A Survey on L^2-torsion

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Introduction

- Atiyah [1] introduced the notion of L^2-Betti numbers. They are the L^2-analogue of Betti numbers. L^2-Betti numbers have many applications to algebra, geometry, and group theory.
- A secondary invariant, the L^2-torsion has been defined analytically by Lott [22] and Mathai [32] and topologically by Lück-Rothenberg [30]. It is the L^2-analogue of Ray-Singer torsion.
- We will discuss basic properties and applications as well as open problems and potential applications of L^2-torsion to various fields in mathematics without going deeply into technical details.
- Hopefully this will be picked up as interesting research projects by some mathematicians.
- We are not planning to go over all the slides in the talk.
- The slides can be downloaded from my homepage.

Topics

1. Basics definitions
2. Basic properties of L^2-torsion
3. Approximation
4. Homological growth
5. Twisting with finite dimensional representations
6. The Thurston norm and the degree of the ϕ-twisted L^2-torsion function
7. An invariant of group automorphisms
8. Simplicial volume and L^2-invariants
9. L^2-torsion and measure equivalence
10. (Generalized) Lehmer's problem
11. References

Basic definitions

- Let G be a (discrete) group. Its group von Neumann algebra

$$\mathcal{N}(G) = \mathcal{B}(L^2(G), L^2(G))^G$$

is the algebra of bounded G-equivariant operators $L^2(G) \to L^2(G)$.

- The von Neumann trace is defined to be

$$\operatorname{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$

- Let $f \colon L^2(G)^m \to L^2(G)^n$ be a G-equivariant bounded operator.
- Then $f^*f \colon L^2(G)^m \to L^2(G)^m$ is a positive G-equivariant bounded operator. Let $\{E_\lambda \mid \lambda \geq 0\}$ be its spectral family and $f^*f = \int_0^\infty \lambda dE_\Lambda$ be its spectral decomposition. Each E_λ is a G-equivariant orthogonal projection $L^2(G)^m \to L^2(G)^m$.

- Note that E_λ can be thought of a (n,n)-matrix $A = (a_{i,i})$ over $\mathcal{N}(G)$ and we can define its von Neumann trace

$$\operatorname{tr}_{\mathcal{N}(G)}(E_\lambda) = \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(a_{i,i}) \in [0,\infty).$$

- Define the spectral density function of f to be

$$F\colon [0,\infty) \to [0,\infty), \quad \lambda \mapsto \operatorname{tr}_{\mathcal{N}(G)}(E_{\lambda^2}).$$

- This is a monotone non-decreasing right-continuous function.
- Define the von Neumann dimension of the kernel of f, which agrees with the kernel of f^*f, to be

$$\dim_{\mathcal{N}(G)}(\ker(f)) = F(0) \in [0,\infty).$$

- Define the Fuglede-Kadison determinant of f to be

$$\det_{\mathcal{N}(G)}(f) = \begin{cases} \exp\left(\int_{0+}^{\infty} \ln(\lambda)\, dF\right) \in (0,\infty) & \text{if } \int_{0+}^{\infty} \ln(\lambda)\, dF > -\infty; \\ 0 & \text{otherwise.} \end{cases}$$

- We have

$$\ln(\det_{\mathcal{N}(G)}(f)) = \ln(a) \cdot (F(a) - F(0)) - \int_{0+}^{a} \frac{F(\lambda) - F(0)}{\lambda}\, d\lambda$$

if $\ln(0) = -\infty$ and $a \geq ||f||$.

- Let $\overline{X} \to X$ be a G-covering of the finite CW-complex X.
- Let $C_*^c(\overline{X})$ be its cellular $\mathbb{Z}[G]$-chain complex. Define the cellular Hilbert $\mathcal{N}(G)$-chain complex

$$C_*^2(\overline{X}) = L^2(G) \otimes_{\mathbb{Z}[G]} C_*^c(\overline{X}).$$

It is of the shape

$$\cdots \xrightarrow{c_{n+2}^{(2)}} L^2(G)^{|I_{n+1}|} \xrightarrow{c_{n+1}^{(2)}} L^2(G)^{|I_n|} \xrightarrow{c_n^{(2)}} L^2(G)^{|I_{n-1}|} \xrightarrow{c_{n-1}^{(2)}} \cdots$$

where I_n is the set of n-cells of X and each $c_n^{(2)}$ is a G-equivariant bounded operator.

