The Farrell-Jones Conjecture for the Hecke algebras of reductive p-adic groups

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Main result

- This is a joint work with Arthur Bartels.
- We formulate a version of the Farrell-Jones Conjecture for the algebraic K-theory of the Hecke algebra of a totally disconnected locally compact second countable Hausdorff group.
- We can prove it for any closed subgroup of a reductive p-adic group.
- This is interesting for the theory of smooth representations of reductive p-adic groups.
- Export of methods from topology to other areas.

Outlook

- Introduction to the Farrell-Jones Conjecture for discrete groups.
- Formulation of the Farrell-Jones Conjecture for the algebraic K-theory of the Hecke algebra of a td-group.
- Explanations why it is much harder to deal with totally disconnected groups than with discrete groups.
- Motivation for trying to extend the Farrell-Jones Conjecture from discrete groups to reductive p-adic groups.
- We will not say much about the actual proof which is very technical and involved, but give some basic ideas.
- I will not cover in this talk all the slides, which can be found on my homepage.

Motivation and Statement of the Farrell-Jones Conjecture for torsionfree groups

- For some time G is a discrete group.
- There are K-groups $K_n(R)$ for every $n \in \mathbb{Z}$.
- Can one identify $K_n(RG)$ with more accessible terms?
- If G_0 and G_1 are torsionfree and R is regular, one gets isomorphisms

$$\begin{aligned} K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R); \\ \widetilde{K}_n(R[G_0 * G_1]) &\cong \widetilde{K}_n(RG_0) \oplus \widetilde{K}_n(RG_1). \end{aligned}$$

- If $\mathcal{H}$ is any (generalized) homology theory, then

$$\begin{aligned} \mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\mathrm{pt}) \oplus \mathcal{H}_{n-1}(\mathrm{pt}); \\ \widetilde{\mathcal{H}}_n(B(G_0 * G_1)) &\cong \widetilde{\mathcal{H}}_n(BG_0) \oplus \widetilde{\mathcal{H}}_n(BG_1). \end{aligned}$$

- Question: Can we find $\mathcal{H}_*$ with $\mathcal{H}_n(BG) \cong K_n(RG)$, provided that G is torsionfree and R is regular?
- Of course such $\mathcal{H}_*$ has to satisfy $\mathcal{H}_n(\mathrm{pt}) = K_n(R)$.
- So the only reasonable candidate is $H_n(-;\mathbf{K}_R)$.

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG;\mathbf{K}_R) \to K_n(RG)$$

is bijective for every $n \in \mathbb{Z}$.

- There is also an *L*-theory version.

Applications of the Farrell-Jones Conjecture

Definition (Topologically rigid)

A closed topological manifold N is called topologically rigid if any homotopy equivalence $f\colon M \to N$ with a closed manifold M as source is homotopic to a homeomorphism.

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts that an aspherical closed manifold with fundamental group G is topologically rigid.

- In particular the Borel Conjecture predicts that two aspherical closed manifolds are homeomorphic if and only if their fundamental groups are isomorphic.
- The Farrell-Jones Conjecture implies the Borel Conjecture in dimensions ≥ 5.

There are many other applications of the Farrell-Jones Conjecture, for instance:

- Vanishing of $\widetilde{K}_n(\mathbb{Z}G)$ for $n \leq 0$ and of $\mathrm{Wh}(G)$ for torsionfree G
- Novikov Conjecture
- Bass Conjecture
- Moody's Induction Conjecture
- Serre's Conjecture
- Classification of manifolds
- Poincaré duality groups
- κ-classes for aspherical manifolds
- Hyperbolic groups with a sphere as boundary
- Rational calculation of the homotopy groups of the space of automorphism of aspherical closed manifolds in a certain range
- Stable Cannon Conjecture

The general version of the Farrell-Jones Conjecture

- One can formulate a version of the Farrell-Jones Conjecture which makes sense for all groups G and all rings R.

Conjecture (K-theoretic Farrell-Jones-Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG).$$

is bijective for every $n \in \mathbb{Z}$.

- $H_*^G(-;\mathbf{K}_R)$ is a G-homology theory satisfying

$$H_n^G(G/H;\mathbf{K}_R) \cong K_n(RH).$$

- The Farrell-Jones Conjecture is equivalent to the homotopy theoretic version that we have a weak homotopy equivalence

$$\operatorname{hocolim}_{V \in \mathcal{S}\mathrm{ub}_{\mathcal{VC}\mathrm{yc}}(G)} \mathbf{K}_R(G/V) \xrightarrow{\simeq} \mathbf{K}_R(G/G).$$

- There is also an L-theory version.

