# Introduction to hyperbolic groups

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# Outline

- We discuss some basic notions of geometric group theory which are interesting in their own right and should be part of the general education of a mathematician.
- Parts of but not the entire material presented in this talk is relevant for the forthcoming talks.
- We will cover the following topics:
  - Quasi-isometry
  - Hyperbolic spaces
  - Hyperbolic groups
  - Some open problems about hyperbolic groups.

# Quasi-isometry

**Definition (Quasi-isometry)**

A map $f\colon X \to Y$ of metric spaces is called a quasi-isometry if there exist real numbers $\lambda, C > 0$ satisfying:

- The inequality
$$\lambda^{-1} \cdot d_X(x_1, x_2) - C \leq d_Y\big(f(x_1), f(x_2)\big) \leq \lambda \cdot d_X(x_1, x_2) + C$$
holds for all $x_1, x_2 \in X$;
- For every $y$ in $Y$ there exists $x \in X$ with $d_Y\big(f(x), y\big) < C$.

- The quasi-isometry condition may be summarized as being Lipschitz up to an additive constant.

## Remark (Quasi-Isometry is an equivalence relation)

- *If $f\colon X_1 \to X_2$ is a quasi-isometry, then there exists a quasi-isometry $g\colon X_2 \to X_1$ such that both composites $g \circ f$ and $f \circ g$ have bounded distance from the identity map.*
- *The composite of two quasi-isometries is again a quasi-isometry.*
- *Hence the notion of quasi-isometry is an equivalence relation on the class of metric spaces.*

- The inclusion $\mathbb{Z} \to \mathbb{R}$ is an quasi-isometry. An inverse quasi-isometry $\mathbb{R} \to \mathbb{Z}$ is given by sending a real number $r$ to the greatest integer which is less or equal to $r$.
- Quasi-isometries are not necessarily continuous.

## Definition (Word-metric)

Let $G$ be a finitely generated group. Let $S$ be a finite set of generators. The word metric

$$d_S\colon G \times G \to \mathbb{R}$$

assigns to $(g, h)$ the minimum over all integers $n \geq 0$ such that $g^{-1}h$ can be written as a product $s_1^{\epsilon_1} s_2^{\epsilon_2} \ldots s_n^{\epsilon_n}$ for elements $s_i \in S$ and $\epsilon_i \in \{\pm 1\}$.

## Lemma

*Let $G$ be a finitely generated group. Let $S_1$ and $S_2$ be two finite sets of generators. Then the identity $\mathrm{id}\colon (G, d_{S_1}) \to (G, d_{S_2})$ is a quasi-isometry.*

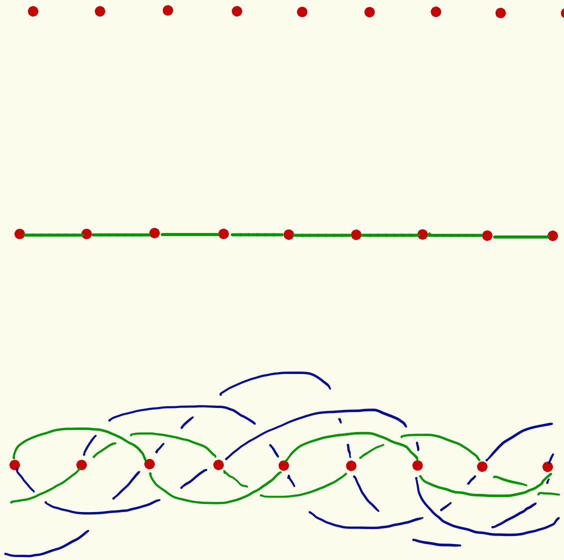
## Proof.

