Introduction to L^2-Betti numbers

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Basic motivation

- Given an invariant for finite CW-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.
- We want to apply this principle to (classical) Betti numbers

$$b_n(X) := \dim_{\mathbb{C}}(H_n(X;\mathbb{C})).$$

- We will use the following successful approach which is essentially due to Atiyah.

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Group von Neumann algebras

Definition

Define the group von Neumann algebra

$$\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G))^G = \overline{\mathbb{C}G}^{\text{weak}}$$

to be the algebra of bounded G-equivariant operators $L^2(G) \to L^2(G)$. The von Neumann trace is defined by

$$\operatorname{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$

Example (Finite G)

If G is finite, then $\mathbb{C}G = L^2(G) = \mathcal{N}(G)$. The trace $\operatorname{tr}_{\mathcal{N}(G)}$ assigns to $\sum_{g \in G} \lambda_g \cdot g$ the coefficient λ_e.

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Example ($G = \mathbb{Z}^n$)

Let G be $\mathbb{Z}^n$. Let $L^2(T^n)$ be the Hilbert space of L^2-integrable functions $T^n \to \mathbb{C}$. Fourier transform yields an isometric $\mathbb{Z}^n$-equivariant isomorphism

$$L^2(\mathbb{Z}^n) \xrightarrow{\cong} L^2(T^n).$$

Let $L^\infty(T^n)$ be the Banach space of essentially bounded measurable functions $f\colon T^n \to \mathbb{C}$. We obtain an isomorphism

$$L^\infty(T^n) \xrightarrow{\cong} \mathcal{N}(\mathbb{Z}^n), \quad f \mapsto M_f$$

where $M_f\colon L^2(T^n) \to L^2(T^n)$ is the bounded $\mathbb{Z}^n$-operator $g \mapsto g \cdot f$.

Under this identification the trace becomes

$$\operatorname{tr}_{\mathcal{N}(\mathbb{Z}^n)}\colon L^\infty(T^n) \to \mathbb{C}, \quad f \mapsto \int_{T^n} f d\mu.$$

von Neumann dimension

Definition (Finitely generated Hilbert module)

A finitely generated Hilbert $\mathcal{N}(G)$-module V is a Hilbert space V together with a linear isometric G-action such that there exists an isometric linear G-embedding of V into $L^2(G)^n$ for some $n \geq 0$. A map of finitely generated Hilbert $\mathcal{N}(G)$-modules $f\colon V \to W$ is a bounded G-equivariant operator.

Definition (von Neumann dimension)

Let V be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a G-equivariant projection $p\colon L^2(G)^n \to L^2(G)^n$ with $\operatorname{im}(p) \cong_{\mathcal{N}(G)} V$. Define the von Neumann dimension of V by

$$\dim_{\mathcal{N}(G)}(V) := \operatorname{tr}_{\mathcal{N}(G)}(p) := \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(p_{i,i}) \quad \in [0,\infty).$$

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For finite G a finitely generated Hilbert $\mathcal{N}(G)$-module V is the same as a unitary finite dimensional G-representation and

$$\dim_{\mathcal{N}(G)}(V) = \frac{1}{|G|} \cdot \dim_{\mathbb{C}}(V).$$

Example ($G = \mathbb{Z}^n$)

Let G be $\mathbb{Z}^n$. Let $X \subset T^n$ be any measurable set with characteristic function $\chi_X \in L^\infty(T^n)$. Let $M_{\chi_X} : L^2(T^n) \to L^2(T^n)$ be the $\mathbb{Z}^n$-equivariant unitary projection given by multiplication with χ_X. Its image V is a Hilbert $\mathcal{N}(\mathbb{Z}^n)$-module with

$$\dim_{\mathcal{N}(\mathbb{Z}^n)}(V) = \mathrm{vol}(X).$$

In particular each $r \in [0, \infty)$ occurs as $r = \dim_{\mathcal{N}(\mathbb{Z}^n)}(V)$.

