Introduction to 3-manifolds

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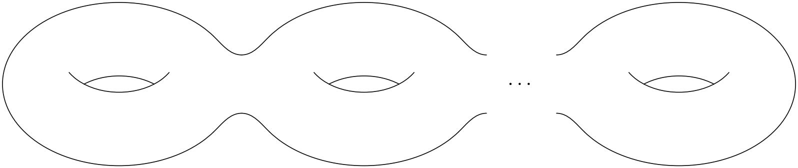
Outline

- We give an introduction and survey about 3-manifolds.
- We cover the following topics:
  - Review of surfaces
  - Prime decomposition and the Kneser Conjecture
  - Jaco-Shalen-Johannsen splitting
  - Thurston's Geometrization Conjecture
  - Fibering 3-manifolds
  - Fundamental groups of 3-manifolds

Some basic facts surfaces

- Surface will mean compact, connected, orientable 2-dimensional manifold possibly with boundary.



- Every surface has a preferred structure of a PL-manifold or smooth manifold which is unique up to PL-homeomorphism or diffeomorphism.
- Every surface is homeomorphic to the standard model F_g^d, which is obtained from S^2 by deleting the interior of d embedded D^2 and taking the connected sum with g-copies of $S^1 \times S^1$.
- The standard models F_g^d and $F_{g'}^{d'}$ are homeomorphic if and only if $g = g'$ and $d = d'$ holds.
- Any homotopy equivalence of closed surfaces is homotopic to a homeomorphism.

- The following assertions for two closed surfaces M and N are equivalent:
  - M and N are homeomorphic;
  - $\pi_1(M) \cong \pi_1(N)$;
  - $H_1(M) \cong H_1(N)$;
  - $\chi(M) = \chi(N)$.

- A closed surface admits a complete Riemannian metric with constant sectional curvature 1, 0 or -1 depending on whether its genus g is 0,1 or ≥ 2. For -1 there are infinitely many such structures on a given surface of genus ≥ 2.

- A closed surface is either simply connected or aspherical.

- A simply connected closed surface is homeomorphic to S^2.

- A closed surface carries a non-trivial S^1-action if and only if it is S^2 or T^2.

- The fundamental group of a compact surface F_g^d is explicitly known.
- The fundamental group of a compact surface F_g^d has the following properties
  - It is either trivial, $\mathbb{Z}^2$, a finitely generated one-relator group, or a finitely generated free group;
  - It is residually finite;
  - Its abelianization is a finitely generated free abelian group;
  - It has a solvable word problem, conjugacy problem and isomorphism problem.

Question

Which of these properties carry over to 3-manifolds?

Unique smooth or PL-structures on 3-manifolds

- 3-manifold will mean compact, connected, orientable 3-dimensional manifold possibly with boundary. We also exclude $I \times F$ for some surface F.
- Every 3-manifold has a preferred structure of a PL-manifold or smooth manifold which is unique up to PL-homeomorphism or diffeomorphism.
- This is not true in general for closed manifolds of dimension ≥ 4.

Prime decomposition and the Kneser Conjecture

Definition (Prime)

A 3-manifold M is called prime if for any decomposition as a connected sum $M_0 \sharp M_1$ one of the summands M_0 or M_1 is homeomorphic to S^3.

Theorem (Prime decomposition)

Every 3-manifold M, which is not homeomorphic to S^3, possesses a prime decomposition

$$M \cong M_1 \sharp M_2 \sharp \cdots \sharp M_r$$

where each M_i is prime and not homeomorphic to S^3. This decomposition is unique up to permutation of the summands and homeomorphism.

Definition (Incompressible)

Given a 3-manifold M, a compact connected orientable surface F which is properly embedded in M, i.e., $\partial M \cap F = \partial F$, or embedded in ∂M, is called incompressible if the following holds:

- The inclusion $F \to M$ induces an injection on the fundamental groups;
- F is not homeomorphic to S^2;
- If $F = D^2$, we do not have $F \subseteq \partial M$ and there is no embedded $D^3 \subseteq M$ with $\partial D^3 \subseteq D^2 \cup \partial M$.

One says that ∂M is incompressible in M if and only if ∂M is empty or any component C of ∂M is incompressible in the sense above.

- $\partial M \subseteq M$ is incompressible if for every component C the inclusion induces an injection $\pi_1(C) \to \pi_1(M)$ and C is not homeomorphic to S^2.

