L^2-invariants

Wolfgang Lück
Bonn
Germany
email wolfgang.lueck@him.uni-bonn.de
http://www.him.uni-bonn.de/lueck/

Matrix Inst., Creswick, Australia

Review of classical L^2-invariants

- Let $G \to \overline{X} \to X$ be a G-covering of a connected finite CW-complex X.
- The cellular chain complex of $\overline{X}$ is a finitely generated free $\mathbb{Z}G$-chain complex:

$$\dots \xrightarrow{c_{n-1}} \bigoplus_{I_n} \mathbb{Z}G \xrightarrow{c_n} \bigoplus_{i_{n-1}} \mathbb{Z}G \xrightarrow{c_{n-1}} \dots$$

- The associated L^2-chain complex

$$C_*^{(2)}(\overline{X}) := L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X})$$

has Hilbert spaces with isometric linear G-action as chain modules and bounded G-equivariant operators as differentials

$$\dots \xrightarrow{c_{n-1}^{(2)}} \bigoplus_{I_n} L^2(G) \xrightarrow{c_n^{(2)}} \bigoplus_{i_{n-1}} L^2(G) \xrightarrow{c_{n-1}^{(2)}} \dots$$

Definition (L^2-homology and L^2-Betti numbers)

- Define the *n-th L^2-homology* to be the Hilbert space

$$H_n^{(2)}(\overline{X}) := \ker(c_n^{(2)})/\overline{\operatorname{im}(c_{n+1}^{(2)})}.$$

- Define the *n-th L^2-Betti number*

$$b_n^{(2)}(\overline{X}) := \dim_{\mathcal{N}(G)}\left(H_n^{(2)}(\overline{X})\right) \quad \in \mathbb{R}^{\geq 0},$$

where $\dim_{\mathcal{N}(G)}$ is the Murray-von Neumann dimension.

- The original notion is due to *Atiyah* [2] and was motivated by index theory. He defined for a G-covering $\overline{M} \to M$ of a closed Riemannian manifold

$$b_n^{(2)}(\overline{M}) := \lim_{t \to \infty} \int_{\mathcal{F}} tr\left(e^{-t \cdot \overline{\Delta}_n}(\overline{x}, \overline{x})\right) d\mathrm{vol}_{\overline{M}}.$$

- If G is finite, we have

$$b_n^{(2)}(\overline{X}) = \frac{1}{|G|} \cdot b_n(\overline{X}).$$

- If $G = \mathbb{Z}$, we have

$$b_n^{(2)}(\overline{X}) = \dim_{\mathbb{C}[\mathbb{Z}]_{(0)}}\left(\mathbb{C}[\mathbb{Z}]_{(0)} \otimes_{\mathbb{C}[\mathbb{Z}]} H_n(\overline{X}; \mathbb{C})\right) \quad \in \mathbb{Z}.$$

- In the sequel 3-manifold means a prime connected compact orientable 3-manifold with infinite fundamental group whose boundary is empty or a union of tori and which is not $S^1 \times D^2$ or $S^1 \times S^2$.

Theorem (Lott-Lück [17])

For every 3-manifold M all L^2-Betti numbers $b_n^{(2)}(\widetilde{M})$ vanish.

- We are interested in the case where all L^2-Betti numbers vanish, since then a very powerful secondary invariant comes into play, the so called L^2-torsion.

- L^2-torsion can be defined analytical in terms of the spectrum of the Laplace operator, generalizing the notion of analytic Ray-Singer torsion. It can also be defined in terms of the cellular $\mathbb{Z}G$-chain complex, generalizing of the Reidemeister torsion.

Definition (L^2-torsion)

Suppose that $\overline{X}$ is L^2-acyclic, i.e., all L^2-Betti numbers $b_n^{(2)}(\overline{X})$ vanish. Let $\Delta_n^{(2)} \colon C_n^{(2)}(\overline{X}) \to C_n^{(2)}(\overline{X})$ be the n-Laplace operator given by $c_{n+1}^{(2)} \circ \left(c_n^{(2)}\right)^* + \left(c_{n-1}^{(2)}\right)^* \circ c_n^{(2)}$.

Define the L^2-torsion

$$\rho^{(2)}(\overline{X}) := \sum_{n \geq 0} (-1)^n \cdot n \cdot \ln\left(\det^{(2)}(\Delta_n^{(2)})\right) \in \mathbb{R}.$$

where $\det^{(2)}$ is the Fuglede-Kadison determinant.

Theorem (Lück-Schick [22])

Let M be a 3-manifold. Let $M_1, M_2, \ldots, M_m$ be the hyperbolic pieces in its Jaco-Shalen decomposition.

