

Euler characteristics, Betti numbers, and their L^2 -analogue

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Topologists, doughnuts, and coffee cups

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- Frequently quoted phrase:

A topologist is someone who cannot tell a doughnut from a coffee cup.

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- Frequently quoted phrase:

A topologist is someone who cannot tell a doughnut from a coffee cup.

- Even better:

A topologist is someone who can tell a doughnut from a coffee cup, but does not necessarily want to.

- My favourite:

Topologists had the brilliant insight that, in a certain sense, the geometric object that describes a doughnut is the same as the geometric object that describes a coffee cup.

Homeomorphism

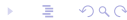
- Let M and N be two geometric objects e.g., a ball, a doughnut, or their surfaces.

Homeomorphism

- Let M and N be two geometric objects e.g., a ball, a doughnut, or their surfaces.
- They are called **homeomorphic**, if there exist continuous maps $f: M \rightarrow N$ and $g: N \rightarrow M$ satisfying $g \circ f = \text{id}_M$ and $f \circ g = \text{id}_N$.

- Informally, this means that by stretching and deforming one object you can transform it into the other without ever cutting, tearing, or breaking it apart and gluing it back together.

- A ball of radius 4000 miles, a ball of radius radius 1.5 inches, and an egg are homeomorphic.
- This is also true for their surfaces.







- A ball and a cube are homeomorphic





- A doughnut and a coffee cup are homeomorphic.





- Surprise:

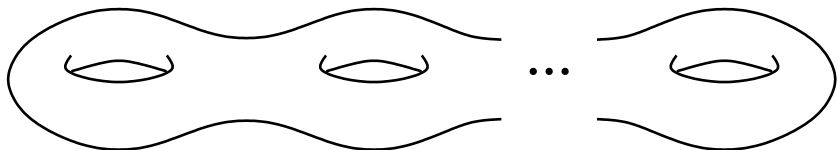
A ball and a doughnut are **not** homeomorphic.

Surfaces

- A **surface** is a geometric closed subspace in three-dimensional space that is locally homeomorphic to two-dimensional space.

- Surfaces of genus g .

(The genus is the number of the holes)



Theorem

Every surface is homeomorphic to exactly one standard surface of genus g .

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- The following images show objects whose surface describes a surface in the mathematically sense above.
- Question: What is their genus?



















Euler-Charakteristik

Euler-Charakteristik

- You can cover any surface with polygons in such a way that any two polygons either do not meet at all or their intersection consists of exactly one edge.



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- The **Euler-Charakteristik** χ of such a covering is defined as

$$\chi = V - E + F.$$

where V , E , and F are the numbers of vertices, edges, and faces, respectively.

Theorem

The Euler characteristic does not depend on the choice of covering.

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Theorem

Two surfaces are homeomorphic if and only if they have the same Euler characteristic since we have

$$\chi(F_g) = 2 - 2g.$$

- The Euler characteristic of the surface of a ball, which is also sometimes called sphere S^2 and agrees with F_0 , ist 2.
- The Euler characteristic of the surface of a doughnut F_1 ist 0.
- The Euler characteristic of the surface of a coffee cup is also 0.

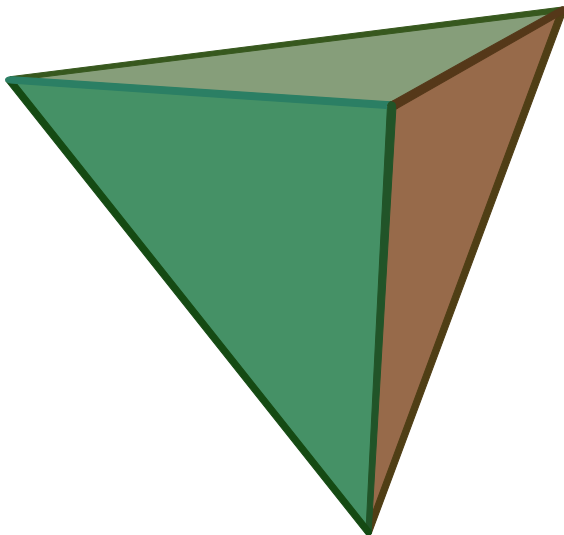
Platonic solids

- A **Platonic solid** is a convex body in space that is bounded by regular n -gons in such a way that the intersection of any two regular n -gons is either empty or consists of exactly one common edge, and exactly m edges meet at each vertex.

