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Aspherical manifolds*

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ABSTRACT. This is a survey on known results and open problems about closed aspherical manifolds, i.e., connected closed manifolds whose universal coverings are contractible. Many examples come from certain kinds of non-positive curvature conditions. The property aspherical, which is a purely homotopy theoretical condition, has many striking implications about the geometry and analysis of the manifold or its universal covering, and about the ring theoretic properties and the K- and L-theory of the group ring associated to its fundamental group.

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1. Introduction

This page is devoted to aspherical closed manifolds.

Definition 1.1. A space X is called *aspherical* if it is path connected and all its higher homotopy groups vanish, i.e., $\pi_n(X)$ is trivial for $n \geq 2$.

Aspherical closed manifolds are very interesting objects since there are many examples, intriguing questions and conjectures about them. For instance:

- Interesting geometric constructions or examples lead to aspherical closed manifolds, e.g., non-positively curved closed manifolds, closed surfaces except S^2 and $\mathbb{RP}^n$, irreducible closed orientable 3-manifolds with infinite fundamental groups, locally symmetric spaces arising from almost connected Lie groups and discrete torsionfree cocompact lattices.
- There are exotic aspherical closed manifolds which do not come from standard constructions and have unexpected properties, e.g., the universal covering is not homeomorphic to $\mathbb{R}^n$, they are not triangulable. The key construction methods are the reflection trick and hyperbolization.
- Which groups occur as fundamental groups of aspherical closed manifolds?
- The Borel Conjecture predicts that aspherical closed topological manifolds are topologically rigid, i.e., any homotopy equivalence of aspherical closed manifolds is homotopic to the identity.
- The condition aspherical is of purely homotopy theoretical nature. Nevertheless there are some interesting questions and conjectures such as the Singer Conjecture and the Zero-in-the-Spectrum Conjecture about the spectrum of the Laplace operator on the universal coverings of aspherical closed Riemannian manifolds.

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2. Homotopy classification of spaces

From the homotopy theory point of view an aspherical CW-complex is completely determined by its fundamental group. Namely,

Theorem 2.1 (Homotopy classification of aspherical spaces)**.** *Two aspherical CW-complexes are homotopy equivalent if and only if their fundamental groups are isomorphic.*

Proof. By Whitehead's Theorem (see [75, Theorem IV.7.15 on page 182]) a map between CW-complexes is a homotopy equivalence if and only if it induces on all homotopy groups bijections. Hence it suffices to construct for two aspherical CW-complexes X and Y together with an isomorphism $\phi\colon \pi_1(X) \to \pi_1(Y)$ a map $f\colon X \to Y$ which induces ϕ on the fundamental groups. Any connected CW-complex is homotopy equivalent to a CW-complex with precisely one 0-cell, otherwise collapse a maximal sub-tree of the 1-skeleton to a point. Hence we can assume without loss of generality that the 1-skeleton of X is a bouquet of 1-dimensional spheres. The map ϕ tells us how to define $f_1\colon X_1 \to Y$, where X_n will denote the n-skeleton of X_n. The composites of the attaching maps for the two-cells of X with f_1 are null-homotopic by the Seifert-van Kampen Theorem. Hence we can extend f_1 to a map $f_2\colon X_2 \to Y$. Since all higher homotopy groups of Y are trivial, we can extend f_2 to a map $f\colon X \to Y$. $\square$

Lemma 2.2. *A CW-complex X is aspherical if and only if it is connected and its universal covering $\widetilde{X}$ is contractible.*

Proof. The projection $p\colon \widetilde{X} \to X$ induces isomorphisms on the homotopy groups π_n for $n \geq 2$ and a connected CW-complex is contractible if and only if all its homotopy groups are trivial (see [75, Theorem IV.7.15 on page 182]). $\square$

An aspherical CW-complex X with fundamental group π is the same as an *Eilenberg Mac-Lane space $K(\pi,1)$ of type $(\pi,1)$* and the same as the *classifying space $B\pi$ for the group π*.

3. Examples of aspherical manifolds

3.1. **Non-positive curvature.**

Let M be a closed smooth manifold. Suppose that it possesses a Riemannian metric whose sectional curvature is non-positive, i.e., is ≤ 0 everywhere. Then the universal covering $\widetilde{M}$ inherits a complete Riemannian metric whose sectional curvature is non-positive. Since $\widetilde{M}$ is simply-connected and has non-positive sectional curvature, the Hadamard-Cartan Theorem (see [36, 3.87 on page 134]) implies that $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^n$ and hence contractible. We conclude that $\widetilde{M}$ and hence M is aspherical.

3.2. **Low-dimensions.**

A connected closed 1-dimensional manifold is homeomorphic to S^1 and hence aspherical.

Let M be a connected closed 2-dimensional manifold. Then M is either aspherical or homeomorphic to S^2 or $\mathbb{RP}^2$. The following statements are equivalent:

(1) M is aspherical.

(2) M admits a Riemannian metric which is *flat*, i.e., with sectional curvature constant 0, or which is *hyperbolic*, i.e., with sectional curvature constant -1.
(3) The universal covering of M is homeomorphic to $\mathbb{R}^2$.