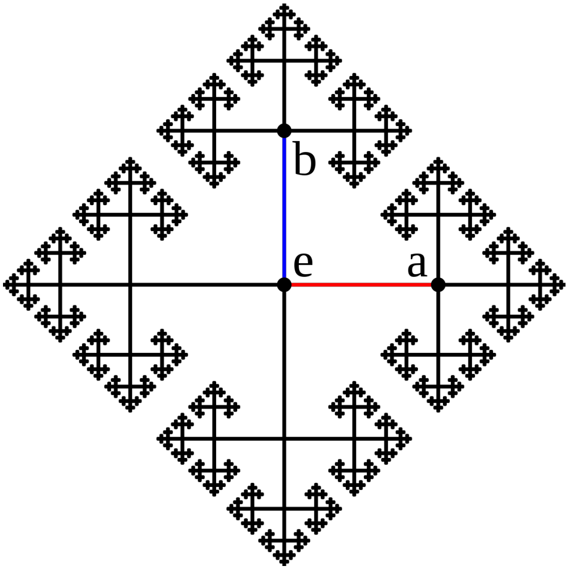
Choose $\lambda$ such that for all $s_1 \in S_1$ we have $d_{S_2}(s_1, e), d_{S_2}(s_1^{-1}, e) \leq \lambda$ and for $s_2 \in S_2$ we have $d_{S_1}(s_2, e), d_{S_1}(s_2^{-1}, e) \leq \lambda$. Take $C = 0$. □

## Definition (Cayley graph)

Let $G$ be a finitely generated group. Consider a finite set $S$ of generators. The Cayley graph $\mathrm{Cay}_S(G)$ is the graph whose set of vertices is $G$ and there is an edge joining $g_1$ and $g_2$ if and only if $g_1 = g_2 s^{\pm 1}$ for some $s \in S$.

- On the next two slides we show the Cayley graph of $\mathbb{Z}$ with respect to the set of generators $S = \{1\}$ and $S = \{2, 3\}$ and the Cayley graph of the free groups with two generators $a$ and $b$.

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- A geodesic in a metric space $(X, d)$ is an isometric embedding $I \to X$, where $I \subset \mathbb{R}$ is an interval equipped with the metric induced from the standard metric on $\mathbb{R}$.

**Definition (Geodesic space)**

A metric space $(X, d)$ is called a geodesic space if for two points $x, y \in X$ there is a geodesic $c\colon [0, d(x, y)] \to X$ with $c(0) = x$ and $c(d(x, y)) = y$.

- Notice that we do not require the unique existence of a geodesic joining two given points.

## Remark (Metric on the Cayley graph)

- *There is an obvious procedure to define a metric on* $\mathrm{Cay}_S(G)$ *such that each edge is isometric to* $[0,1]$ *and such that the distance of two points in* $\mathrm{Cay}_S(G)$ *is the infimum over the length over all piecewise linear paths joining these two points.*
- *This metric restricted to G is just the word metric* $d_S$ *and turns* $\mathrm{Cay}_S(G)$ *into a geodesic space.*
- *Obviously the inclusion* $(G, d_S) \to \mathrm{Cay}_S(G)$ *is a quasi-isometry. In particular, the quasi-isometry class of the geodesic space* $\mathrm{Cay}_S(G)$ *is independent of S.*
- *The Cayley graph allows to translate properties of a finitely generated group to properties of a geodesic metric space and thus allows to use geometry to investigate groups.*

## Definition (Proper $G$-action)

A $G$-space $X$ is called proper if for each pair of points $x$ and $y$ in $X$ there are open neighborhoods $V_x$ of $x$ and $W_y$ of $y$ in $X$ such that set $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$ is finite.

## Definition (Cocompact $G$-action)

A $G$-space $X$ is called cocompact if $X/G$ is compact.

## Lemma (Švarc-Milnor Lemma)

*Let $X$ be a geodesic space. Suppose that the finitely generated group $G$ acts properly, cocompactly and isometrically on $X$. Choose a base point $x \in X$. Then the map*

$$f\colon G \to X, \quad g \mapsto gx$$

*is a quasi-isometry.*

## Definition (Commensurable)

Two groups $G_1$ and $G_2$ are commensurable if there are subgroups $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ such that the indices $[G_1 : H_1]$ and $[G_2 : H_2]$ are finite and $H_1$ and $H_2$ are isomorphic.