Tetrahedron

Tetrahedron.svg

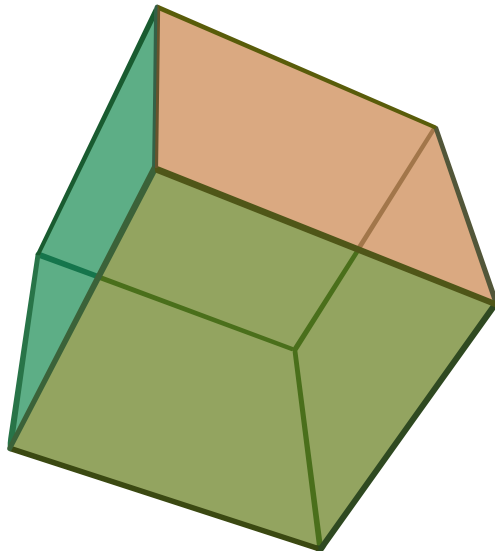
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Hexahedron

Hexahedron.svg

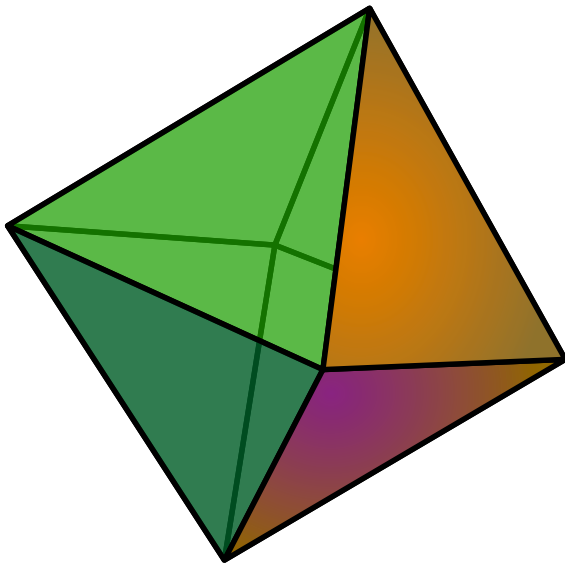
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Octahedron

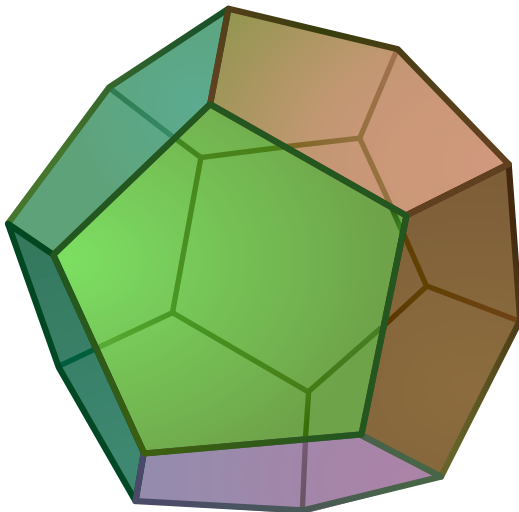
Octahedron.svg

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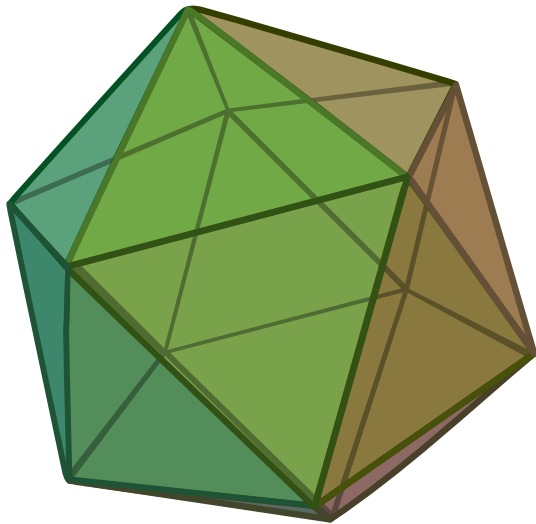
Dodecahedron

POV-Ray-Dodecahedron.svg



Icosahedron

Icosahedron.svg



- Although there are infinitely many regular polygons, there exist only five Platonic solids:

Tetrahedron,

Hexahedron,

Octahedron,

Dodecahedron,

Icosahedron.

- We will prove this using the Euler characteristic.

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- The surface of a Platonic solid is homeomorphic to sphere F_0 .
- Hence we get

$$V - E + F = 2.$$

- Obviously, we also have

$$mV = 2E;$$

$$nF = 2E.$$

- This implies

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{E} + \frac{1}{2}.$$

- Obviously we must have $m, n \geq 3$.

