

The Stable Cannon Conjecture

Wolfgang Lück

Bonn

Germany

email wolfgang.lueck@him.uni-bonn.de

<http://www.him.uni-bonn.de/lueck/>

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Poincaré duality groups

Definition (Finite Poincaré complex)

A (connected) finite n -dimensional CW -complex X is a **finite n -dimensional Poincaré complex** if there is $[X] \in H_n(X; \mathbb{Z}^w)$ such that the induced $\mathbb{Z}\pi$ -chain map

$$-\cap [X]: C^{n-*}(\tilde{X}) \rightarrow C_*(\tilde{X})$$

is a simple $\mathbb{Z}\pi$ -chain homotopy equivalence.

Theorem (Closed manifolds are Poincaré complexes)

A closed n -dimensional manifold M is a finite n -dimensional Poincaré complex with $w = w_1(X)$.

Definition (Poincaré duality group)

A **Poincaré duality group** G of dimension n is a finitely presented group satisfying:

- G is of type FP.
- $H^i(G; \mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

Corollary

If M is a closed aspherical manifold of dimension d , then $\pi_1(X)$ is a d -dimensional Poincaré duality group.

Theorem (Wall)

If G is a d -dimensional Poincaré duality group for $d \geq 3$ and $\tilde{K}_0(\mathbb{Z}G) = 0$, then there is a model for BG which is a finite Poincaré complex of dimension d .

- Recall that the K -theoretic Farrell-Jones Conjecture implies that $K_n(\mathbb{Z}G)$ for $n \leq 1$, $\tilde{K}_0(\mathbb{Z}G)$, and $\text{Wh}(G)$ vanish for a torsionfree group G .
- Moreover, the Farrell-Jones Conjecture is known to be true for hyperbolic groups and fundamental groups of 3-manifolds.
- In particular we can ignore in the sequel the difference between simple homotopy equivalence and homotopy equivalence.

Hyperbolic groups

Definition (hyperbolic group)

A **hyperbolic group** G is a finitely generated group such that for one (and hence all) choice of symmetric finite set of generators the Cayley graph with the associated word metric is a hyperbolic geodesic metric space.

- A geodesic metric space is called **hyperbolic** if geodesic triangles are thin in comparison with geodesic triangles in \mathbb{R}^2 .
- Every torsionfree hyperbolic group has a finite model for BG .
- One can assign to a hyperbolic group a topological space called **boundary** ∂G such that for any geodesic hyperbolic metric space X on which G acts properly and cocompactly by isometries there is a compactification $\bar{X} = X \amalg \partial G$ such that ∂G is a Z -set in \bar{X} . This applies in particular to the Cayley graph. Notice that ∂G is independent of X .

Theorem (Hadamard)

If M is a closed smooth Riemannian manifold of dimension n , whose section curvature is negative, then $\pi = \pi_1(M)$ is a torsionfree hyperbolic group with $\partial\pi = S^{n-1}$.

- Actually \tilde{M} is a geodesic metric space on which π acts freely, properly and cocompactly by isometries.
- There is a diffeomorphism $\tilde{M} \xrightarrow{\cong} \mathbb{R}^n$ and $\partial\pi$ is the sphere S^{n-1} at infinity.

The main conjectures

Conjecture (Wall)

Every Poincaré duality group is the fundamental group of an aspherical closed manifold.

Theorem (Eckmann-Müller, Linnell)

Every 2-dimensional Poincaré duality group is the fundamental group of a closed surface.

Theorem (Bestvina)

Let G be a hyperbolic 3-dimensional Poincaré duality group. Then its boundary is homeomorphic to S^2 .

Theorem (Cannon-Cooper, Eskin-Fisher-Whyte, Kapovich-Leeb)

A Poincaré duality group G of dimension 3 is the fundamental group of an aspherical closed 3-manifold if and only if it is quasi-isometric to the fundamental group of an aspherical closed 3-manifold.

- A closed 3-manifold is a **Seifert manifold** if it admits a finite covering $\overline{M} \rightarrow M$ such that there exists a S^1 -principal bundle $S^1 \rightarrow \overline{M} \rightarrow S$ for some closed orientable surface S .

Theorem (Bowditch)

If a Poincaré duality group of dimension 3 contains an infinite normal cyclic subgroup, then it is the fundamental group of a closed Seifert 3-manifold.

Theorem (Bestvina-Mess)

A torsionfree hyperbolic G is a Poincaré duality group of dimension n if and only if its boundary ∂G and S^{n-1} have the same Čech cohomology.