- Define the combinatorial n-th Laplace operator to be the positive G-equivariant bounded operator

$$\Delta_n^{(2)} = c_{n+1}^{(2)} \circ (c_n^{(2)})^* + (c_{n-1}^{(2)})^* \circ c_n^{(2)} \colon L^2(G)^{|I_n|} \to L^2(G)^{|I_n|}.$$

- Define the *n-th L^2 Betti number* of $\overline{X}$ to be

$$b_n^{(2)}(\overline{X};\mathcal{N}(G)) = \dim_{\mathcal{N}(G)}(\ker(\Delta_n^{(2)})) \in [0,\infty).$$

- We say that $\overline{X}$ is *L^2-acyclic* if $b_n^{(2)}(\overline{X};\mathcal{N}(G)) = 0$ holds for $n \geq 0$.
- We say that $\overline{X}$ is of *det-class* if $\det_{\mathcal{N}(G)}(\Delta_n^{(2)}) > 0$ holds for $n \geq 0$.
- If $\overline{X}$ is of det-class, then we define the *L^2-torsion* of $\overline{X}$ to be

$$\rho^{(2)}(\overline{X};\mathcal{N}(G)) = \sum_{n\geq 0}(-1)^n \cdot n \cdot \ln(\det_{\mathcal{N}(G)}(\Delta_n^{(2)})) \quad \in \mathbb{R}.$$

- The condition det-class is satisfied automatically if G belongs to a very large class of groups which contains all sofic groups and all groups we will be interested in. Therefore we will from now on assume tacitly that $\overline{X}$ is of det-class and not discuss this notion any further.

- $b_n^{(2)}(\overline{X})$ is a G-homotopy invariant of $\overline{X}$.
- Note that $b_0^{(2)}(\overline{X}) = |G|^{-1}$ and hence zero if G is infinite, whereas $b_0(X) = 1$. Hence $\overline{X}$ can be L^2-acyclic (and will be in many interesting cases), whereas X is never acyclic.
- If $\overline{X}$ is L^2-acyclic, then $\rho^{(2)}(\overline{X})$ is a simple G-homotopy invariant of $\overline{X}$. We can drop simple if G satisfies the Farrell-Jones Conjecture which is known to be true for a large class of groups, see [29].
- If $\overline{X}$ is not L^2-acyclic, then $\rho^{(2)}(\overline{X})$ depends on the structure of a finite CW-complex.
- If we assume that X is a closed Riemannian manifold, then one can modify the definition by taking the L^2-Hodge-deRham isomorphism into account, so that $\rho^{(2)}(\overline{X})$ becomes independent of the choice of a smooth triangulation of M but depends on the Riemannian metric.
- If X is a closed Riemannian manifold and $\overline{X}$ is L^2-acyclic, then this modification does not occur and $\rho^{(2)}(\overline{X})$ is a (simple) G-homotopy invariant.

- If M is a closed Riemannian manifold, the L^2-Betti numbers can be defined analytically in terms of the heat kernel on $\widetilde{M}$

$$b_n^{(2)}(\overline{M};\mathcal{N}(G)) = \lim_{t\to\infty}\int_{\mathcal{F}} \operatorname{tr}_{\mathbb{R}}(e^{-t\Delta_n^{(2)}}(\overline{x},\overline{x}))\; d\mathrm{vol}_{\overline{M}},$$

where $\mathcal{F}$ is a fundamental domain for the G-action on $\overline{M}$.

- If M is a closed Riemannian manifold, its L^2-torsion has an analytic expression in terms of the heat kernel on $\overline{M}$, namely for any choice of $\epsilon > 0$ we have

$$\begin{aligned}\rho^{(2)}(\overline{M};\mathcal{N}(G)) \;&=\; \frac{1}{2}\cdot\sum_{n\geq 0}(-1)^n\cdot n\cdot\Biggl(\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^{\epsilon} t^{s-1}\cdot\theta_n(t)dt\;\Bigg|_{s=0}\\ &\qquad + \int_{\epsilon}^{\infty} t^{-1}\cdot\theta_n(t)dt\Biggr)\end{aligned}$$

for $\theta_n(t) = \int_{\mathcal{F}} \operatorname{tr}(e^{-t\Delta_n^{(2)}}(\overline{x},\overline{x}))\; d\mathrm{vol}_{\widetilde{M}} - b_n^{(2)}(\overline{M};\mathcal{N}(G))$.