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L^2-homology and L^2-Betti numbers

Definition (L^2-homology and L^2-Betti numbers)

Let X be a connected CW-complex of finite type. Let $\widetilde{X}$ be its universal covering and $\pi = \pi_1(M)$. Denote by $C_*(\widetilde{X})$ its cellular $\mathbb{Z}\pi$-chain complex.

Define its cellular L^2-chain complex to be the Hilbert $\mathcal{N}(\pi)$-chain complex

$$C_*^{(2)}(\widetilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) = \overline{C_*(\widetilde{X})}.$$

Define its n-th L^2-homology to be the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_n^{(2)}(\widetilde{X}) := \ker(c_n^{(2)})/\overline{\operatorname{im}(c_{n+1}^{(2)})}.$$

Define its n-th L^2-Betti number

$$b_n^{(2)}(\widetilde{X}) := \dim_{\mathcal{N}(\pi)}\left(H_n^{(2)}(\widetilde{X})\right) \quad \in \mathbb{R}^{\geq 0}.$$

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Theorem (Main properties of Betti numbers)

Let X and Y be connected CW-complexes of finite type.

- *Homotopy invariance*

  If X and Y are homotopy equivalent, then

  $$b_n(X) = b_n(Y);$$

- *Euler-Poincaré formula*

  If X is finite, we have

  $$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n(X);$$

- *Poincaré duality*

  Let M be an oriented closed manifold of dimension d. Then

  $$b_n(M) = b_{d-n}(M);$$

Theorem (Main properties of L^2-Betti numbers)

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Theorem (Continued)

- *Künneth formula*

$$b_n(X \times Y) = \sum_{p+q=n} b_p(X) \cdot b_q(Y);$$

- *Zero-th L^2-Betti number*

  We have

$$b_0(X) = 1;$$

Theorem (Continued)

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$$b_0^{(2)}(\widetilde{X}) = \frac{1}{|\pi|};$$

- *Finite coverings*

  If $X \to Y$ is a finite covering with d sheets, then

$$b_n^{(2)}(\widetilde{X}) = d \cdot b_n^{(2)}(\widetilde{Y}).$$

Some computations and results

Example (Finite π)

If π is finite then

$$b_n^{(2)}(\widetilde{X}) = \frac{b_n(\widetilde{X})}{|\pi|}.$$

Example (Finite self coverings)

We get for a connected CW-complex X of finite type, for which there is a selfcovering $X \to X$ with d-sheets for some integer $d \geq 2$,

$$b_n^{(2)}(\widetilde{X}) = 0 \quad \text{for } n \geq 0.$$

This implies for each connected CW-complex Y of finite type

$$b_n^{(2)}(\widetilde{S^1 \times Y}) = 0 \quad \text{for } n \geq 0.$$

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Theorem (S^1-actions, Lück)

Let M be a connected compact manifold with S^1-action. Suppose that for one (and hence all) $x \in X$ the map $S^1 \to M, \quad z \mapsto zx$ is π_1-injective. Then we get for all $n \geq 0$

$$b_n^{(2)}(\widetilde{M}) = 0.$$

Theorem (S^1-actions on aspherical manifolds, Lück)

Let M be an aspherical closed manifold with non-trivial S^1-action. Then

1. *The action has no fixed points;*
2. *The map $S^1 \to M, \quad z \mapsto zx$ is π_1-injective for $x \in M$;*
3. *$b_n^{(2)}(\widetilde{M}) = 0$ for $n \geq 0$ and $\chi(M) = 0$.*

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Example (L^2-Betti number of surfaces)

