Introduction to the Farrell-Jones Conjecture

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$K_0(R)$ and the Idempotent Conjecture

- Given a ring R and a group G, denote by RG or $R[G]$ the group ring.
- An RG-module is the same as G-representation with coefficients in R, i.e., an R-module with G-action by R-linear maps.
- If $\overline{X} \to X$ is a G-covering of a CW-complex X, then the cellular chain complex of $\overline{X}$ is a free $\mathbb{Z}G$-chain complex.

- If g has finite order $|g|$ and F is a field of characteristic zero, then we get an idempotent in FG by

$$x = \frac{1}{|g|} \cdot \sum_{i=0}^{|g|-1} g^i.$$

- Are there other idempotents?

Conjecture (Idempotent Conjecture)

The Kaplansky Conjecture says that for a torsionfree group G and a field F of characteristic zero the elements 0 and 1 are the only idempotents in FG.

Definition (Projective class group $K_0(R)$)

Define the projective class group of a ring R

$$K_0(R)$$

to be the following abelian group:

- Generators are isomorphism classes $[P]$ of finitely generated projective R-modules P;
- The relations are $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

- The assignment $P \mapsto [P] \in K_0(R)$ is the universal additive invariant or dimension function for finitely generated projective R-modules.

Definition (Reduced Projective class group $\widetilde{K}_0(R)$)

The reduced projective class group

$$\widetilde{K}_0(R) = \operatorname{cok}\big(K_0(\mathbb{Z}) \to K_0(R)\big)$$

is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R-modules.

- Let P be a finitely generated projective R-module. It is stably free, i.e., $P \oplus R^m \cong R^n$ for some $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$.

Conjecture (Vanishing of reduced projective class group for torsionfree G)

If G is torsionfree, then $\widetilde{K}_0(\mathbb{Z}G)$ and $\widetilde{K}_0(FG)$ for a field F of characteristic zero vanish.

- The last conjecture implies the Idempotent Conjecture.

Definition (K_1-group $K_1(R)$)

Define the K_1-group of a ring R

$$K_1(R)$$

to be the abelian group whose generators are conjugacy classes $[f]$ of automorphisms $f\colon P \to P$ of finitely generated projective R-modules with the following relations:

- Given an exact sequence $0 \to (P_0, f_0) \to (P_1, f_1) \to (P_2, f_2) \to 0$ of automorphisms of finitely generated projective R-modules, we get $[f_0] + [f_2] = [f_1]$;
- $[g \circ f] = [f] + [g]$.

- Put $GL(R) := \bigcup_{n \geq 1} GL_n(R)$. The obvious maps $GL_n(R) \to K_1(R)$ induce an isomorphism

$$GL(R)/[GL(R), GL(R)] \xrightarrow{\cong} K_1(R).$$

- An invertible matrix $A \in GL(R)$ can be reduced by elementary row and column operations and (de-)stabilization to the trivial empty matrix if and only if $[A] = 0$ holds in the reduced K_1-group

$$\widetilde{K}_1(R) := K_1(R)/\{\pm 1\} = \operatorname{cok}\left(K_1(\mathbb{Z}) \to K_1(R)\right).$$

- The assignment $A \mapsto [A] \in K_1(R)$ can be thought of as the universal determinant for R.

Definition (Whitehead group)

The Whitehead group of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

Theorem (*s*-Cobordism Theorem, Barden, Mazur, Stallings, Kirby-Siebenmann)

Let M be a closed smooth or topological manifold of dimension ≥ 5. Then the so called Whitehead torsion yields a bijection

$$\tau\colon \mathcal{H}(M) \xrightarrow{\cong} \mathrm{Wh}(\pi_1(M))$$

where $\mathcal{H}(M)$ is the set of h-cobordisms over M modulo diffeomorphisms or homeomorphisms relative M.

Conjecture (Vanishing of $\mathrm{Wh}(G)$ for torsionfree G)

If G is torsionfree, then

$$\mathrm{Wh}(G) = \{0\}.$$

Lemma

Let G be finitely presented and $d \geq 5$ be any natural number. Then the following statements are equivalent:

- *The Whitehead group $\mathrm{Wh}(G)$ vanishes;*
- *For one closed manifold M of dimension d with $G \cong \pi_1(M)$ every h-cobordism over M is trivial;*
- *For every closed manifold M of dimension d with $G \cong \pi_1(M)$ every h-cobordism over M is trivial.*

Conjecture (Poincaré Conjecture)

Let M be an n-dimensional topological manifold which is a homotopy sphere, i.e., homotopy equivalent to S^n.