A connected closed 3-manifold M is called *prime* if for any decomposition as a connected sum $M \cong M_0 \sharp M_1$ one of the summands M_0 or M_1 is homeomorphic to S^3. It is called *irreducible* if any embedded sphere S^2 bounds a disk D^3. Every irreducible closed 3-manifold is prime. A prime closed 3-manifold is either irreducible or an S^2-bundle over S^1 (see [41, Lemma 3.13 on page 28]). A closed orientable 3-manifold is aspherical if and only if it is irreducible and has infinite fundamental group. This follows from the Sphere Theorem [41, Theorem 4.3 on page 40]. Thurston's Geometrization Conjecture implies that a closed 3-manifold is aspherical if and only if its universal covering is homeomorphic to $\mathbb{R}^3$. This follows from [41, Theorem 13.4 on page 142] and the fact that the 3-dimensional geometries which have compact quotients and whose underlying topological spaces are contractible have as underlying smooth manifold $\mathbb{R}^3$ (see [72]). A proof of Thurston's Geometrization Conjecture is given in [62] following ideas of Perelman. There are examples of closed orientable 3-manifolds that are aspherical but do not support a Riemannian metric with non-positive sectional curvature (see [52]). For more information about 3-manifolds we refer for instance to [41, 72].

3.3. **Torsionfree discrete subgroups of almost connected Lie groups.** Let L be a Lie group with finitely many path components. Let $K \subseteq L$ be a maximal compact subgroup. Let $G \subseteq L$ be a discrete torsionfree subgroup. Then $M = G\backslash L/K$ is an aspherical closed manifold with fundamental group G since its universal covering L/K is diffeomorphic to $\mathbb{R}^n$ for appropriate n (see [40, Theorem 1. in Chapter VI]).

3.4. **Products and fibrations.** Obviously the product $X \times Y$ of two aspherical spaces is again aspherical. More generally, if $F \to E \to B$ is a fibration for aspherical spaces B and F, then the long homotopy sequence associated to it shows that E is aspherical.

3.5. **Pushouts.** Let X be a CW-complex with sub-CW-complexes X_0, X_1 and X_2 such that $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Suppose that X_0, X_1 and X_2 are aspherical and that for $i = 0, 1, 2$ and each base point $x_i \in X_i$ the inclusion induces an injection $\pi_1(X_i, x_i) \to \pi_1(X, x_i)$. Then X is aspherical. The idea of the proof is to check by a Mayer-Vietoris argument that the reduced homology of $\widetilde{X}$ is trivial as $\widetilde{X}$ is the union of $\pi_1(X) \times_{\pi_1(X_1)} \widetilde{X_1}$ and $\pi_1(X) \times_{\pi_1(X_2)} \widetilde{X_2}$, and $\pi_1(X) \times_{\pi_1(X_0)} \widetilde{X_0}$ is the intersection of $\pi_1(X) \times_{\pi_1(X_1)} \widetilde{X_1}$ and $\pi_1(X) \times_{\pi_1(X_2)} \widetilde{X_2}$. Hence $\widetilde{X}$ is contractible by the Hurewicz Theorem (see [75, Theorem IV.7.15 on page 182]).

3.6. **Hyperbolization.** A very important construction of aspherical closed manifolds comes from the *hyperbolization technique* due to Gromov [38]. It turns a cell complex into a non-positively curved (and hence aspherical) polyhedron. The rough idea is to define this procedure for simplices such that it is natural under inclusions of simplices and then define the hyperbolization of a simplicial complex by gluing the results for the simplices together as described by the combinatorics of the simplicial complex. The goal is to achieve that the result shares some of the properties of the simplicial complexes one has started with, but additionally to produce a non-positively curved and hence aspherical polyhedron. Since this construction preserves local structures, it turns manifolds into manifolds. We briefly explain what the *orientable hyperbolization procedure* gives. Further expositions of this construction can be found in [15, 20, 21, 18]. We start with a finite-dimensional simplicial complex Σ and assign to it a cubical cell complex $h(\Sigma)$ and a natural map $c\colon h(\Sigma) \to \Sigma$ with the following properties:

(1) $h(\Sigma)$ is non-positively curved and in particular aspherical;
(2) The natural map $c\colon h(\Sigma) \to \Sigma$ induces a surjection on the integral homology;
(3) $\pi_1(f)\colon \pi_1(h(\Sigma)) \to \pi_1(\Sigma)$ is surjective;
(4) If Σ is an orientable manifold, then
(5) $h(\Sigma)$ is a manifold;
(6) The natural map $c\colon h(\Sigma) \to \Sigma$ has degree one;
(7) There is a stable isomorphism between the tangent bundle $Th(\Sigma)$ and the pullback $c^*T\Sigma$;

3.7. **Exotic aspherical closed manifolds.** The following result is taken from Davis-Januszkiewicz [18, Theorem 5a.1].

Theorem 3.1. *There is an aspherical closed 4-manifold N with the following properties:*

(1) *N is not homotopy equivalent to a PL-manifold;*
(2) *N is not triangulable, i.e., not homeomorphic to a simplicial complex;*
(3) *The universal covering $\widetilde{N}$ is not homeomorphic to $\mathbb{R}^4$;*
(4) *N is homotopy equivalent to a piecewise flat, non-positively curved polyhedron.*

The next result is due to Davis-Januszkiewicz [18, Theorem 5a.4].

