The Stable Cannon Conjecture

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Poincaré duality groups

Definition (Finite Poincaré complex)

A (connected) finite n-dimensional CW-complex X is a finite n-dimensional Poincaré complex if there is $[X] \in H_n(X; \mathbb{Z}^w)$ such that the induced $\mathbb{Z}\pi$-chain map

$$- \cap [X] \colon C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$$

is a simple $\mathbb{Z}\pi$-chain homotopy equivalence.

Theorem (Closed manifolds are Poincaré complexes)

A closed n-dimensional manifold M is a finite n-dimensional Poincaré complex with $w = w_1(X)$.

Definition (Poincaré duality group)

A Poincaré duality group G of dimension n is a finitely presented group satisfying:

- G is of type FP.
- $H^i(G;\mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

Corollary

If M is a closed aspherical manifold of dimension d, then $\pi_1(X)$ is a d-dimensional Poincaré duality group.

Theorem (Wall)

If G is a d-dimensional Poincaré duality group for $d \geq 3$ and $\widetilde{K}_0(\mathbb{Z}G) = 0$, then there is a model for BG which is a finite Poincaré complex of dimension d.

- Recall that the K-theoretic Farrell-Jones Conjecture implies that $K_n(\mathbb{Z}G)$ for $n \leq 1$, $\widetilde{K}_0(\mathbb{Z}G)$, and $\mathrm{Wh}(G)$ vanish for a torsionfree group G.
- Moreover, the Farrell-Jones Conjecture is known to be true for hyperbolic groups and fundamental groups of 3-manifolds.
- In particular we can ignore in the sequel the difference between simple homotopy equivalence and homotopy equivalence.

The main conjectures

Conjecture (Wall)

Every Poincaré duality group is the fundamental group of an aspherical closed manifold.

Theorem (Eckmann-Müller, Linnell)

Every 2-dimensional Poincaré duality group is the fundamental group of a closed surface.

Theorem (Bestvina)

Let G be a hyperbolic 3-dimensional Poincaré duality group. Then its boundary is homeomorphic to S^2.

Theorem (Cannon-Cooper, Eskin-Fisher-Whyte, Kapovich-Leeb)

A Poincaré duality group G of dimension 3 is the fundamental group of an aspherical closed 3-manifold if and only if it is quasi-isometric to the fundamental group of an aspherical closed 3-manifold.

- A closed 3-manifold is a Seifert manifold if it admits a finite covering $\overline{M} \to M$ such that there exists a S^1-principal bundle $S^1 \to \overline{M} \to S$ for some closed orientable surface S.

Theorem (Bowditch)

If a Poincaré duality group of dimension 3 contains an infinite normal cyclic subgroup, then it is the fundamental group of a closed Seifert 3-manifold.

Theorem (Bestvina-Mess)

A torsionfree hyperbolic G is a Poincaré duality group of dimension n if and only if its boundary ∂G and S^{n-1} have the same Čech cohomology.

Theorem (Bartels-Lück-Weinberger)

Let G be a torsion-free group which satisfies the Farrell-Jones Conjecture. Then for $n \geq 5$ the following are equivalent:

- *G is a Poincaré duality group of formal dimension n;*
- *There exists a closed* ANR*-homology manifold M which has (DDP) and satisfies $\pi_1(M) \cong G$;*

- We will deal with ANR-homology manifolds and the question when they are homotopy equivalent to closed manifolds later.

Conjecture (Gromov)

Let G be a torsionfree hyperbolic group whose boundary is a sphere S^{n-1}. Then there is a closed aspherical manifold M with $\pi_1(M) \cong G$.

Theorem (Bartels-Lück-Weinberger)

Gromov's Conjecture is true for $n \geq 5$.

Conjecture (Cannon's Conjecture in the torsionfree case)

A torsionfree hyperbolic group G has S^2 as boundary if and only if it is the fundamental group of a closed hyperbolic 3-manifold.

Theorem (Bestvina-Mess)

Let G be an infinite torsionfree hyperbolic group which is prime, not infinite cyclic, and the fundamental group of a closed 3-manifold M. Then M is hyperbolic and G satisfies the Cannon Conjecture.

- In order to prove the Cannon Conjecture, it suffices to show for a hyperbolic group G, whose boundary is S^2, that it is quasi-isometric to the fundamental group of some closed 3-manifold.

Theorem (Fundamental groups of aspherical oriented closed 3-manifolds)

Let G be the fundamental group of an aspherical oriented closed 3-manifold. Then G satisfies:

- *G is residually finite and Hopfian;*
- *All its L^2-Betti numbers $b_n^{(2)}(G)$ vanish;*
- *Its deficiency is 0. In particular it possesses a presentation with the same number of generators and relations;*
- *Suppose that M is hyperbolic. Then G is virtually compact special and linear over $\mathbb{Z}$. It contains a subgroup of finite index G' which can be written as an extension $1 \to \pi_1(S) \to G \to \mathbb{Z} \to 1$ for some closed orientable surface S.*

- Recall that any finitely presented group occurs as the fundamental group of a closed d-dimensional smooth manifold for every $d \geq 4$.