Theorem (Bartels-Lück-Weinberger)

Let G be a torsion-free group which satisfies the Farrell-Jones Conjecture. Then for $n \geq 5$ the following are equivalent:

- *G is a Poincaré duality group of formal dimension n ;*
 - *There exists an aspherical closed ANR-homology manifold M which has (DDP) and satisfies $\pi_1(M) \cong G$;*
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- We will deal with ANR-homology manifolds and the question when they are homotopy equivalent to closed manifolds later.

Conjecture (Gromov)

Let G be a torsionfree hyperbolic group whose boundary is a sphere S^{n-1} . Then there is a closed aspherical manifold M with $\pi_1(M) \cong G$.

Theorem (Bartels-Lück-Weinberger)

Gromov's Conjecture is true for $n \geq 5$.

Conjecture (Cannon's Conjecture in the torsionfree case)

A torsionfree hyperbolic group G has S^2 as boundary if and only if it is the fundamental group of a closed hyperbolic 3-manifold.

Theorem (Bestvina-Mess)

Let G be an infinite torsionfree hyperbolic group which is prime, not infinite cyclic, and the fundamental group of a closed 3-manifold M . Then M is hyperbolic and G satisfies the Cannon Conjecture.

- In order to prove the Cannon Conjecture, it suffices to show for a hyperbolic group G , whose boundary is S^2 , that it is quasi-isometric to the fundamental group of some closed 3-manifold.

Theorem (Fundamental groups of aspherical oriented closed 3-manifolds)

Let G be the fundamental group of an aspherical oriented closed 3-manifold. Then G satisfies:

- G is residually finite and Hopfian;
 - All its L^2 -Betti numbers $b_n^{(2)}(G)$ vanish;
 - Its deficiency is 0. In particular it possesses a presentation with the same number of generators and relations;
 - Suppose that M is hyperbolic. Then G is virtually compact special and linear over \mathbb{Z} . It contains a subgroup of finite index G' which can be written as an extension $1 \rightarrow \pi_1(S) \rightarrow G' \rightarrow \mathbb{Z} \rightarrow 1$ for some closed orientable surface S .
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- Recall that any finitely presented group occurs as the fundamental group of a closed d -dimensional smooth manifold for every $d \geq 4$.

- The following result illustrates what the strategy of proof for the Cannon Conjecture by experts on 3-manifolds is.
- The boundary ∂G of a hyperbolic group G is metrizable but the metric is not determined by G .
- However, the induced **quasi-conformal structure** and the induced **quasi-Möbius structure** associated to some visual metric on ∂G of a hyperbolic group G are canonical, i.e., independent of the choice of a visual metric.
- These structures are quasi-isometry invariants.

- The **Ahlfors regular conformal dimension** of a metric space Z is the infimal Hausdorff dimension of all Ahlfors regular metric spaces quasi-symmetrically homeomorphic to Z .

Theorem (Bonk-Kleiner)

*The Cannon Conjecture is equivalent to the following statement:
If G is a hyperbolic group G with boundary S^2 , then the Ahlfors regular conformal dimension of ∂G is attained.*

The main results

Theorem (Ferry-Lück-Weinberger, (preprint, 2018), Vanishing of the surgery obstruction)

Let G be a hyperbolic 3-dimensional Poincaré duality group. Then there is a normal map of degree one (in the sense of surgery theory)

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}}^a & \xrightarrow{\bar{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \end{array}$$

satisfying

- 1 The space BG is a finite 3-dimensional CW-complex;
- 2 The map $H_n(f, \mathbb{Z}): H_n(M; \mathbb{Z}) \xrightarrow{\cong} H_n(BG; \mathbb{Z})$ is bijective for all $n \geq 0$;
- 3 The simple algebraic surgery obstruction $\sigma(f, \bar{f}) \in L_3^S(\mathbb{Z}G)$ vanishes.

Theorem (Ferry-Lück-Weinberger, (preprint, 2018), Stable Cannon Conjecture)

Let G be a hyperbolic 3-dimensional Poincaré duality group. Let N be any smooth, PL or topological manifold respectively which is closed and whose dimension is ≥ 2 .

Then there is a closed smooth, PL or topological manifold M and a normal map of degree one

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{f} & \xi \times TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & BG \times N \end{array}$$

such that the map f is a simple homotopy equivalence.

Theorem (Stable Cannon Conjecture, continued)

Moreover:

Let $\widehat{M} \rightarrow M$ be the G -covering associated to the composite of the isomorphism $\pi_1(f): \pi_1(M) \xrightarrow{\cong} G \times \pi_1(N)$ with the projection $G \times \pi_1(N) \rightarrow G$. Suppose additionally that N is aspherical and $\dim(N) \geq 3$.