Theorem 3.2 (Non-PL-example)**.** *For every $n \geq 4$ there exists an aspherical closed n-manifold which is not homotopy equivalent to a PL-manifold*

The proof of the following theorem can be found in [19], [18, Theorem 5b.1].

Theorem 3.3 (Exotic universal covering)**.** *For each $n \geq 4$ there exists an aspherical closed n-dimensional manifold such that its universal covering is not homeomorphic to $\mathbb{R}^n$.*

By the Hadamard-Cartan Theorem (see [36, 3.87 on page 134]) the manifold appearing in Theorem 3.3 above cannot be homeomorphic to a smooth manifold with Riemannian metric with non-positive sectional curvature. The following theorem is proved in [18, Theorem 5c.1 and Remark on page 386] by considering the ideal boundary, which is a quasiisometry invariant in the negatively curved case.

Theorem 3.4 (Exotic example with hyperbolic fundamental group)**.** *For every $n \geq 5$ there exists an aspherical closed smooth n-dimensional manifold N which is homeomorphic to a strictly negatively curved polyhedron and has in particular a hyperbolic fundamental group such that the universal covering is homeomorphic to $\mathbb{R}^n$ but N is not homeomorphic to a smooth manifold with Riemannian metric with negative sectional curvature.*

The next results are due to Belegradek [8, Corollary 5.1], Mess [60] and Weinberger (see [20, Section 13]).

Theorem 3.5 (Exotic fundamental groups)**.** (1) *For every $n \geq 4$ there is an aspherical closed manifold of dimension n whose fundamental group contains an infinite divisible abelian group;*
(2) *For every $n \geq 4$ there is an aspherical closed manifold of dimension n whose fundamental group has an unsolvable word problem and whose simplicial volume is non-zero.*

Notice that a finitely presented group with unsolvable word problem is not a CAT(0)-group, not hyperbolic, not automatic, not asynchronously automatic, not residually finite and not linear over any commutative ring (see [8, Remark 5.2]). The proof of Theorem 3.5 is based on the *reflection group trick* as it appears for instance in [20, Sections 8, 10 and 13]. It can be summarized as follows.

Theorem 3.6 (Reflection group trick)**.** *Let G be a group which possesses a finite model for BG. Then there is an aspherical closed manifold M and two maps $i\colon BG \to M$ and $r\colon M \to BG$ such that $r \circ i = \mathrm{id}_{BG}$.*

Remark 3.7 (Reflection group trick and various conjectures)**.** Another interesting immediate consequence of the reflection group trick is (see also [20, Sections 11]) that many well-known conjectures about groups hold for every group which possesses a finite model for BG if and only if it holds for the fundamental group of every aspherical closed manifold. This applies for instance to the Kaplansky Conjecture, Unit Conjecture, Zero-divisor-conjecture, Baum-Connes Conjecture, Farrell-Jones Conjecture for algebraic K-theory for regular R, Farrell-Jones Conjecture for algebraic L-theory, the vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ and of $\mathrm{Wh}(G) = 0$, For information about these conjectures and their links we refer for instance to [5], [56] and [54]. Further similar consequences of the reflection group trick can be found in Belegradek [8].

4. Non-aspherical closed manifolds

A closed manifold of dimension ≥ 1 with finite fundamental group is never aspherical. So prominent non-aspherical closed manifolds are spheres, lens spaces, real projective spaces and complex projective spaces.

Lemma 4.1. *The fundamental group of an aspherical finite-dimensional CW-complex X is torsionfree.*

Proof. Let $C \subseteq \pi_1(X)$ be a finite cyclic subgroup of $\pi_1(X)$. We have to show that C is trivial. Since X is aspherical, $C\backslash\widetilde{X}$ is a finite-dimensional model for BC. Hence $H_k(BC) = 0$ for large k. This implies that C is trivial. □

We mention without proof:

Lemma 4.2. *If M is a connected sum $M_1 \sharp M_2$ of two closed manifolds M_1 and M_2 of dimension $n \geq 3$ which are not homotopy equivalent to a sphere, then M is not aspherical.*

5. Characteristic classes and bordisms of aspherical closed manifolds

Suppose that M is a closed manifold. Then the pullbacks of the characteristic classes of M under the natural map $c\colon h(M) \to M$ appearing in the Section 3.6 about hyperbolization yield the characteristic classes of $h(M)$ and M and $h(M)$ have the same characteristic numbers. This shows that the condition aspherical does not impose any restrictions on the characteristic numbers of a manifold. Consider a bordism theory Ω_* for PL-manifolds or smooth manifolds which is given by imposing conditions on the stable tangent bundle. Examples are unoriented bordism, oriented bordism, framed bordism. Then any bordism class can be represented by an aspherical closed manifold. If two aspherical closed manifolds represent the same bordism class, then one can find an aspherical bordism between them. See [20, Remarks 15.1], [18, Theorem B], and [17].