- The following result illustrates what the strategy of proof for the Cannon Conjecture by experts on 3-manifolds is.

- The boundary ∂G of a hyperbolic group G is metrizable but the metric is not determined by G.

- However, the induced quasi-conformal structure and the induced quasi-Möbius structure associated to some visual metric on ∂G of a hyperbolic group G are canonical, i.e., independent of the choice of a visual metric.

- These structures are quasi-isometry invariants.

- The Ahlfors regular conformal dimension of a metric space Z is the infimal Hausdorff dimension of all Ahlfors regular metric spaces quasi-symmetrically homeomorphic to Z.

Theorem (Bonk-Kleiner)

The Cannon Conjecture is equivalent to the following statement:

If G is a hyperbolic group G with boundary S^2, then the Ahlfors regular conformal dimension of ∂G is attained.

The main results

Theorem (Ferry-Lück-Weinberger, (preprint, 2018), Vanishing of the surgery obstruction)

Let G be a hyperbolic 3-dimensional Poincaré duality group.
Then there is a normal map of degree one (in the sense of surgery theory)

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\overline{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \end{array}$$

satisfying

1. *The space BG is a finite 3-dimensional CW-complex;*
2. *The map $H_n(f,\mathbb{Z})\colon H_n(M;\mathbb{Z}) \xrightarrow{\cong} H_n(BG;\mathbb{Z})$ is bijective for all $n \geq 0$;*
3. *The simple algebraic surgery obstruction $\sigma(f,\overline{f}) \in L_3^s(\mathbb{Z}G)$ vanishes.*

Theorem (Ferry-Lück-Weinberger, (preprint, 2018), Stable Cannon Conjecture)

Let G be a hyperbolic 3-dimensional Poincaré duality group. Let N be any smooth, PL or topological manifold respectively which is closed and whose dimension is ≥ 2.

Then there is a closed smooth, PL or topological manifold M and a normal map of degree one

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\underline{f}} & \xi \times TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \times N \end{array}$$

such that the map f is a simple homotopy equivalence.

Theorem (Stable Cannon Conjecture, continued)

Moreover:

Let $\widehat{M} \to M$ *be the G-covering associated to the composite of the isomorphism* $\pi_1(f)\colon \pi_1(M) \xrightarrow{\cong} G \times \pi_1(N)$ *with the projection* $G \times \pi_1(N) \to G$*. Suppose additionally that N is aspherical and* $\dim(N) \geq 3$*.*

Then $\widehat{M}$ *is homeomorphic to* $\mathbb{R}^3 \times N$*. Moreover, there is a compact topological manifold* $\overline{\widehat{M}}$ *whose interior is homeomorphic to* $\widehat{M}$ *and for which there exists a homeomorphism of pairs*

$$(\overline{\widehat{M}}, \partial\overline{\widehat{M}}) \to (D^3 \times N, S^2 \times N).$$

- The last two theorems follow from the Cannon Conjecture.

- By the product formula for surgery theory and the technique of pulling back the boundary the second last theorem implies the last theorem.

- The manifold M appearing in the last theorem is unique up to homeomorphism by the Borel Conjecture, provided that $\pi_1(N)$ satisfies the Farrell-Jones Conjecture.

- If we take $N = T^k$ for some $k \geq 2$, then the Cannon Conjecture is equivalent to the statement that this M is homeomorphic to $M' \times T^k$ for some closed 3-manifold M'.

The existence of a normal map

Theorem (Existence of a normal map)

Let X be a connected oriented finite 3-dimensional Poincaré complex. Then there are an integer $a \geq 0$ and a vector bundle ξ over BG and a normal map of degree one

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\overline{f}} & \underline{\xi} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

Proof.

- Stable vector bundles over X are classified by the first and second Stiefel-Whitney class $w_1(\xi)$ and $w_2(\xi)$ in $H^*(X;\mathbb{Z}/2)$.

- Let ξ be a k-dimensional vector bundle over X such that $w_1(\xi) = w_1(X)$ and $w_2(\xi) = w_1(\xi) \cup w_1(\xi)$ holds.

- A spectral sequence argument applied to $\Omega_3(X, w_1(X))$ shows that there is a closed 3-manifold M together with a map $f\colon M \to X$ of degree one such that $f^* w_1(X) = w_1(M)$.