Then M is homeomorphic to S^n.

Theorem (Freedman, Perelman, Smale)

The Poincaré Conjecture is true.

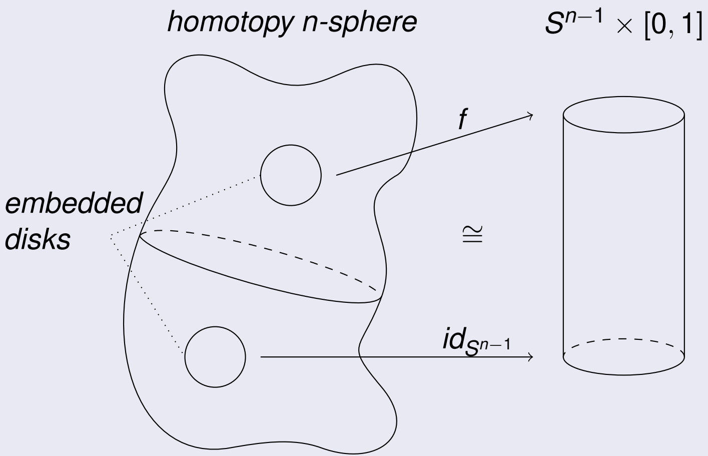
Proof.

We sketch the proof for $n \geq 6$.

- Let M be a n-dimensional homotopy sphere.
- Let W be obtained from M by deleting the interior of two disjoint embedded disks D_1^n and D_2^n. Then W is a simply connected h-cobordism.
- Since $\mathrm{Wh}(\{1\})$ is trivial, we can find a homeomorphism $f\colon W \xrightarrow{\cong} \partial D_1^n \times [0,1]$ which is the identity on $\partial D_1^n = D_1^n \times \{0\}$.
- By the Alexander trick we can extend the homeomorphism $f|_{D_1^n \times \{1\}}\colon \partial D_2^n \xrightarrow{\cong} \partial D_1^n \times \{1\}$ to a homeomorphism $g\colon D_1^n \to D_2^n$.
- The three homeomorphisms $id_{D_1^n}$, f and g fit together to a homeomorphism $h\colon M \to D_1^n \cup_{\partial D_1^n \times \{0\}} \partial D_1^n \times [0,1] \cup_{\partial D_1^n \times \{1\}} D_1^n$. The target is obviously homeomorphic to S^n.

$\square$

Figure (Proof of the Poincaré Conjecture)



- The argument above does not imply that for a smooth manifold M we obtain a diffeomorphism $g\colon M \to S^n$ since the Alexander trick does not work smoothly.
- Indeed, there exist so called exotic spheres, i.e., closed smooth manifolds which are homeomorphic but not diffeomorphic to S^n.
- The s-cobordism theorem is a key ingredient in the Surgery Program for the classification of closed manifolds due to Browder, Novikov, Sullivan and Wall.

Motivation and Statement of the Farrell-Jones Conjecture for torsionfree groups

- There are K-groups $K_n(R)$ for every $n \in \mathbb{Z}$.
- Can one identify $K_n(RG)$ with more accessible terms?
- If G_0 and G_1 are torsionfree and R is regular, one gets isomorphisms

$$\begin{aligned} K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R); \\ \widetilde{K}_n(R[G_0 * G_1]) &\cong \widetilde{K}_n(RG_0) \oplus \widetilde{K}_n(RG_1). \end{aligned}$$

- If $\mathcal{H}$ is any (generalized) homology theory, then

$$\begin{aligned} \mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\mathrm{pt}) \oplus \mathcal{H}_{n-1}(\mathrm{pt}); \\ \widetilde{\mathcal{H}}_n(B(G_0 * G_1)) &\cong \widetilde{\mathcal{H}}_n(BG_0) \oplus \widetilde{\mathcal{H}}_n(BG_1). \end{aligned}$$

- Question: Can we find $\mathcal{H}_*$ with $\mathcal{H}_n(BG) \cong K_n(RG)$, provided that G is torsionfree and R is regular.
- Of course such $\mathcal{H}_*$ has to satisfy $\mathcal{H}_n(\mathrm{pt}) = K_n(R)$.
- So the only reasonable candidate is $H_n(-;\mathbf{K}_R)$.

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG;\mathbf{K}_R) \to K_n(RG)$$

is bijective for every $n \in \mathbb{Z}$.

- There is also an *L*-theory version.