Then \widehat{M} is homeomorphic to $\mathbb{R}^3 \times N$. Moreover, there is a compact topological manifold $\overline{\widehat{M}}$ whose interior is homeomorphic to \widehat{M} and for which there exists a homeomorphism of pairs

$$(\overline{\widehat{M}}, \partial\overline{\widehat{M}}) \rightarrow (D^3 \times N, S^2 \times N).$$

- The last two theorems follow from the Cannon Conjecture.
- By the **product formula for surgery theory** and the technique of **pulling back the boundary** the second last theorem implies the last theorem.
- The manifold M appearing in the last theorem is unique up to homeomorphism by the **Borel Conjecture**, provided that $\pi_1(N)$ satisfies the Farrell-Jones Conjecture.
- If we take $N = T^k$ for some $k \geq 2$, then the Cannon Conjecture is equivalent to the statement that this M is homeomorphic to $M' \times T^k$ for some closed 3-manifold M' .

The existence of a normal map

Theorem (Existence of a normal map)

Let X be a connected oriented finite 3-dimensional Poincaré complex. Then there are an integer $a \geq 0$ and a vector bundle ξ over BG and a normal map of degree one

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}}^a & \xrightarrow{\bar{f}} & \underline{\xi} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

Proof.

- Stable vector bundles over X are classified by the first and second Stiefel-Whitney class $w_1(\xi)$ and $w_2(\xi)$ in $H^*(X; \mathbb{Z}/2)$.
- Let ξ be a k -dimensional vector bundle over X such that $w_1(\xi) = w_1(X)$ and $w_2(\xi) = w_1(\xi) \cup w_1(\xi)$ holds.
- A spectral sequence argument applied to $\Omega_3(X, w_1(X))$ shows that there is a closed 3-manifold M together with a map $f: M \rightarrow X$ of degree one such that $f^* w_1(X) = w_1(M)$.
- Then $w_1(f^*\xi) = w_1(M)$ and the Wu formula implies $w_2(M) = w_1(f^*\xi) \cup w_1(f^*\xi)$.
- Hence $f^*\xi$ is stably isomorphic to the stable tangent bundle of M and we get the desired normal map.



The total surgery obstruction

- Consider an aspherical finite n -dimensional Poincaré complex X such that $G = \pi_1(X)$ is a **Farrell-Jones group**, i.e., satisfies both the K -theoretic and the L -theoretic Farrell-Jones Conjecture with coefficients in additive categories, and $\mathcal{N}(X)$ is non-empty. (For simplicity we assume $w_1(X) = 0$ in the sequel.)
- We want to find one normal map of degree one

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}}^a & \xrightarrow{\bar{f}} & \underline{\xi} \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

whose simple surgery obstruction $\sigma^s(f, \bar{f}) \in L_n^s(\mathbb{Z}G)$ vanishes.

- Recall that the simple surgery obstruction defines a map

$$\sigma^S: \mathcal{N}(X) \rightarrow L_n^S(\mathbb{Z}G).$$

- Fix a normal map (f_0, \bar{f}_0) .
- Then there is a commutative diagram

$$\begin{array}{ccc}
 \mathcal{N}(X) & \xrightarrow{\sigma^S(-, -) - \sigma^S(f_0, \bar{f}_0)} & L_n^S(\mathbb{Z}G) \\
 s_0 \downarrow \cong & & \cong \uparrow \text{asmb}_n^S(X) \\
 H_n(X; \mathbf{L}_{\mathbb{Z}}^S\langle 1 \rangle) & \xrightarrow{H_n^G(\text{id}_X; \mathbf{i})} & H_n(X; \mathbf{L}_{\mathbb{Z}}^S)
 \end{array}$$

whose vertical arrows are bijections thanks to the Farrell-Jones Conjecture and the upper arrow sends the class of (f, \bar{f}) to the difference $\sigma^S(f, \bar{f}) - \sigma^S(f, \bar{f}_0)$ of simple surgery obstructions.