Then

$$\rho^{(2)}(\widetilde{M}) := -\frac{1}{3\pi} \cdot \sum_{i=1}^{m} \mathrm{vol}(M_i).$$

- The L^2-torsion is a homotopy invariant and depends only on $\pi_1(M)$ in the theorem above. This is not at all obvious for the sum of the volumes of the hyperbolic pieces.

- What we have presented so far is the very beginning of the long success story of L^2-invariants in various areas of mathematics such as topology, differential geometry, geometric and measured group theory, algebra, and operator theory. We do not have the time to go along this road at this occasion.

Universal L^2-torsion

Definition ($K_1^w(\mathbb{Z}G)$)

Let $K_1^w(\mathbb{Z}G)$ be the abelian group given by:

- generators

  If $f\colon \mathbb{Z}G^m \to \mathbb{Z}G^m$ is a $\mathbb{Z}G$-map such that the induced bounded G-equivariant $L^2(G)^m \to L^2(G)^m$ map is a weak isomorphism, i.e., the dimensions of its kernel and cokernel are trivial, then it determines a generator $[f]$ in $K_1^w(\mathbb{Z}G)$.

- relations

$$\begin{aligned} \left[\begin{pmatrix} f_1 & * \\ 0 & f_2 \end{pmatrix}\right] &= [f_1] + [f_2]; \\ [g \circ f] &= [f] + [g]. \end{aligned}$$

Define $\mathrm{Wh}^w(G) := K_1^w(\mathbb{Z}G)/\{\pm g \mid g \in G\}$.

Definition (Universal L^2-torsion)

Let $G \to \overline{X} \to X$ be a G-covering of a finite CW-complex. Suppose that $\overline{X}$ is L^2-acyclic, i.e., $b_n^{(2)}(\overline{X})$ vanishes for all $n \in \mathbb{Z}$.

Then its universal L^2-torsion is defined to be an element

$$\rho_u^{(2)}(\overline{X}) \in K_1^w(\mathbb{Z}G).$$

- The universal L^2-torsion is defined by the same expression as the L^2-torsion, but now using the fact that the combinatorial Laplace operator can be thought of as an element in $K_1^w(\mathbb{Z}[G])$, namely by

$$\rho_u^{(2)}(\overline{X}) := \sum_{n \geq 0} (-1)^n \cdot n \cdot [\Delta_n^c] \quad \in K_1^w(\mathbb{Z}G).$$

for $\Delta_n^c := c_{n+1} \circ c_n^* + c_{n-1}^* \circ c_n$.

- The universal L^2-torsion is a simple homotopy invariant.
- It satisfies useful sum formulas and product formulas. There are also formulas for appropriate fibrations and S^1-actions.
- If G is finite, we rediscover essentially the classical Reidemeister torsion.

- Many other invariants come from the universal L^2-torsion by applying a homomorphism $K_1^w(\mathbb{Z}G) \to A$ of abelian groups.

- For instance, the Fuglede-Kadison determinant defines a homomorphism
$$\det^{(2)}\colon \operatorname{Wh}^w(\mathbb{Z}G) \to \mathbb{R}$$
which maps the universal L^2-torsion $\rho_u^{(2)}(\overline{X})$ to the (classical) L^2-torsion $\rho^{(2)}(\overline{X})$.

- In this talk we want to illustrate the surprizing fact that a prominent invariant for 3-manifolds can be rediscovered from the universal L^2-torsion, namely the Thurston norm and the Turston polytope.

The fundamental square and the Atiyah Conjecture

- The fundamental square is given by the following inclusions of rings

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G) \end{array}$$

- $\mathcal{N}(G)$ is the group von Neumann algebra $\mathcal{B}(L^2(G))^G$.
- $\mathcal{U}(G)$ is the algebra of affiliated operators. Algebraically it is just the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is a unit in $\mathcal{U}(G)$, is already a unit in $\mathcal{D}(G)$ itself.

- If G is finite, it is given by

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathbb{C}G \\ \downarrow & & \downarrow \mathrm{id} \\ \mathbb{Q}G & \longrightarrow & \mathbb{C}G \end{array}$$

- If $G = \mathbb{Z}$, it is given by

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}] & \longrightarrow & L^\infty(S^1) \\ \downarrow & & \downarrow \\ \mathbb{Q}[\mathbb{Z}]_{(0)} & \longrightarrow & L(S^1) \end{array}$$

- If G is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.
- In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.

Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- Fix a natural number $d \geq 5$. Then a finitely generated torsionfree group G satisfies the Atiyah Conjecture if and only if for any G-covering $\overline{M} \to M$ of a closed Riemannian manifold of dimension d we have $b_n^{(2)}(\overline{M}) \in \mathbb{Z}$ for every $n \geq 0$.
- The Atiyah Conjecture implies for instance that the complex group ring of a torsionfree group embeds into a skewfield and in particular has no zero-divisors besides 0 and 1.

