Survey on L^2-invariants

Wolfgang Lück
Bonn
Germany
email wolfgang.lueck@him.uni-bonn.de
http://131.220.77.52/lueck/

Bonn, November 2015

Basic motivation

- Given an invariant for finite CW-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.
- We want to apply this principle to (classical) Betti numbers

$$b_n(X) := \dim_{\mathbb{C}}(H_n(X; \mathbb{C})).$$

- Certain naive attempts aiming at $H_n(\widetilde{X})$ regarded as module over the group ring $\mathbb{Z}G$ fail since this ring is too complicated.
- We will use the following successful approach which is essentially due to Atiyah.

Basic motivation

- Given an invariant for finite CW-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.
- We want to apply this principle to (classical) Betti numbers

$$b_n(X) := \dim_{\mathbb{C}}(H_n(X;\mathbb{C})).$$

- Certain naive attempts aiming at $H_n(\widetilde{X})$ regarded as module over the group ring $\mathbb{Z}G$ fail since this ring is too complicated.
- We will use the following successful approach which is essentially due to Atiyah.

Basic motivation

- Given an invariant for finite CW-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.
- We want to apply this principle to (classical) Betti numbers

$$b_n(X) := \dim_{\mathbb{C}}(H_n(X;\mathbb{C})).$$

- Certain naive attempts aiming at $H_n(\widetilde{X})$ regarded as module over the group ring $\mathbb{Z}G$ fail since this ring is too complicated.
- We will use the following successful approach which is essentially due to Atiyah.

Basic motivation

- Given an invariant for finite CW-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.
- We want to apply this principle to (classical) Betti numbers

$$b_n(X) := \dim_{\mathbb{C}}(H_n(X;\mathbb{C})).$$

- Certain naive attempts aiming at $H_n(\widetilde{X})$ regarded as module over the group ring $\mathbb{Z}G$ fail since this ring is too complicated.
- We will use the following successful approach which is essentially due to Atiyah.

Basic motivation

- Given an invariant for finite CW-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.
- We want to apply this principle to (classical) Betti numbers

$$b_n(X) := \dim_{\mathbb{C}}(H_n(X;\mathbb{C})).$$

- Certain naive attempts aiming at $H_n(\widetilde{X})$ regarded as module over the group ring $\mathbb{Z}G$ fail since this ring is too complicated.
- We will use the following successful approach which is essentially due to Atiyah.

Group von Neumann algebras

- Denote by $L^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$.

Definition (Group von Neumann algebra)

Define the group von Neumann algebra

$$\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G))^G = \overline{\mathbb{C}G}^{\text{weak}}$$

to be the algebra of bounded G-equivariant operators $L^2(G) \to L^2(G)$. The von Neumann trace is defined by

$$\operatorname{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$

Group von Neumann algebras

- Denote by $L^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$.

Definition (Group von Neumann algebra)

Define the group von Neumann algebra

$$\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G))^G = \overline{\mathbb{C}G}^{\text{weak}}$$

to be the algebra of bounded G-equivariant operators $L^2(G) \to L^2(G)$. The von Neumann trace is defined by

$$\operatorname{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$

Definition (Finitely generated Hilbert module)

A finitely generated Hilbert $\mathcal{N}(G)$-module V is a Hilbert space V together with a linear isometric G-action such that there exists an isometric linear G-embedding of V into $L^2(G)^n$ for some $n \geq 0$.
A map of finitely generated Hilbert $\mathcal{N}(G)$-modules $f\colon V \to W$ is a bounded G-equivariant operator.

Definition (von Neumann dimension)

Let V be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a G-equivariant projection $p\colon L^2(G)^n \to L^2(G)^n$ with $\operatorname{im}(p) \cong_{\mathcal{N}(G)} V$. Define the von Neumann dimension of V by

$$\dim_{\mathcal{N}(G)}(V) := \operatorname{tr}_{\mathcal{N}(G)}(p) := \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(p_{i,i}) \quad \in [0,\infty).$$

Definition (Finitely generated Hilbert module)

A finitely generated Hilbert $\mathcal{N}(G)$-module V is a Hilbert space V together with a linear isometric G-action such that there exists an isometric linear G-embedding of V into $L^2(G)^n$ for some $n \geq 0$.
A map of finitely generated Hilbert $\mathcal{N}(G)$-modules $f\colon V \to W$ is a bounded G-equivariant operator.

