Aspherical manifolds: what we know and what we do not know

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What is an aspherical manifold?

- Manifold means connected closed topological manifold unless stated explicitly differently.

Definition (Aspherical manifold)

An aspherical manifold is a manifold such that one of the following equivalent conditions is satisfied:

- M is a model for $B\pi_1(M)$;
- $\pi_k(M)$ is trivial for $k \geq 2$;
- The universal covering $\widetilde{M}$ is contractible.

- The homotopy type type and the homology of an aspherical manifold and of maps between them depends only on the fundamental group.

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What are examples of aspherical manifolds?

- A smooth Riemannian manifold with non-positive sectional curvature is aspherical;
- Let G be connected Lie group with maximal compact subgroup $K \subseteq G$. Let $L \subseteq G$ be a torsionfree cocompact lattice.

  Then $M = L\backslash G/K$ is aspherical;
- A surface, which is different from S^2 and $\mathbb{RP}^2$, is aspherical;
- A prime 3-manifold, which is not an S^1-bundle over S^2 and has infinite fundamental group, is aspherical.

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The Farrell-Jones Conjecture

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for every $n \in \mathbb{Z}$.

- There is also an *L*-theory version.
- There is also a version, the Full Farrell-Jones Conjecture, which works for all groups and rings and where one can even allow twisted group rings and non-trivial orientation homomorphisms in the *L*-theory case.

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Theorem (Bartels, Farrell, Kammeyer, Lück, Reich, Rüping, Wegner)

Let $\mathcal{FJ}$ be the class of groups for which the Full Farrell-Jones Conjecture holds. Then $\mathcal{FJ}$ contains the following groups:

- *Hyperbolic groups;*
- CAT(0)*-groups;*
- *Solvable groups,*
- *(Not necessarily uniform) lattices in almost connected Lie groups;*
- *Fundamental groups of (not necessarily compact) d-dimensional manifolds (possibly with boundary) for $d \leq 3$.*
- *Subgroups of $GL_n(\mathbb{Q})$ and of $GL_n(F[t])$ for a finite field F.*
- *All S-arithmetic groups.*

Theorem (continued)

Moreover, $\mathcal{FJ}$ has the following inheritance properties:

- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *If $H \subseteq G$ is a subgroup of G with $[G : H] < \infty$ and $H \in \mathcal{FJ}$, then $G \in \mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*

To which extend does the fundamental group determine an aspherical manifold?

Conjecture (Borel Conjecture)

The Borel Conjecture for a group G predicts that for two aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ any homotopy equivalence $M \to N$ is homotopic to a homeomorphism.

- In particular the Borel Conjecture predicts that two aspherical manifolds are homeomorphic if and only if their fundamental groups are isomorphic.

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- The Borel Conjecture can be viewed as the topological version of Mostow rigidity.

  A special case of Mostow rigidity says that any homotopy equivalence between hyperbolic manifolds of dimension ≥ 3 is homotopic to an isometric diffeomorphism.

- The Borel Conjecture is not true in the smooth category. Namely, Farrell-Jones show that for any $\epsilon > 0$ and $n \geq 5$ there exists a hyperbolic n-manifold N and a Riemannian n-manifold M with $-1 - \epsilon \leq \sec(M) \leq -1$ such that M and N are homeomorphic, but not diffeomorphic.

- The Borel Conjecture implies the Novikov Conjecture about the homotopy invariance of higher signatures, which in turns implies the conjecture that an aspherical smooth manifold does not carry a Riemannian metric with positive scalar curvature.

- The Borel Conjecture is true in dimensions ≤ 3.

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Theorem (Farrell-Jones implies Borel)

Let G be a group for which there exists an aspherical manifold M with $G \cong \pi_1(M)$ *and* $\dim(M) \geq 5$ *and which belongs to* $\mathcal{FJ}$*.*

Then the Borel Conjecture holds for G.

Question

Do the Farrell-Jones Conjecture, the Borel Conjecture and the Novikov Conjecture hold in general?

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Theorem (Projections of block bundles, Farrell-Lück-Steimle)

Let B be an aspherical triangulable manifold with hyperbolic fundamental group. Let M be a manifold. Assume that $\dim(M) - \dim(B)$ is greater or equal to 5. Suppose $\pi_1(M)$ is torsionfree and belongs to $\mathcal{FJ}$.

Then a map $M \to B$ is homotopic to the projection of a block bundle if and only if the homotopy fiber of p is finitely dominated.

Which groups occur as fundamental groups of aspherical manifolds?

Definition (Poincaré duality group)

A Poincaré duality group G of dimension n is a finitely presented group satisfying:

- G is of type FP;
- $H^i(G;\mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

Lemma

Let X be an aspherical ANR-homology manifold of dimension n. Then its fundamental group is a Poincaré duality group of dimension n.

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Let X be an aspherical ANR-homology manifold of dimension n. Then its fundamental group is a Poincaré duality group of dimension n.

