomorphism on the interior of N by considering the preimage of a regular value.

- The normal bundle is stably independent of the choice of the embedding.

- Next we describe the homotopy theoretic analog of the normal bundle for a finite n-dimensional Poincaré complex X.

Definition (Spivak normal structure)

A Spivak normal $(k-1)$-structure is a pair (p, c) where $p\colon E \to X$ is a $(k-1)$-spherical fibration called the Spivak normal fibration, and $c\colon S^{n+k} \to \mathrm{Th}(p)$ is a map such that the Hurewicz homomorphism $h\colon \pi_{n+k}(\mathrm{Th}(p)) \to H_{n+k}(\mathrm{Th}(p))$ sends $[c]$ to a generator of the infinite cyclic group $H_{n+k}(\mathrm{Th}(p))$.

Theorem (Existence and Uniqueness of Spivak Normal Fibrations)

1. *If k is a natural number satisfying $k \geq n+1$, then there exists a Spivak normal $(k-1)$-structure (p, c);*

2. *For $i = 0, 1$ let $p_i \colon E_i \to X$ and $c_i \colon S^{n+k_i} \to \mathrm{Th}(p_i)$ be Spivak normal (k_i-1)-structures for X. Then there exists an integer k*

   with $k \geq k_0, k_1$ such that there is up to strong fibre homotopy precisely one strong fibre homotopy equivalence

   $$(\mathrm{id}, \overline{f}) \colon p_0 * \underline{S^{k-k_0}} \to p_1 * \underline{S^{k-k_1}}$$

   for which $\pi_{n+k}(\mathrm{Th}(\overline{f}))(\Sigma^{k-k_0}([c_0])) = \Sigma^{k-k_1}([c_1])$ holds.

- The Pontryagin-Thom construction yields a Spivak normal $(k-1)$-structure on a closed manifold M with the sphere bundle $S\nu(M)$ as the spherical $(k-1)$ fibration.
- Hence a finite n-dimensional Poincaré complex is homotopy equivalent to a closed manifold only if the Spivak normal fibration has (stably) a vector bundle reduction.
- There exists a finite n-dimensional Poincaré complex whose Spivak normal fibration does not possess a vector bundle reduction and which therefore is not homotopy equivalent to a closed manifold.
- Hence we assume from now on that X is a (connected oriented) finite n-dimensional Poincaré complex which comes with a vector bundle reduction ξ of the Spivak normal fibration.

Normal maps

Definition (Normal map of degree one)

A normal map of degree one with target X consists of:

- A closed (oriented) n-dimensional manifold M;
- A map of degree one $f\colon M \to X$;
- A $(k+n)$-dimensional vector bundle ξ over X;
- A bundle map $\overline{f}\colon TM \oplus \underline{\mathbb{R}^k} \to \xi$ covering f.

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\overline{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

- A vector bundle reduction yields a normal map of degree one with X as target as explained next.

- Let η be a vector bundle reduction of the Spivak normal fibration.

- Let $c\colon S^{n+k} \to \mathrm{Th}(p)$ be the associated collaps map. Make it transversal to the zero-section in $\mathrm{Th}(p)$.

- Let M be the preimage of the zero-section. This is a closed submanifold of S^{n+k} and comes with a map $f\colon M \to X$ of degree one covered by a bundle map $\nu(M \subseteq S^{n+k}) \to \eta$.

- Since $TM \oplus \nu(M \subseteq S^{n+k})$ is stably trivial, we can construct from these data a normal map of degree one from M to X.

Problem (Surgery Problem)

Let $(f, \overline{f})\colon M \to X$ be a normal map of degree one. Can we modify it without changing the target such that f becomes a homotopy equivalence?

- Suppose that X is homotopy equivalent to a closed manifold M.
- Then there exists a normal map of degree one from M to X whose underlying map $f\colon M \to X$ is a homotopy equivalence. Just take $\xi = f^{-1}TM$ for some homotopy inverse f^{-1} of f.

The surgery step

- Suppose that M is a closed manifold of dimension n, X is a CW-complex and $f\colon M \to X$ is a k-connected map. Consider $\omega \in \pi_{k+1}(f)$ represented by a diagram

$$\begin{array}{ccc} S^k & \xrightarrow{q} & M \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle f} \\ D^{k+1} & \xrightarrow{Q} & X. \end{array}$$

  We want to kill ω.