## Lemma

*Let $G_1$ and $G_2$ be finitely generated groups. Then:*

- *A group homomorphism $G_1 \to G_2$ is a quasi-isometry if and only if its kernel is finite and its image has finite index in $G_2$;*
- *If $G_1$ and $G_2$ are commensurable, then they are quasi-isometric.*

## Example (Quasi-Isometry does not imply commensurable)

- Consider a semi-direct product $G_\phi = \mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ for an isomorphism $\phi\colon \mathbb{Z}^2 \to \mathbb{Z}^2$.
- These groups act properly and cocompactly by isometries on precisely one of the 3-dimensional simply connected geometries $\mathbb{R}^3$, Nil or Sol.
-
    - If $\phi$ has finite order, then the geometry is $\mathbb{R}^3$.
    - If $\phi$ has infinite order and the eigenvalues of the induced $\mathbb{C}$-linear map $\mathbb{C}^2 \to \mathbb{C}^2$ have absolute value 1, then the geometry is Nil.
    - If $\phi$ has infinite order and one of the eigenvalues of the induced $\mathbb{C}$-linear map $\mathbb{C}^2 \to \mathbb{C}^2$ has absolute value $> 1$, then the geometry is Sol.
- These metric spaces given by the geometries $\mathbb{R}^3$, Nil or Sol are mutually distinct under quasi-isometry.

## Example (Continued)

- Two groups of the shape $G_\phi$ are quasi-isometric if and only if they belong to the same geometry.
- Two groups $G_\phi$ and $G_{\phi'}$ belonging to the same geometry $\mathbb{R}^3$ or Nil are always commensurable.
- However, suppose that $G_\phi$ and $G_{\phi'}$ belong to Sol. Then they are commensurable if and only if the eigenvalues $\Lambda$ and $\Lambda'$ with absolute value $> 1$ of $\phi$ and $\phi'$, respectively, have a common power.
- This obviously yields examples of groups $G_\phi$ and $G_{\phi'}$ that belong to the geometry Sol and are quasi-isometric but are not commensurable.

## Theorem (Group properties invariant under quasi-isometry)

*The following properties of a group are quasi-isometry invariants:*

- *Finite;*
- *Infinite virtually cyclic;*
- *Finitely presented;*
- *Virtually abelian;*
- *Virtually nilpotent;*
- *Virtually free;*
- *Amenable;*
- *Hyperbolic;*
- *The existence of a model for the classifying space BG with finite n-skeleton for given $n \geq 2$;*
- *The existence of a model for BG of finite type, i.e., all skeletons are finite.*

## Theorem (Invariants under quasi-isometry)

*Let $G_1$ and $G_2$ be two finitely generated groups which are quasi-isometric. Then:*

- *They have the same number of ends;*
- *Let $R$ be a commutative ring. Then we get*

  $$\mathrm{cd}_R(G_1) = \mathrm{cd}_R(G_2)$$

  *if one of the following assumptions is satisfied:*
  - *The cohomological dimensions $\mathrm{cd}_R(G_1)$ and $\mathrm{cd}_R(G_2)$ are both finite;*
  - *One of the groups $G_1$ and $G_2$ is amenable and $\mathbb{Q} \subseteq R$;*
- *If they are solvable, then they have the same Hirsch length;*

## Theorem (Continued)

- *Suppose that $G_1$ has polynomial growth of degree not greater than $d$, intermediate growth, or exponential growth, respectively. Then the same is true for $G_2$;*
- *Let $G_1$ and $G_2$ be nilpotent. Then their real cohomology rings $H^*(G_1;\mathbb{R})$ and $H^*(G_2;\mathbb{R})$ are isomorphic as graded rings. In particular the Betti numbers of $G_1$ and $G_2$ agree.*

## Remark (Free products)

- *Let $G_1$, $G'_1$, $G_2$ and $G'_2$ be finitely generated groups. Suppose that $G_i$ and $G'_i$ are quasi-isometric for $i = 1, 2$. Assume that none of the groups $G_1$, $G'_1$, $G_2$ and $G'_2$ is trivial or $\mathbb{Z}/2$.*
- *Then the free products $G_1 * G_2$ and $G'_1 * G'_2$ are quasi-isometric.*
- *The corresponding statement is false if one replaces quasi-isometric by commensurable.*

## Remark (Property (T))

*Kazhdan's Property (T) is not a quasi-isometry invariant.*

## Remark (The sign of the Euler characteristic)

*The sign of the Euler characteristic of a group with a finite model for $BG$ is not a quasi-isometry invariant.*