Theorem (The Kneser Conjecture is true)

Let M be a compact 3-manifold with incompressible boundary. Suppose that there are groups G_0 and G_1 together with an isomorphism $\alpha\colon G_0 * G_1 \xrightarrow{\cong} \pi_1(M)$.

Then there are 3-manifolds M_0 and M_1 coming with isomorphisms $u_i\colon G_i \xrightarrow{\cong} \pi_1(M_i)$ and a homeomorphism

$$h\colon M_0 \sharp M_1 \xrightarrow{\cong} M$$

such that the following diagram of group isomorphisms commutes up to inner automorphisms

$$\begin{array}{ccc} \pi_1(M_0) * \pi_1(M_1) & \xrightarrow{\cong} & \pi_1(M_0 \sharp M_1) \\ \uparrow{\scriptstyle u_0 * u_1} & & \downarrow{\scriptstyle \pi_1(h)} \\ G_0 * G_1 & \xrightarrow[\alpha]{\cong} & \pi_1(M) \end{array}$$

Definition (Irreducible)

A 3-manifold is called irreducible if every embedded two-sphere $S^2 \subseteq M$ bounds an embedded disk $D^3 \subseteq M$.

Theorem (Prime versus irreducible)

A prime 3-manifold M is either homeomorphic to $S^1 \times S^2$ or is irreducible.

Theorem (Knot complement)

The complement of a non-trivial knot in S^3 is an irreducible 3-manifold with incompressible toroidal boundary.

Theorem (Aspherical irreducible manifolds)

An irreducible 3-manifold is aspherical if and only if it is homeomorphic to D^3 or its fundamental group is infinite.

Haken manifolds

Definition (Haken manifold)

An irreducible 3-manifold is Haken if it contains an incompressible embedded surface.

Lemma

If the first Betti number $b_1(M)$ is non-zero, which is implied if ∂M contains a surface other than S^2, and M is irreducible, then M is Haken.

- A lot of conjectures for 3-manifolds could be proved for Haken manifolds first using an inductive procedure which is based on cutting a Haken manifold into pieces of smaller complexity using the incompressible surface.

Conjecture (Waldhausen's Virtually Haken Conjecture)

Every irreducible 3-manifold with infinite fundamental group has a finite covering which is a Haken manifold.

Theorem (Agol, [1])

The Virtually Haken Conjecture is true.

- Agol shows that there is a finite covering with non-trivial first Betti number.

Seifert and hyperbolic 3-manifolds

- We use the definition of a Seifert manifold given in the survey article by Scott [8], which we recommend as a reference on Seifert manifolds besides the book of Hempel [4].

Lemma

If a 3-manifold M has infinite fundamental group and empty or incompressible boundary, then it is Seifert if and only if it admits a finite covering $\overline{M}$ which is the total space of a S^1-principal bundle over a compact orientable surface.

Theorem (Gabai [3])

An irreducible 3-manifold M with infinite fundamental group π is Seifert if and only if π contains a normal infinite cyclic subgroup.

Definition (Hyperbolic)

A compact manifold (possible with boundary) is called hyperbolic if its interior admits a complete Riemannian metric whose sectional curvature is constant -1.

Lemma

Let M be a hyperbolic 3-manifold. Then its interior has finite volume if and only if ∂M is empty or a disjoint union of incompressible tori.

Geometries

Definition (Geometry)

A geometry on a 3-manifold M is a complete locally homogeneous Riemannian metric on its interior.

- Locally homogeneous means that for any two points there exist open neighbourhoods which are isometrically diffeomorphic.
- The universal cover of the interior has a complete homogeneous Riemannian metric, meaning that the isometry group acts transitively. This action is automatically proper.
- Thurston has shown that there are precisely eight maximal simply connected 3-dimensional geometries having compact quotients, which often come from left invariant Riemannian metrics on connected Lie groups.

- S^3, $\mathrm{Isom}(S^3) = O(4)$;
- $\mathbb{R}^3$, $1 \to \mathbb{R}^3 \to \mathrm{Isom}(\mathbb{R}^3) \to O(3) \to 1$;
- $S^2 \times \mathbb{R}$, $\mathrm{Isom}(S^2 \times \mathbb{R}) = \mathrm{Isom}(S^2) \times \mathrm{Isom}(\mathbb{R})$;
- $\mathbb{H}^2 \times \mathbb{R}$, $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{R})$;
- $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, $1 \to \mathbb{R} \to \mathrm{Isom}(\widetilde{\mathrm{SL}_2(\mathbb{R})}) \to \mathrm{PSL}_2(\mathbb{R}) \to 1$;
- $\mathrm{Nil} := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$, $1 \to \mathbb{R} \to \mathrm{Isom}(\mathrm{Nil}) \to \mathrm{Isom}(\mathbb{R}^2) \to 1$;
- $\mathrm{Sol} := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$; $1 \to \mathrm{Sol} \to \mathrm{Isom}(\mathrm{Sol}) \to D_{2 \cdot 4} \to 1$;
- $\mathbb{H}^3$, $\mathrm{Isom}(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$.