- The Farrell-Jones Conjecture and the Baum-Connes Conjecture share some analogies and are indeed related.

- One can also allow twisted group rings and orientation characters.

- In the sequel the Full Farrell-Jones Conjecture refers to the most general version for both K-theory and L-theory, namely, with coefficients in additive G-categories (with involution) and finite wreath products.

Status of the Full Farrell-Jones Conjecture

Theorem (Bartels, Bestvina, Farrell, Kammeyer, Lück, Reich, Rüping, Wegner)

Let $\mathcal{FJ}$ be the class of groups for which the Full Farrell-Jones Conjecture holds. Then $\mathcal{FJ}$ contains the following groups:

- *Hyperbolic groups;*
- $\mathrm{CAT}(0)$*-groups;*
- *Solvable groups;*
- *(Not necessarily uniform) lattices in almost connected Lie groups;*
- *Fundamental groups of (not necessarily compact) d-dimensional manifolds (possibly with boundary) for $d \leq 3$;*
- *Subgroups of $GL_n(\mathbb{Q})$ and of $GL_n(F[t])$ for a finite field F;*
- *All S-arithmetic groups;*
- *mapping class groups.*

Theorem (continued)

Moreover, $\mathcal{FJ}$ has the following inheritance properties:

- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *If $H \subseteq G$ is a subgroup of G with $[G : H] < \infty$ and $H \in \mathcal{FJ}$, then $G \in \mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*

- Many more mathematicians have made important contributions to the Farrell-Jones Conjecture, e.g., Bökstedt, Bunke, Carlsson, Farrell, Ferry, Goodwillie, Hsiang, Jones, Kasprowski, Linnell, Madsen, Pedersen, Quinn, Ranicki, Reich, Rognes, Varisco, Weinberger, Winges, Yu, Wu.

- The Farrell-Jones Conjecture is open for:
  - $\mathrm{Out}(F_n)$;
  - amenable groups;
  - Thompson's groups.

- It would be great to find a counterexample.

- The Farrell-Jones Conjecture is true for all groups, if and only if it is true for one specific finitely presented group, namely the universal finitely presented group due to Higman. It can be constructed as the complement of an embedded S^2 in S^4.

Towards the Farrell-Jones Conjecture for reductive p-adic groups

- For fields of characteristic zero and discrete G one can replace the family $\mathcal{VC}$yc by the smaller family $\mathcal{FIN}$.

Conjecture (Farrell-Jones Conjecture for fields of characteristic zero)

Let F be a field of characteristics zero. Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_F) \xrightarrow{\cong} K_n(FG)$$

is an isomorphism for $n \in \mathbb{Z}$.

Conjecture (Farrell-Jones Conjecture for td-groups)

For any td-group G and any uniformly regular ring R satisfying $\mathbb{Q} \subseteq R$ and $n \in \mathbb{Z}$ the assembly map

$$H_n^G(E_{\mathcal{COM}}(G); \mathbf{K}_R) \xrightarrow{\cong} H_n^G(G/G; \mathbf{K}_R) = K_n(\mathcal{H}(G; R))$$

is an isomorphism.

- $H_*^G(-; \mathbf{K}_R)$ is a smooth G-homology theory satisfying $H_n^G(G/H; \mathbf{K}_R) \cong K_n(\mathcal{H}(H; R))$ for every open subgroup $H \subseteq G$.

Theorem (Bartels-Lück (2023))

The Farrell-Jones Conjecture holds for G if G is a closed subgroup of a reductive p-adic group.

Dictionary

td-groups	discrete groups
Smooth G-representations over R	G-representations over R
Hecke algebra $\mathcal{H}(G;R)$	group ring RG
$\exists$ approximate unit	$\exists$ unit
{smooth G-representations} = {n.d. $\mathcal{H}(G;R)$-modules}	$\{G$-representations$\}$ = $\{RG$-modules$\}$

reductive p-adic groups	CAT(0)-groups
Examples: $\mathrm{GL}_n(\mathbb{Q}_p)$, $\mathrm{SL}_n(\mathbb{Q}_p)$	Examples: Fundamental groups of closed manifolds with non-positive sectional curvature
Cocompact proper smooth action on the associated Bruhat-Tits building	Cocompact proper action on a CAT(0)-space
Family $\mathcal{COM}$ of compact open subgroups	Family $\mathcal{FIN}$ of finite subgroups