6. The Borel Conjecture

Definition 6.1 (Topologically rigid)**.** We call a closed manifold N *topologically rigid* if any homotopy equivalence $M \to N$ with a closed manifold M as source is homotopic to a homeomorphism.

The Poincaré Conjecture is equivalent to the statement that any sphere S^n is topologically rigid.

Conjecture 6.2 (Borel Conjecture)**.** Every aspherical closed manifold is topologically rigid.

In particular the Borel Conjecture 6.2 implies because of Theorem 2.1 that two aspherical closed manifolds are homeomorphic if and only if their fundamental groups are isomorphic.

Remark 6.3 (The Borel Conjecture in low dimensions)**.** The Borel Conjecture is true in dimension ≤ 2 by the classification of closed manifolds of dimension 2. It is true in dimension 3 if Thurston's Geometrization Conjecture is true. This follows from results of Waldhausen (see Hempel [41, Lemma 10.1 and Corollary 13.7]) and Turaev (see [73]) as explained for instance in [50, Section 5]. A proof of Thurston's Geometrization Conjecture is given in [62] following ideas of Perelman.

Remark 6.4 (Topological rigidity for non-aspherical manifolds)**.** Topological rigidity phenomenons do hold also for some non-aspherical closed manifolds. For instance the sphere S^n is topologically rigid by the Poincaré Conjecture. The Poincaré Conjecture is known to be true in all dimensions. This follows in high dimensions from the h-cobordism theorem, in dimension four from the work of Freedman [34], in dimension three from the work of Perelman as explained in [48] and [61] and in dimension two from the classification of surfaces. Many more examples of classes of manifolds which are topologically rigid are given and analyzed in Kreck-Lück [50]. For instance the connected sum of closed manifolds of dimension ≥ 5 which are topologically rigid and whose fundamental groups do not contain elements of order two, is again topologically rigid and the connected sum of two manifolds is in general not aspherical (see Lemma 4.2). The product $S^k \times S^n$ is topologically rigid if and only if k and n are odd.

Remark 6.5 (The Borel Conjecture does not hold in the smooth category)**.** The Borel Conjecture 6.2 is false in the smooth category, i.e., if one replaces topological manifold by smooth manifold and homeomorphism by diffeomorphism. The torus T^n for $n \geq 5$ is an example (see [74, 15A]).

Other interesting counterexamples involving negatively curved manifolds are given by Farrell-Jones [24, Theorem 0.1]. They construct for every $\delta > 0$ and $d \geq 5$ a d-dimensional closed hyperbolic manifold M and a closed Riemannian manifold N such that the sectional curvature of N is pinched between $-1-\delta$ and $-1+\delta$ and the manifolds M and N are homeomorphic but not diffeomorphic.

Remark 6.6 (The Borel Conjecture versus Mostow rigidity)**.** The examples of Farrell-Jones [24, Theorem 0.1] give actually more. Namely, they yield for given $\epsilon > 0$ a closed Riemannian manifold M_0 whose sectional curvature lies in the interval $[1-\epsilon, -1+\epsilon]$ and a closed hyperbolic manifold M_1 such that M_0 and M_1 are homeomorphic but no diffeomorphic. The idea of the construction is essentially to take the connected sum of M_1 with exotic spheres. Notice that by definition M_0 were hyperbolic if we would take $\epsilon = 0$. Hence this example is remarkable in view of *Mostow rigidity*, which predicts for two closed hyperbolic manifolds N_0 and N_1 that they are isometrically diffeomorphic if and only if $\pi_1(N_0) \cong \pi_1(N_1)$ and any homotopy equivalence $N_0 \to N_1$ is homotopic to an isometric diffeomorphism. One may view the Borel Conjecture as the topological version of Mostow rigidity. The conclusion in the Borel Conjecture is weaker, one gets only homeomorphisms and not isometric diffeomorphisms, but the assumption is also weaker, since there are many more aspherical closed topological manifolds than hyperbolic closed manifolds.

Remark 6.7 (The work of Farrell-Jones)**.** Farrell-Jones have made deep contributions to the Borel Conjecture. They have proved it in dimension ≥ 5 for non-positively curved closed Riemannian manifolds, for compact complete affine flat manifolds and for aspherical closed manifolds whose fundamental group is isomorphic to the fundamental group of a complete non-positively curved Riemannian manifold which is A-regular (see [25, 26, 27, 28]).

The following result is a consequence of [3, 7, 4].

Theorem 6.8. *Let $\mathcal{B}$ be the smallest class of groups satisfying:*

- *Every hyperbolic group belongs to $\mathcal{B}$;*
- *Every* $\mathrm{CAT}(0)$*-group, i.e., a group that acts properly, isometrically and cocompactly on a complete proper* $\mathrm{CAT}(0)$*-space, belongs to $\mathcal{B}$;*
- *Every cocompact lattice in an almost connected Lie group belongs to $\mathcal{B}$;*
- *Every arithmetic group over an algebraic number field belongs to $\mathcal{B}$;*
- *If G_1 and G_2 belong to $\mathcal{B}$, then both $G_1 * G_2$ and $G_1 \times G_2$ belong to $\mathcal{B}$;*
- *If H is a subgroup of G and $G \in \mathcal{B}$, then $H \in \mathcal{B}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{B}$ for every $i \in I$. Then the directed colimit $\operatorname{colim}_{i\in I} G_i$ belongs to $\mathcal{B}$.*

Then every aspherical closed manifold of dimension ≥ 5 whose fundamental group belongs to $\mathcal{B}$ is topologically rigid.