Theorem (Linnell [15], Schick [23])

1. *Let $\mathcal{C}$ be the smallest class of groups, which contains all free groups, and which is closed under extensions with elementary amenable groups as quotients and directed unions.*
   Then every torsionfree group G which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture.

2. *If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.*

- This theorem implies that the fundamental group of a 3-manifold (with the exception of some graph manifolds) satisfies the Atiyah Conjecture.

Identifying $K_1^w(\mathbb{Z}G)$ and $K_1(\mathcal{D}(G))$

Theorem (Linnell-Lück [14])

If G belongs to $\mathcal{C}$, then the natural map

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G))$$

is an isomorphism.

- Its proof is based on identifying $\mathcal{D}(G)$ as an appropriate Cohn localization of $\mathbb{Z}G$ and the investigating localization sequences in algebraic K-theory.
- There is a Dieudonné determinant which induces an isomorphism

$$\det_D \colon K_1(\mathcal{D}(G)) \xrightarrow{\cong} \left(\mathcal{D}(G)^\times\right)_{\text{abel}}$$

- In particular we get for $G = \mathbb{Z}$

$$K_1^w(\mathbb{Z}[\mathbb{Z}]) \cong \mathbb{Q}[\mathbb{Z}]_{(0)} \setminus \{0\}.$$

- It turns out that then the universal torsion is the same as the Alexander polynomial of an infinite cyclic covering, as it occurs for instance in knot theory.

- Hence already for the rather simple group $G = \mathbb{Z}$ we get a very rich invariant.

- In some sense all these new invariants are designed to extend classical invariants for abelian coverings to the universal coverings, or, equivalently, to pass from abelian groups to arbitrary groups.

Thurston norm

Definition (Thurston norm)

Let M be a 3-manifold and $\phi \in H^1(M;\mathbb{Z})$ be a class. Define its Thurston norm

$$x_M(\phi) = \min\{\chi_-(F) \mid F \text{ embedded surface in } M \text{ dual to } \phi\}$$

where

$$\chi_-(F) = \sum_{C \in \pi_0(M)} \max\{-\chi(C), 0\}.$$

- Thurston showed that this definition extends to the real vector space $H^1(M;\mathbb{R})$ and defines a seminorm on it.

- If $F \to M \xrightarrow{p} S^1$ is a fiber bundle with connected closed surface $F \ncong S^2$ and $\phi = \pi_1(p)$, then

$$x_M(\phi) = -\chi(F).$$

- Let $K \subset S^3$ be a knot with knot complement C_K and genus $g(K)$. Then for any generator ϕ of $H^1(X_K;\mathbb{Z}) \cong \mathbb{Z}$ we have

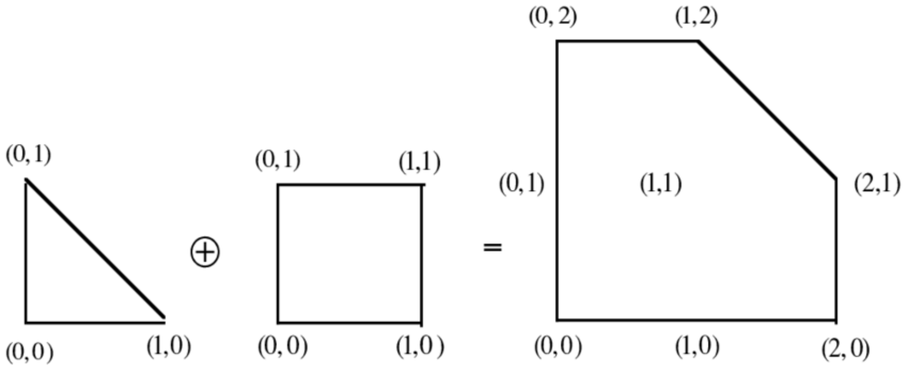
$$x_{C_K}(\phi) \;=\; \max\{2g(K)-1, 0\}.$$

Polytopes

- Consider a finitely generated free abelian group A. Let $A_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} A$ be the real vector space containing A as a spanning lattice;
- A polytope $P \subseteq A_{\mathbb{R}}$ is a convex bounded subset which is the convex hull of a finite subset S;
- It is called integral, if S is contained in A;
- The Minkowski sum of two polytopes P and Q is defined by

$$P + Q = \{p + q \mid p \in P, q \in Q\};$$

- It is cancellative, i.e., it satisfies $P_0 + Q = P_1 + Q \implies P_0 = P_1$;


(0,1)
(0,0)
(1,0)
⊕
(0,1)
(1,1)
(0,0)
(1,0)
=
(0,2)
(1,2)
(0,1)
(1,1)
(2,1)
(0,0)
(1,0)
(2,0)