- In the category of CW-complexes this can be achieved by attaching cells. But attaching a cell destroys in general the structure of a closed manifold, so we have to do a more sophisticated modification.

- Suppose that the map $q\colon S^k \to M$ extends to an embedding

$$q^{\mathrm{th}}\colon S^k \times D^{n-k} \hookrightarrow M.$$

- Let $\mathrm{int}(\mathrm{im}(q^{\mathrm{th}}))$ be the interior of the image of q^{th}. Then $M - \mathrm{int}(\mathrm{im}(q^{\mathrm{th}}))$ is a manifold with boundary $\mathrm{im}(q^{\mathrm{th}}|_{S^k \times S^{n-k-1}})$.

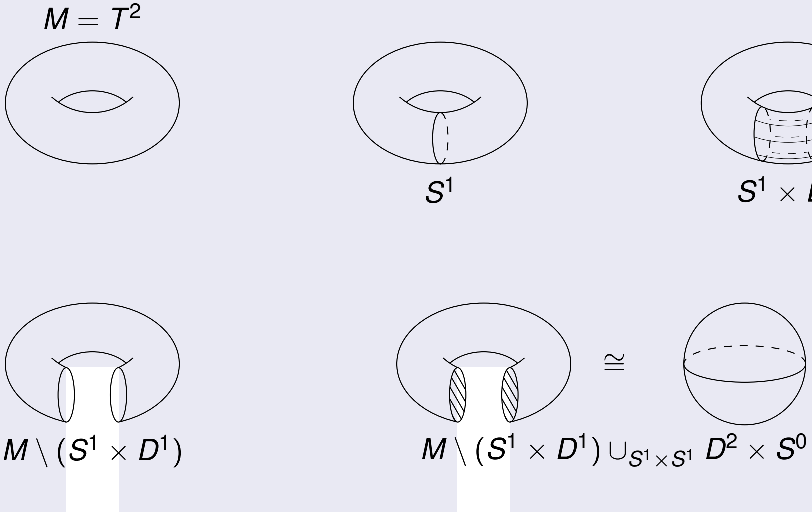
- We can get rid of the boundary by attaching $D^{k+1} \times S^{n-k-1}$ along $q^{\mathrm{th}}|_{S^k \times S^{n-k-1}}$. Denote the resulting manifold

$$M' := \left(D^{k+1} \times S^{n-k-1}\right) \cup_{q^{\mathrm{th}}|_{S^k \times S^{n-k-1}}} \left(M - \mathrm{int}(\mathrm{im}(q^{\mathrm{th}}))\right).$$

- The manifold M' is said to be obtained from M by surgery along q^{th}.

- Let $f\colon T^2 \to S^2$ be a Hopf collapse map. We fix $y_0 \in S^1$ so that $S^1 := S^1 \times \{y_0\} \subset T^2$ satisfies $f(S^1) = x_0$. We define $\omega \in \pi_2(f)$ by extending $f|_{S^1}$ to the constant map at x_0 on all of D^2.
- The following figure illustrates the effect of surgery on the source.

Figure (Source of a surgery step for $M = T^2$)



- The map $f' \colon S^2 \to S^2$ obtained by carrying out the surgery step on the Hopf collapse map $f \colon T^2 \to S^2$ as described above is a homotopy equivalence since it is a map $S^2 \to S^2$ of degree one.

- Consider a map $f \colon M \to X$ from a closed n-dimensional manifold M to a finite CW-complex X. Suppose that it can be converted by a finite sequence of surgery steps to a homotopy equivalence $f' \colon M' \to X$. Then

$$\chi(M) - \chi(X) \equiv 0 \mod 2$$

  by the additivity and homotopy invariance of the Euler characteristic.

- Hence in general there are obstructions to solve the Surgery Problem.