Theorem (Poincaré duality groups and ANR-homology manifolds, Bartels-Lück-Weinberger)

Let G be a torsionfree group. Suppose that it belongs to $\mathcal{FJ}$. Consider $n \geq 6$.

Then the following statements are equivalent:

1. *G is a Poincaré duality group of dimension n;*
2. *There exists an aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$;*
3. *There exists an aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$ which has the DDP (Disjoint Disk Property).*

If the first statements holds, then the homology ANR-manifold M appearing above is unique up to s-cobordism of ANR-homology manifolds.

Theorem (Quinn (1987))

There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ for (not necessarily compact) homology ANR-manifolds with the following properties:

- *if $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;*
- *$i(M \times N) = i(M) \cdot i(N)$;*
- *Let M be a homology ANR-manifold of dimension ≥ 5. Then M is a topological manifold if and only if M has the DDP and $\iota(M) = 1$.*

Question

Does the Quinn obstruction always vanish for aspherical homology ANR-manifolds?

- If the answer is yes, we can replace “ANR-homology manifold”by “manifold” in the theorem above.
- In general the Quinn obstruction is not a homotopy invariant but it is a homotopy invariant for aspherical ANR-homology manifolds, provided that the Farrell-Jones Conjecture holds.
- However, some experts expect the answer no.
- I am not an expert and hope that the answer is yes.

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Conjecture (Gromov (1994))

Let G be a hyperbolic group whose boundary is a sphere S^{n-1}. Then there is an aspherical manifold M with $\pi_1(M) \cong G$.

Theorem (Bartels-Lück-Weinberger)

Let G be a torsionfree hyperbolic group and let n be an integer ≥ 6.

Then following statements are equivalent:

- *The boundary ∂G is homeomorphic to S^{n-1};*
- *There is an aspherical manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.*

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Theorem (Casson-Jungreis, Freden, Gabai)

A hyperbolic group has S^1 as boundary if and only if it acts properly, cocompactly and isometrically on $\mathbb{H}^2$.

Conjecture (Cannon's Conjecture)

A hyperbolic group G has S^2 as boundary if and only if it acts properly, cocompactly and isometrically on $\mathbb{H}^3$.

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How exotic can aspherical manifolds be?

By hyperbolization techniques due to Charney, Davis,Januskiewicz one can find the following examples:

Examples (Exotic universal coverings)

Given $n \geq 5$, there are aspherical manifolds M of dimension n with hyperbolic fundamental group $G = \pi_1(M)$ satisfying:

- The universal covering $\widetilde{M}$ is not homeomorphic to $\mathbb{R}^n$ and ∂G is not homeomorphic to S^{n-1}.
- M is smooth and $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ but ∂G is not S^{n-1}.

Example (No smooth structures)

For every $k \geq 2$ there exists a torsionfree hyperbolic group G with $\partial G \cong S^{4k-1}$ such that there is no aspherical closed smooth manifold M with $\pi_1(M) \cong G$. In particular G is not the fundamental group of a closed smooth Riemannian manifold with $\sec(M) < 0$.

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Theorem (Davis-Fowler-Lafont, based on Manolescu)

For every $n \geq 6$ there exists an aspherical manifold with hyperbolic fundamental group which is not triangulable.

Theorem (Bartels-Lück)

For every $n \geq 5$ aspherical topological manifolds with hyperbolic fundamental groups are topologically rigid.

Corollary

For any $n \geq 6$ there exists a hyperbolic group which is the fundamental group of an aspherical topological manifold but not the fundamental group of an aspherical triangulable topological manifold.

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Theorem (Exotic fundamental groups, Belegradek, Mess, Weinberger)

1. *For every $n \geq 4$ there is an aspherical manifold of dimension n whose fundamental group contains an infinite divisible abelian group;*
2. *For every $n \geq 4$ there is an aspherical manifold of dimension n whose fundamental group has an unsolvable word problem.*

- A finitely presented group with unsolvable word problem is not a CAT(0)-group, not hyperbolic, not automatic, not asynchronously automatic, not residually finite and not linear over any commutative ring.

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What can be said about the automorphism groups of an aspherical manifold?

Theorem (Homotopy groups of automorphism groups of aspherical manifolds)

Let M be an orientable aspherical (smooth) manifold of dimension > 10 with fundamental group G. Suppose $G \in \mathcal{FJ}$.