Applications of the Farrell-Jones Conjecture

- The conjectures above about the vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ for torsionfree G do follow from the Farrell-Jones Conjecture above.
- The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG;\mathbf{K}_R)$ whose E^2-term is given by

  $$E^2_{p,q} = H_p(BG, K_q(R)),$$

  using

  $$K_n(\mathbb{Z}) = \begin{cases} \{0\} & n \leq -1; \\ \mathbb{Z} & n = 0; \\ \{\pm 1\} & n = 1. \end{cases}$$

Definition (Topologically rigid)

A closed topological manifold N is called topologically rigid if any homotopy equivalence $f\colon M \to N$ with a closed manifold M as source is homotopic to a homeomorphism.

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts that an aspherical closed manifold with fundamental group G is topologically rigid.

- In particular the Borel Conjecture predicts that two aspherical closed manifolds are homeomorphic if and only if their fundamental groups are isomorphic.
- The Poincaré Conjecture is equivalent to the statement that S^n is topologically rigid.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity.

  A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds of dimension ≥ 3 is homotopic to an isometric diffeomorphism.

- The Borel Conjecture is not true in the smooth category by results of Farrell-Jones.

- The Borel Conjecture follows in dimension ≥ 5 from the Farrell-Jones Conjecture as briefly explained next.

Definition (The structure set)

Let N be a closed topological manifold of dimension n. We call two homotopy equivalences $f_i \colon M_i \to N$ from closed topological manifolds M_i of dimension n to N for $i = 0, 1$ equivalent if there exists a homeomorphism $g \colon M_0 \to M_1$ such that $f_1 \circ g$ is homotopic to f_0.

The structure set $\mathcal{S}_n(N)$ of N is the set of equivalence classes of homotopy equivalences $M \to X$ from closed topological manifolds of dimension n to N. This set has a preferred base point, namely the class of the identity $\mathrm{id} \colon N \to N$.

- A closed topological manifold M is topologically rigid if and only if the structure set $\mathcal{S}_n(M)$ consists of exactly one point.

Theorem (The topological Surgery Exact Sequence, Browder, Novikov, Sullivan, Wall)

For a closed n-dimensional topological manifold N with $n \geq 5$, there is an exact sequence of abelian groups, called surgery exact sequence,

$$\ldots \xrightarrow{\eta} \mathcal{N}^{\mathrm{top}}_{n+1}(N \times [0,1], N \times \{0,1\}) \xrightarrow{\sigma_{n+1}} L^s_{n+1}(\mathbb{Z}\pi) \xrightarrow{\partial} \mathcal{S}^{\mathrm{top}}_n(N) \xrightarrow{\eta} \mathcal{N}^{\mathrm{top}}_n(N) \xrightarrow{\sigma_n} L^s_n(\mathbb{Z}\pi)$$

- It is the main ingredient in showing that the Farrell-Jones Conjecture implies the Borel Conjecture in dimension ≥ 5.
- The key idea is to use the Farrell-Jones Conjecture to show that the map σ_{n+1} is surjective and the map σ_n is injective.

Theorem (Bartels-Lück-Weinberger)

Let G be a torsionfree hyperbolic group and let n be an integer ≥ 6.

Then the following statements are equivalent:

- *The boundary ∂G is homeomorphic to S^{n-1};*
- *There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.*

The manifold above is unique up to homeomorphism.

Theorem (Homotopy groups of automorphism groups of aspherical manifolds)

Let M be an orientable closed aspherical (smooth) manifold of dimension > 10 with fundamental group G. Suppose that G satisfies the K-and the L-theoretic Farrell Jones Conjecture.

Then for $1 \leq i \leq (\dim M - 7)/3$ one has

$$\pi_i(\mathrm{Top}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathrm{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ 0 & \text{if } i > 1, \end{cases}$$

and

$$\pi_i(\mathrm{Diff}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathrm{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ \bigoplus_{j=1}^{\infty} H_{(i+1)-4j}(M;\mathbb{Q}) & \text{if } i > 1,\ \dim M \text{ odd}; \\ 0 & \text{if } i > 1,\ \dim M \text{ even}. \end{cases}$$

There are many other applications of the Farrell-Jones Conjecture, for instance:

- Novikov Conjecture.
- Bass Conjecture.
- Moody's Induction Conjecture.
- Serre's Conjecture.
- Classification of certain classes of manifolds with infinite fundamental group.
- Classification of Poincaré duality groups.
- κ-classes for aspherical manifolds.
- Stable Cannon Conjecture.

The general version the Farrell-Jones Conjecture

- One can formulate a version of the Farrell-Jones Conjecture which makes sense for all groups G and all rings R.

Conjecture (K-theoretic Farrell-Jones-Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG).$$

is bijective for every $n \in \mathbb{Z}$.