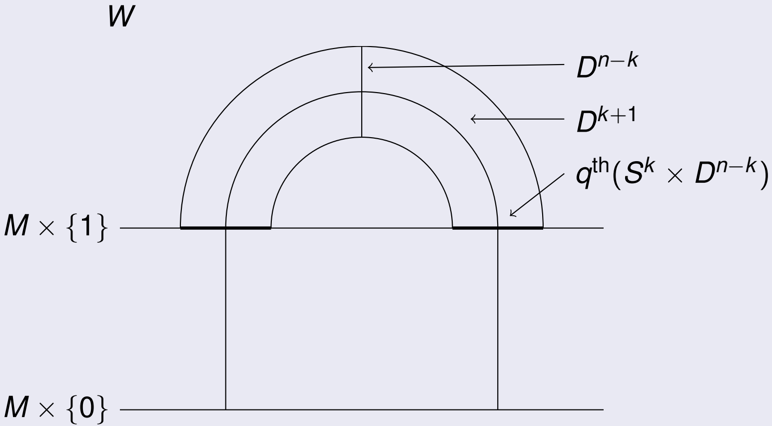
- It is important to notice that the maps $f\colon M \to X$ and $f'\colon M' \to X$ are bordant as manifolds with reference map to X.
- The relevant bordism is given by

$$W = \left(D^{k+1} \times D^{n-k}\right) \cup_{q^{\mathrm{th}}} (M \times [0,1]),$$

where we think of q^{th} as an embedding $S^k \times D^{n-k} \to M \times \{1\}$. In other words, W is obtained from $M \times [0,1]$ by attaching a handle $D^{k+1} \times D^{n-k}$ to $M \times \{1\}$.
- Then M appears in W as $M \times \{0\}$ and M' as other component of the boundary of W.

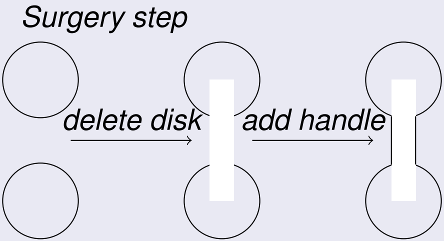
- The manifold W is called the trace of surgery along the embedding q^{th}.
- The next figure below gives a schematic representation of the trace of a surgery. For obvious reasons, this fundamental image in surgery theory is often called the surgeon's suitcase.

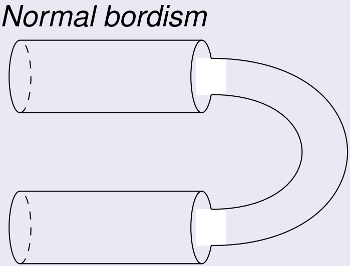
Figure (Surgeon's suitcase)



- The next figure displays the surgery step and its trace for the special case $M = S^1$ and $k = 0$, where we start from an embedding $S^0 \times S^1 \hookrightarrow S^1$.

Figure (Surgery along $S^0 \times D^1 \hookrightarrow S^1$)





- Notice that the inclusion $M - \text{int}(\text{im}(q^{\text{th}})) \to M$ is $(n-k-1)$-connected since $S^k \times S^{n-k-1} \to S^k \times D^{n-k}$ is $(n-k-1)$-connected. Hence $\pi_l(f) = \pi_l(f')$ for $l \leq k$ and there is an epimorphism $\pi_{k+1}(f) \to \pi_{k+1}(f')$ whose kernel contains ω, provided that $2(k+1) \leq n$.

- The condition $2(k+1) \leq n$ can be viewed as a consequence of Poincaré duality. Roughly speaking, if we change something in a manifold in dimension l, Poincaré duality also forces a change in dimension $(n-l)$. This phenomenon is the reason why there are surgery obstructions to converting any map $f \colon M \to X$ into a homotopy equivalence in a finite number of surgery steps for odd dimension n.

- The bundle data ensure that the thickening q^{th} exists when we are doing surgery below the middle dimension. If one carries out the thickening in a specific way, the bundle data extend to the resulting normal map of degree one and we can continue the process.

Theorem (Making a normal map highly connected)

Given a normal map of degree one, we can carry out a finite sequence of surgery steps so that the resulting $f' \colon N \to X$ *is* k*-connected, where* $n = 2k$ *or* $n = 2k + 1$*.*

Lemma

A normal map of degree one which is $(k + 1)$*-connected, where* $n = 2k$ *or* $n = 2k + 1$*, is a homotopy equivalence.*

- Hence we have to make a normal map, which is already k-connected, $(k+1)$-connected in order to achieve a homotopy equivalence, where $n = 2k$ or $n = 2k+1$. Exactly here the surgery obstruction occurs.

- In odd dimension $n = 2k+1$ the surgery obstruction comes from the previous observation that by Poincare duality modifications in the $(k+1)$-th homology cause automatically (undesired) changes in the k-th homology.