- Note that

$$\begin{aligned} b_n^{(2)}(\overline{M};\mathcal{N}(G)) &= B_n \cdot \mathrm{Vol}(M); \\ \rho^{(2)}(\overline{M};\mathcal{N}(G)) &= T_n \cdot \mathrm{Vol}(M), \end{aligned}$$

  hold for constants B_n and C_n if $\mathrm{tr}(e^{-t\Delta_n^{(2)}}(\overline{x},\overline{x}))$ is independent of $\overline{x}$, e.g., if M is hyperbolic or a locally symmetric space.
- The L^2-version of the Cheeger-Müller Theorem is proved by Burghelea-Friedlander-Kappeler-McDonald [6].
- The main idea of their proof is to perform the Witten deformation of the Laplacian with a suitable Morse function and investigate the splitting of the de Rham complex according to small and large eigenvalues.
- Our favourite case is when X is a connected finite CW-complex, G is the fundamental group $\pi = \pi_1(X)$, and $\overline{X}$ is the universal covering of $\widetilde{X}$ of X. In this case we abbreviate

$$\begin{aligned} b_n^{(2)}(\widetilde{X}) &= b_n^{(2)}(\widetilde{X};\mathcal{N}(\pi)); \\ \rho^{(2)}(\widetilde{X}) &= \rho^{(2)}(\widetilde{X};\mathcal{N}(\pi)). \end{aligned}$$

The special case where G is finite

- Let us discuss the special case where G is finite.
- Then $\overline{X}$ is a finite CW-complex and is a closed Riemannian manifold if X is a closed Riemannian manifold.
- $L^2(G)$ and $\mathcal{N}(G)$ agree with $\mathbb{C}G$ and $\mathrm{tr}_{\mathcal{N}(G)}(\sum_{g \in G} \lambda_g \cdot g) = \lambda_e$.
- The spectral density function of the G-equivariant linear map $f \colon \mathbb{C}G^m \to \mathbb{C}G^n$ is a right continuous step function which jumps at any eigenvalue μ of f^*f by the multiplicity of this eigenvalue μ scaled by $[G]^{-1}$.
- We have

$$\dim_{\mathcal{N}(G)}(\ker(f)) = \dim_{\mathcal{N}(G)}(\ker(f^*f)) = \frac{\dim_{\mathbb{C}}(\ker(f))}{|G|}.$$

- We get
$$\det_{\mathcal{N}(G)}(f) = \det_{\mathbb{C}}(f^*f^{\perp})^{\frac{1}{2|G|}}$$
for $f^*f^{\perp}\colon \ker(f^*f)^{\perp} \xrightarrow{\cong} \ker(f^*f)^{\perp}$ the automorphism induced by f^*f.
If f is injective, this boils down to $\det_{\mathcal{N}(G)}(f) = \det_{\mathbb{C}}(f^*f)^{\frac{1}{2|G|}}$.
If f is a selfadjoint automorphism, we get $\det_{\mathcal{N}(G)}(f) = |\det_{\mathbb{C}}(f)|^{\frac{1}{|G|}}$.
- We have
$$b_n^{(2)}(\overline{X};\mathcal{N}(G)) = \frac{b_n(\overline{X})}{|G|}.$$
- The L^2-torsion $\rho^{(2)}(\overline{X};\mathcal{N}(G))$ is the Ray-Singer torsion $\rho_{RS}(\overline{X})$ of $\overline{X}$ scaled by $|G|^{-1}$.

Upshot

- The upshot of the discussion above is that whenever for a connected finite CW-complex X its universal covering $\widetilde{X}$ is L^2-acyclic, then a secondary invariant, its L^2-torsion $\rho(\widetilde{X}) \in \mathbb{R}$, can be considered and is a (simple) homotopy invariant of X.
- The relation of the L^2-torsion to L^2-Betti numbers can be viewed as the L^2-analogue of the relation of the classical Reidemeister torsion to classical Betti numbers.
- L^2-torsion is the L^2-analogue of Ray-Singer torsion.
- Next we want to convince the reader about the high potential of L^2-torsion.
- No knowledge about the constructions above is needed for the rest of the talk which will be much less technical.