- The Newton polytope

$$N(p) \subseteq \mathbb{R}^n$$

of a polynomial

$$p(t_1, t_2, \ldots, t_n) = \sum_{i_1,\ldots,i_n} a_{i_1,i_2,\ldots,i_n} \cdot t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$$

in n variables $t_1, t_2, \ldots, t_n$ is defined to be the convex hull of the elements $\{(i_1, i_2, \ldots i_n) \in \mathbb{Z}^n \mid a_{i_1,i_2,\ldots,i_n} \neq 0\}$;

- One has

$$N(p \cdot q) = N(p) + N(q).$$

Definition (Polytope group)

- Let $\mathcal{P}_{\mathbb{Z}}(A)$ be the Grothendieck group of the abelian monoid of integral polytopes in $A_{\mathbb{R}}$.
- Denote by $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$ the quotient of $\mathcal{P}_{\mathbb{Z}}(A)$ by the canonical homomorphism $A \to \mathcal{P}_{\mathbb{Z}}(A)$ sending a to the class of the polytope $\{a\}$.

- In $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$ we consider polytopes up to translation with an element in A.
- Given a homomorphism of finitely generated abelian groups $f\colon A \to A'$, we obtain a homomorphism of abelian groups

$$\mathcal{P}_{\mathbb{Z}}(f)\colon \mathcal{P}_{\mathbb{Z}}(A) \to \mathcal{P}_{\mathbb{Z}}(A'), \quad [P] \mapsto [\mathrm{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f(P)];$$

  and analogously for $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$.

Example ($A = \mathbb{Z}$)

- An integral polytope in $\mathbb{Z}_\mathbb{R}$ is just an interval $[m, n]$ for $m, n \in \mathbb{Z}$ satisfying $m \leq n$.
- The Minkowski sum becomes $[m_1, n_1] + [m_2, n_2] = [m_1 + m_2, n_1 + n_2]$.
- One obtains isomorphisms of abelian groups

$$\begin{array}{rcll} \mathcal{P}_\mathbb{Z}(\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z}^2 & [[m,n]] \mapsto (n-m, m). \\ \mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z}, & [[m,n]] \mapsto n-m. \end{array}$$

Theorem (Funke [9])

$\mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n)$ is a free abelian group possessing an explicite basis.

- We obtain an injection, useful for detection,

$$\mathcal{P}_{\mathbb{Z}}(A) \to \prod_{\phi \in \hom_{\mathbb{Z}}(A,\mathbb{Z})} \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}), \quad x \mapsto \big(\phi(x)\big)_{\phi}.$$

- We obtain a well-defined homomorphism of abelian groups

$$\big(\mathbb{Q}[\mathbb{Z}^n]_{(0)}\big)^{\times} \to \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n), \quad \frac{p}{q} \mapsto [N(p)] - [N(q)].$$

We want to generalize it to the so called polytope homomorphism.

Polytope homomorphism

- Consider the projection

$$\mathrm{pr}\colon G \to H_1(G)_f := H_1(G)/\operatorname{tors}(H_1(G)).$$

  Let K be its kernel.

- After a choice of a set-theoretic section of pr we get isomorphisms

$$\begin{aligned} \mathbb{Z}K * H_1(G)_f &\xrightarrow{\cong} \mathbb{Z}G; \\ S^{-1}\big(\mathcal{D}(K) * H_1(G)_f\big) &\xrightarrow{\cong} \mathcal{D}(G), \end{aligned}$$

  where here and in the sequel S^{-1} denotes Ore localization with respect to the multiplicative closed set of non-trivial elements.

- Given $x = \sum_{h \in H_1(G)_f} u_h \cdot h \in \mathcal{D}(K) * H_1(G)_f$, define its support

$$\operatorname{supp}(x) := \{h \in H_1(G)_f \mid h \in H_1(G)_f, u_h \neq 0\}.$$

- The convex hull of $\operatorname{supp}(x)$ defines a polytope

$$P(x) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f.$$

- We have $P(x \cdot y) = P(x) + P(y)$ for $x, y \in \mathcal{D}(K) * H_1(G)_f$.

- Hence we can define a homomorphism of abelian groups

$$P' \colon \Big(\big(S^{-1}\big(\mathcal{D}(K) * H_1(G)_f\big)\big)^{\times}\Big)_{\text{abel}} \to \mathcal{P}_{\mathbb{Z}}(H_1(G)_f),$$

by sending $x \cdot y^{-1}$ to $[P(x)] - [P(y)]$.