Actually, Bartels and Lück [7] prove the Farrell-Jones Conjecture about the algebraic K- and L-theory of group rings which does imply the claim appearing in Theorem 6.8 by surgery theory.

Remark 6.9 (Exotic aspherical closed manifolds)**.** Theorem 6.8 implies that the exotic aspherical manifolds mentioned in Subsection 3.7 satisfy the Borel Conjecture in dimension ≥ 5 since their universal coverings are CAT(0)-spaces.

Remark 6.10 (Directed colimits of hyperbolic groups)**.** There are also a variety of interesting groups such as *lacunary groups* in the sense of Olshanskii-Osin-Sapir [**?**] or *groups with expanders* as they appear in the counterexample to the Baum-Connes Conjecture with coefficients due to Higson-Lafforgue-Skandalis [42] and which have been constructed by Arzhantseva-Delzant [1, Theorem 7.11 and Theorem 7.12] following ideas of Gromov [39]. Since these arise as colimits of directed systems of hyperbolic groups, they do satisfy the Farrell-Jones Conjecture and the Borel Conjecture in dimension ≥ 5 by Bartels and Lück [7]. The *Bost Conjecture* has also been proved for colimits of hyperbolic groups by Bartels-Echterhoff-Lück [2].

7. Poincaré duality groups

In this section we deal with the question when a group G is the fundamental group of an aspherical closed manifold. The following definition is due to Johnson-Wall [46].

Definition 7.1 (Poincaré duality group)**.** A group G is called a *Poincaré duality group of dimension n* if the following conditions holds:

(1) The group G is of type FP, i.e., the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ possesses a finite-dimensional projective $\mathbb{Z}G$-resolution by finitely generated projective $\mathbb{Z}G$-modules;
(2) We get an isomorphism of abelian groups

$$H^i(G;\mathbb{Z}G) \cong \begin{cases} \{0\} & \text{for } i \neq n; \\ \mathbb{Z} & \text{for } i = n. \end{cases}$$

Conjecture 7.2 (Poincaré duality groups)**.** A finitely presented group is a n-dimensional Poincaré duality group if and only if it is the fundamental group of an aspherical closed n-dimensional topological manifold.

A topological space X is called an *absolute neighborhood retract* or briefly ANR if for every normal space Z, every closed subset $Y \subseteq Z$ and every (continuous) map $f\colon Y \to X$ there exists an open neighborhood U of Y in Z together with an extension $F\colon U \to Z$ of f to U. A *compact n-dimensional homology* ANR-*manifold* X is a compact absolute neighborhood retract such that it has a countable basis for its topology, has finite topological dimension and for every $x \in X$ the abelian group $H_i(X, X - \{x\})$ is trivial for $i \neq n$ and infinite cyclic for $i = n$. A closed n-dimensional topological manifold is an example of a compact n-dimensional homology ANR-manifold (see [16, Corollary 1A in V.26 page 191]). For a proof of the next result we refer to [58, Section 5].

Theorem 7.3. *Suppose that the torsionfree group G belongs to the class $\mathcal{B}$ occurring in Theorem 6.8 and its cohomological dimension is ≥ 6. Then G is the fundamental group of an aspherical compact homology* ANR*-manifold.*

Remark 7.4 (Compact homology ANR-manifolds versus closed topological manifolds)**.** One would prefer if in the conclusion of Theorem 7.3 one could replace 'compact homology ANR-manifold' by 'closed topological manifold'. There are compact homology ANR-manifolds that are not homotopy equivalent to closed manifolds. But no example of an aspherical compact homology ANR-manifold that is not homotopy equivalent to a closed topological manifold is known.

The Borel Conjecture about the topologically rigidity of closed topological manifolds and the fact that it is implied by the Farrell-Jones Conjecture indimensions ≥ 5 carry over to compact homology ANR-manifolds if one replaces 'being homotopic to a homeomorphism' by 'being s-cobordant to a homeomorphism'.

We refer for instance to [12, 29, 68, 69, 70] for more information about this topic.

8. PRODUCT DECOMPOSITIONS

In this section we show that, roughly speaking, an aspherical closed manifold M is a product $M_1 \times M_2$ if and only if its fundamental group is a product $\pi_1(M) = G_1 \times G_2$ and that such a decomposition is unique up to homeomorphism. A proof of the next result can be found in [58, Section 6].