- Then $w_1(f^*\xi) = w_1(M)$ and the Wu formula implies $w_2(M) = w_1(f^*\xi) \cup w_1(f^*\xi)$.

- Hence $f^*\xi$ is stably isomorphic to the stable tangent bundle of M and we get the desired normal map.

□

The total surgery obstruction

- Consider an aspherical finite n-dimensional Poincaré complex X such that $G = \pi_1(X)$ is a Farrell-Jones group, i.e., satisfies both the K-theoretic and the L-theoretic Farrell-Jones Conjecture with coefficients in additive categories, and $\mathcal{N}(X)$ is non-empty. (For simplicity we assume $w_1(X) = 0$ in the sequel.)

- We want to find one normal map of degree one

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\overline{f}} & \underline{\xi} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

whose simple surgery obstruction $\sigma^s(f, \overline{f}) \in L_n^s(\mathbb{Z}G)$ vanishes.

- Recall that the simple surgery obstruction defines a map

$$\sigma^s \colon \mathcal{N}(X) \to L_n^s(\mathbb{Z}G).$$

- Fix a normal map $(f_0, \overline{f_0})$.
- Then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{N}(X) & \xrightarrow{\sigma^s(-,-)-\sigma^s(f_0,\overline{f_0})} & L_n^s(\mathbb{Z}G) \\ s_0 \downarrow \cong & & \cong \uparrow \operatorname{asmb}_n^s(X) \\ H_n(X; \mathbf{L}_{\mathbb{Z}}^s\langle 1 \rangle) & \xrightarrow[H_n^G(\mathrm{id}_X;\mathbf{i})]{} & H_n(X; \mathbf{L}_{\mathbb{Z}}^s) \end{array}$$

whose vertical arrows are bijections thanks to the Farrell-Jones Conjecture and the upper arrow sends the class of $(f, \overline{f})$ to the difference $\sigma^s(f, \overline{f}) - \sigma^s(f, \overline{f_0})$ of simple surgery obstructions.

- An easy spectral sequence argument yields a short exact sequence

$$0 \to H_n(X; \mathbf{L}^s_{\mathbb{Z}}\langle 1\rangle) \xrightarrow{H_n(\mathrm{id}_X;\mathbf{i})} H_n(X; \mathbf{L}^s_{\mathbb{Z}}) \xrightarrow{\lambda^s_n(X)} L_0(\mathbb{Z}).$$

- Consider the composite

$$\mu^s_n(X)\colon \mathcal{N}(X) \xrightarrow{\sigma^s} L^s_n(\mathbb{Z}G, w) \xrightarrow{\mathrm{asmb}^s_n(X)^{-1}} H_n(X; \mathbf{L}^s_{\mathbb{Z}}) \xrightarrow{\lambda^s_n(X)} L_0(\mathbb{Z}).$$

- We conclude that there is precisely one element, called the total surgery obstruction,

$$s(X) \in L_0(\mathbb{Z}) \cong \mathbb{Z}$$

such that for any element $[(f, \overline{f})]$ in $\mathcal{N}(X)$ its image under $\mu_n^s(X)$ is $s(X)$.

Theorem (Total surgery obstruction)

- *There exists a normal map of degree one $(f, \overline{f})$ with target X and vanishing simple surgery obstruction $\sigma^s(f, \overline{f}) \in L_n^s(\mathbb{Z}G)$ if and only if $s(X) \in L_0(\mathbb{Z}) \cong \mathbb{Z}$ vanishes.*
- *The total surgery obstruction is a homotopy invariant of X and hence depends only on G.*

Definition (Absolute Neighborhood Retract (ANR))

A topological space X is called an absolute neighborhood retract or briefly an ANR if it is normal and for every normal space Z, every closed subset $Y \subseteq Z$ and every map $f\colon Y \to X$ there exists an open neighborhood U of Y in Z together with an extension $F\colon U \to X$ of f to U.

Definition (Homology ANR-manifold)

A homology ANR-manifold X is an ANR satisfying:

- X has a countable basis for its topology;
- The topological dimension of X is finite;
- X is locally compact;
- for every $x \in X$ we have for the singular homology

$$H_i(X, X - \{x\}; \mathbb{Z}) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$$

If X is additionally compact, it is called a closed ANR-homology manifold.

Definition (Disjoint disk property (DDP))

An ANR homology manifold M has the disjoint disk property (DDP), if for one (and hence any) choice of metric on M, any $\epsilon > 0$ and any maps $f, g \colon D^2 \to M$, there are maps $f', g' \colon D^2 \to M$ so that f' is ϵ-close to f, g' is ϵ-close to g and $f'(D^2) \cap g'(D^2) = \emptyset$,

- Every closed topological manifold is a closed ANR-homology manifold having (DDP).
- Let M be homology sphere with non-trivial fundamental group. Then its suspension ΣM is a closed ANR-homology manifold but not a topological manifold.