- An easy spectral sequence argument yields a short exact sequence

$$0 \rightarrow H_n(X; \mathbf{L}_{\mathbb{Z}}^S \langle 1 \rangle) \xrightarrow{H_n(\text{id}_X; \mathbf{i})} H_n(X; \mathbf{L}_{\mathbb{Z}}^S) \xrightarrow{\lambda_n^S(X)} L_0(\mathbb{Z}).$$

- Consider the composite

$$\mu_n^S(X): \mathcal{N}(X) \xrightarrow{\sigma^S} L_n^S(\mathbb{Z}G, w) \xrightarrow{\text{asmb}_n^S(X)^{-1}} H_n(X; \mathbf{L}_{\mathbb{Z}}^S) \xrightarrow{\lambda_n^S(X)} L_0(\mathbb{Z}).$$

- We conclude that there is precisely one element, called the **total surgery obstruction**,

$$s(X) \in L_0(\mathbb{Z}) \cong \mathbb{Z}$$

such that for any element $[(f, \bar{f})]$ in $\mathcal{N}(X)$ its image under $\mu_n^s(X)$ is $s(X)$.

Theorem (Total surgery obstruction)

- *There exists a normal map of degree one (f, \bar{f}) with target X and vanishing simple surgery obstruction $\sigma^s(f, \bar{f}) \in L_n^s(\mathbb{Z}G)$ if and only if $s(X) \in L_0(\mathbb{Z}) \cong \mathbb{Z}$ vanishes.*
- *The total surgery obstruction is a homotopy invariant of X and hence depends only on G .*

Definition (Absolute Neighborhood Retract (ANR))

A topological space X is called an **absolute neighborhood retract** or briefly an ANR if it is normal and for every normal space Z , every closed subset $Y \subseteq Z$ and every map $f: Y \rightarrow X$ there exists an open neighborhood U of Y in Z together with an extension $F: U \rightarrow X$ of f to U .

Definition (Homology ANR-manifold)

A **homology ANR-manifold** X is an ANR satisfying:

- X has a countable basis for its topology;
- The topological dimension of X is finite;
- X is locally compact;
- for every $x \in X$ we have for the singular homology

$$H_i(X, X - \{x\}; \mathbb{Z}) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$$

If X is additionally compact, it is called a **closed ANR-homology manifold**.

Definition (Disjoint disk property (DDP))

An ANR homology manifold M has the **disjoint disk property (DDP)**, if for one (and hence any) choice of metric on M , any $\epsilon > 0$ and any maps $f, g: D^2 \rightarrow M$, there are maps $f', g': D^2 \rightarrow M$ so that f' is ϵ -close to f , g' is ϵ -close to g and $f'(D^2) \cap g'(D^2) = \emptyset$,

- Every closed topological manifold is a closed ANR-homology manifold having (DDP).
- Let M be homology sphere with non-trivial fundamental group. Then its suspension ΣM is a closed ANR-homology manifold but not a topological manifold.

Quinn's resolution obstruction

Theorem (Quinn (1987))

There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ for homology ANR-manifolds with the following properties:

- *if $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;*
- *$i(M \times N) = i(M) \cdot i(N)$;*
- *Let M be a homology ANR-manifold of dimension ≥ 5 having (DDP). Then M is a topological manifold if and only if $\iota(M) = 1$;*
- *The Quinn obstruction and the total surgery obstruction are related for an aspherical closed ANR-homology manifold M of dimension ≥ 5 by*

$$\iota(M) = 8 \cdot s(X) + 1,$$

if $\pi_1(M)$ is a Farrell-Jones group.

Proof of the Theorem about the vanishing of the surgery obstruction

Proof.

- We have to show for the aspherical finite 3-dimensional Poincaré complex X that its total surgery obstruction vanishes.
- The total surgery obstruction satisfies a product formula

$$8 \cdot s(X \times Y) + 1 = (8 \cdot s(X) + 1) \cdot (8 \cdot s(Y) + 1).$$

- This implies

$$s(X \times T^3) = s(X).$$

- Hence it suffices to show that $s(X \times T^3)$ vanishes.

Proof (continued).

- There is a Z -compactification \widetilde{X} of \widetilde{X} by the boundary $\partial G = S^2$.
- One constructs an appropriate Z -compactification \widetilde{M} of \widetilde{M} so that we get a ANR-homology manifold \widetilde{M} whose boundary is a topological manifold and whose interior is \widetilde{M} . This is based on the technique **pulling back the boundary**.
- By adding a collar to \widetilde{M} one obtains a ANR-homology manifold Y which contains \widetilde{M} as an open subset and contains an open subset U which is homeomorphic to \mathbb{R}^6 .



Proof (continued).

- Hence we get

$$\begin{aligned} 8s(X \times T^3) + 1 &= 8s(M) + 1 = i(M) = i(\tilde{M}) \\ &= i(Y) = i(U) = i(\mathbb{R}^6) = 1. \end{aligned}$$

- This implies $s(X \times T^3) = 0$ and hence $s(X) = 0$.