- Let F_g be the orientable closed surface of genus $g \geq 1$.
- Then $|\pi_1(F_g)| = \infty$ and hence $b_0^{(2)}(\widetilde{F_g}) = 0$.
- By Poincaré duality $b_2^{(2)}(\widetilde{F_g}) = 0$.
- $\dim(F_g) = 2$, we get $b_n^{(2)}(\widetilde{F_g}) = 0$ for $n \geq 3$.
- The Euler-Poincaré formula shows

$$\begin{aligned} b_1^{(2)}(\widetilde{F_g}) &= -\chi(F_g) = 2g-2; \\ b_n^{(2)}(\widetilde{F_0}) &= 0 \quad \text{for} \quad n \neq 1. \end{aligned}$$

Theorem (Hodge - de Rham Theorem)

Let M be an oriented closed Riemannian manifold. Put

$$\mathcal{H}^n(M) = \{\omega \in \Omega^n(M) \mid \Delta_n(\omega) = 0\}$$

Then integration defines an isomorphism of real vector spaces

$$\mathcal{H}^n(M) \xrightarrow{\cong} H^n(M;\mathbb{R}).$$

Corollary (Betti numbers and heat kernels)

$$b_n(M) = \lim_{t\to\infty} \int_M \operatorname{tr}_{\mathbb{R}}(e^{-t\Delta_n}(x,x))\, d\mathrm{vol}\,.$$

where $e^{-t\Delta_n}(x,y)$ is the heat kernel on M.

Theorem (L^2-Hodge - de Rham Theorem, Dodziuk)

Let M be an oriented closed Riemannian manifold. Put

$$\mathcal{H}^n_{(2)}(\widetilde{M}) = \{\widetilde{\omega} \in \Omega^n(\widetilde{M}) \mid \widetilde{\Delta}_n(\widetilde{\omega}) = 0,\ ||\widetilde{\omega}||_{L^2} < \infty\}$$

Then integration defines an isomorphism of finitely generated Hilbert $\mathcal{N}(\pi)$-modules

$$\mathcal{H}^n_{(2)}(\widetilde{M}) \xrightarrow{\cong} H^n_{(2)}(\widetilde{M}).$$

Corollary (L^2-Betti numbers and heat kernels)

$$b_n^{(2)}(\widetilde{M}) = \lim_{t\to\infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{R}}(e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{x}))\ dvol\,.$$

where $e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{y})$ is the heat kernel on $\widetilde{M}$ and $\mathcal{F}$ is a fundamental domain for the π-action.

Theorem (Hyperbolic manifolds, Dodziuk)

Let M be a hyperbolic closed Riemannian manifold of dimension d. Then:

$$b_n^{(2)}(\widetilde{M}) = \begin{cases} = 0 & \text{, if } 2n \neq d; \\ > 0 & \text{, if } 2n = d. \end{cases}$$

Proof.

A direct computation shows that $\mathcal{H}^p_{(2)}(\mathbb{H}^d)$ is not zero if and only if $2n = d$. Notice that M is hyperbolic if and only if $\widetilde{M}$ is isometrically diffeomorphic to the standard hyperbolic space $\mathbb{H}^d$. □

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Corollary

Let M be a hyperbolic closed manifold of dimension d. Then

1. *If $d = 2m$ is even, then*

$$(-1)^m \cdot \chi(M) > 0;$$

2. *M carries no non-trivial S^1-action.*

Proof.

(1) We get from the Euler-Poincaré formula and the last result

$$(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M}) > 0.$$

(2) We give the proof only for $d = 2m$ even. Then $b_m^{(2)}(\widetilde{M}) > 0$. Since $\widetilde{M} = \mathbb{H}^d$ is contractible, M is aspherical. Now apply a previous result about S^1-actions. □

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Theorem (3-manifolds, Lott-Lück)

Let the 3-manifold M be the connected sum $M_1 \sharp \ldots \sharp M_r$ of (compact connected orientable) prime 3-manifolds M_j. Assume that $\pi_1(M)$ is infinite. Then

$$\begin{aligned}
b_1^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} - \chi(M) \\
&\quad + \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right|; \\
b_2^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} \\
&\quad + \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right|; \\
b_n^{(2)}(\widetilde{M}) &= 0 \quad \text{for } n \neq 1, 2.
\end{aligned}$$