Definition (von Neumann dimension)

Let V be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a G-equivariant projection $p\colon L^2(G)^n \to L^2(G)^n$ with $\operatorname{im}(p) \cong_{\mathcal{N}(G)} V$. Define the von Neumann dimension of V by

$$\dim_{\mathcal{N}(G)}(V) := \operatorname{tr}_{\mathcal{N}(G)}(p) := \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(p_{i,i}) \quad \in [0, \infty).$$

L^2-homology and L^2-Betti numbers

Definition (L^2-homology and L^2-Betti numbers)

Let X be a connected CW-complex of finite type. Let $\widetilde{X}$ be its universal covering and $\pi = \pi_1(M)$. Denote by $C_*(\widetilde{X})$ its cellular $\mathbb{Z}\pi$-chain complex.

Define its cellular L^2-chain complex to be the Hilbert $\mathcal{N}(\pi)$-chain complex

$$C_*^{(2)}(\widetilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) = \overline{C_*(\widetilde{X})}.$$

Define its n-th L^2-homology to be the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_n^{(2)}(\widetilde{X}) := \ker(c_n^{(2)})/\overline{\operatorname{im}(c_{n+1}^{(2)})}.$$

Define its n-th L^2-Betti number

$$b_n^{(2)}(\widetilde{X}) := \dim_{\mathcal{N}(\pi)}\left(H_n^{(2)}(\widetilde{X})\right) \quad \in \mathbb{R}^{\geq 0}.$$

L^2-homology and L^2-Betti numbers

Definition (L^2-homology and L^2-Betti numbers)

Let X be a connected CW-complex of finite type. Let $\widetilde{X}$ be its universal covering and $\pi = \pi_1(M)$. Denote by $C_*(\widetilde{X})$ its cellular $\mathbb{Z}\pi$-chain complex.

Define its cellular L^2-chain complex to be the Hilbert $\mathcal{N}(\pi)$-chain complex

$$C_*^{(2)}(\widetilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) = \overline{C_*(\widetilde{X})}.$$

Define its n-th L^2-homology to be the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_n^{(2)}(\widetilde{X}) := \ker(c_n^{(2)})/\overline{\operatorname{im}(c_{n+1}^{(2)})}.$$

Define its n-th L^2-Betti number

$$b_n^{(2)}(\widetilde{X}) := \dim_{\mathcal{N}(\pi)}\left(H_n^{(2)}(\widetilde{X})\right) \quad \in \mathbb{R}^{\geq 0}.$$

Theorem (Main properties of Betti numbers)

Let X and Y be connected CW-complexes of finite type.

- *Homotopy invariance*

  If X and Y are homotopy equivalent, then

  $$b_n(X) = b_n(Y);$$

- *Euler-Poincaré formula*

  We have

  $$\chi(X) = \sum_{n\geq 0}(-1)^n \cdot b_n(X);$$

- *Poincaré duality*

  Let M be a closed manifold of dimension d. Then

  $$b_n(M) = b_{d-n}(M);$$

Theorem (Main properties of L^2-Betti numbers)

Let X and Y be connected CW-complexes of finite type.

- *Homotopy invariance*

  If X and Y are homotopy equivalent, then

  $$b_n^{(2)}(\widetilde{X}) = b_n^{(2)}(\widetilde{Y});$$

- *Euler-Poincaré formula*

  We have

  $$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{X});$$

- *Poincaré duality*

  Let M be a closed manifold of dimension d. Then

  $$b_n^{(2)}(\widetilde{M}) = b_{d-n}^{(2)}(\widetilde{M});$$

Theorem (Continued)

- *Künneth formula*

$$b_n(X \times Y) = \sum_{p+q=n} b_p(X) \cdot b_q(Y);$$

- *Zero-th L^2-Betti number*

  We have

$$b_0(X) = 1;$$

Theorem (Continued)

- *Künneth formula*

$$b_n^{(2)}(\widetilde{X \times Y}) = \sum_{p+q=n} b_p^{(2)}(\widetilde{X}) \cdot b_q^{(2)}(\widetilde{Y});$$

- *Zero-th L^2-Betti number*

  We have

$$b_0^{(2)}(\widetilde{X}) = \frac{1}{|\pi|};$$

Theorem (Continued)

