Survey on the Farrell-Jones Conjecture

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Outline and goal

- We present a list of prominent conjectures such as the one due to Bass, Borel, Gromov, Moody, Kaplansky and Novikov.
- We briefly introduce the Farrell-Jones Conjecture and explain that it implies all the other conjectures mentioned above.
- We state our main theorem which is joint work with Bartels. It says that the Farrell-Jones Conjecture and hence also all the other conjecture above are true for an interesting large class of groups including word-hyperbolic groups and CAT(0)-groups.
- We discuss consequences and open cases.
- We make a few comments about the proof if time allows.

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Some prominent Conjectures

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Conjecture (Projective class groups)

Let R be a regular ring. Suppose that G is torsionfree. Then:

- $K_n(RG) = 0$ *for* $n \leq -1$;
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective;*
- *If R is a principal ideal domain, then* $\widetilde{K}_0(RG) = 0$.

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- The vanishing of $\widetilde{K}_0(RG)$ is equivalent to the statement that any finitely generated projective RG-module P is stably free, i.e., there are $m, n \geq 0$ with $P \oplus RG^m \cong RG^n$;
- Let G be a finitely presented group. The vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ is equivalent to the geometric statement that any finitely dominated space X with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW-complex.

Conjecture (Whitehead group)

If G is torsionfree, then the Whitehead group $\mathrm{Wh}(G)$ *vanishes.*

- Fix $n \geq 6$. The vanishing of $\mathrm{Wh}(G)$ is equivalent to the following geometric statement:

  Every compact n-dimensional h-cobordism W with $G \cong \pi_1(W)$ is trivial.

- The vanishing of $\widetilde{K}_0(RG)$ is equivalent to the statement that any finitely generated projective RG-module P is stably free, i.e., there are $m, n \geq 0$ with $P \oplus RG^m \cong RG^n$;
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Conjecture (Moody's Induction Conjecture)

- *Let R be a regular ring with $\mathbb{Q} \subseteq R$. Then the map given by induction from finite subgroups of G*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(RH) \to K_0(RG)$$

is bijective;

- *Let F be a field of characteristic p for a prime number p. Then the map*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(FH)[1/p] \to K_0(FG)[1/p]$$

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- The various versions of the Bass Conjecture fit into this context as well.

- Roughly speaking, the Bass Conjecture extends basic facts of the representation theory of finite groups to the projective class group of infinite groups.

Conjecture (Novikov Conjecture)

Higher signatures are homotopy invariant.

- More precisely, it predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\operatorname{sign}_x(M,f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M,f), i.e., for every orientation preserving homotopy equivalence of closed oriented manifolds $g\colon M_0 \to M_1$ and homotopy equivalence $f_i\colon M_i \to BG$ with $f_1 \circ g \simeq f_2$ we have

$$\operatorname{sign}_x(M_0,f_0) = \operatorname{sign}_x(M_1,f_1).$$

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- For $x = 1$ this follows from Hirzebruch's signature formula

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- For a homotopy equivalence $f \colon M \to N$ of closed aspherical manifolds the Novikov Conjecture predicts $f^*\mathcal{L}(N) = \mathcal{L}(M)$.
- There are examples of orientation preserving homotopy equivalences $f \colon M \to N$ for which $f^*\mathcal{L}(N) = \mathcal{L}(M)$ does not hold, e.g., fake complex projective spaces.
- Obviously we get $f^*\mathcal{L}(N) = \mathcal{L}(M)$ for an orientation preserving diffeomorphism.
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The Borel Conjecture

Conjecture (Borel Conjecture)

Aspherical closed manifolds are topologically rigid.

- More precisely, it predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism.
  In particular M and N are homeomorphic.
- This is the topological version of Mostow rigidity. One version of Mostow rigidity says that any homotopy equivalence between hyperbolic closed Riemannian manifolds is homotopic to an isometric diffeomorphism. In particular they are isometrically diffeomorphic if and only if their fundamental groups are isomorphic.
- Examples due to Farrell-Jones (1989) show that the Borel Conjecture becomes definitely false if one replaces homeomorphism by diffeomorphism.

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- In some sense the Borel Conjecture is opposed to the Poincaré Conjecture.

  Namely, in the Borel Conjecture the fundamental group can be complicated but there are no higher homotopy groups, whereas in the Poincaré Conjecture there is no fundamental group but complicated higher homotopy groups.

- A systematic study of topologically rigid manifolds is presented in a paper by Kreck-Lück (2006), where a kind of interpolation between the Poincaré Conjecture and the Borel Conjecture is studied.

- There is also an existence part of the Borel Conjecture.

  Namely, if X is an aspherical finite Poincaré complex, then X is homotopy equivalent to an ANR-homology manifold.

- One may also hope that X is homotopy equivalent to a closed manifold. But then one runs into Quinn's resolutions obstruction which seem to be a completely different story (see Byrant-Ferry-Mio-Weinberger (1995)). The question is whether it vanishes for closed aspherical manifolds.