Theorem (Mapping tori, Lück)

Let $f\colon X \to X$ be a cellular selfhomotopy equivalence of a connected CW-complex X of finite type. Let T_f be the mapping torus. Then

$$b_n^{(2)}(\widetilde{T_f}) = 0 \quad \textit{for } n \geq 0.$$

Proof:

- As $T_{f^d} \to T_f$ is a d-sheeted covering (up to homotopy), we get

$$b_n^{(2)}(\widetilde{T_f}) = \frac{b_n^{(2)}(\widetilde{T_{f^d}})}{d}.$$

- If $\beta_n(X)$ is the number of n-cells, then there is a CW-structure on T_{f^d} with $\beta_n(T_{f^d}) = \beta_n(X) + \beta_{n-1}(X)$.
- We have

$$b_n^{(2)}(\widetilde{T_{f^d}}) \leq \beta_n(T_{f^d}).$$

- This implies for all $d \geq 1$

$$b_n^{(2)}(\widetilde{T_f}) \ \leq \ \frac{\beta_n(X) + \beta_{n-1}(X)}{d}.$$

- Taking the limit for $d \to \infty$ yields the claim.

The fundamental square and the Atiyah Conjecture

Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)

Let G be a torsionfree finitely presented group. We say that G satisfies the ***Atiyah Conjecture*** *if for any closed Riemannian manifold M with $\pi_1(M) \cong G$ we have for every $n \geq 0$*

$$b_n^{(2)}(\widetilde{M}) \in \mathbb{Z}.$$

- All computations presented above support the Atiyah Conjecture.

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- The fundamental square is given by the following inclusions of rings

$$\begin{tikzcd} \mathbb{Z}G \arrow[r] \arrow[d] & \mathcal{N}(G) \arrow[d] \\ \mathcal{D}(G) \arrow[r] & \mathcal{U}(G) \end{tikzcd}$$

- $\mathcal{U}(G)$ is the algebra of affiliated operators. Algebraically it is just the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is a unit in $\mathcal{U}(G)$, is already a unit in $\mathcal{D}(G)$ itself.

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- If G is finite, its is given by

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathbb{C}G \\ \downarrow & & \downarrow \mathrm{id} \\ \mathbb{Q}G & \longrightarrow & \mathbb{C}G \end{array}$$

- If $G = \mathbb{Z}$, it is given by

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}] & \longrightarrow & L^\infty(S^1) \\ \downarrow & & \downarrow \\ \mathbb{Q}[\mathbb{Z}]^{(0)} & \longrightarrow & L(S^1) \end{array}$$

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Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- A torsionfree group G satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Z}G)$ the von Neumann dimension

$$\dim_{\mathcal{N}(G)}\big(\ker\big(r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n\big)\big)$$

is an integer. In this case this dimension agrees with

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- The general version above is equivalent to the one stated before if G is finitely presented.

Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- A torsionfree group G satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Z}G)$ the von Neumann dimension

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- The Atiyah Conjecture implies the Zero-divisor Conjecture due to Kaplansky saying that for any torsionfree group and field of characteristic zero F the group ring FG has no non-trivial zero-divisors.
- There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.
- However, there exist closed Riemannian manifolds whose universal coverings have an L^2-Betti number which is irrational, see Austin, Grabowski.

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Theorem (Linnell, Schick)

1. *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under extensions with elementary amenable groups as quotients and directed unions. Then every torsionfree group G which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture.*
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Approximation

Theorem (Approximation Theorem, Lück)

Let X be a connected CW-complex of finite type. Suppose that π is residually finite, i.e., there is a nested sequence

$$\pi = G_0 \supset G_1 \supset G_2 \supset \ldots$$

of normal subgroups of finite index with $\cap_{i \geq 1} G_i = \{1\}$. Let X_i be the finite $[\pi : G_i]$-sheeted covering of X associated to G_i.