- *Künneth formula*

$$b_n^{(2)}(\widetilde{X \times Y}) = \sum_{p+q=n} b_p(X) \cdot b_q(Y);$$

- *Zero-th L^2-Betti number*

  We have

$$b_0^{(2)}(\widetilde{X}) = \frac{1}{|\pi|};$$

- *Finite coverings*

  If $X \to Y$ is a finite covering with d sheets, then

$$b_n^{(2)}(\widetilde{X}) = d \cdot b_n^{(2)}(\widetilde{Y}).$$

Some computations and results

Example (Finite self coverings)

We get for a connected CW-complex X of finite type, for which there is a selfcovering $X \to X$ with d-sheets for some integer $d \geq 2$,

$$b_n^{(2)}(\widetilde{X}) = 0 \quad \text{for } n \geq 0.$$

This implies for each connected CW-complex Y of finite type

$$b_n^{(2)}(\widetilde{S^1 \times Y}) = 0 \quad \text{for } n \geq 0.$$

Example (L^2-Betti number of surfaces)

- Let F_g be the orientable closed surface of genus $g \geq 1$.
- Then $|\pi_1(F_g)| = \infty$ and hence $b_0^{(2)}(\widetilde{F_g}) = 0$.
- By Poincaré duality $b_2^{(2)}(\widetilde{F_g}) = 0$.
- $\dim(F_g) = 2$, we get $b_n^{(2)}(\widetilde{F_g}) = 0$ for $n \geq 3$.
- The Euler-Poincaré formula shows

$$\begin{aligned} b_1^{(2)}(\widetilde{F_g}) &= -\chi(F_g) = 2g - 2; \\ b_n^{(2)}(\widetilde{F_g}) &= 0 \quad \text{for } n \neq 1. \end{aligned}$$

Theorem (Hodge - de Rham Theorem)

Let M be a closed Riemannian manifold. Put

$$\mathcal{H}^n(M) = \{\omega \in \Omega^n(M) \mid \Delta_n(\omega) = 0\}$$

Then integration defines an isomorphism of real vector spaces

$$\mathcal{H}^n(M) \xrightarrow{\cong} H^n(M;\mathbb{R}).$$

Corollary (Betti numbers and heat kernels)

$$b_n(M) = \lim_{t\to\infty} \int_M \operatorname{tr}_{\mathbb{R}}(e^{-t\Delta_n}(x,x)) \, dvol\,.$$

where $e^{-t\Delta_n}(x,y)$ is the heat kernel on M.

Theorem (L^2-Hodge - de Rham Theorem, Dodziuk)

Let M be a closed Riemannian manifold. Put

$$\mathcal{H}^n_{(2)}(\widetilde{M}) = \{\widetilde{\omega} \in \Omega^n(\widetilde{M}) \mid \widetilde{\Delta}_n(\widetilde{\omega}) = 0,\ ||\widetilde{\omega}||_{L^2} < \infty\}$$

Then integration defines an isomorphism of finitely generated Hilbert $\mathcal{N}(\pi)$-modules

$$\mathcal{H}^n_{(2)}(\widetilde{M}) \xrightarrow{\cong} H^n_{(2)}(\widetilde{M}).$$

Corollary (L^2-Betti numbers and heat kernels)

$$b_n^{(2)}(\widetilde{M}) = \lim_{t\to\infty} \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{R}}(e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{x}))\, dvol\,.$$

where $e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{y})$ is the heat kernel on $\widetilde{M}$ and $\mathcal{F}$ is a fundamental domain for the π-action.

Theorem (Hyperbolic manifolds, Dodziuk)

Let M be a hyperbolic closed Riemannian manifold of dimension d. Then:

$$b_n^{(2)}(\widetilde{M}) = \begin{cases} = 0 & , \text{if } 2n \neq d; \\ > 0 & , \text{if } 2n = d. \end{cases}$$

Proof.

A direct computation shows that $\mathcal{H}^p_{(2)}(\mathbb{H}^d)$ is not zero if and only if $2n = d$. Notice that M is hyperbolic if and only if $\widetilde{M}$ is isometrically diffeomorphic to the standard hyperbolic space $\mathbb{H}^d$. □

Theorem (Hyperbolic manifolds, Dodziuk)

Let M be a hyperbolic closed Riemannian manifold of dimension d. Then:

$$b_n^{(2)}(\widetilde{M}) = \begin{cases} = 0 & , \text{if } 2n \neq d; \\ > 0 & , \text{if } 2n = d. \end{cases}$$

Proof.