## Remark ($L^2$-Betti numbers)

- *If the finitely generated groups $G_1$ and $G_2$ are quasi-isometric and there exist finite models for $BG_1$ and $BG_2$ then we have*

$$b_p^{(2)}(G_1) = 0 \Leftrightarrow b_p^{(2)}(G_2) = 0.$$

- *However, it is general not true that in the situation above there exists a constant $C > 0$ such that $b_p^{(2)}(G_1) = C \cdot b_p^{(2)}(G_2)$ holds for all $p \geq 0$.*

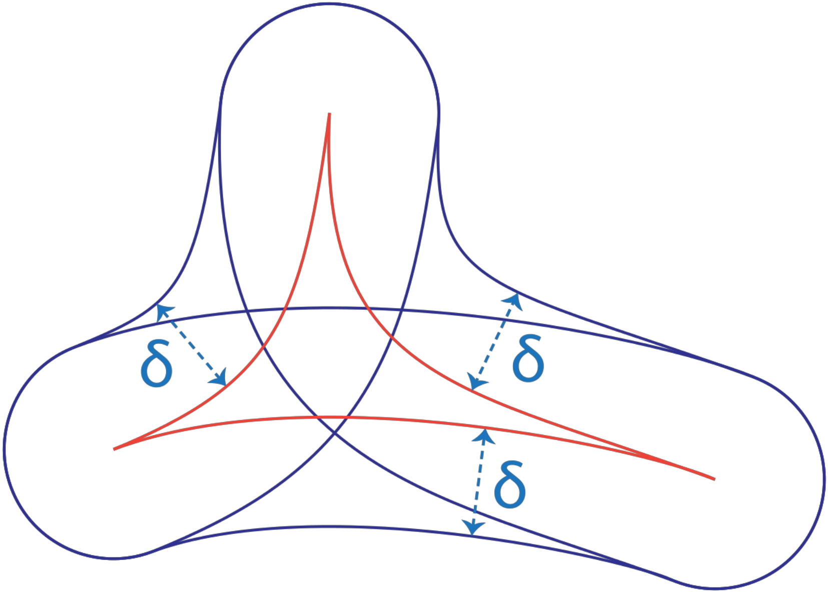
# Hyperbolic spaces

## Definition (Thin triangles)

A geodesic triangle in a geodesic space $X$ is a configuration of three points $x_1$, $x_2$ and $x_3$ in $X$ together with a choice of three geodesics $g_1$, $g_2$ and $g_3$ such that $g_1$ joins $x_2$ to $x_3$, $g_2$ joins $x_1$ to $x_3$ and $g_3$ joins $x_1$ to $x_2$. For $\delta > 0$ a geodesic triangle is called *$\delta$-thin* if each edge is contained in the closed $\delta$-neighborhood of the union of the other two edges.

## Definition (Hyperbolic space)

Consider $\delta \geq 0$. A $\delta$-hyperbolic space is a geodesic space whose geodesic triangles are all $\delta$-thin.
A geodesic space is called hyperbolic it is $\delta$-hyperbolic for some $\delta > 0$.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- A geodesic space with bounded diameter is hyperbolic.
- A tree is 0-hyperbolic.
- The hyperbolic space $\mathbb{H}^n$ is hyperbolic.
- More generally, a simply connected complete Riemannian manifold $M$, whose sectional curvature satisfies $\sec(M) \leq \kappa$ for some $\kappa < 0$, is hyperbolic as a metric space.
- $\mathbb{R}^n$ is hyperbolic if and only if $n \leq 1$.

## Definition (Boundary of a hyperbolic space)

Let $X$ be a hyperbolic space. Define its boundary $\partial X$ to be the set of equivalence classes of geodesic rays. Put

$$\overline{X} := X \amalg \partial X.$$

- Two geodesic rays $c_1, c_2 \colon [0, \infty) \to X$ are called equivalent if there exists $C > 0$ satisfying $d_X\big(c_1(t), c_2(t)\big) \leq C$ for $t \in [0, \infty)$.