Theorem 8.1 (Product decomposition)**.** *Let M be an aspherical closed manifold of dimension n with fundamental group $G = \pi_1(M)$. Suppose we have a product decomposition*

$$p_1 \times p_2 \colon G \xrightarrow{\cong} G_1 \times G_2.$$

Suppose that G, G_1 and G_2 belong to the class $\mathcal{B}$ occurring in Theorem 6.8. Assume that the cohomological dimension $\mathrm{cd}(G_i)$ is different from 3, 4 and 5 for $i = 1, 2$ and $n \neq 4$. Then:

(1) *There are aspherical closed topological manifolds M_1 and M_2 together with isomorphisms*

$$v_i \colon \pi_1(M_i) \xrightarrow{\cong} G_i$$

and maps

$$f_i \colon M \to M_i$$

for $i = 1, 2$ such that

$$f = f_1 \times f_2 \colon M \to M_1 \times M_2$$

is a homeomorphism and $v_i \circ \pi_1(f_i) = p_i$ (up to inner automorphisms) for $i = 1, 2$;

(2) *Suppose we have another such choice of aspherical closed manifolds M_1' and M_2' together with isomorphisms*

$$v_i' \colon \pi_1(M_i') \xrightarrow{\cong} G_i$$

and maps

$$f_i' \colon M \to M_i'$$

for $i = 1, 2$ such that the map $f' = f'_1 \times f'_2$ is a homotopy equivalence and $v'_i \circ \pi_1(f'_i) = p_i$ (up to inner automorphisms) for $i = 1, 2$. Then there are for $i = 1, 2$ homeomorphisms $h_i \colon M_i \to M'_i$ such that $h_i \circ f_i \simeq f'_i$ and $v_i \circ \pi_1(h_i) = v'_i$ holds for $i = 1, 2$.

Remark 8.2 (Product decompositions and non-positive sectional curvature)**.** The following result has been proved independently by Gromoll-Wolf [37, Theorem 2] and Lawson-Yau [51]. Let M be a closed Riemannian manifold with non-positive sectional curvature. Suppose that we are given a splitting of its fundamental group $\pi_1(M) = G_1 \times G_2$ and that the center of $\pi_1(M)$ is trivial. Then this splitting comes from an isometric product decomposition of closed Riemannian manifolds of non-positive sectional curvature $M = M_1 \times M_2$.

9. The Novikov Conjecture

Let G be a group and let $u \colon M \to BG$ be a map from a closed oriented smooth manifold M to BG. Let

$$\mathcal{L}(M) \in \bigoplus_{k \in \mathbb{Z}, k \geq 0} H^{4k}(M; \mathbb{Q})$$

be the *L-class of M*. Its k-th entry $\mathcal{L}(M)_k \in H^{4k}(M; \mathbb{Q})$ is a certain homogeneous polynomial of degree k in the rational Pontrjagin classes $p_i(M; \mathbb{Q}) \in H^{4i}(M; \mathbb{Q})$ for $i = 1, 2, \ldots, k$ such that the coefficient s_k of the monomial $p_k(M; \mathbb{Q})$ is different from zero. The L-class $\mathcal{L}(M)$ is determined by all the rational Pontrjagin classes and vice versa. The L-class depends on the tangent bundle and thus on the differentiable structure of M. For $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$ define the *higher signature of M associated to x and u* to be the integer

$$\operatorname{sign}_x(M, u) \quad := \quad \langle \mathcal{L}(M) \cup f^*x, [M] \rangle.$$

We say that sign_x for $x \in H^*(BG; \mathbb{Q})$ is *homotopy invariant* if for two closed oriented smooth manifolds M and N with reference maps $u \colon M \to BG$ and $v \colon N \to BG$ we have

$$\operatorname{sign}_x(M, u) = \operatorname{sign}_x(N, v),$$

whenever there is an orientation preserving homotopy equivalence $f \colon M \to N$ such that $v \circ f$ and u are homotopic. If $x = 1 \in H^0(BG)$, then the higher signature $\operatorname{sign}_x(M, u)$ is by the Hirzebruch signature formula (see [44, 45]) the signature of M itself and hence an invariant of the oriented homotopy type. This is one motivation for the following conjecture.

Conjecture 9.1 (Novikov Conjecture)**.** Let G be a group. Then sign_x is homotopy invariant for all $x \in \prod_{k \in \mathbb{Z}, k \geq 0} H^k(BG; \mathbb{Q})$.

This conjecture appears for the first time in the paper by Novikov [66, § 11]. A survey about its history can be found in [32]. More information can be found for instance in [30, 31, 49].

Remark 9.2 (The Novikov Conjecture and aspherical closed manifolds)**.** Let the map $f \colon M \to N$ be a homotopy equivalence of aspherical closed oriented manifolds. Then the Novikov Conjecture 9.1 implies that $f_*\mathcal{L}(M) = \mathcal{L}(N)$. This is certainly true if f is a diffeomorphism. On the other hand, in general the rational Pontrjagin classes are not homotopy invariants and the integral Pontrjagin classes $p_k(M)$ are not homeomorphism invariants (see for instance [49, Example 1.6 and Theorem 4.8]). This seems to shed doubts about the Novikov Conjecture. However, if the Borel Conjecture is true, the map $f\colon M \to N$ is homotopic to a homeomorphism and the conclusion $f_*\mathcal{L}(M) = \mathcal{L}(N)$ does follow from the following deep result due to Novikov [64, 63, 65].