A direct computation shows that $\mathcal{H}^p_{(2)}(\mathbb{H}^d)$ is not zero if and only if $2n = d$. Notice that M is hyperbolic if and only if $\widetilde{M}$ is isometrically diffeomorphic to the standard hyperbolic space $\mathbb{H}^d$. □

Theorem (3-manifolds, Lott-Lück)

Let the 3*-manifold* M *be the connected sum* $M_1 \sharp \ldots \sharp M_r$ *of (compact connected orientable) prime* 3*-manifolds* M_j*. Assume that* $\pi_1(M)$ *is infinite. Then*

$$\begin{aligned}
b_1^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} - \chi(M) \\
&\quad + \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right|; \\
b_2^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} \\
&\quad + \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right|; \\
b_n^{(2)}(\widetilde{M}) &= 0 \quad \textit{for } n \neq 1, 2.
\end{aligned}$$

The Atiyah Conjecture

Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)

Let G be a torsionfree finitely presented group. We say that G satisfies the Atiyah Conjecture if for any closed Riemannian manifold M with $\pi_1(M) \cong G$ we have for every $n \geq 0$

$$b_n^{(2)}(\widetilde{M}) \in \mathbb{Z}.$$

- All computations presented above support the Atiyah Conjecture.

The Atiyah Conjecture

Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)

Let G be a torsionfree finitely presented group. We say that G satisfies the Atiyah Conjecture if for any closed Riemannian manifold M with $\pi_1(M) \cong G$ we have for every $n \geq 0$

$$b_n^{(2)}(\widetilde{M}) \in \mathbb{Z}.$$

- All computations presented above support the Atiyah Conjecture.

Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)

Let G be a torsionfree finitely presented group. We say that G satisfies the Atiyah Conjecture if for any closed Riemannian manifold M with $\pi_1(M) \cong G$ we have for every $n \geq 0$

$$b_n^{(2)}(\widetilde{M}) \in \mathbb{Z}.$$

- All computations presented above support the Atiyah Conjecture.

- The Atiyah Conjecture implies the Zero-divisor Conjecture of Kaplanski which predicts for a torsionfree group G that the group ring $\mathbb{C}G$ has no non-trivial zero-divisors.

Theorem (Linnell, Schick)

1. *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions. Then every torsionfree group G which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture.*
2. *If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.*

- The Atiyah Conjecture implies the Zero-divisor Conjecture of Kaplanski which predicts for a torsionfree group G that the group ring $\mathbb{C}G$ has no non-trivial zero-divisors.

Theorem (Linnell, Schick)

1. *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions. Then every torsionfree group G which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture.*
2. *If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.*

Approximation

Theorem (Approximation Theorem, Lück)

Let X be a connected CW-complex of finite type. Suppose that π is residually finite, i.e., there is a nested sequence

$$\pi = G_0 \supset G_1 \supset G_2 \supset \ldots$$

of normal subgroups of finite index with $\cap_{i \geq 1} G_i = \{1\}$. Let X_i be the finite $[\pi : G_i]$-sheeted covering of X associated to G_i.

Then for any such sequence $(G_i)_{i \geq 1}$

$$b_n^{(2)}(\widetilde{X}) = \lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}.$$

Applications to deficiency and signature

Definition (Deficiency)

Let G be a finitely presented group. Define its deficiency

$$\text{deficiency}(G) := \max\{g(P) - r(P)\}$$

where P runs over all presentations P of G and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation P.

Lemma

Let G be a finitely presented group. Then

$$\text{deficiency}(G) \;\leq\; 1 - |G|^{-1} + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Applications to deficiency and signature

Definition (Deficiency)

Let G be a finitely presented group. Define its deficiency

$$\text{deficiency}(G) := \max\{g(P) - r(P)\}$$

where P runs over all presentations P of G and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation P.

Lemma

Let G be a finitely presented group. Then

$$\text{deficiency}(G) \;\leq\; 1 - |G|^{-1} + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Applications to deficiency and signature

Definition (Deficiency)

Let G be a finitely presented group. Define its deficiency

$$\text{deficiency}(G) := \max\{g(P) - r(P)\}$$

where P runs over all presentations P of G and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation P.

Lemma

Let G be a finitely presented group. Then

$$\text{deficiency}(G) \;\leq\; 1 - |G|^{-1} + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Proof.