## Lemma

*There is a topology on $\overline{X}$ with the properties:*

- *$\overline{X}$ is compact and metrizable;*
- *The subspace topology $X \subseteq \overline{X}$ is the given one;*
- *$X$ is open and dense in $\overline{X}$.*

- Let $M$ be a simply connected complete Riemannian manifold $M$ with $\sec(M) \le \kappa$ for some $\kappa < 0$. Then $M$ is hyperbolic as a metric space and $\partial M = S^{\dim(M)-1}$. The latter claim follows from the Cartan-Hadamard Theorem.

### Lemma (Quasi-isometry invariance of being hyperbolic)

*The property "hyperbolic" is a quasi-isometry invariant of geodesic spaces.*

### Lemma (Quasi-isometry invariance of the boundary)

*A quasi-isometry $f\colon X_1 \to X_2$ of hyperbolic spaces induces a homeomorphism*

$$\partial X_1 \xrightarrow{\cong} \partial X_2.$$

# Hyperbolic groups

**Definition (Hyperbolic group)**

A finitely generated group is called hyperbolic if its Cayley graph is hyperbolic.

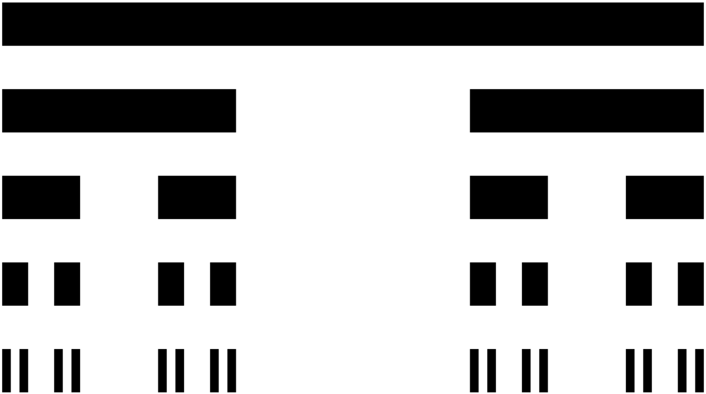
**Definition (Boundary of a hyperbolic group)**

Define the boundary $\partial G$ of a hyperbolic group to be the boundary of its Cayley graph.

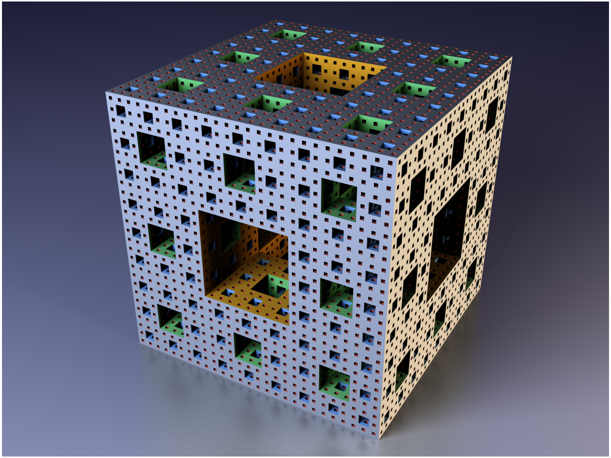
# Basic properties of hyperbolic groups

- A group $G$ is hyperbolic if and only if it acts properly, cocompactly and isometrically on some hyperbolic space. In this case $\partial G = \partial X$.
- Let $M$ be a closed Riemannian manifold with $\sec(M) < 0$. Then $\pi_1(M)$ is hyperbolic with $S^{\dim(M)-1}$ as boundary.
- If $G$ is virtually torsionfree and hyperbolic, then $\operatorname{vcd}(G) = \dim(\partial G) + 1$.
- If the boundary of a hyperbolic group contains an open subset homeomorphic to $\mathbb{R}^n$, then the boundary is homeomorphic to $S^n$.

- The boundary of a free group is a Cantor set.



- A typical boundary of a hyperbolic group looks like a Menger sponge which is a three-dimensional generalization of the one-dimensional Cantor set.