Theorem 9.3 (Topological invariance of rational Pontrjagin classes)**.** *The rational Pontrjagin classes $p_k(M,\mathbb{Q}) \in H^{4k}(M;\mathbb{Q})$ are topological invariants, i.e. for a homeomorphism $f\colon M \to N$ of closed smooth manifolds we have*

$$H_{4k}(f;\mathbb{Q})\Big(p_k(M;\mathbb{Q})\Big) = p_k(N;\mathbb{Q})$$

for all $k \geq 0$ and in particular $H_*(f;\mathbb{Q})(\mathcal{L}(M)) = \mathcal{L}(N)$.

Remark 9.4 (Positive scalar curvature)**.** There is the conjecture that a closed aspherical smooth manifold does not carry a metric of positive scalar curvature. One evidence for it is the fact that it is implied by the (strong) Novikov Conjecture see [71, Theorem 3.5].

10. BOUNDARIES OF HYPERBOLIC GROUPS

We mention the following result of Bartels-Lück-Weinberger [6]. For the notion of the boundary of a hyperbolic group and its main properties we refer for instance to [47].

Theorem 10.1. *Let G be a torsion-free hyperbolic group and let n be an integer ≥ 6. Then the following statements are equivalent:*

(1) *The boundary ∂G is homeomorphic to S^{n-1};*
(2) *There is an aspherical closed topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n;*
(3) *The aspherical closed topological manifold M appearing in the assertion above is unique up to homeomorphism.*

In general the boundary of a hyperbolic group is not locally a Euclidean space but has a fractal behavior. If the boundary ∂G of an infinite hyperbolic group G contains an open subset homeomorphic to Euclidean n-space, then it is homeomorphic to S^n. This is proved in [47, Theorem 4.4], where more information about the boundaries of hyperbolic groups can be found. For every $n \geq 5$ there exists a strictly negatively curved polyhedron of dimension n whose fundamental group G is hyperbolic, which is homeomorphic to an aspherical closed smooth manifold and whose universal covering is homeomorphic to $\mathbb{R}^n$, but the boundary ∂G is not homeomorphic to S^{n-1}, see [18, Theorem 5c.1 on page 384 and Remark on page 386]. Thus the condition that ∂G is a sphere for a torsion-free hyperbolic group is (in high dimensions) not equivalent to the existence of an aspherical closed manifold whose fundamental group is G.

Remark 10.2 (The Cannon Conjecture)**.** We do not get information in dimensions $n \leq 4$ for the usual problems about surgery. In the case $n = 3$ there is the conjecture of Cannon [13] that a group G acts properly, isometrically and cocompactly on the 3-dimensional hyperbolic plane $\mathcal{H}^3$ if and only if it is a hyperbolic group whose boundary is homeomorphic to S^2. Provided that the infinite hyperbolic group G occurs as the fundamental group of a closed irreducible 3-manifold, Bestvina-Mess [11, Theorem 4.1] have shown that its universal covering is homeomorphic to $\mathbb{R}^3$ and its compactification by ∂G is homeomorphic to D^3, and the Geometrization Conjecture of Thurston implies that M is hyperbolic and G satisfies Cannon's conjecture. The problem is solved in the case $n = 2$, namely, for a hyperbolic group G its boundary ∂G is homeomorphic to S^1 if and only if G is a Fuchsian group (see [14, 33, 35]).

11. L^2-INVARIANTS

Next we mention some prominent conjectures about aspherical closed manifolds and L^2-invariants of their universal coverings. For more information about these conjectures and their status we refer to [56] and [57].

11.1. **The Hopf and the Singer Conjectures.**

Conjecture 11.1 (Hopf Conjecture)**.** If M is an aspherical closed manifold of even dimension, then

$$(-1)^{\dim(M)/2} \cdot \chi(M) \geq 0.$$

If M is a closed Riemannian manifold of even dimension with sectional curvature $\sec(M)$, then

$$\begin{array}{rclcrcl}
(-1)^{\dim(M)/2} \cdot \chi(M) & > & 0 & \text{if} & \sec(M) & < & 0; \\
(-1)^{\dim(M)/2} \cdot \chi(M) & \geq & 0 & \text{if} & \sec(M) & \leq & 0; \\
\chi(M) & = & 0 & \text{if} & \sec(M) & = & 0; \\
\chi(M) & \geq & 0 & \text{if} & \sec(M) & \geq & 0; \\
\chi(M) & > & 0 & \text{if} & \sec(M) & > & 0.
\end{array}$$

Conjecture 11.2 (Singer Conjecture)**.** If M is an aspherical closed manifold, then

$$b_n^{(2)}(\widetilde{M}) = 0 \qquad \text{if } 2n \neq \dim(M).$$

If M is a closed connected Riemannian manifold with negative sectional curvature, then

$$b_n^{(2)}(\widetilde{M}) \begin{cases} = 0 & \text{if } 2n \neq \dim(M); \\ > 0 & \text{if } 2n = \dim(M). \end{cases}$$

11.2. L^2–**torsion and aspherical closed manifolds.**

Conjecture 11.3 (L^2-torsion for aspherical closed manifolds)**.** If M is an aspherical closed manifold of odd dimension, then $\widetilde{M}$ is det-L^2-acyclic and

$$(-1)^{\frac{\dim(M)-1}{2}} \cdot \rho^{(2)}(\widetilde{M}) \geq 0.$$

If M is a closed connected Riemannian manifold of odd dimension with negative sectional curvature, then $\widetilde{M}$ is det-L^2-acyclic and

$$(-1)^{\frac{\dim(M)-1}{2}} \cdot \rho^{(2)}(\widetilde{M}) > 0.$$

If M is an aspherical closed manifold whose fundamental group contains an amenable infinite normal subgroup, then $\widetilde{M}$ is det-L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) = 0.$$

11.3. **Homological growth and L^2-torsion for closed aspherical manifolds.** The following conjecture is motivated by [56, Conjecture 11.3 on page 418] and in particular by the preprint of Bergeron and Venkatesh [10, Conjecture 1.3].