We have to show for any presentation P that

$$g(P) - r(P) \;\leq\; 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Let X be a CW-complex realizing P. Then

$$\chi(X) = 1 - g(P) + r(P) = b_0^{(2)}(\widetilde{X}) + b_1^{(2)}(\widetilde{X}) - b_2^{(2)}(\widetilde{X}).$$

Since the classifying map $X \to BG$ is 2-connected, we get

$$\begin{array}{rcll} b_n^{(2)}(\widetilde{X}) & = & b_n^{(2)}(G) & \quad \text{for } n = 0, 1; \\ b_2^{(2)}(\widetilde{X}) & \geq & b_2^{(2)}(G). & \end{array}$$

$\square$

Theorem (Deficiency and extensions, Lück)

Let $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ *be an exact sequence of infinite groups. Suppose that* G *is finitely presented and* H *is finitely generated. Then:*

1. $b_1^{(2)}(G) = 0$;
2. $\mathrm{deficiency}(G) \le 1$;
3. *Let* M *be a closed oriented* 4*-manifold with* G *as fundamental group. Then*
$$|\operatorname{sign}(M)| \le \chi(M).$$

Definition (L^2-torsion, Lück-Rothenberg, Mathai, Lott)

Let X be a connected finite CW-complex. Then we define the L^2-torsion

$$\rho^{(2)}(\widetilde{X}) := -\frac{1}{2} \cdot \sum_{n \geq 0} (-1)^n \cdot n \cdot \ln(\det(\Delta_n)) \quad \in \mathbb{R}.$$

where $\Delta_n = (c_n^{(2)})^* \circ c_n^{(2)} + c_{n+1}^{(2)} \circ (c_n^{(2)})^*$ is the nth Laplace operator and det denotes the Fuglede-Kadison determinant.

Theorem (Homotopy invariance)

Let $f\colon X \to Y$ be a homotopy equivalence of finite CW-complexes. Suppose that $\widetilde{X}$ and hence also $\widetilde{Y}$ are L^2-acyclic.

Then

$$\rho^{(2)}(\widetilde{Y}) - \rho^{(2)}(\widetilde{X}).$$

Theorem (Sum formula)

Let X be a finite CW-complex with subcomplexes X_0, X_1 and X_2 satisfying $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Suppose $\widetilde{X_0}$, $\widetilde{X_1}$ and $\widetilde{X_2}$ are L^2-acyclic and the inclusions $X_i \to X$ are π-injective.

Then $\widetilde{X}$ is L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{X}) = \rho^{(2)}(\widetilde{X_1}) + \rho^{(2)}(\widetilde{X_2}) - \rho^{(2)}(\widetilde{X_0}).$$

Theorem (Homotopy invariance)

Let $f\colon X \to Y$ be a homotopy equivalence of finite CW-complexes. Suppose that $\widetilde{X}$ and hence also $\widetilde{Y}$ are L^2-acyclic.

Then

$$\rho^{(2)}(\widetilde{Y}) - \rho^{(2)}(\widetilde{X}).$$

Theorem (Sum formula)

Let X be a finite CW-complex with subcomplexes X_0, X_1 and X_2 satisfying $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Suppose $\widetilde{X_0}$, $\widetilde{X_1}$ and $\widetilde{X_2}$ are L^2-acyclic and the inclusions $X_i \to X$ are π-injective.

Then $\widetilde{X}$ is L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{X}) = \rho^{(2)}(\widetilde{X_1}) + \rho^{(2)}(\widetilde{X_2}) - \rho^{(2)}(\widetilde{X_0}).$$

Theorem (Fibration formula)

Let $F \to E \to B$ be a fibration of connected finite CW-complexes such that $\widetilde{F}$ is L^2-acyclic and the inclusion $F \to E$ is π-injective.

Then $\widetilde{E}$ is L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{E}) = \chi(B) \cdot \rho^{(2)}(\widetilde{F}).$$

- The L^2-torsion is multiplicative under finite coverings, i.e., if $X \to Y$ is a d-sheeted covering of connected finite CW-complexes and $\widetilde{X}$ is L^2-acyclic, then $\widetilde{Y}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{X}) = d \cdot \rho^{(2)}(\widetilde{Y}).$$

- In particular $\widetilde{S^1}$ is L^2-acyclic and $\rho^{(2)}(\widetilde{S^1}) = 0$.