- In even dimension $n = 2k$ one encounters the problem that the bundle data only guarantee that one can find an immersion with finitely many self-intersection points

$$q^{\mathrm{th}} \colon S^k \times D^k \to M.$$

  The surgery obstruction is the algebraic obstruction to get rid of the self-intersection points. If $n \geq 5$, its vanishing is indeed sufficient to convert q^{th} into an embedding.

- One prominent necessary surgery obstruction is given in the case $n = 4k$ by the difference of the signatures $\operatorname{sign}(X) - \operatorname{sign}(M)$ since the signature is a bordism invariant and a homotopy invariant.

- If $\pi_1(M)$ is simply connected and $n = 4k$ for $k \geq 2$, then the vanishing of $\operatorname{sign}(X) - \operatorname{sign}(M)$ is indeed sufficient.

- If $\pi_1(M)$ is simply connected and n is odd and $n \geq 5$, there are no surgery obstructions.

Theorem (Existence problem in the simply connected case)

Let X be a simply connected finite Poincaré complex of dimension n

1. *Suppose that n is odd and $n \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction to a vector bundle.*
2. *Suppose $n = 4k \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction to a vector bundle $\xi \colon E \to X$ such that*

$$\langle \mathcal{L}(\xi), [X] \rangle = \operatorname{sign}(X).$$

3. *Suppose that $n = 4k + 2 \geq 5$. Then X is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a reduction to a vector bundle such that the Arf invariant of the associated surgery problem, which takes values in $\mathbb{Z}/2$, vanishes.*

Algebraic L-groups

- In general there are surgery obstructions taking values in the so called L-groups $L_n(\mathbb{Z}[\pi_1(M)])$.
- In even dimensions $L_n(R)$ is defined for a ring with involution in terms of quadratic forms over R, where the hyperbolic quadratic forms always represent zero. In odd dimensions $L_n(R)$ is defined in terms of automorphisms of hyperbolic quadratic forms, or, equivalently, in terms of so called formations.
- The L-groups are easier to compute than K-groups since they are 4-periodic, i.e., $L_n(R) \cong L_{n+4}(R)$.
- We have

$$L_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 4k; \\ \mathbb{Z}/2 & \text{if } n = 4k+2; \\ \{0\} & \text{if } n = 2k+1. \end{cases}$$

- The surgery obstruction is defined in all dimensions and is always a necessary condition to solve the surgery problem.
- In dimension $n \geq 5$ the vanishing of the surgery obstruction is sufficient.
- In dimension 4 the methods of proof of sufficiency break down because the so called Whitney trick is not available anymore which relies in higher dimensions on the fact that two embedded 2-disks can be made disjoint by transversality.
- In dimension 3 problems occur concerning the effect of surgery on the fundamental group.

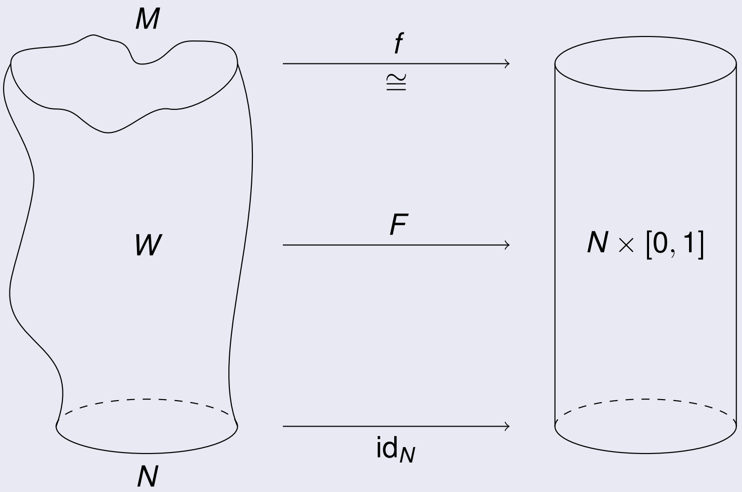
The Surgery Program

- The surgery Program addresses the uniqueness problem whether two closed manifolds M and N are diffeomorphic.

- The idea is to construct an h-cobordism $(W; M, N)$ with vanishing Whitehead torsion. Then W is diffeomorphic to the trivial h-cobordism over M and hence M and N are diffeomorphic.

