Topological Rigidity of Aspherical Manifolds

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Outline

- Present a list of prominent conjectures such as the one due to Borel, Farrell-Jones, Kaplansky and Novikov.
- State our main theorem which is joint work with Bartels. It says that these conjectures are true for an interesting class of groups including word-hyperbolic groups and CAT(0)-groups.
- Discuss consequences and open cases.

The Borel Conjecture

Definition (Topologically rigid)

A closed topological manifold M is called *topologically rigid* if any homotopy equivalence $N \to M$ with some manifold N as source and M as target is homotopic to a homeomorphism.

- The Poincaré Conjecture in dimension n is equivalent to the statement that S^n is topologically rigid.

Theorem (Kreck-Lück (2006))

- *Suppose that $k + d \neq 3$. Then $S^k \times S^d$ is topologically rigid if and only if both k and d are odd.*
- *If Thurston's Geometrization Conjecture is true, then every closed 3-manifold with torsionfree fundamental group is topologically rigid.*
- *Let M and N be closed manifolds of the same dimension $n \geq 5$ such that neither $\pi_1(M)$ nor $\pi_1(N)$ contains elements of order 2. If both M and N are topologically rigid, then the same is true for their connected sum $M \# N$.*

Theorem (Chang-Weinberger (2003))

Let M^{4k+3} be a closed oriented smooth manifold for $k \geq 1$ whose fundamental group has torsion. Then M is not topologically rigid.

- Hence in most cases the fundamental group of a topologically rigid manifold is torsionfree.

Definition (Aspherical manifold)

A manifold M is called *aspherical* if $\pi_n(M) = 0$ for $n \geq 2$, or, equivalently, $\widetilde{M}$ is contractible.

- If M is a closed smooth Riemannian manifold with non-positive sectional curvature, then it is aspherical.
- Let L be a connected Lie group, $K \subseteq L$ a maximal compact Lie group and $G \subseteq L$ a discrete torsionfree group. Then $G \backslash L/K$ is an aspherical closed smooth manifold.

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts that a closed aspherical manifold M with $\pi_1(M) \cong G$ is topologically rigid.

- Two aspherical manifolds are homotopy equivalent if and only if their fundamental groups are isomorphic.
- The Borel Conjecture predicts that two aspherical manifolds have isomorphic fundamental groups if and only if they are homeomorphic.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity.
- One version of Mostow rigidity says that any homotopy equivalence between hyperbolic closed Riemannian manifolds is homotopic to an isometric diffeomorphism.
- In particular they are isometrically diffeomorphic if and only if their fundamental groups are isomorphic.

- The Borel Conjecture becomes definitely false if one replaces homeomorphism by diffeomorphism.
- For instance, there are smooth manifolds M which are homeomorphic to T^n but not diffeomorphic to T^n.

Other prominent Conjectures

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Conjecture (Reduced projective class group)

If R is a principal ideal domain and G is torsionfree, then $\widetilde{K}_0(RG) = 0$.

- The vanishing of $\widetilde{K}_0(RG)$ is equivalent to the statement that any finitely generated projective RG-module P is stably free, i.e., there are $m, n \geq 0$ with $P \oplus RG^m \cong RG^n$;
- Let G be a finitely presented group. The vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ is equivalent to the geometric statement that any finitely dominated space X with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex.
- The last conjecture implies the Conjecture due to Serre that a group of type FP is already of type FF.

Conjecture (Whitehead group)

If G is torsionfree, then the Whitehead group $\mathrm{Wh}(G)$ vanishes.

- Fix $n \geq 6$. The vanishing of $\mathrm{Wh}(G)$ is equivalent to the following geometric statement: Every compact n-dimensional h-cobordism W with $G \cong \pi_1(W)$ is trivial.

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f \colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\operatorname{sign}_x(M, f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M, f).

Definition (Poincaré duality group)

A group is called a *Poincaré duality group of dimension n* if it of type FP and

$$H^i(G;\mathbb{Z}G) \cong \begin{cases} \{0\} & \text{for } i \neq n; \\ \mathbb{Z} & \text{for } i = n. \end{cases}$$

Conjecture (Poincaré duality groups)

Let G be a finitely presented Poincaré duality group. Then there is a closed ANR-homology manifold with $\pi_1(M) \cong G$.

- One may also hope that M can be choosen to be a closed manifold.
- But then one runs into Quinn's resolutions obstruction.

The Farrell-Jones Conjecture and its consequences

Conjecture (K-theoretic Farrell-Jones Conjecture for regular rings and torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- There is an L-theoretic version of the Farrell-Jones Conjecture.
- Both the K-theoretic and the L-theoretic Farell-Jones Conjecture can be formulated for arbitrary groups G and arbitrary rings R allowing also a G-twist on R.