Basic properties of L^2-torsion

- For more information about L^2-Betti numbers and L^2-torsion, the proofs of the following results, and the relevant references in the literature, we refer for instance to [24].
- (Simple) homotopy invariance

  If X and Y are simple homotopy equivalent and X is L^2-acyclic, then Y is L^2-acyclic and we get

  $$\rho^{(2)}(\widetilde{X}) = \rho^{(2)}(\widetilde{Y}).$$

  If the Farrell-Jones Conjecture holds, we can drop simple.
- Sum formula

  If $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$, X_i is L^2-acyclic for $i = 0, 1, 2$, and the inclusions $X_i \to X$ are π_1-injective, then X is L^2-acyclic and we get

  $$\rho^{(2)}(\widetilde{X}) = \rho^{(2)}(\widetilde{X_1}) + \rho^{(2)}(\widetilde{X_2}) - \rho^{(2)}(\widetilde{X_0}).$$

- **Product formula**

  If X is L^2-acyclic, then $X \times Y$ is L^2-acyclic and we get

  $$\rho^{(2)}(\widetilde{X \times Y}) = \chi(Y) \cdot \rho^{(2)}(\widetilde{Y}).$$

- **Fibration formula**

  Let $F \to E \to B$ be a fibration of connected finite CW-complexes such that F is L^2-acyclic and the inclusion $F \to E$ is π_1-injective.

  Then E is L^2-acyclic and we get

  $$\rho^{(2)}(\widetilde{E}) = \chi(B) \cdot \rho^{(2)}(\widetilde{F}).$$

- **Poincaré duality**

  If M is a closed manifold which is L^2-acyclic and of even dimension, then

  $$\rho^{(2)}(\widetilde{M}) = 0.$$

- Multiplicativity
  Let $Y \to X$ be a finite covering with d-sheets. Then
  $$b_n^{(2)}(\widetilde{Y}) = d \cdot b_n^{(2)}(\widetilde{X}).$$
  Suppose that X or Y is L^2-acyclic. Then both are L^2-acyclic and
  $$\rho^{(2)}(\widetilde{Y}) = d \cdot \rho^{(2)}(\widetilde{X}).$$
- We conclude that $\widetilde{X}$ is L^2-acyclic and satisfies
  $$\rho^{(2)}(\widetilde{X}) = 0,$$
  provided that there exists a d-sheeted covering $X \to X$ for $d \geq 2$.
- Hence $\widetilde{X \times S^1}$ is L^2-acyclic and satisfies
  $$\rho^{(2)}(\widetilde{X \times S^1}) = 0.$$

- Hyperbolic manifolds of odd dimension

  If M is a closed hyperbolic manifold of odd dimension $2k+1$, then M is L^2-acyclic and there is a rational number $r_k > 0$ (depending only on k) satisfying

$$\rho^{(2)}(\widetilde{M}) = (-1)^k \cdot \pi^{-k} \cdot r_k \cdot \mathrm{Vol}(M).$$

- Hyperbolic manifolds of even dimension

  If M is a closed hyperbolic manifold of even dimension $2k$, then

$$b_2^{(2)}(\widetilde{M}) = \begin{cases} (-1)^k \cdot \chi(M) > 0 & n = k; \\ 0 & \text{otherwise.} \end{cases}$$

- Locally symmetric spaces

  There are analogous formulas for locally symmetric spaces of non-compact type.

- Aspherical closed manifolds

  Let M be a closed manifold which is aspherical, i.e., its universal covering is contractible. Assume one of the following conditions:

  1. M carries a non-trivial S^1-action;
  2. The fundamental group $\pi_1(M)$ contains an infinite normal elementary amenable subgroup.

  Then $\widetilde{M}$ is L^2-acyclic and $\rho^{(2)}(\widetilde{M})$ vanishes.

- This implies that every S^1-action on a hyperbolic manifold M is trivial and that its Euler characteristic satisfies $(-1)^k \cdot \chi(M) > 0$ if $\dim(M) = 2k$.

- 3-manifolds

  Let M be a compact connected irreducible 3-manifold with infinite π whose boundary is empty or a union of incompressible tori. Let $M_1, M_2, \ldots, M_r$ be the hyperbolic pieces in its JSJ-decomposition. Define $\operatorname{Vol}(M)$ to be $\sum_{i=1}^{r} \operatorname{Vol}(M_i)$.

  Then M is L^2-acyclic and

$$\rho^{(2)}(\widetilde{M}) = \frac{-1}{6\pi} \cdot \operatorname{Vol}(M).$$

- Knots

  Let $K \subseteq S^3$ be a knot and $M(K)$ be its knot complement which is the complement of an open regular neighborhood of K. Then

  1. $M(K)$ is L^2-acyclic and we can define the L^2-torsion $\rho^{(2)}(K) := \rho^{(2)}(\widetilde{M(K)})$.
  2. We have $\rho^{(2)}(K) = 0$ if and only if K is obtained from the trivial knot by applying a finite number of times the operation "connected sum" and "cabling".
  3. A knot is trivial if and only if both its L^2-torsion $\rho^{(2)}(K)$ and its Alexander polynomial $\Delta(K)$ are trivial.