- A geometry on a 3-manifold M modelled on S^3, IR^3 or $\mathbb{H}^3$ is the same as a complete Riemannian metric on the interior of constant section curvature with value 1, 0 or -1.
- If a closed 3-manifold admits a geometric structure modelled on one of these eight geometries, then the geometry involved is unique.
- The geometric structure on a fixed 3-manifold is in general not unique. For instance, one can scale the standard flat Riemannian metric on the torus T^3 by a real number and just gets a new geometry with different volume which of course still is a $\mathbb{R}^3$-geometry.

Theorem (Mostow Rigidity)

Let M and N be two hyperbolic n-manifolds with finite volume for $n \geq 3$. Then for any isomorphism $\alpha\colon \pi_1(M) \xrightarrow{\cong} \pi_1(N)$ there exists an isometric diffeomorphism $f\colon M \to N$ such that up to inner automorphism $\pi_1(f) = \alpha$ holds.

- This is not true in dimension 2, see Teichmüller space.

- A 3-manifold is a Seifert manifold if and only if it carries one of the geometries $S^2 \times \mathbb{R}$, $\mathbb{R}^3$, $H^2 \times \mathbb{R}$, S^3, Nil, or $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. In terms of the Euler class e of the Seifert bundle and the Euler characteristic χ of the base orbifold, the geometric structure of a closed Seifert manifold M is determined as follows

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbb{R}$	$\mathbb{R}^3$	$H^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil	$\widetilde{\mathrm{SL}_2(\mathbb{R})}$

The JSJ-splitting

Theorem (Jaco-Shalen [5], Johannson [6])

Let M be an irreducible 3-manifold M with incompressible boundary.

1. *There is a finite family of disjoint, pairwise-nonisotopic incompressible tori in M which are not isotopic to boundary components and which split M into pieces that are Seifert manifolds or are geometrically atoroidal, i.e., they admit no embedded incompressible torus (except possibly parallel to the boundary).*
2. *A minimal family of such tori is unique up to isotopy.*

Definition (Toral splitting or JSJ-decomposition)

We will say that the minimal family of such tori gives a toral splitting or a JSJ-decomposition.

We call the toral splitting a geometric toral splitting if the geometrically atoroidal pieces which do not admit a Seifert structure are hyperbolic.

Thurston's Geometrization Conjecture

Conjecture (Thurston's Geometrization Conjecture)

- *An irreducible 3-manifold with infinite fundamental group has a geometric toral splitting;*
- *For a closed 3-manifold with finite fundamental group, its universal covering is homeomorphic to S^3, the fundamental group of M is a subgroup of $SO(4)$ and the action of it on the universal covering is conjugated by a homeomorphism to the restriction of the obvious $SO(4)$-action on S^3.*

Theorem (Perelmann, see Morgan-Tian [7])

Thurston's Geometrization Conjecture is true.

- Thurston's Geometrization Conjecture implies the 3-dimensional Poincaré Conjecture.
- Thurston's Geometrization Conjecture implies:
  - The fundamental group of a 3-manifold M is residually finite, Hopfian and has a solvable word, conjugacy and membership problem.
  - If M is closed, $\pi_1(M)$ has a solvable isomorphism problem.
  - Every closed 3-manifold has a solvable homeomorphism problem.
- Thanks to the proof of the Geometrization Conjecture, there is a complete list of those finite groups which occur as fundamental groups of closed 3-manifolds. They all are subgroups of $SO(4)$.
- Recall that, for every $n \geq 4$ and any finitely presented group G, there exists a closed n-dimensional smooth manifold M with $\pi_1(M) \cong G$.

- Thurston's Geometrization Conjecture implies the Borel Conjecture in dimension 3 stating that every homotopy equivalence of aspherical closed 3-manifolds is homotopic to a homeomorphism.

- There are irreducible 3-manifolds with finite fundamental group which are homotopy equivalent but not homeomorphic, namely the lens spaces $L(7;1,1)$ and $L(7;1,2)$.