- So the Surgery Program due to Browder, Novikov, Sullivan and Wall is:
  1. Construct a homotopy equivalence $f \colon M \to N$;
  2. Construct a cobordism $(W; M, N)$ and a map $(F, f, \mathrm{id}) \colon (W; M, N) \to (N \times [0,1]; N \times \{0\}, N \times \{1\})$;
  3. Modify W and F relative boundary by surgery such that F becomes a homotopy equivalence and thus W becomes an h-cobordism;
  4. During these processes one should make certain that the Whitehead torsion of the resulting h-cobordism is trivial. Or one knows already that $\mathrm{Wh}(\pi_1(M))$ vanishes.

Figure (Surgery Program)



Definition (The structure set)

Let N be a closed topological manifold of dimension n. We call two simple homotopy equivalences $f_i \colon M_i \to N$ from closed topological manifolds M_i of dimension n to N for $i = 0, 1$ equivalent if there exists a homeomorphism $g \colon M_0 \to M_1$ such that $f_1 \circ g$ is homotopic to f_0.

The structure set $\mathcal{S}_n^{\mathrm{top}}(N)$ of N is the set of equivalence classes of simple homotopy equivalences $M \to X$ from closed topological manifolds of dimension n to N. This set has a preferred base point, namely the class of the identity $\mathrm{id} \colon N \to N$.

- If we assume $\mathrm{Wh}(\pi_1(N)) = 0$, then every homotopy equivalence with target N is automatically simple.
- There is an obvious version, where topological and homeomorphism are replaced by smooth and diffeomorphism.

Definition (Topological rigid)

A closed topological manifold N is called topologically rigid if any homotopy equivalence $f\colon M \to N$ with a closed manifold M as source is homotopic to a homeomorphism.

Lemma

A closed topological manifold M is topologically rigid if and only if the structure set $\mathcal{S}_n^{\mathrm{top}}(M)$ consists of exactly one point.

Lemma

The Poincaré Conjecture implies that S^n is topologically rigid.

Theorem (The topological Surgery Exact Sequence)

For a closed n-dimensional topological manifold N with $n \geq 5$, there is an exact sequence of abelian groups, called surgery exact sequence,

$$\cdots \xrightarrow{\eta} \mathcal{N}_{n+1}^{\mathrm{top}}(N \times [0,1], N \times \{0,1\}) \xrightarrow{\sigma} L_{n+1}^{s}(\mathbb{Z}\pi) \xrightarrow{\partial} \mathcal{S}_{n}^{\mathrm{top}}(N) \xrightarrow{\eta} \mathcal{N}_{n}^{\mathrm{top}}(N) \xrightarrow{\sigma} L_{n}^{s}(\mathbb{Z}\pi)$$

- $L_n^s(\mathbb{Z}\pi)$ is the algebraic L-group of the group ring $\mathbb{Z}\pi$ for $pi = \pi_1(N)$ (with decoration s).
- $\mathcal{N}_n^{\mathrm{top}}(N)$ is the set of normal bordism classes of normal maps of degree one with target N.
- $\mathcal{N}_{n+1}^{\mathrm{top}}(N \times [0,1], N \times \{0,1\})$ is the set of normal bordism classes of normal maps $(M, \partial M) \to (N \times [0,1], N \times \{0,1\})$ of degree one with target $N \times [0,1]$ which are simple homotopy equivalences on the boundary.

- The map σ is given by the surgery obstruction.
- The map η sends $f\colon M \to N$ to the normal map of degree one for which $\xi = (f^{-1})^* TN$.
- The map ∂ sends an element $x \in L_{n+1}(\mathbb{Z}\pi)$ to $f\colon M \to N$ if there exists a normal map $F\colon (W, \partial W) \to (N \times [0,1], N \times \{0,1\})$ of degree one with target $N \times [0,1]$ such that $\partial W = N \amalg M$, $F|_N = \mathrm{id}_N$, $F|_M = f$, and the surgery obstruction of F is x.
- There is a space $\mathrm{G/TOP}$ together with bijections

$$\begin{aligned}
[N, \mathrm{G/TOP}] &\xrightarrow{\cong} \mathcal{N}_n^{\mathrm{top}}(N); \\
[N \times [0,1]/N \times \{0,1\}, \mathrm{G/TOP}] &\xrightarrow{\cong} \mathcal{N}_{n+1}^{\mathrm{top}}(N \times [0,1], N \times \{0,1\}).
\end{aligned}$$

- There is an analog of the Surgery Exact Sequence in the smooth category except that it is only an exact sequence of pointed sets and not of abelian groups.