- One sees that L^2-torsion has much nicer global properties than Ray-Singer torsion if the fundamental group is infinite and L^2-acyclicity holds, which is quite often the case.

Approximation

- In general there are no relations between the Betti numbers $b_n(X)$ and the L^2-Betti numbers $b_n^{(2)}(\widetilde{X})$ for a connected finite CW-complex X except for the Euler Poincaré formula

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{X}) = \sum_{n \geq 0} (-1)^n \cdot b_n(X).$$

- But there is an approximate relation described next.

- A normal chain $\{G_i\}$ of the group G is a descending chain of subgroups
$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$
such that G_i is normal in G and $\bigcap_{i \geq 0} G_i = \{1\}$.

- A normal chain is a finite index normal chain, if and only if $[G : G_i]$ is finite for each i.

- Put $X[i] =:= \overline{X}/G_i$. Then the projection $X[i] \to X$ is a G/G_i-covering.

- The basic intuition is that the tower of coverings $X[i] \to X$ approximates the G-covering $\overline{X} \to X$.

- For $A \in M_{s,t}(R[G])$, let $A[i] \in M_{s,t}(R[G/G_i])$ be obtained from A by applying the projection $R[G] \to R[G/G_i]$ to each entry of A.

Theorem (Approximation Theorem, Lück)

Let $\overline{X} \to X$ be a G-covering of the finite CW-complex X.

Then for any finite index normal chain sequence $\{G_i\}$

$$b_n^{(2)}(\overline{X}; \mathcal{N}(G)) = \lim_{i \to \infty} \frac{b_n(X[i])}{[G : G_i]}.$$

- Next we explain what happens if we drop the condition finite index.

Conjecture (Approximation Conjecture for L^2-Betti numbers)

A group G together with normal chain $\{G_i \mid i \in \mathbb{N}\}$ satisfies the Approximation Conjecture for L^2-Betti numbers if one of the following equivalent conditions hold:

1. *Matrix version*

   Let $A \in M_{r,s}(\mathbb{Q}G)$ be a matrix. Then

$$\begin{aligned}\dim_{\mathcal{N}(G)}\bigl(\ker\bigl(r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s\bigr)\bigr) \\ = \lim_{i \to \infty} \dim_{\mathcal{N}(G/G_i)}\bigl(\ker\bigl(r_{A[i]}^{(2)} \colon L^2(G/G_i)^r \to L^2(G/G_i)^s\bigr)\bigr);\end{aligned}$$

2. *CW-complex version*

   Let X be a finite CW-complex and $\overline{X} \to X$ be a G-covering. Then

$$b_n^{(2)}(X;\mathcal{N}(G)) = \lim_{i\to\infty} b_n^{(2)}(X[i];\mathcal{N}(G/G_i)).$$

- The basic pattern of an Approximation Theorem or Approximation Conjecture is a formula of the shape

$$\alpha^{(2)}(\overline{X};\mathcal{N}(G)) = \lim_{i\to\infty} \alpha^{(2)}(X[i];\mathcal{N}(G/G_i))$$

for an L^2-invariant $\alpha^{(2)}$ such for finite $[G:G_i]$ we have

$$\alpha^{(2)}(X[i];\mathcal{N}(G/G_i)) = \frac{\alpha(X[i])}{[G:G_i]}$$

for some classical term α.

Conjecture (Determinant Conjecture for a group G)

For any matrix $A \in M_{r,s}(\mathbb{Z}[G])$, the Fuglede-Kadison determinant of the G-equivariant bounded operator $r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s$ given by right multiplication with A satisfies

$$\det^{(2)}_{\mathcal{N}(G)}\bigl(r_A^{(2)}\bigr) \geq 1.$$

- The Determinant Conjecture implies the Approximation Conjecture for L^2-Betti numbers.
- The Determinant Conjecture holds for a very large class of groups which contains all sofic groups.
- The Determinant Conjecture implies the condition of $\det$-class.
- Next we deal with the obvious question whether analogous Approximation Conjectures make sense for the Fuglede-Kadison determinant and the L^2-torsion.