Theorem (Fibration formula)

Let $F \to E \to B$ be a fibration of connected finite CW-complexes such that $\widetilde{F}$ is L^2-acyclic and the inclusion $F \to E$ is π-injective.

Then $\widetilde{E}$ is L^2-acyclic and we get

$$\rho^{(2)}(\widetilde{E}) = \chi(B) \cdot \rho^{(2)}(\widetilde{F}).$$

- The L^2-torsion is multiplicative under finite coverings, i.e., if $X \to Y$ is a d-sheeted covering of connected finite CW-complexes and $\widetilde{X}$ is L^2-acyclic, then $\widetilde{Y}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{X}) = d \cdot \rho^{(2)}(\widetilde{Y}).$$

- In particular $\widetilde{S^1}$ is L^2-acyclic and $\rho^{(2)}(\widetilde{S^1}) = 0$.

Theorem (S^1-actions on aspherical manifolds (Lück))

Let M be an aspherical closed manifold with non-trivial S^1-action.

Then $\widetilde{M}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) = 0.$$

Theorem (L^2-torsion and aspherical CW-complexes (Lück))

Let X be an aspherical finite CW-complex. Suppose that its fundamental group $\pi_1(X)$ contains an elementary amenable infinite normal subgroup.

Then $\widetilde{X}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{X}) = 0.$$

Theorem (S^1-actions on aspherical manifolds (Lück))

Let M be an aspherical closed manifold with non-trivial S^1-action.

Then $\widetilde{M}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) = 0.$$

Theorem (L^2-torsion and aspherical CW-complexes (Lück))

Let X be an aspherical finite CW-complex. Suppose that its fundamental group $\pi_1(X)$ contains an elementary amenable infinite normal subgroup.

Then $\widetilde{X}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{X}) = 0.$$

Theorem (Hyperbolic manifolds (Hess-Schick))

There are (computable) rational numbers $r_n > 0$ such that for every hyperbolic closed manifold M of odd dimension $2n+1$ the universal covering $\widetilde{M}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) = (-1)^n \cdot \pi^{-n} \cdot r_n \cdot \mathrm{vol}(M).$$

- We rediscover the fact that the volume of an odd-dimensional hyperbolic closed manifold depends only on $\pi_1(M)$.
- We also rediscover the theorem that any S^1-action on a closed hyperbolic manifold is trivial.

Theorem (Hyperbolic manifolds (Hess-Schick))

There are (computable) rational numbers $r_n > 0$ such that for every hyperbolic closed manifold M of odd dimension $2n+1$ the universal covering $\widetilde{M}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) = (-1)^n \cdot \pi^{-n} \cdot r_n \cdot \mathrm{vol}(M).$$

- We rediscover the fact that the volume of an odd-dimensional hyperbolic closed manifold depends only on $\pi_1(M)$.
- We also rediscover the theorem that any S^1-action on a closed hyperbolic manifold is trivial.

- The proof is based on the fact that the analytic version of L^2-torsion is of the shape

$$\rho^{(2)}(\widetilde{M}) = \int_{\mathcal{F}} f(x)\ d\mathrm{vol}_{\mathbb{H}^{2n+1}}$$

  where $\mathcal{F}$ is a fundamental domain of the π-action on the hyperbolic space $\mathbb{H}^{2n+1}$ and $f(x)$ is an expression in terms of the heat kernel $k(x,x)(t)$.

- By the symmetry of $\mathbb{H}^{2n+1}$ this function $k(x,x)(t)$ is independent of x and hence $f(x)$ is independent of x.

- If we take $r_n = (-1)^n \cdot \pi^n \cdot f(x)$ for any $x \in \mathbb{H}^{2n+1}$, we get

$$\int_{\mathcal{F}} f(x)\ d\mathrm{vol}_{\mathbb{H}^{2n+1}} = (-1)^n \cdot \pi^{-n} \cdot r_n \cdot \mathrm{vol}(\mathcal{F}) = (-1)^n \cdot \pi^{-n} \cdot r_n \cdot \mathrm{vol}(M).$$

- We have $r_1 = \frac{1}{6}$, $r_2 = \frac{31}{45}$, $r_7 = \frac{221}{70}$.

Theorem (Lück-Schick)

Let M be an irreducible closed 3-manifold with infinite fundamental group. Let $M_1, M_2, \ldots, M_m$ be the hyperbolic pieces in its Jaco-Shalen decomposition.