- The composite

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G)) \xrightarrow{\cong} \left(\mathcal{D}(G)^\times\right)_{\text{abel}}$$
$$\xrightarrow{\cong} \left(\left(S^{-1}\left(\mathcal{D}(K) * H_1(G)_f\right)\right)^\times\right)_{\text{abel}} \xrightarrow{P'} \mathcal{P}_{\mathbb{Z}}(H_1(G)_f)$$

factories to the polytope homomorphism

$$P\colon\ \mathrm{Wh}^w(G) \to \mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(H_1(G)_f).$$

Example ($G = \mathbb{Z}$)

The polytope homomorphism reduces for $G = \mathbb{Z}$ with generator t to the map

$$\mathbb{Q}[\mathbb{Z}]^{(0)}/\{r \cdot t^n \mid r \in \mathbb{Q}, n \in \mathbb{Z}\} \to \mathbb{Z}, \quad [f/g] \mapsto \mathrm{width}(f) - \mathrm{width}(g)$$

for $\mathrm{width}\left(\sum_{n\in\mathbb{Z}} a_n t^n\right) := \max\{n - m \mid m \leq n, a_m \neq 0, a_n \neq 0\}$.

- Let M be a compact oriented 3-manifold.
- In the sequel we will identify $\mathbb{R} \otimes_{\mathbb{Z}} H_1(M)_f = H_1(M;\mathbb{R})$ and $H^1(M;\mathbb{R}) = H_1(M;\mathbb{R})^*$ by the obvious isomorphisms.

Definition (Dual Thurston polytope)

We refer to

$$T(M)^* := \{v \in H_1(M;\mathbb{R}) \mid \phi(v) \leq x_M(\phi) \text{ for all } \phi \in H^1(M;\mathbb{R})\}.$$

as the dual Thurston polytope.

- Thurston [24] has shown that $T(M)^*$ is an integral polytope.
- Moreover, Thurston [24] showed that we can find a (possibly empty) marking on the vertices of $T(M)^*$ so that a cohomology class $\phi \in H^1(M;\mathbb{R})$ fibers, i.e., is represented by a non-degenerate closed 1-form, if and only if it pairs maximally with a marked vertex, i.e., if and only if there exists a marked vertex v such that $\phi(v) > \phi(w)$ for all $w \in TM^*$ with $v \neq w$. If ϕ lies in $H^1(X;\mathbb{Z})$, then fibers means that ϕ is induced by a surface bundle $F \to M \to S^1$.
- The Thurston seminorm x_M obviously determines the dual Thurston polytope.
- The converse is also true, namely, we have

$$x_M(\phi) := \frac{1}{2} \cdot \sup\{\phi(x_0) - \phi(x_1) \mid x_0, x_1 \in T(M)^*\}.$$

Definition (L^2-polytope)

Let X be a connected finite CW-complex such that $b_n^{(2)}(\widetilde{X}) = 0$ holds for all $n \geq 0$. Define its L^2-polytope $P^{(2)}(X)$ to be the element of the universal L^2-torsion $\rho_u^{(2)}(\widetilde{M})$ under the polytope homomorphism

$$P\colon \operatorname{Wh}^w(\pi_1(M)) \to \mathcal{P}_{\mathbb{Z},\operatorname{Wh}}(H_1(\pi_1(M))_f).$$

Theorem (Friedl-Lück [6])

Let M be an admissible 3-manifold

Then the L^2-polytope $P^{(2)}(X)$ is represented by the dual Thurston polytope $T(M)^*$.

- The L^2-polytope $P^{(2)}(X)$ determines $T(M)^*$ uniquely since the dual Thurston polytope $T(M)^*$ satisfies $T(M)^* = -T(M)^*$.
- Its proof is based on the proof of the Fibering Conjecture by Agol [1].

Appendix A: Higher order Alexander polynomials

- Higher order Alexander polynomials were introduced for a covering $G \to \overline{M} \to M$ of a 3-manifold by Harvey [11] and Cochran [3], provided that G occurs in the rational derived series of $\pi_1(M)$.

- At least the degree of these polynomials is a well-defined invariant of M and G.

- We can extend this notion of degree also to the universal covering of M and can prove the conjecture that the degree coincides with the Thurston norm, see Friedl-Lück [7].

Appendix B: Group automorphisms

Theorem (Lück [18])

Let $f\colon X \to X$ be a self homotopy equivalence of a finite connected CW-complex. Let T_f be its mapping torus.

Then all L^2-Betti numbers $b_n^{(2)}(\widetilde{T_f})$ vanish.

Definition (Universal torsion for group automorphisms)

Let $f\colon G \to G$ be a group automorphism of the group G. Suppose that there is a finite model for BG, the Whitehead group $\mathrm{Wh}(G)$ vanishes, and G satisfies the Atiyah Conjecture. Then we can define the universal L^2-torsion of f by

$$\rho_u^{(2)}(f) := \rho^{(2)}(\widetilde{T_f}; \mathcal{N}(G \rtimes_f \mathbb{Z})) \in \mathrm{Wh}^w(G \rtimes_f \mathbb{Z})$$

- This seems to be a very powerful invariant which needs to be investigated further.