Then for any such sequence $(G_i)_{i \geq 1}$

$$b_n^{(2)}(\widetilde{X}) = \lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}.$$

- Ordinary Betti numbers are not multiplicative under finite coverings, whereas the L^2-Betti numbers are. With the expression

$$\lim_{i\to\infty} \frac{b_n(X_i)}{[G:G_i]},$$

  we try to force the Betti numbers to be multiplicative by a limit process.

- The theorem above says that L^2-Betti numbers are asymptotic Betti numbers. It was conjectured by Gromov.

Applications to deficiency and signature

Definition (Deficiency)

Let G be a finitely presented group. Define its deficiency

$$\operatorname{defi}(G) := \max\{g(P) - r(P)\}$$

where P runs over all presentations P of G and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation P.

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Lemma

Let G be a finitely presented group. Then

$$\operatorname{defi}(G) \;\leq\; 1 - |G|^{-1} + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Proof.

We have to show for any presentation P that

$$g(P) - r(P) \;\leq\; 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Let X be a CW-complex realizing P. Then

$$\chi(X) = 1 - g(P) + r(P) = b_0^{(2)}(\widetilde{X}) + b_1^{(2)}(\widetilde{X}) - b_2^{(2)}(\widetilde{X}).$$

Since the classifying map $X \to BG$ is 2-connected, we get

$$\begin{array}{lcll} b_n^{(2)}(\widetilde{X}) & = & b_n^{(2)}(G) & \quad \text{for } n = 0, 1; \\ b_2^{(2)}(\widetilde{X}) & \geq & b_2^{(2)}(G). & \end{array}$$

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Theorem (Deficiency and extensions, Lück)

Let $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ *be an exact sequence of infinite groups. Suppose that G is finitely presented H is finitely generated. Then:*

1. $b_1^{(2)}(G) = 0$;
2. $\operatorname{defi}(G) \leq 1$;
3. *Let M be a closed oriented 4-manifold with G as fundamental group. Then*
$$|\operatorname{sign}(M)| \leq \chi(M).$$

The Singer Conjecture

Conjecture (Singer Conjecture)

If M is an aspherical closed manifold, then

$$b_n^{(2)}(\widetilde{M}) = 0 \quad \text{if } 2n \neq \dim(M).$$

If M is a closed Riemannian manifold with negative sectional curvature, then

$$b_n^{(2)}(\widetilde{M}) \begin{cases} = 0 & \text{if } 2n \neq \dim(M); \\ > 0 & \text{if } 2n = \dim(M). \end{cases}$$

- The computations presented above do support the Singer Conjecture.
- Under certain negative pinching conditions the Singer Conjecture has been proved by Ballmann-Brüning, Donnelly-Xavier, Jost-Xin.

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- Because of the Euler-Poincaré formula

$$\chi(M) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{M})$$

the Singer Conjecture implies the following conjecture provided that M has non-positive sectional curvature.

Conjecture (Hopf Conjecture)

If M is a closed Riemannian manifold of even dimension with sectional curvature $\operatorname{sec}(M)$*, then*

$$\begin{array}{rclcrcl} (-1)^{\dim(M)/2} \cdot \chi(M) & > & 0 & \textit{if} & \operatorname{sec}(M) & < & 0; \\ (-1)^{\dim(M)/2} \cdot \chi(M) & \geq & 0 & \textit{if} & \operatorname{sec}(M) & \leq & 0; \\ \chi(M) & = & 0 & \textit{if} & \operatorname{sec}(M) & = & 0; \\ \chi(M) & \geq & 0 & \textit{if} & \operatorname{sec}(M) & \geq & 0; \\ \chi(M) & > & 0 & \textit{if} & \operatorname{sec}(M) & > & 0. \end{array}$$