Quinn's resolution obstruction

Theorem (Quinn (1987))

There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ for homology ANR*-manifolds with the following properties:*

- *if $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;*
- *$i(M \times N) = i(M) \cdot i(N)$;*
- *Let M be a homology* ANR*-manifold of dimension ≥ 5 having (DDP). Then M is a topological manifold if and only if $\iota(M) = 1$;*
- *The Quinn obstruction and the total surgery obstruction are related for an aspherical closed* ANR*-homology manifold M of dimension ≥ 5 by*

  $$\iota(M) = 8 \cdot s(X) + 1,$$

  if $\pi_1(M)$ is a Farrell-Jones group.

Proof of the Theorem about the vanishing of the surgery obstruction

Proof.

- We have to show for the aspherical finite 3-dimensional Poincaré complex X that its total surgery obstruction vanishes.
- The total surgery obstruction satisfies a product formula

$$8 \cdot s(X \times Y) + 1 = (8 \cdot s(X) + 1) \cdot (8 \cdot s(Y) + 1).$$

- This implies

$$s(X \times T^3) = s(X).$$

- Hence it suffices to show that $s(X \times T^3)$ vanishes.

Proof (continued).

- There exists an aspherical closed ANR-homology manifold M having (DDP) and a homotopy equivalence $f\colon M \to X \times T^3$.
- This follows from the surgery exact sequence of Bryant-Ferry-Mio-Weinberger for ANR-homology manifolds

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M;\mathbf{L}_{\mathbb{Z}}) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}} \mathcal{S}^{\mathrm{ANR}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}_{\mathbb{Z}}) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

by similar arguments as they were presented in the proof that the Farrell-Jones Conjecture implies the Borel Conjecture.

□

Proof (continued).

- There is a Z-compactification $\overline{\widetilde{X}}$ of $\widetilde{X}$ by the boundary $\partial G = S^2$.
- One constructs an appropriate Z-compactification $\overline{\widetilde{M}}$ of $\widetilde{M}$ so that we get a ANR-homology manifold $\overline{\widetilde{M}}$ whose boundary is a topological manifold and whose interior is $\widetilde{M}$. This is based on the technique pulling back the boundary.
- By adding a collar to $\overline{\widetilde{M}}$ one obtains a ANR-homology manifold Y which contains $\widetilde{M}$ as an open subset and contains an open subset U which is homeomorphic to $\mathbb{R}^6$.

□

Proof (continued).

- Hence we get

$$\begin{aligned} 8s(X \times T^3) + 1 = 8s(M) + 1 = i(M) = i(\widetilde{M}) \\ = i(Y) = i(U) = i(\mathbb{R}^6) = 1. \end{aligned}$$

- This implies $s(X \times T^3) = 0$ and hence $s(X) = 0$.

□

Appendix: Pulling back the boundary

Notation

Let $(\overline{Y}, Y)$ be a topological pair. Put $\partial Y := \overline{Y} \setminus Y$. Let X be a topological space and $f\colon X \to Y$ be a continuous map. Pulling back the boundary is a construction of a topological pair $(\overline{X}, X)$ and a continuous map $\overline{f}\colon \overline{X} \to X$

- It has the desired universal property which we will not state here.
- Its basic properties are:

Lemma

- *$Y \subseteq \overline{Y}$ is dense and the closure of the image of f in $\overline{Y}$ contains ∂Y, then $X \subseteq \overline{X}$ is dense;*
- *$\overline{Y}$ is compact, $Y \subseteq \overline{Y}$ is open and $f\colon X \to Y$ is proper. Then $\overline{X}$ is compact;*
- *We have for the topological dimension of $\overline{X}$*

$$\dim(\overline{X}) \leq \dim(X) + \dim(\overline{Y}) + 1;$$

- *The induced map $\overline{f}$ induces a homeomorphism $\partial f\colon \partial X \to \partial Y$.*

Definition (Z-set)

A closed subset Z of a compact ANR X is called a Z-set if for every open subset U of X the inclusion $U \setminus (U \cap Z) \to U$ is a homotopy equivalence.

- The boundary of a manifold is a Z-set in the manifold.

Lemma

Consider a pair $(\overline{Y}, Y)$ of spaces such that $\overline{Y}$ is an ANR *and ∂Y is a Z-set in $\overline{Y}$. Consider a homotopy equivalence $f\colon X \to Y$ which is continuously controlled. Let $(\overline{f}, f)\colon (\overline{X}, X) \to (\overline{Y}, Y)$ be obtained by pulling back the boundary along f.*

Then $\overline{X}$ is an ANR *and $\partial X \subseteq \overline{X}$ is a Z-set.*