Corollary

A topological manifold of dimension $n \geq 5$ is topologically rigid if and only if the map $\mathcal{N}_{n+1}^{\mathrm{top}}(N \times [0,1], N \times \{0,1\}) \to L_{n+1}^s(\mathbb{Z}\pi)$ is surjective and the map $\mathcal{N}_n^{\mathrm{top}}(N) \to L_n^s(\mathbb{Z}\pi)$ is injective.

Conjecture (Borel Conjecture)

An aspherical closed manifold is topologically rigid.

- The Surgery Exact Sequence is the main tool for the classification of closed manifolds.
- The proof of the Borel Conjecture for a large class of groups and the classification of exotic spheres are prominent examples.
- For a certain class of fundamental groups called good fundamental groups, the Surgery Exact Sequence works also in dimension 4 by the work of Freedman.
- For more information about surgery theory we refer for instance to [1, 2, 3, 4].

The definition of the even dimensional L-groups

- Let R be a ring with involution. Fix $\epsilon \in \{\pm 1\}$.
- For a finitely generated projective R-module P, let

$$e(P)\colon P \to (P^*)^*$$

be the canonical isomorphism sending $p \in P$ to the element in $(P^*)^*$ given by the homomorphism $P^* \to R,\ f \mapsto \overline{f(p)}$.

Definition (Symmetric form)

An ϵ-symmetric form (P, ϕ) is a finitely generated projective R-module P together with an R-homomorphism $\phi\colon P \to P^*$ such that the composition $P \xrightarrow{e(P)} (P^*)^* \xrightarrow{\phi^*} P^*$ agrees with $\epsilon \cdot \phi$. We call (P, ϕ) non-singular if ϕ is an isomorphism.

- We can rewrite (P, ϕ) as a pairing

$$\lambda \colon P \times P \to R, \quad (p, q) \mapsto \phi(p)(q).$$

- Then the condition that ϕ is R-linear becomes the conditions

$$\begin{array}{rcl} \lambda(p, r_1 \cdot q_1 + r_2 \cdot q_2,) & = & r_1 \cdot \lambda(p, q_1) + r_2 \cdot \lambda(p, q_2); \\ \lambda(r_1 \cdot p_1 + r_2 \cdot p_2, q) & = & \lambda(p_1, q) \cdot \overline{r_1} + \lambda(p_2, q) \cdot \overline{r_2}. \end{array}$$

- The condition $\phi = \epsilon \cdot \phi^* \circ e(P)$ translates to $\lambda(q, p) = \epsilon \cdot \overline{\lambda(p, q)}$.

Example (Standard hyperbolic symmetric form)

- Let P be a finitely generated projective R-module.
- The standard hyperbolic ϵ-symmetric form $H^{\epsilon}(P)$ is given by the R-module $P \oplus P^*$ and the R-isomorphism

$$\phi\colon (P \oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}} P^* \oplus P \xrightarrow{\mathrm{id} \oplus e(P)} P^* \oplus (P^*)^* = (P \oplus P^*)^*.$$

- If we write it as a pairing we obtain

$$(P \oplus P^*) \times (P \oplus P^*) \to R, \quad ((p,f),(p',f')) \mapsto f(p') + \epsilon \cdot f'(p).$$

- Let P be a finitely generated projective R-module
- Define an involution of R-modules

$$T\colon \hom_R(P,P^*) \to \hom(P,P^*), \quad f \mapsto f^* \circ e(P).$$

- Define abelian groups

$$\begin{aligned} Q^\epsilon(P) &:= \ker\left((1-\epsilon\cdot T)\colon \hom_R(P,P^*) \to \hom_R(P,P^*)\right); \\ Q_\epsilon(P) &:= \operatorname{coker}\left((1-\epsilon\cdot T)\colon \hom_R(P,P^*) \to \hom_R(P,P^*)\right). \end{aligned}$$

- Let

$$(1+\epsilon\cdot T)\colon Q_\epsilon(P) \to Q^\epsilon(P)$$

be the homomorphism which sends the class represented by $f\colon P \to P^*$ to the element $f + \epsilon \cdot T(f)$

Definition (Quadratic form)

An ϵ-quadratic form (P, ψ) is a finitely generated projective R-module P together with an element $\psi \in Q_\epsilon(P)$. It is called non-singular if the associated ϵ-symmetric form $(P, (1 + \epsilon \cdot T)(\psi))$ is non-singular, i.e., $(1 + \epsilon \cdot T)(\psi) \colon P \to P^*$ is bijective.