- The last equation implies

$$\frac{1}{m} + \frac{1}{n} > \frac{1}{2}.$$

- Therefore only the following cases can occur:

$$m = 3 \quad n = 3;$$

$$m = 4 \quad n = 3;$$

$$m = 3 \quad n = 4;$$

$$m = 3 \quad n = 5;$$

$$m = 5 \quad n = 3;$$

solid	m	n	V	E	F
Tetrahedron	3	3	6	4	4
Hexahedron	3	4	12	8	6
Octahedron	4	3	12	6	8
Dodecahedron	3	5	30	20	12
Icosahedron	5	3	30	12	20

A geometric proof of a combinatorial formula

- We want to give a geometric proof of the formula for any natural number d :

$$1 = \sum_{i=0}^d (-1)^i \cdot \binom{d+1}{i+1}.$$

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$$1 = \sum_{i=0}^d (-1)^i \cdot \binom{d+1}{i+1}.$$

- The d -simplex Δ_d is defined to be the subspace of the $(d+1)$ -dimensional Euclidean space \mathbb{R}^{d+1} given by the convex hull of standard basis.

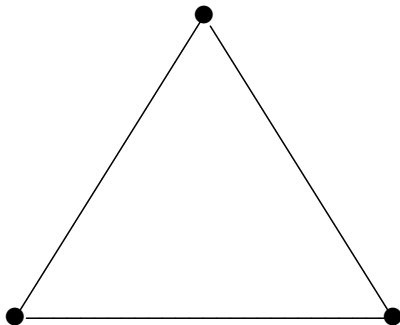
- Δ_0 is just a point.



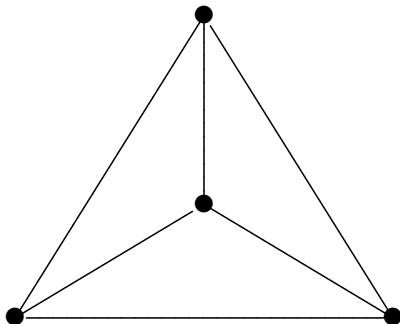
- Δ_1 is the interval $[0, 1]$.



- Δ_2 is the solid triangle



- Δ_3 is the solid tetrahedron



- Define the **Euler characteristic** $\chi(\Delta_d)$ for Δ_d by counting the number of n -dimensional faces with alternating signs:

$$\chi(\Delta_d) = \sum_{n \geq 0} (-1)^n \cdot |\{n - \text{faces}\}|.$$

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$$\chi(\Delta_d) = \sum_{n \geq 0} (-1)^n \cdot |\{n - \text{faces}\}|.$$

- Δ_d and Δ_0 are not homeomorphic, but they are equal in an even weaker sense, namely they are **homotopy equivalent**.

- This implies

$$\chi(\Delta_d) = \chi(\Delta_0) = 1.$$

- The number of the i -dimensional faces of Δ_d ist

$$\binom{d+1}{i+1}.$$

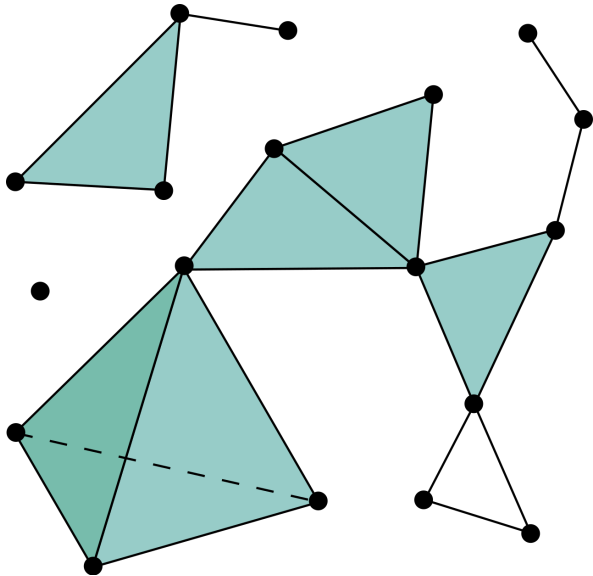
- Hence we get

$$1 = \chi(\Delta_0) = \chi(\Delta_d) = \sum_{i=0}^d (-1)^i \cdot \binom{d+1}{i+1}$$

- This is a geometric proof of a combinatorial formula.

Simplicial complex

- A **simplicial complex** is a topological space built by gluing finitely many simplices in Euclidean space. It is a collection of these simplices where any two intersect only at a common face or not at all, and all faces of a simplex are included in the collection.