## Remark (The boundary of a hyperbolic group)

- *The boundary $\partial X$ of a hyperbolic space and in particular the boundary $\partial G$ of a hyperbolic group $G$ are metrizable.*
- *Any compact metric space can be realized as the boundary of a hyperbolic space.*
- *However, not every compact metrizable space can occur as the boundary of a hyperbolic group.*
- *Namely, exactly one of the following three cases occurs:*
  - *$G$ is finite and $\partial G$ is empty;*
  - *$G$ is infinite virtually cyclic and $\partial G$ consists of two points;*
  - *$G$ contains a free group of rank two as subgroup and $\partial G$ is an infinite perfect, (i.e., without isolated points) compact metric space.*

- A subgroup of a hyperbolic group is either virtually cyclic or contains $\mathbb{Z} * \mathbb{Z}$ as subgroup. In particular $\mathbb{Z}^2$ is not a subgroup of a hyperbolic group.
- A free product of two hyperbolic groups is again hyperbolic.
- A direct product of two finitely generated groups is hyperbolic if and only if one of the two groups is finite and the other is hyperbolic.

## Definition (Rips complex)

Let G be a finitely generated group generated by a finite set $S$ of generators. For any $d \geq 0$ the simplicial complex $P_d(G, S)$, called the Rips complex, is defined as follows.

The vertices are elements of $G$. A finite collection of distinct elements $g_0, g_1, \ldots, g_k$ in $G$ spans a $k$-simplex if and only $d_S(g_i, g_j) \leq d$ holds for all $0 \leq i, j \leq k$.

## Theorem (Finiteness properties of hyperbolic groups)

*Let G be a group with a finite set of generators.*

- *Suppose that* $(G, S)$ *is* $\delta$*-hyperbolic for the real number* $\delta \geq 0$*. Let* $d$ *be a natural number with* $d \geq 16\delta + 8$*.*
  *Then the barycentric subdivision of the Rips complex* $P_d(G, S)'$ *is a finite* $G$*-*$CW$*-model for* $\underline{E}G$*;*
- *There is a model of finite type for BG, if G is hyperbolic;*
- *There is a finite model for BG, if G is torsionfree hyperbolic.*

- A finitely generated torsion group is hyperbolic if and only if it is finite.

- Given $r$ elements $g_1, g_2, \ldots, g_r$ in a hyperbolic group, then there exists an integer $n \geq 1$, such that $\{g_1^n, g_2^n, \ldots, g_r^n\}$ generates a free subgroup of rank at most $r$.

- The word-problem and the conjugation-problem are solvable for a hyperbolic group.

- The isomorphism-problem is solvable for torsionfree hyperbolic groups.

- Let $G$ be a hyperbolic group. Let $S$ be a finite set of generators. For the integer $n \geq 0$ let $\sigma(n)$ be the number of elements $g \in G$ with $d_S(g, e) = n$;
  Then the formal power series $\sum_{n=0}^{\infty} \sigma(n) \cdot t^n$ is a rational function.
- The same is true if one replaces $\sigma(n)$ by the number $\beta(n)$ of elements $g \in G$ with $d_S(g, e) \leq n$;
- A random finitely presented group is hyperbolic.

## Remark (Construction of groups with exotic properties)

*Colimits of directed systems of hyperbolic groups which come from adding more and more relations have been used to construct exotic groups. Here are two prominent examples:*

- *Let $G$ be a torsionfree hyperbolic group which is not virtually cyclic. Then there exists a quotient of $G$ which is an infinite torsion-group whose proper subgroups are all finite (or cyclic).*
- *There exist groups with expanders. They play a role in the construction of counterexamples to the Baum-Connes Conjecture with coefficients due to Higson, Lafforgue and Skandalis.*

# Some open problems about hyperbolic groups

- Is every hyperbolic group virtually torsionfree?
- Is every hyperbolic group residually finite?
- Cannon Conjecture: Suppose that the space at infinity of a hyperbolic group is homeomorphic to $S^2$. Does this imply that it acts properly isometrically and cocompactly on the 3-dimensional hyperbolic space?
- Has the boundary of a hyperbolic group the integral Čech cohomology of a sphere if and only if it occurs as the fundamental group of an aspherical closed manifold $M$?
- Is every hyperbolic group a CAT(0)-group?
- Is the complex group ring $\mathbb{C}G$ of a torsionfree hyperbolic group an integral domain?