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Conjecture (*K*-theoretic Farrell-Jones Conjecture for regular rings and torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(RG)$ is the algebraic K-theory of the group ring RG.
- $\mathbf{K}_R$ is the (non-connective) algebraic K-theory spectrum of the ring R.
- $H_n(\mathrm{pt}; \mathbf{K}_R) \cong \pi_n(\mathbf{K}_R) \cong K_n(R)$.
- BG is the classifying space of the group G.

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- BG is the classifying space of the group G.

Lemma

Let R be a regular ring and let G be a torsionfree group such that K-theoretic Farrell-Jones Conjecture holds. Then

- $K_n(RG) = 0$ *for* $n \leq -1$;
- *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective. In particular* $\widetilde{K}_0(RG)$ *is trivial if and only if* $\widetilde{K}_0(R)$ *is trivial;*
- *The Whitehead group* $\mathrm{Wh}(G)$ *is trivial.*

- The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG; \mathbf{K}_R)$ whose E^2-term is given by

$$E^2_{p,q} = H_p(BG, K_q(R)).$$

- Since R is regular by assumption, we get $K_q(R) = 0$ for $q \leq -1$.
- Hence the edge homomorphism yields an isomorphism

$$K_0(R) = H_0(\mathrm{pt}, K_0(R)) \xrightarrow{\cong} H_0(BG; \mathbf{K}_R) \cong K_0(RG).$$

- A similar argument works for $\mathrm{Wh}(G) = 0$.

Conjecture (*L*-theoretic Farrell-Jones Conjecture for torsionfree groups)

The L-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution R for the torsionfree group G predicts that the assembly map

$$H_n\big(BG; \mathbf{L}_R^{\langle -\infty \rangle}\big) \to L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Definition (Structure set)

The *structure set* $S^{top}(M)$ of a manifold M consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold N as source.

Two such homotopy equivalences $f_0 \colon N_0 \to M$ and $f_1 \colon N_1 \to M$ are equivalent if there exists a homeomorphism $g \colon N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

Theorem

The Borel Conjecture holds for a closed manifold M if and only if $\mathcal{S}^{\mathrm{top}}(M)$ consists of one element.

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Theorem (Algebraic surgery sequence Ranicki (1992))

There is an exact sequence of abelian groups called algebraic surgery exact sequence *for an n-dimensional closed manifold M*

$$\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}} \mathcal{S}^{\text{top}}(M) \xrightarrow{\sigma_n} H_n(M;\mathbf{L}\langle 1\rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots$$

It can be identified with the classical geometric surgery sequence due to Browder, Novikov, Sullivan and Wall in high dimensions.

- $\mathcal{S}^{\text{top}}(M)$ consist of one element if and only if A_{n+1} is surjective and A_n is injective.
- $H_k(M;\mathbf{L}\langle 1\rangle) \to H_k(M;\mathbf{L}^{\langle -\infty\rangle})$ is bijective for $k \geq n+1$ and injective for $k = n$ if both the K-theoretic and L-theoretic Farrell-Jones Conjectures hold for $G = \pi_1(M)$ and $R = \mathbb{Z}$.
- Hence the Farrell-Jones Conjecture implies the Borel Conjecture in dimensions ≥ 5.

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Poincaré duality groups

Definition (Poincaré duality group)

A Poincaré duality group G of dimension n is a finitely presented group satisfying:

- G is of type FP;
- $H^i(G;\mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

Lemma

Let X be a closed aspherical ANR-homology manifold of dimension n. Then its fundamental group is a Poincaré duality group of dimension n.

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Theorem (Poincaré duality groups and ANR-homology manifolds Bartels-Lück-Weinberger (2011))

Let G be a torsionfree group. Suppose that its satisfies the K- and L-theoretic Farrell-Jones Conjecture. Consider $n \geq 6$.

Then the following statements are equivalent:

1. *G is a Poincaré duality group of dimension n;*
2. *There exists a closed aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$.*

If the first statements holds, then the homology ANR-manifold M appearing above is unique up to s-cobordism of ANR-homology manifolds.

Gromov's Conjecture about hyperbolic groups with spheres as boundary

Conjecture (Gromov (1994))

Let G be a hyperbolic group whose boundary is a sphere S^{n-1}. Then there is a closed aspherical manifold M with $\pi_1(M) \cong G$.

Theorem (Hyperbolic groups with spheres as boundary Bartels-Lück-Weinberger(2011))

Let G be a torsionfree hyperbolic group and let n be an integer ≥ 6. Then the following statements are equivalent:

1. *The boundary ∂G is homeomorphic to S^{n-1};*
2. *There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.*

The manifold appearing above is unique up to homeomorphism.

Gromov's Conjecture about hyperbolic groups with spheres as boundary

Conjecture (Gromov (1994))

Let G be a hyperbolic group whose boundary is a sphere S^{n-1}. Then there is a closed aspherical manifold M with $\pi_1(M) \cong G$.