- Define the **Euler characteristic** $\chi(X)$ of a simplicial complex X by counting the number of n -dimensional faces with alternating signs:

$$\chi(X) = \sum_{i \geq 0} (-1)^i \cdot |\{i\text{-simplices}\}|.$$

- For instance we get for the simplicial complex X above

$$\chi(X) = 18 - 23 + 8 - 1 = 2.$$

- One can assign to a simplicial complex X its **simplicial chain complex** $C_*(X)$, i.e., a sequence of real vector spaces

$$\begin{aligned} \dots &\xrightarrow{c_{n+1}} C_n \xrightarrow{c_n} C_{n-1} \xrightarrow{c_{n-1}} \dots \\ &\dots \xrightarrow{c_2} C_1 \xrightarrow{c_1} C_0 \xrightarrow{c_{-1}} \{0\} \end{aligned}$$

such that $c_{n-1} \circ c_n = 0$ holds for $n \geq 0$ and each C_n is a real vector space with the set of n -dimensional simplices as basis.

Definition

Define for $n \geq 0$ the n -th homology module of X to be

$$H_n(X) = \ker(c_n) / \operatorname{im}(c_{n+1}).$$

Definition

Define for $n \geq 0$ the n -th Betti number of X to be

$$b_n(X) = \dim_{\mathbb{R}}(H_n(X)).$$

Theorem

We have

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n(X).$$

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- We indicate the proof in the case, where X has only 0 and 1-simplices.

- The simplicial chain complex looks like

$$0 \rightarrow C_1 \xrightarrow{c_1} C_0 \rightarrow 0.$$

- This implies

$$H_1(X) = \ker(c_1);$$

$$H_0(X) = C_0 / \operatorname{im}(c_1).$$

- We compute

$$\begin{aligned} b_0(X) - b_1(X) &= \dim_{\mathbb{R}}(H_0(X)) - \dim_{\mathbb{R}}(H_1(X)) \\ &= \dim_{\mathbb{R}}(\mathcal{C}_0 / \text{im}(\mathcal{C}_1)) - \dim_{\mathbb{R}}(\ker(\mathcal{C}_1)) \\ &= \dim_{\mathbb{R}}(\mathcal{C}_0) - \dim_{\mathbb{R}}(\text{im}(\mathcal{C}_1)) - \dim_{\mathbb{R}}(\ker(\mathcal{C}_1)) \\ &= \dim_{\mathbb{R}}(\mathcal{C}_0) - \dim_{\mathbb{R}}(\mathcal{C}_1) \\ &= |\{0 - \text{simplices}\}| - |\{1 - \text{simplices}\}| \\ &= \chi(X). \end{aligned}$$

- Let us consider the case $X = \Delta_d$ for $d = 0, 1, 2$.
- The simplicial chain complex $C_*(\Delta_d)$ looks like

$$0 \rightarrow 0 \xrightarrow{0} \mathbb{R} \rightarrow 0$$

for $d = 0$ and like

$$0 \rightarrow \mathbb{R} \xrightarrow{\begin{pmatrix} 1 & -1 \end{pmatrix}} \mathbb{R}^2 \rightarrow 0$$

for $d = 1$.

- The simplicial chain complex $C_*(\Delta_d)$ looks like

$$0 \rightarrow \mathbb{R} \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}} \mathbb{R} \rightarrow 0.$$

for $d = 2$.

- We get for the homology

$$H_n(\Delta_d) = \begin{cases} \mathbb{R} & n = 0; \\ 0 & n \neq 0. \end{cases}$$

- We get for the Betti numbers

$$b_n(\Delta_d) = \begin{cases} 1 & n = 0; \\ 0 & n \neq 0. \end{cases}$$

- Hence we obtain for $d = 0, 1, 2$

$$\chi(\Delta_d) = 1 = \sum_{n \geq 0} (-1)^n \cdot b_n(\Delta_d).$$

Theorem

Let X and Y be simplicial complexes which are homotopy equivalent. Then:

- 1 We get for $n \geq 0$

$$b_n(X) = b_n(Y).$$

- 2 We get

$$\chi(X) = \chi(Y).$$

Corollary

The following statements are equivalent for two surfaces S and S' ;

- 1 S and S' are homeomorphic.*
- 2 S and S' are homotopy equivalent.*
- 3 They have the same Betti numbers.*
- 4 They have the same Euler characteristic.*
- 5 They have the same genus.*

L^2 -Betti numbers

- Following **Atiyah** one can generalize the notion of Betti number to the notion of L^2 -Betti number as follows.
- Let π be the fundamental group of the simplicial complex X and $p: \tilde{X} \rightarrow X$ be its universal covering.
- Then the simplicial chain complex $C_*(\tilde{X})$ becomes a free $\mathbb{R}[\pi]$ -chain complex such that $C_n(\tilde{X})$ has the set of n -simplices as $\mathbb{R}[\pi]$ -basis.