Then $\widetilde{M}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) := -\frac{1}{6\pi} \cdot \sum_{i=1}^{m} \mathrm{vol}(M_i).$$

- The proof of the result above is based on the meanwhile approved Thurston Geometrization Conjecture. It reduces the claim to Seifert manifolds with incompressible torus boundary and to hyperbolic manifolds with incompressible torus boundary using the sum formula. The Seifert pieces are treated analogously to aspherical closed manifolds with S^1-action. The hyperbolic pieces require a careful analysis of the cusps.

Theorem (Lück-Schick)

Let M be an irreducible closed 3-manifold with infinite fundamental group. Let $M_1, M_2, \ldots, M_m$ be the hyperbolic pieces in its Jaco-Shalen decomposition.

Then $\widetilde{M}$ is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) := -\frac{1}{6\pi} \cdot \sum_{i=1}^{m} \operatorname{vol}(M_i).$$

- The proof of the result above is based on the meanwhile approved Thurston Geometrization Conjecture. It reduces the claim to Seifert manifolds with incompressible torus boundary and to hyperbolic manifolds with incompressible torus boundary using the sum formula. The Seifert pieces are treated analogously to aspherical closed manifolds with S^1-action. The hyperbolic pieces require a careful analysis of the cusps.

Approximation

- The following conjecture combines and generalizes Conjectures by Bergeron-Venkatesh, Hopf, Singer, Lück, and Shalen.
- A chain for a group G is a sequence of in G normal subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

such that $[G : G_i] < \infty$ and $\bigcap_{i \geq 0} G_i = \{1\}$.

Conjecture (Homological growth and L^2-invariants for aspherical closed manifolds)

Let M be an aspherical closed manifold of dimension d and fundamental group $G = \pi_1(M)$. Let $\widetilde{M}$ be its universal covering. Then

1.) *For any natural number n with $2n \neq d$ we get*

$$b_n^{(2)}(\widetilde{M}) = 0.$$

If $d = 2n$, we have

$$(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M}) \geq 0.$$

Conjecture (Continued)

2.) *Let $(G_i)_{i \geq 0}$ be any chain. Put $M[i] = G_i \backslash \widetilde{M}$.*
Then we get for any natural number n and any field F

$$b_n^{(2)}(\widetilde{M}) = \lim_{i \to \infty} \frac{b_n(M[i];F)}{[G:G_i]};$$

3.) *Let $(G_i)_{i \geq 0}$ be a chain. Put $M[i] = G_i \backslash \widetilde{M}$.*
Then we get for any natural number n with $2n+1 \neq d$

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i])\right)\right|\right)}{[G:G_i]} = 0,$$

and we get in the case $d = 2n+1$

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i])\right)\right|\right)}{[G:G_i]} = (-1)^n \cdot \rho^{(2)}(\widetilde{M}) \geq 0.$$

- The conjecture above is very optimistic, but we do not know a counterexample.
- It is related to the Approximation Conjecture for Fuglede-Kadison determinants.
- The main issue here are uniform estimates about the spectrum of the n-th Laplace operators on $M[i]$ which are independent of i.
- Some evidence for parts of this conjecture come from the work of Abert-Nikolov, Abert-Gelander-Nikolov, Ballmann-Brüning, Bergeron-Linnell-Lück-Sauer, Bergeron-Venkatesh, Bridson-Kochloukova, Cheeger-Gromov, Davis-Okun, Donnelly-Xavier, Gromov, Jost-Xin, Kar-Kropholler-Nikolov , Li-Thom, Lück, Linnell-Lück-Sauer.

- Let M be a closed hyperbolic 3-manifold. Then the conjecture above predicts for any chain $(G_i)_{i \geq 0}$

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_1(G_i)\right)\right|\right)}{[G : G_i]} = \frac{1}{6\pi} \cdot \operatorname{vol}(M).$$

Since the volume is always positive, the equation above implies that $\left|\operatorname{tors}\left(H_1(G_i)\right)\right|$ grows exponentially in $[G : G_i]$.

Further major applications of L^2-invariants

- Measure and orbit equivalence of groups
- Theory of von Neumann algebras
- Entropy
- Geometric group theory
- Group rings
- Algebraic K-theory
- 3-manifolds

Further major applications of L^2-invariants

- Measure and orbit equivalence of groups
- Theory of von Neumann algebras
- Entropy
- Geometric group theory
- Group rings
- Algebraic K-theory
- 3-manifolds