- There is an obvious notion of direct sum of two ϵ-quadratic forms.
- An isomorphism $f \colon (P, \psi) \to (P', \psi')$ of two ϵ-quadratic forms is an R-isomorphism $f \colon P \xrightarrow{\cong} P'$ such that the induced map $Q_\epsilon(f) \colon Q_\epsilon(P') \to Q_\epsilon(P)$ sends ψ' to ψ.

- We can rewrite (P, ψ) as a triple (P, λ, μ) consisting of a pairing

$$\lambda\colon P \times P \to R$$

satisfying

$$\begin{aligned}
\lambda(p, r_1 \cdot q_1 + r_2 \cdot q_2) &= r_1 \cdot \lambda(p, q_1) + r_2 \cdot \lambda(p, q_2);\\
\lambda(r_1 \cdot p_1 + r_2 \cdot p_2, q) &= \lambda(p_1, q) \cdot \overline{r_1} + \lambda(p_2, q) \cdot \overline{r_2};\\
\lambda(q, p) &= \epsilon \cdot \overline{\lambda(p, q)},
\end{aligned}$$

and a map

$$\mu\colon P \to Q_\epsilon(R) = R/\{r - \epsilon \cdot \overline{r} \mid r \in R\}$$

satisfying

$$\begin{aligned}
\mu(rp) &= \rho(r, \mu(p));\\
\mu(p+q) - \mu(p) - \mu(q) &= \mathrm{pr}(\lambda(p, q));\\
\lambda(p, p) &= (1 + \epsilon \cdot T)(\mu(p)),
\end{aligned}$$

where $\mathrm{pr}\colon R \to Q_\epsilon(R)$ is the projection and $(1 + \epsilon \cdot T)\colon Q_\epsilon(R) \to R$ the map sending the class of r to $r + \epsilon \cdot \overline{r}$.

Example (Standard hyperbolic quadratic form)

- Let P be a finitely generated projective R-module.
- The standard hyperbolic ϵ-quadratic form $H_\epsilon(P)$ is given by the $\mathbb{Z}\pi$-module $P \oplus P^*$ and the class in $Q_\epsilon(P \oplus P^*)$ of the R-homomorphism

$$\phi \colon (P \oplus P^*) \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} P^* \oplus P \xrightarrow{\mathrm{id} \oplus e(P)} P^* \oplus (P^*)^* = (P \oplus P^*)^*.$$

- The ϵ-symmetric form associated to $H_\epsilon(P)$ is $H^\epsilon(P)$.
- If we rewrite it as a triple (P, λ, μ), we get

$$\begin{array}{rcll} (P \oplus P^*) \times (P \oplus P^*) & \to & R, & ((p,f),(p',f')) \mapsto f(p') + \epsilon \cdot f'(p); \\ \mu \colon P \oplus P^* & \to & Q_\epsilon(R), & (x,f) \mapsto [f(p)]. \end{array}$$

- We call two non-singular $(-1)^k$-quadratic forms (P, ψ) and (P', ψ') equivalent if and only if there exists a finitely generated projective R-modules Q and Q' and and an isomorphism of non-singular ϵ-quadratic forms

$$(P, \psi) \oplus H_\epsilon(Q) \cong (P', \psi') \oplus H_\epsilon(Q').$$

Definition (Quadratic L-groups in even dimensions)

Define the abelian group $L^p_{2k}(R)$ called the projective $2k$-th quadratic L-group to be the abelian group of equivalence classes $[(P,\psi)]$ of non-singular $(-1)^k$-quadratic forms (P,ψ)

- Addition is given by the sum of two ϵ-quadratic forms. The zero element is represented by $[H_\epsilon(Q]$ for any finitely generated projective R-module Q. The inverse of $[(P,\psi)]$ is given by $[(P,-\psi)]$.
- If one takes P, P', Q and Q' above to be finitely generated free, one obtains the $2k$-th quadratic L-group $L^h_{2k}(R)$.

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