Then for $1 \leq i \leq (\dim M - 7)/3$ one has

$$\pi_i(\mathrm{Top}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathrm{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ 0 & \text{if } i > 1, \end{cases}$$

and

$$\pi_i(\mathrm{Diff}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathrm{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ \bigoplus_{j=1}^{\infty} H_{(i+1)-4j}(M; \mathbb{Q}) & \text{if } i > 1, \dim M \text{ odd}; \\ 0 & \text{if } i > 1, \dim M \text{ even}. \end{cases}$$

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- There is a canoncial map

$$\pi_1(\mathrm{Top}(M), \mathrm{id}) \to G_1(M) \subseteq \pi_1(M).$$

- Suppose from now on that M is an orientable aspherical manifold of dimension > 10 with $G := \pi_1(M) \in \mathcal{FJ}$.
- Then $G_1(M) = \mathrm{center}(G)$ and the induced map

$$B\,\mathrm{Top}(M)^\circ \to K(\mathrm{center}(G), 2)$$

is a map of simply connected spaces inducing isomorphism on the rationalized homotopy groups in a range.
- This implies that in this range we get an isomorphism

$$H^*(K(\mathrm{center}(G), 2); \mathbb{Q}) \xrightarrow{\cong} H^*(B\,\mathrm{Top}(M)^\circ; \mathbb{Q}).$$

- Notice the canonical epimorphism which is rationally bijective

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What can be said about the L^2-invariants of the universal covering of an aspherical manifold?

- Given a smooth Riemannian manifold M, Atiyah gave an analytic definition of the *n*th L^2-Betti number of its universal covering

$$b_n^{(2)}(\widetilde{M}) = \lim_{t \to \infty} \int_{\mathcal{F}} \operatorname{tr}\left(e^{t\Delta_n(\widetilde{x},\widetilde{x})}\right) \, d\mathrm{vol}_{\widetilde{M}} \, .$$

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L^2-Betti numbers are VOODOO!!

- The following conjecture combines and generalizes Conjectures by Bergeron-Venkatesh, Hopf, Singer, Lück, and Lück-Shalen.

Conjecture (Homological growth and L^2-invariants for aspherical manifolds)

Let M be an aspherical manifold of dimension d and fundamental group $G = \pi_1(M)$. Let $\widetilde{M}$ be its universal covering. Then

1.) *For any natural number n with $2n \neq d$ we get*

$$b_n^{(2)}(\widetilde{M}) = 0.$$

If $d = 2n$, we have

$$(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M}) \geq 0.$$

If $d = 2n$ and $\sec(M) < 0$, then

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Conjecture (Continued)

2.) *Let* $(G_i)_{i \geq 0}$ *be a chain, i.e., a sequence of in G normal subgroups*

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

such that $[G : G_i] < \infty$ *and* $\bigcap_{i \geq 0} G_i = \{1\}$*. Put* $M[i] = G_i \backslash \widetilde{M}$*. Then we get for any natural number n and any field F*

$$b_n^{(2)}(\widetilde{M}) = \lim_{i \to \infty} \frac{b_n(M[i]; F)}{[G : G_i]};$$

Conjecture (Continued)

3.) *Let $(G_i)_{i\geq 0}$ be any chain. Put $M[i] = G_i\backslash\widetilde{M}$. Then we get for any natural number n with $2n+1 \neq d$*

$$\lim_{i\to\infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i])\right)\right|\right)}{[G : G_i]} = 0,$$

and we get in the case $d = 2n+1$

$$\lim_{i\to\infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i])\right)\right|\right)}{[G : G_i]} = (-1)^n \cdot \rho^{(2)}(\widetilde{M}) \geq 0.$$

If M is hyperbolic of dimension 3, this boils down to

$$\lim_{i\to\infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_1(G_i)\right)\right|\right)}{[G : G_i]} = \frac{\operatorname{vol}(M)}{6\pi}.$$

How many aspherical manifolds are there?

Slogan

A random manifold is aspherical and topologically rigid (and asymmetric).

Question

What is a random manifold?

- Such a notion exists for finite presented groups and had many application, in particular to find groups which have exotic properties or are counterexamples to prominent conjecture and questions.
- A random finitely presented group is hyperbolic and torsionfree, is not a free product, and has trivial center.

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- Here is some mild evidence for the slogan above.

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- The condition aspherical does not impose any restrictions on the characteristic numbers of a manifold.

- Consider a bordism theory Ω_* for PL-manifolds or smooth manifolds which is given by imposing conditions on the stable tangent bundle. Examples are unoriented bordism, oriented bordism, framed bordism. Then any bordism class can be represented by an aspherical manifold. If two aspherical manifolds represent the same bordism class, then one can find an aspherical bordism between them.

- Borel has shown that an aspherical manifold is asymmetric, if its fundamental group is centerless and its outer automorphism group is torsionfree.

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The universe of manifolds and our universe

- The slogan above is — at least on the first glance — surprising since often our favorite manifolds are not asymmetric and not determined by their fundamental group, e.g., lens spaces and simply connected manifolds.
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- If one asks people for the most prominent manifold, most people name the standard sphere.

- It is interesting that the n-dimensional standard sphere S^n can be characterized among (simply connected) Riemannian manifolds of dimension n by the property that its isometry group has maximal dimension.

- It is likely that the human taste whether a geometric object is beautiful is closely related to the question how many symmetries it admits. In general it seems to be the case that a human being is attracted by unusual representatives among mathematical objects.

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- Here is an interesting parallel to our actual universe.
- If you materialize at a random point in the universe, it will be very cold and nothing will be there. There is no interaction between different random points, i.e., it is rigid.
- A human being will not like this place, actually even worse, it cannot exist at such a random place.
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