- It has nice properties, e.g., it depends only on the conjugacy class of f, satisfies a sum formula and a formula for exact sequences.

- If G is amenable, it vanishes.

- If G is the fundamental group of a compact surface F and f comes from an automorphism $a \colon F \to F$, then T_f is a 3-manifold and a lot of the material above applies.

- For instance, if a is irreducible, $\rho_u^{(2)}(f)$ detects whether a is pseudo-Anosov since we can read off the sum of the volumes of the hyperbolic pieces in the Jaco-Shalen decomposition of T_f.

- Suppose that $H_1(f) = \mathrm{id}$. Then there is an obvious projection

$$\mathrm{pr}\colon H_1(G \rtimes_f \mathbb{Z})_f = H_1(G)_f \times \mathbb{Z} \to H_1(G)_f.$$

  Let

$$P(f) \in \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

  be the image of $\rho_u^{(2)}(f)$ under the composite

$$\mathrm{Wh}^w(G \rtimes \mathbb{Z}) \xrightarrow{P} \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G \rtimes_f \mathbb{Z})) \xrightarrow{\mathcal{P}(\mathrm{pr})} \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f).$$

- What are the main properties of this polytope? In which situations can it be explicitly computed? The case, where F is a finitely generated free group, is of particular interest, see for example Funke-Kielak [10], Kielak [13].

Appendix C: Twisting L^2-invariants

- Consider a CW-complex X with $\pi = \pi_1(M)$.
- Fix an element $\phi \in H^1(X;\mathbb{Z}) = \hom(\pi;\mathbb{Z})$.
- For $t \in (0,\infty)$, let $\phi^*\mathbb{C}_t$ be the 1-dimensional π-representation given by
$$w \cdot \lambda := t^{\phi(w)} \cdot \lambda \quad \text{for } w \in \pi, \lambda \in \mathbb{C}.$$
- One can twist the L^2-chain complex of X with this representation, or, equivalently, apply the following ring homomorphism to the cellular $\mathbb{Z}G$-chain complex before passing to the Hilbert space completion
$$\mathbb{C}G \to \mathbb{C}G, \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \lambda \cdot t^{\phi(g)} \cdot g.$$

- Define the ϕ-twisted L^2-torsion function

$$\rho(\widetilde{X};\phi)\colon (0,\infty) \to \mathbb{R}$$

by sending t to the $\mathbb{C}_t$-twisted L^2-torsion.

- Its value at $t = 1$ is independent of ϕ and just the L^2-torsion $\rho^{(2)}(\widetilde{M})$. Recall that for an (irreducible) 3-manifold M (with empty or incompressible toroidal boundary and infinite fundamental group) this is up to the factor $-1/6\pi$ the sum of the volumes of the hyperbolic pieces in the Jaco-Shalen decomposition.

- It is not at all obvious that this definition makes sense.
- Notice that for irrational t the relevant chain complexes do not have coefficients in $\mathbb{Q}G$ anymore and the Determinant Conjecture does not apply. Moreover, the Fuglede-Kadison determinant is in general not continuous.
- On the analytic side this corresponds for a closed Riemannian manifold M to twisting with the flat line bundle $\widetilde{M} \times_{\pi} \mathbb{C}_t \to M$. It is obvious that some work is necessary to show that this is a well-defined invariant since the π-action on $\mathbb{C}_t$ is not isometric.

Theorem (Lück [21])

Suppose that $\widetilde{X}$ is L^2-acyclic.

1. *The L^2 torsion function $\rho^{(2)} := \rho^{(2)}(\widetilde{X};\phi)\colon (0,\infty) \to \mathbb{R}$ is well-defined;*
2. *The limits $\limsup_{t\to\infty} \frac{\rho^{(2)}(t)}{\ln(t)}$ and $\liminf_{t\to 0} \frac{\rho^{(2)}(t)}{\ln(t)}$ exist and we can define the degree of ϕ*

$$\deg(X;\phi) \in \mathbb{R}$$

to be their difference.

Theorem (Friedl-Lück [5], Liu [16])

Let M be an admissible 3-manifold. Then for every $\phi \in H^1(M;\mathbb{Z})$ we get the equality

$$\deg(M;\phi) = x_M(\phi).$$

- One can define by twisted Fuglede-Kadison determinants a group homomorphism

  $$\mathrm{Wh}^w(\pi) \to \mathrm{map}((0,\infty),\mathbb{R})$$

  such that the universal L^2-torsion $\rho_u^{(2)}(\widetilde{M})$ is mapped to the L^2-torsion function.

- Hence there are sum formulas, fibration formulas and Poincaré duality for the L^2-torsion function, and the Thurston norm is determined by the universal L^2-torsion $\rho_u^{(2)}(\widetilde{M})$.