Theorem (The Farrell-Jones Conjecture implies (nearly) everything)

If G satisfies both the K-theoretic and L-theoretic Farrell-Jones Conjecture (for any additive G-category as coefficients), then all the conjectures mentioned above follow for G, i.e., for the Borel Conjecture (for $\dim \geq 5$), Kaplansky Conjecture, Vanishing of $\widetilde{K}_0(RG)$ and $\mathrm{Wh}(G)$, Novikov Conjecture (for $\dim \geq 5$), Serre's Conjecture, Conjecture about Poincaré duality groups, and other conjecture as well.

- We want to explain this for the Borel Conjecture.

Definition (Structure set)

The *structure set* $S^{top}(M)$ of a manifold M consists of equivalence classes of homotopy equivalences $N \longrightarrow M$ with a manifold N as source.
Two such homotopy equivalences $f_0 \colon N_0 \longrightarrow M$ and $f_1 \colon N_1 \longrightarrow M$ are equivalent if there exists a homeomorphism $g \colon N_0 \longrightarrow N_1$ with $f_1 \circ g \simeq f_0$.

Theorem

A closed manifold M is topologically rigid if and only if $\mathcal{S}^{\text{top}}(M)$ consists of one element.

Theorem (Algebraic surgery sequence Ranicki (1992))

There is an exact sequence of abelian groups called algebraic surgery exact sequence for an n-dimensional closed manifold M

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}} \mathcal{S}^{\mathrm{top}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

It can be identified with the classical geometric surgery sequence due to Browder, Novikov, Sullivan and Wall in high dimensions.

- $\mathcal{S}^{\text{top}}(M)$ consist of one element if and only if A_{n+1} is surjective and A_n is injective.
- $H_k(M; \mathbf{L}\langle 1\rangle) \to L_k(\mathbb{Z}G)$ is bijective for $k \geq n+1$ and injective for $k = n$ if $M = BG$ and both the K-theoretic and L-theoretic Farrell-Jones Conjectures hold for $G = \pi_1(M)$ and $R = \mathbb{Z}$.

The status of the Farrell-Jones Conjecture

Theorem (Main Theorem Bartels-Lück (2008))

Let $\mathcal{FJ}$ be the class of groups for which both the K-theoretic and the L-theoretic Farrell-Jones Conjectures holds (in his most general form, namely with coefficients in any additive G-category) has the following properties:

- *Hyperbolic group and virtually nilpotent groups belongs to $\mathcal{FJ}$;*
- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}$;*

Theorem (Continued)

- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*
- *If we demand on the K-theory version only that the assembly map is 1-connected and keep the full L-theory version, then the properties above remain valid and the class $\mathcal{FJ}$ contains also all CAT(0)-groups.*

- Limit groups in the sense of Zela are CAT(0)-groups (Alibegovic-Bestvina (2005)).
- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.
- One example is the construction of groups with expanders due to Gromov. These yield counterexamples to the Baum-Connes Conjecture with coefficients (see Higson-Lafforgue-Skandalis (2002)).

- However, our results show that these groups do satisfy the Farrell-Jones Conjecture in its most general form and hence also the other conjectures mentioned above.
- Bartels-Echterhoff-Lück (2007) show that the Bost Conjecture with coefficients in C^*-algebras is true for colimits of hyperbolic groups. Thus the failure of the Baum-Connes Conjecture with coefficients comes from the fact that the change of rings map

$$K_0\big(\mathcal{A} \rtimes_{l^1} G\big) \to K_0\big(\mathcal{A} \rtimes_{C_r^*} G\big)$$

is not bijective for all G-C^*-algebras $\mathcal{A}$.

- Mike Davis (1983) has constructed exotic closed aspherical manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.
- However, in all cases the universal coverings are CAT(0)-spaces and hence the fundamental groups are CAT(0)-groups.
- Hence by our main theorem they satisfy the Farrell-Jones Conjecture and hence the Borel Conjecture in dimension ≥ 5.

- There are still many interesting groups for which the Farrell-Jones Conjecture in its most general form is open. Examples are:
  - Amenable groups;
  - $Sl_n(\mathbb{Z})$ for $n \geq 3$;
  - Mapping class groups;
  - $\mathrm{Out}(F_n)$;
  - Thompson groups.
- If one looks for a counterexample, there seems to be no good candidates which do not fall under our main theorems.

Computational aspects

Theorem (The K- and L-theory of torsionfree hyperbolic groups)

Let G be a torsionfree hyperbolic group and let R be a ring. Then we get isomorphisms

$$H_n(BG;\mathbf{K}_R) \oplus \left(\bigoplus_{\substack{(C), C \subseteq G, C \neq 1 \\ C \text{ maximal cyclic}}} NK_n(R) \right) \xrightarrow{\cong} K_n(RG)$$

and

$$H_n(BG;\mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG);$$

Boundaries of hyperbolic groups

Theorem (Bartels-Lück-Weinberger (in progress))

Let G be a torsionfree hyperbolic group and let n be an integer ≥ 5. Then the following statements are equivalent:

- *The boundary ∂G is homeomorphic to S^{n-1};*
- *There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.*