Fibering

Theorem (Stallings [9])

The following assertions are equivalent for an irreducible 3*-manifold* M *and an exact sequence* $1 \to K \to \pi_1(M) \to \mathbb{Z} \to 1$*:*

- K *is finitely generated;*
- K *is the fundamental group of a surface* F*;*
- *There is a locally trivial fiber bundle* $F \to M \to S^1$ *with a surface* F *as fiber such that the induced sequence*

  $$1 \to \pi_1(F) \to \pi_1(E) \to \pi_1(S^1) \to 1$$

  can be identified with the given sequence.

Conjecture (Thurston's Virtual Fibering Conjecture)

Let M be a closed hyperbolic 3-manifold. Then a finite covering of M fibers over S^1, i.e., is the total space of a surface bundle over S^1.

- A locally compact surface bundle $F \to E \to S^1$ is the same as a selfhomeomorphism of the surface F by the mapping torus construction.
- Two surface homeomorphisms are isotopic if and only if they induce the same automorphism on $\pi_1(F)$ up to inner automorphisms.
- Therefore mapping class groups play an important role for 3-manifolds.

Theorem (Agol [1])

The Virtually Fibering Conjecture is true.

Definition (Graph manifold)

An irreducible 3-manifold is called graph manifold if its JSJ-splitting contains no hyperbolic pieces.

- There are aspherical closed graph manifolds which do not virtually fiber over S^1.
- There are closed graph manifolds, which are aspherical, but do not admit a Riemannian metric of non-positive sectional curvature.
- Agol proved the Virtually Fibering Conjecture for any irreducible manifold with infinite fundamental group and empty or incompressible toral boundary which is not a closed graph manifold.

- Actually, Agol, based on work of Wise, showed much more, namely that the fundamental group of a hyperbolic 3-manifold is virtually compact special.
- This implies in particular that they occur as subgroups of RAAG-s (right Artin angled groups) and that they are linear over $\mathbb{Z}$ and LERF (locally extended residually finite).
- For the definition of these notions and much more information we refer for instance to Aschenbrenner-Friedl-Wilton [2].

On the fundamental groups of 3-manifolds

- The fundamental group plays a dominant role for 3-manifolds what we want to illustrate by many examples and theorems.
- A 3-manifold is prime if and only if $\pi_1(M)$ is prime in the sense that $\pi_1(M) \cong G_0 * G_1$ implies that G_0 or G_1 are trivial.
- A 3-manifold is irreducible if and only if $\pi_1(M)$ is prime and $\pi_1(M)$ is not infinite cyclic.
- A 3-manifold is aspherical if and only if its fundamental group is infinite, prime and not cyclic.
- A 3-manifold has infinite cyclic fundamental group if and only if it is homeomorphic to $S^1 \times S^2$.

- Let M and N be two prime closed 3-manifolds whose fundamental groups are infinite. Then:
  - M and N are homeomorphic if and only if $\pi_1(M)$ and $\pi_1(N)$ are isomorphic.
  - Any isomorphism $\pi_1(M) \xrightarrow{\cong} \pi_1(N)$ is induced by a homeomorphism.
- Let M be a closed irreducible 3-manifold with infinite fundamental group. Then M is hyperbolic if and only if $\pi_1(M)$ does not contain $\mathbb{Z} \oplus \mathbb{Z}$ as subgroup.
- Let M be a closed irreducible 3-manifold with infinite fundamental group. Then M is a Seifert manifold if and only if $\pi_1(M)$ contains a normal infinite cyclic subgroup.

- A closed Seifert 3-manifold carries precisely one geometry and one can read off from $\pi_1(M)$ which one it is:
  - S^3
    $\pi_1(M)$ is finite.
  - $\mathbb{R}^3$
    $\pi_1(M)$ contains $\mathbb{Z}^3$ as subgroup of finite index.
  - $S^2 \times \mathbb{R}$
    $\pi_1(M)$ is virtually cyclic.
  - $\mathbb{H}^2 \times \mathbb{R}$
    $\pi_1(M)$ contains a subgroup of finite index which is isomorphic to $\mathbb{Z} \times \pi_1(F)$ for some closed surface F of genus 2.
  - $\widetilde{SL_2(\mathbb{R})}$
    $\pi_1(M)$ contains a subgroup of finite index G which can be written as a non-trivial central extension $1 \to \mathbb{Z} \to G \to \pi_1(F) \to 1$ for a surface F of genus ≥ 2.
  - Nil
    $\pi_1(M)$ contains a subgroup of finite index G which can be written as a non- trivial central extension $1 \to \mathbb{Z} \to G \to \mathbb{Z}^2 \to 1$.

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