- The proof of the result above is based on approximation techniques and a profound understanding of Mahler measures.
- Actually, if we consider the function

$$f\colon (0,\infty) \to (0,\infty), \quad t \mapsto \exp \circ \rho^{(2)}(\widetilde{M};\phi)$$

  for an admissible 3-manifold M, then f is multiplicatively convex, i.e., $f(t_o^{\lambda} \cdot t_1^{1-\lambda}) \le f(t_0)^{\lambda} \cdot f(t_1)^{1-\lambda}$ for $t_0, t_1 \in (0,\infty)$ and $\lambda \in (0,1)$.
- In particular f and $\rho^{(2)}(\widetilde{M};\phi)$ are monotone increasing and continuous for a 3-manifold M.

Definition (L^2-Euler characteristic)

Let Y be a G-space. Suppose that

$$h^{(2)}(Y;\mathcal{N}(G)) := \sum_{n\geq 0} b_n^{(2)}(Y;\mathcal{N}(G)) < \infty.$$

Then we define its L^2-Euler characteristic

$$\chi^{(2)}(Y;\mathcal{N}(G)) := \sum_{n\geq 0}(-1)^n \cdot b_n^{(2)}(Y;\mathcal{N}(G)) \quad \in \mathbb{R}.$$

Definition (ϕ-L^2-Euler characteristic)

Let X be a connected CW-complex. Suppose that $\widetilde{X}$ is L^2-acyclic. Consider an epimorphism $\phi\colon \pi = \pi_1(M) \to \mathbb{Z}$. Let K be its kernel. Suppose that G is torsionfree and satisfies the Atiyah Conjecture.

Define the ϕ-L^2-Euler characteristic

$$\chi^{(2)}(\widetilde{X};\phi) := \chi^{(2)}(\widetilde{X};\mathcal{N}(K)) \in \mathbb{R}.$$

- Notice that $\widetilde{X}/K$ is not a finite CW-complex. Hence it is not obvious but true that $h^{(2)}(\widetilde{X};\mathcal{N}(K)) < \infty$ and $\chi^{(2)}(\widetilde{X};\phi)$ is a well-defined real number.
- The ϕ-L^2-Euler characteristic has a bunch of good properties, it satisfies for instance a sum formula, product formula and is multiplicative under finite coverings.
- It turns out that the ϕ-L^2-Euler characteristic is always an integer.

- Let $f\colon X \to X$ be a selfhomotopy equivalence of a connected finite CW-complex. Let T_f be its mapping torus. The projection $T_f \to S^1$ induces an epimorphism $\phi\colon \pi_1(T_f) \to \mathbb{Z} = \pi_1(S^1)$.

  Then $\widetilde{T_f}$ is L^2-acyclic and we get

$$\chi^{(2)}(\widetilde{T_f};\phi) = \chi(X).$$

Theorem (Friedl-Lück [6])

Let M be a 3-manifold and $\phi\colon \pi_1(M) \to \mathbb{Z}$ be an epimorphism. Then

$$-\chi^{(2)}(\widetilde{M};\phi) = x_M(\phi).$$

- Suppose that G is torsionfree and satisfies the Atiyah Conjecture. Consider $\phi\colon G \to \mathbb{Z}$.

  Then there is a homomorphism

  $$\chi_\phi^{(2)}\colon \mathrm{Wh}^w(G) \to \mathbb{Z}$$

  which sends the universal L^2-torsion $\rho_u^{(2)}(\widetilde{X})$ to $\chi^{(2)}(\widetilde{X};\phi)$.

- Recall: Let $K \subset S^3$ be a knot with knot complement C_K and genus $g(K)$. Then for any generator ϕ of $H^1(X_K;\mathbb{Z}) \cong \mathbb{Z}$ we have

$$x_{C_K}(\phi) \; = \; \max\{2g(K) - 1, 0\}.$$

- One of the key motivations for developing the theory of L^2-Euler characteristics is the following question by Simon.

Question (Simon)

Let K and K' be two knots. If there is an epimorphism from the knot group of K to the knot group of K', does this imply that the genus of K is greater than or equal to the genus of K'?

- We propose the following conjecture.

Conjecture (Inequality of the Thurston norm)

Let $f \colon M \to N$ be a map between admissible 3-manifolds. Suppose that it induces an epimorphism $\pi_1(M) \to \pi_1(N)$ and an isomorphism $H_n(M;\mathbb{Q}) \to H_n(N;\mathbb{Q})$ for $n \geq 0$.

Then we get for any $\phi \in H^1(N;\mathbb{R})$ that

$$x_M(f^*\phi) \;\geq\; x_N(\phi).$$

- A proof of the conjecture above would give an affirmative answer to Simon's question.

- The condition on the induced map on rational homology cannot be dropped.