- There is also an L-theory version.
- One can also allow twisted group rings and orientation characters.
- In the sequel the Full Farrell-Jones Conjecture refers to the most general version for both K-theory and L-theory, namely, with coefficients in additive G-categories (with involution) and finite wreath products.
- All conjectures or results mentioned in this talk follow from the Full Farrell-Jones Conjecture.

Status of the Full Farrell-Jones Conjecture

Theorem (Bartels, Bestvina, Farrell, Kammeyer, Lück, Reich, Rüping, Wegner)

Let $\mathcal{FJ}$ be the class of groups for which the Full Farrell-Jones Conjecture holds. Then $\mathcal{FJ}$ contains the following groups:

- *Hyperbolic groups;*
- $\mathrm{CAT}(0)$*-groups;*
- *Solvable groups;*
- *(Not necessarily uniform) lattices in almost connected Lie groups;*
- *Fundamental groups of (not necessarily compact) d-dimensional manifolds (possibly with boundary) for $d \leq 3$;*
- *Subgroups of $GL_n(\mathbb{Q})$ and of $GL_n(F[t])$ for a finite field F;*
- *All S-arithmetic groups;*
- *mapping class groups.*

Theorem (continued)

Moreover, $\mathcal{FJ}$ has the following inheritance properties:

- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *If $H \subseteq G$ is a subgroup of G with $[G : H] < \infty$ and $H \in \mathcal{FJ}$, then $G \in \mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*

- Many more mathematicians have made important contributions to the Farrell-Jones Conjecture, e.g., Bökstedt, Carlsson, Jim Davis, Ferry, Hambleton, Gandini, Hsiang, Jones, Kasprowski, Linnell, Madsen, Nicas, Pedersen, Quinn, Ranicki, Rognes, Roushon, Rosenthal, Stark, Tessera, Varisco, Weinberger, Yu, Wu.

The Farrell-Jones Conjecture is open for:

- $\mathrm{Out}(F_n)$;
- amenable groups;
- Thompson's groups;
- $G = F_n \rtimes \mathbb{Z}$.

- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.
- One example is the construction of groups with expanders due to Gromov, see Arzhantseva-Delzant. These yield counterexamples to the Baum-Connes Conjecture with coefficients due to Higson-Lafforgue-Skandalis.
- However, our results show that these groups do satisfy the Full Farrell-Jones Conjecture and hence also the other conjectures mentioned above.
- Many groups of the region 'Hic abundant leones' in the universe of groups in the sense of Bridson do satisfy the Full Farrell-Jones Conjecture.
- We have no good candidate for a group (or for a property of groups) for which the Farrell-Jones Conjecture may fail.

- Davis-Januszkiewicz have constructed exotic aspherical closed manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.

- However, in all cases the universal coverings are CAT(0)-spaces and the fundamental groups are CAT(0)-groups. Hence they satisfy the Full Farrell-Jones Conjecture and in particular the Borel Conjecture in dimension ≥ 5.

Ideas of proofs

- The assembly map can be thought of an approximation of the algebraic K- or L-theory by a homology theory.
- The basic feature between the left and right side of the assembly map is that on the left side one has excision which is not present on the right side.
- In general excision is available if one can make representing cycles small.
- A best illustration for this is the proof of excision for simplicial or singular homology based on subdivision whose effect is to make the support of cycles arbitrary small.

- Then the basic goal of the proof is obvious: Find a procedure to make the support of a representing cocycle as small as possible without changing its class.
- Suppose that $G = \pi_1(M)$ for a closed Riemannian manifold with negative sectional curvature.
- The idea is to use the geodesic flow on the universal covering to gain the necessary control.
- We will briefly explain this in the case, where the universal covering is the two-dimensional hyperbolic space $\mathbb{H}^2$.

- Consider two points with coordinates (x_1, y_1) and (x_2, y_2) in the upper half plane model of two-dimensional hyperbolic space. We want to use the geodesic flow to make their distance smaller in a functorial fashion. This is achieved by letting these points flow towards the boundary at infinity along the geodesic given by the vertical line through these points, i.e., towards infinity in the y-direction.

- There is a fundamental problem: if $x_1 = x_2$, then the distance between these points is unchanged. Therefore we make the following prearrangement. Suppose that $y_1 < y_2$. Then we first let the point (x_1, y_1) flow so that it reaches a position where $y_1 = y_2$. Inspecting the hyperbolic metric, one sees that the distance between the two points (x_1, τ) and (x_2, τ) goes to zero if τ goes to infinity. This is the basic idea n the negatively curved case to make the cycles small, or in other words, to gain control.