A sequence of subgroups $K \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ of a compact td-group does in general *not* stabilize after finitely many steps.	A sequence of subgroups $F \supseteq F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ of a finite group F stabilizes after finitely many steps.
The space $G/H \times G/K$ with the diagonal G-action is in general *not* G-homotopy equivalent to a G-CW-complex.	The space $G/H \times G/K$ with the diagonal G-action is a zero-dimensional G-CW-complex.
The classifying spaces $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$ are *not* G-homotopy equivalent in general. Fortunately $E_{\mathcal{COM}}(G)$ and $J_{\mathcal{COM}}(G)$ are G-homotopy equivalent.	The classifying spaces $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$ are G-homotopy equivalent.

The Hecke algebra is only functorial under open group homomorphisms. In particular it is *not* functorial under inclusions of subgroups $H \subseteq G$, unless H is open in G.	The group ring RG is functorial under any group homomorphism.
$\exists$ unit $\Leftrightarrow$ G discrete	$\exists$ unit
Reductive p-adic groups contain interesting closed but not open subgroups such as the Borel subgroup.	not applicable

Applications to the theory of smooth representations over p-adic groups

Theorem (Bernstein, Dat)

Let G be a reductive p-adic group. Then the canoncial map

$$\operatorname*{colim}_{K \in \operatorname{Sub}_{\mathcal{COM}}(G)} K_0(\mathcal{H}(K;\mathbb{C})) \xrightarrow{\cong} K_0(\mathcal{H}(G;\mathbb{C}))$$

is rationally an isomorphism.

Theorem (Bartels-Lück)

Let G be a closed subgroup of a reductive p-adic group. Let R be a uniformly regular ring satisfying $\mathbb{Q} \subseteq R$. Then the canoncial map

$$\operatorname*{colim}_{K \in \operatorname{Sub}_{\mathcal{COM}}(G)} K_0(\mathcal{H}(K;R)) \xrightarrow{\cong} K_0(\mathcal{H}(G;R))$$

is an isomorphism.

- A result by Schneider and Stuhler says that any admissible smooth representation over a reductive p-adic group with complex coefficients admits a finite projective resolution by compactly induced finitely generated projective modules.
- Our result implies that this holds in full generality, namely, one can replace admissible by finitely generated.

- There is the conjecture, often attributed to Bernstein, that every irreducible super-cuspidal representation of a reductive p-adic group is (compactly) induced from some compact open subgroup.
- Fintzen has made substantial progress about this conjecture.
- Our results can be viewed as a K-theoretic analogue proved in much more generality.

Strategy of the proof in the discrete case

- The assembly map can be thought of as an approximation of the algebraic K- or L-theory by a homology theory.
- The basic feature between the left and right side of the assembly map is that on the left side one has excision which is not present on the right side.
- In general excision is available if one can make representing cycles small.
- A best illustration for this is the proof of excision for simplicial or singular homology based on subdivision whose effect is to make the support of cycles arbitrary small.

- Then the basic goal of the proof is obvious: Find a procedure to make the support of a representing cocycle as small as possible without changing its class.
- Suppose that $G = \pi_1(M)$ for a closed Riemannian manifold with negative sectional curvature.
- The idea is to use the geodesic flow on the sphere tangent bundle of the universal covering to gain the necessary control and to use transfer methods which allow to work on the sphere tangent bundle.
- The obvious strategy to construct a direct inverse map has never worked out and led only to rational injective detection maps, often with topological cyclic homology groups as target, see Böksteadt-Hsiang-Madsen and Lück-Reich-Rognes-Varisco.

Challenges in the td-case

- It is not enough to consider open subgroups. In particular the reduction from covirtually cyclic subgroups to compact open subgroups is very difficult.
- One needs to pass from strict G-actions to homotopy G-actions up to infinite coherence.
- Extension of the functor $G/H \mapsto \mathbf{K}(\mathcal{H}(H))$ to subgroups $H \subseteq G$ which are not open.
- The classifying spaces $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$ are *not* G-homotopy equivalent in general.
- One needs to understand the geometry of the Bruhat-Tits building and its compactification and understand the associated geodesic flow space.
- Lack of Devissage Theorems for non-connective K-theory of exact categories.