- A group G is called locally indicable if any finitely generated non-trivial subgroup of G admits an epimorphism onto $\mathbb{Z}$. For example Howie [12] showed that the fundamental group of any admissible 3-manifold with non-trivial boundary is locally indicable.

- The next theorem is based on the theorem above about the equality of the Thurston norm and the L^2-Euler characteristic.

Theorem (Inequality of the Thurston norm, Friedl-Lück [6])

Let $f\colon M \to N$ be a map of admissible 3-manifolds which is surjective on $\pi_1(N)$ and induces an isomorphism $f_*\colon H_n(M;\mathbb{Q}) \to H_n(N;\mathbb{Q})$ for $n \geq 0$. Suppose that $\pi_1(N)$ is residually locally indicable elementary amenable. Then we get for any $\phi \in H^1(N;\mathbb{R})$ that

$$x_M(f^*\phi) \;\geq\; x_N(\phi).$$

Conjecture

The fundamental group of any admissible 3-manifold M with $b_1(M) \geq 1$ is residually locally indicable elementary amenable.

- A proof of the conjecture above together with the theorem above implies the Conjecture about the Inequality of the Thurston norm and in particular an affirmative answer to Simon's Question.

I. Agol.
The virtual Haken conjecture.
Doc. Math., 18:1045–1087, 2013.
With an appendix by Agol, Groves, and Manning.

M. F. Atiyah.
Elliptic operators, discrete groups and von Neumann algebras.
Astérisque, 32-33:43–72, 1976.

T. D. Cochran.
Noncommutative knot theory.
Algebr. Geom. Topol., 4:347–398, 2004.

J. Dubois, S. Friedl, and W. Lück.
The L^2-Alexander torsion of 3-manifolds.
J. Topol., 9(3):889–926, 2016.

S. Friedl and W. Lück.
The L^2-torsion function and the Thurston norm of 3-manifolds.
Preprint, arXiv:1510.00264 [math.GT], to appear in Commentarii Mathematici Helvetici, 2015.

S. Friedl and W. Lück.
L^2-Euler characteristics and the Thurston norm.
Preprint, arXiv:1609.07805 [math.GT], to appear in Proc. of the LMS, 2016.

S. Friedl and W. Lück.
Universal L^2-torsion, polytopes and applications to 3-manifolds.
Proc. Lond. Math. Soc. (3), 114(6):1114–1151, 2017.

S. Friedl, W. Lück, and S. Tillman.
Groups and polytopes.
Preprint arXiv:1611.01857 [math.GT], to appear in the Proceedings of the 2017 Georgia International Topology Conference, 2016.

F. Funke.
The integral polytope group.
Preprint, arXiv:1605.01217 [math.MG], 2016.

F. Funke and D. Kielak.

Alexander and Thurston norms, and the Bier-Neumann-Strebel invariants for free-by-cyclic groups.
Preprint, arXiv:1605.09069 [math.RA], 2016.

S. L. Harvey.
Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm.
Topology, 44(5):895–945, 2005.

J. Howie.
On locally indicable groups.
Math. Z., 180(4):445–461, 1982.

D. Kielak.
The Bieri-Neumann-Strebel invariants via Newton polytopes.
Preprint, arXiv:1802.07049 [math.GR], 2018.

P. Linnell and W. Lück.
Localization, Whitehead groups and the Atiyah conjecture.
Ann. K-Theory, 3(1):33–53, 2018.

P. A. Linnell.
Division rings and group von Neumann algebras.
Forum Math., 5(6):561–576, 1993.

Y. Liu.
Degree of L^2-Alexander torsion for 3-manifolds.
Invent. Math., 207(3):981–1030, 2017.

J. Lott and W. Lück.
L^2-topological invariants of 3-manifolds.
Invent. Math., 120(1):15–60, 1995.

W. Lück.
L^2-Betti numbers of mapping tori and groups.
Topology, 33(2):203–214, 1994.

W. Lück.
L^2-Invariants: Theory and Applications to Geometry and K-Theory, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in*

Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].
Springer-Verlag, Berlin, 2002.

W. Lück.
Approximating L^2-invariants by their classical counterparts.
EMS Surv. Math. Sci., 3(2):269–344, 2016.

W. Lück.
Twisting l^2-invariants with finite-dimensional representations.
J. Topol. Anal., 10(4):723–816, 2018.

W. Lück and T. Schick.
L^2-torsion of hyperbolic manifolds of finite volume.
GAFA, 9:518–567, 1999.

T. Schick.
L^2-determinant class and approximation of L^2-Betti numbers.
Trans. Amer. Math. Soc., 353(8):3247–3265 (electronic), 2001.

W. P. Thurston.
A norm for the homology of 3-manifolds.

Mem. Amer. Math. Soc., 59(339):i–vi and 99–130, 1986.