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Status report of the Farrell-Jones Conjecture

- There are certain generalizations of the Farrell-Jones Conjectures.
- One can allow coefficients in additive categories or consider fibered versions or the version with finite wreath products.
- In what follows, the Full Farrell-Jones Conjecture will mean the most general form with coefficients in additive categories and with finite wreath products and require it for both K and L-theory.
- The strong version encompasses twisted group rings $R_\Phi G$, or even crossed product rings $R * G$, and includes orientation characters $w \colon G \to \{\pm 1\}$ in the L-theory setting.
- We think of it as an advanced induction theorem (such as Artin's or Brower's induction theorem for representations of finite groups).

Theorem (The Farrell-Jones Conjecture implies (nearly) everything)

If G satisfies both the K-theoretic and L-theoretic Farrell-Jones Conjecture for any additive G-category $\mathcal{A}$, then all the conjectures mentioned above follow for G.

Theorem (Bartels, Echterhoff, Farrell, Lück, Reich, Roushon, Rüping, Wegner (2008 - 2013))

Let $\mathcal{FJ}$ be the class of groups for which the Full Farrell-Jones Conjecture holds. Then $\mathcal{FJ}$ contains the following groups:

- *Hyperbolic groups belong to $\mathcal{FJ}$;*
- CAT(0)*-groups belong to $\mathcal{FJ}$;*
- *Virtually poly-cyclic groups belong to $\mathcal{FJ}$;*
- *Solvable groups belong to $\mathcal{FJ}$;*
- *Cocompact lattices in almost connected Lie groups belong to $\mathcal{FJ}$;*
- *All* 3*-manifold groups belong to $\mathcal{FJ}$;*
- *If R is a ring whose underlying abelian group is finitely generated free, then $SL_n(R)$ and $GL_n(R)$ belong to $\mathcal{FJ}$ for all $n \geq 2$;*
- *All arithmetic groups belong to $\mathcal{FJ}$.*

Theorem (continued)

Moreover, $\mathcal{FJ}$ has the following inheritance properties:

- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *If $H \subseteq G$ is a subgroup of G with $[G : H] < \infty$ and $H \in \mathcal{FJ}$, then $G \in \mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*

- Many more mathematicians have made important contributions to the Farrell-Jones Conjecture, e.g., Bökstedt, Cappell, Carlsson, Davis, Ferry, Hambleton, Hsiang, Jones, Linnell, Madsen, Pedersen, Quinn, Ranicki, Rognes, Rosenthal, Tessera, Varisco, Weinberger, Yu.

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- Limit groups in the sense of Zela are CAT(0)-groups (Alibegovic-Bestvina).
- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.
- One example is the construction of groups with expanders due to Gromov, see Arzhantseva-Delzant. These yield counterexamples to the Baum-Connes Conjecture with coefficients due to Higson-Lafforgue-Skandalis.
- However, our results show that these groups do satisfy the Full Farrell-Jones Conjecture and hence also the other conjectures mentioned above.
- Many groups of the region 'Hic abundant leones' in the universe of groups in the sense of Bridson do satisfy the Full Farrell-Jones Conjecture.

- Davis-Januszkiewicz have constructed exotic closed aspherical manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.
- However, in all cases the universal coverings are CAT(0)-spaces and the fundamental groups are CAT(0)-groups. Hence they satisfy the Full Farrell-Jones Conjecture and in particular the Borel Conjecture in dimension ≥ 5.
- The Baum-Connes Conjecture is open for CAT(0)-groups, cocompact lattices in almost connected Lie groups and $SL_n(\mathbb{Z})$ for $n \geq 3$, but known, for instance, for all a-T-menable groups due to work of Higson-Kasparov.

Computational aspects

Theorem (The algebraic K-theory of torsionfree hyperbolic groups L.-Rosenthal (2013))

Let G be a torsionfree hyperbolic group and let R be a ring (with involution). Then we get an isomorphisms

$$H_n(BG;\mathbf{K}_R) \oplus \left(\bigoplus_{\substack{(C), C \subseteq G, C \neq 1 \\ C \text{ maximal cyclic}}} NK_n(R) \right) \xrightarrow{\cong} K_n(RG);$$

and

$$H_n(BG;\mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG);$$

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Theorem (L. (2002))

Let G be a group. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. There is a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\
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\end{array}$$

- The vertical arrows come from the obvious change of rings and of K-theory maps.
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Open problems

- There are still many interesting groups for which the Farrell-Jones Conjecture is open. Examples are:
  - Amenable groups;
  - Mapping class groups;
  - $\mathrm{Out}(F_n)$;
  - Thompson groups.
  - Extension of a free group by $\mathbb{Z}$.
- We have no good candidate for a group (or for a property of groups) for which the Farrell-Jones Conjecture may fail.
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