- However, the ring $\mathbb{R}[\pi]$ is very complicated. It is in general not Noetherian and hence $H_n(C_*(X))$ is not necessarily finitely generated as $\mathbb{R}[\pi]$ -module.
- The way out is to complete $\mathbb{R}[\pi]$ to the so called **group von Neumann algebra** $\mathcal{N}(G)$ which has better properties.
- In particular one can assign to any $\mathcal{N}(G)$ -module M a dimension $\dim(M) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.

Definition

Define the n - L^2 -Betti number $b_n^{(2)}(\tilde{X})$ to be

$$\dim_{\mathcal{N}(\pi)}(H_n(\mathcal{N}(G) \otimes_{\mathbb{R}[G]} C_*(\tilde{X}))) \in \mathbb{R}_{\geq 0}.$$

- Note that the L^2 -Betti numbers are not necessarily integers.
- The L^2 -Betti numbers have all the useful properties of the classical Betti numbers.

- They are homotopy invariants of X .
- They satisfy

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\tilde{X})$$

- They have an analytic interpretation for a closed Riemannian manifold M in terms of the heat kernel $e^{-t\tilde{\Delta}_n}(x, y)$ on the universal covering

$$b_n^{(2)}(\tilde{X}) = \lim_{t \rightarrow \infty} \int_{\mathcal{F}} \text{tr}(e^{-t\tilde{\Delta}_n}(x, x)) d \text{vol}_M.$$

- They have even further good properties which classical Betti numbers, do not have.
- One has

$$b_0^{(2)}(\tilde{X}) = \begin{cases} |\pi|^{-1} & |\pi| < \infty; \\ 0 & \text{otherwise.} \end{cases}$$

- They are multiplicative under finite coverings.
- The latter implies $b_0^{(2)}(\tilde{S^1}) = 0$ for $n \geq 0$.

- L^2 -Betti numbers have many applications to problems which on the first glance are not related to L^2 -Betti numbers at all.
- Here are some examples.

Theorem (Signatures of 4-manifolds and group extensions)

Let M be a closed oriented 4-manifold. Suppose that $\pi_1(M)$ contains an infinite normal finitely generated subgroup of infinite index.

Then

$$|\operatorname{sign}(M)| \leq \chi(M).$$

Theorem (Hopf's Conjecture holds for hyperbolic manifolds)

Let M be a hyperbolic closed manifold of even dimension $2m$. Then

$$(-1)^m \cdot \chi(M) > 0.$$

Theorem (S^1 -actions and hyperbolic manifolds)

Any S^1 -action on a hyperbolic closed manifold is trivial.

Theorem (Euler characteristic of amenable groups)

Let G be a group which contains a normal infinite amenable subgroup. Suppose that there is a finite model for BG .

Then its Euler characteristic

$$\chi(BG) := \sum_{n \geq 0} (-1)^n \dim_{\mathbb{C}}(H_n(BG; \mathbb{C})).$$

vanishes.

Theorem (Deficiency and group extensions)

Let $1 \rightarrow H \xrightarrow{i} G \xrightarrow{q} K \rightarrow 1$ be an exact sequence of infinite groups. Suppose that G is finitely presented and H is finitely generated. Then:

$$\text{deficiency}(G) \leq 1.$$

Theorem (Kähler manifolds and projective algebraic varieties,)

Let M be a closed Kähler manifold.

Suppose that $\pi_1(M)$ is hyperbolic (in the sense of Gromov) and $\pi_2(M)$ is trivial.

Then M is a projective algebraic variety.

Conjecture (Atiyah Conjecture for torsionfree fundamental groups)

If M is a closed Riemannian manifold whose fundamental group $\pi_1(M)$ is torsionfree, then all L^2 -Betti numbers $b_n^{(2)}(\tilde{M})$ are integers.

- The Atiyah Conjecture is known to be true for a rather large class of groups and implies the following prominent and old conjecture.

Conjecture (Zero-divisor Conjecture)

Let G be a torsionfree group and F a field of characteristic zero.

Then the group ring $F[G]$ has no zero-divisors.

Thank you for your attention!