Conjecture 11.4 (Homological growth and L^2-torsion for aspherical manifolds)**.** Let M be a closed aspherical manifold of dimension n. Let

$$\pi_1(M) = G_0 \supseteq G_1 \supseteq G_1 \supseteq \cdots$$

be a nested sequence of in G normal subgroups of finite index $[G : G_i]$ such that their intersection $\bigcap_{i\geq 0} G_i$ is the trivial subgroup. Then:

$$\begin{array}{rcll} \lim_{i\in I} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(G_i\backslash\widetilde{M};\mathbb{Z})\right)\right|\right)}{[G:G_i]} & = & 0 & \text{if } 2n+1 \neq \dim(M); \\ \lim_{i\in I} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(G_i\backslash\widetilde{M};\mathbb{Z})\right)\right|\right)}{[G:G_i]} & = & (-1)^p\cdot\rho^{(2)}\left(\widetilde{M}\right) & \text{if } 2n+1 = \dim(M). \end{array}$$

If $\pi_1(M)$ is residually finite, then Conjecture 11.4 implies Conjecture 11.3. Conjecture 11.4 has been proved in the special case, where $\pi_1(M)$ contains an infinite normal elementary amenable subgroup or M carries a non-trivial S^1-action, in [59]. A very interesting open case is the one of a closed hyperbolic 3-manifold.

11.4. $\mathbb{Q}$ **versus $\mathbb{F}_p$-approximation.**

Conjecture 11.5 (Approximation by Betti numbers)**.** Let M be a closed aspherical manifold of dimension n. Let

$$\pi_1(M) = G_0 \supseteq G_1 \supseteq G_1 \supseteq \cdots$$

be a nested sequence of in G normal subgroups of finite index $[G : G_i]$ such that their intersection $\bigcap_{i\geq 0} G_i$ is the trivial subgroup. Let K be any field. Then we get for every $n \geq 0$

$$b_n^{(2)}(\widetilde{M}) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};K)}{[G:G_i]}.$$

Remark 11.6. Conjecture 11.5 follows from [55] in the case that K has characteristic zero, actually without the assumption that M is aspherical. The interesting and open case is the case of the prime characteristic p, where the assumption 'aspherical' is definitely necessary, see for instance [9], [22] and [53], and one may additionally demand that each index $[G : G_i]$ is a p-power.

11.5. **Simplicial volume and L^2-invariants.**

Conjecture 11.7 (Simplicial volume and L^2-invariants)**.** Let M be an aspherical closed orientable manifold. Suppose that its simplicial volume $||M||$ vanishes. Then $\widetilde{M}$ is of determinant class and

$$\begin{array}{rcll} b_n^{(2)}(\widetilde{M}) & = & 0 & \text{for } n \geq 0; \\ \rho^{(2)}(\widetilde{M}) & = & 0. & \end{array}$$

11.6. **The Zero-in-the-Spectrum Conjecture.**

Conjecture 11.8 (Zero-in-the-spectrum Conjecture)**.** Let $\widetilde{M}$ be a complete Riemannian manifold. Suppose that $\widetilde{M}$ is the universal covering of an aspherical closed Riemannian manifold M (with the Riemannian metric coming from M). Then for some $p \geq 0$ zero is in the Spectrum of the minimal closure

$$(\Delta_p)_{\min} \colon \operatorname{dom}\bigl((\Delta_p)_{\min}\bigr) \subset L^2\Omega^p(\widetilde{M}) \to L^2\Omega^p(\widetilde{M})$$

of the Laplacian acting on smooth p-forms on $\widetilde{M}$.

Remark 11.9 (Non-aspherical counterexamples to the Zero-in-the-Spectrum Conjecture)**.** For all of the conjectures about aspherical spaces stated in this article it is obvious that they cannot be true if one drops the condition aspherical except for the zero-in-the-Spectrum Conjecture 11.8. Farber and Weinberger [23] gave the first example of a closed Riemannian manifold for which zero is not in the spectrum of the minimal closure $(\Delta_p)_{\min} \colon \operatorname{dom}((\Delta_p)_{\min}) \subset L^2\Omega^p(\widetilde{M}) \to L^2\Omega^p(\widetilde{M})$ of the Laplacian acting on smooth p-forms on $\widetilde{M}$ for each $p \geq 0$. The construction by Higson, Roe and Schick [43] yields plenty of such counterexamples. But there are no aspherical counterexamples known.

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