Conjecture (Approximation Conjecture for Fuglede-Kadison determinants)

A group G satisfies the Approximation Conjecture for Fuglede-Kadison determinants if for any normal chain $\{G_i\}$ and any matrix $A \in M_{r,s}(\mathbb{Q}G)$ we get for the Fuglede-Kadison determinant

$$\begin{aligned} &\det_{\mathcal{N}(G)}\bigl(r_A^{(2)}\colon L^2(G)^r \to L^2(G)^s\bigr) \\ &\quad = \lim_{i\to\infty} \det_{\mathcal{N}(G/G_i)}\bigl(r_{A[i]}^{(2)}\colon L^2(G/G_i)^r \to L^2(G/G_i)^s\bigr). \end{aligned}$$

- The Approximation Conjecture for Fuglede-Kadison determinants is known to be true for $G = \mathbb{Z}$ and hence for any infinite virtually cyclic group G but to the author's knowledge not for any other infinite group G which is not virtually cyclic.
- Nevertheless we are optimistic that it holds for many interesting groups.

Theorem (Uniform Integrability Condition, Lück)

Let $A \in M_{r,s}(\mathbb{Z}G)$ be a matrix. Let F and $F[i]$ be the spectral density functions of $r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s$. and $r_{A[i]}^{(2)} \colon L^2(G/G_i)^r \to L^2(G/G_i)^s$.

Suppose that the Uniform Integrability Condition is satisfied, i.e., there exists $\epsilon > 0$ satisfying

$$\int_{0+}^{\epsilon} \sup \left\{ \frac{F[i](\lambda) - F[i](0)}{\lambda} \;\middle|\; i \in \mathbb{N} \right\} d\lambda \quad < \quad \infty.$$

Then:

$$\begin{aligned} & \det_{\mathcal{N}(G)}(r_A^{(2)} \colon L^2(G)^r \to L^2(G)^s) \\ & \quad = \quad \lim_{i \to \infty} \det_{\mathcal{N}(G/G_i)}(r_{A[i]}^{(2)} \colon L^2(G/G_i)^r \to L^2(G/G_i)^s). \end{aligned}$$

- If there is a uniform gap at zero for the operators $r^{(2)}_{A[i]}$, i.e., there exists $\epsilon > 0$ satisfying $F[i](\epsilon) = F[i](0)$ for almost all $i \in \mathbb{N}$, then the Uniform Integrability Condition is obviously satisfied.
- There is some evidence that the Uniform Integrability Condition is true for many interesting groups G.
- One reason is that there is a countable set $S \subseteq [0, \infty)$ such that for all $\lambda \in [0, \infty) \setminus S$ we have

$$F(\lambda) = \lim_{i \to \infty} F[i](\lambda)$$

and often F is well-understood and behaves well, e.g., there are constants $\epsilon > 0$ and $\delta > 0$ satisfying

$$F(\lambda) - F(0) \leq C \cdot \lambda^{\delta} \quad \text{for all} \quad \lambda \in (0, \epsilon)$$

which implies

$$\int_{0+}^{\epsilon} \frac{F(\lambda) - F(0)}{\lambda} \, d\lambda \quad < \quad \infty.$$

Theorem (Lück)

Suppose that the Determinant Conjecture holds which is true if the normal chain $\{G_i\}$ is of finite index.

Then there are constants $C > 0$ and $\epsilon > 0$ (depending on A only) such that

$$F[i](\lambda) - F[i](0) \leq \frac{C}{-\ln(\lambda)}$$

holds for all $\lambda \in (0, \epsilon)$ and $i \in \mathbb{N}$.

- This is not enough to ensure the Uniform Integrability Condition.
- The Uniform Integrability Condition does follows if we can find additionally $\mu > 0$ such that the stronger inequality

  $$\frac{F[i](\lambda) - F[i](0)}{\lambda} \leq \frac{C}{(-\ln(\lambda))^{1+\mu}}$$

  holds for all $\lambda \in (0, \epsilon)$ and $i \in \mathbb{N}$.

Conjecture (Approximation Conjecture for L^2-torsion)

Let $\overline{M} \to M$ be a G-covering of the closed Riemannian manifold M.

Then we get for any normal chain $\{G_i\}$

$$\rho^{(2)}(\overline{M}; \mathcal{N}(G)) = \lim_{i \to \infty} \rho^{(2)}(M[i]; \mathcal{N}(G/G_i)).$$

Theorem (Lück)

Let $\overline{M} \to M$ be a G-covering of the closed Riemannian manifold M. Suppose that G satisfies the Approximation Conjecture for Fuglede-Kadison determinants and we assume that $\overline{M}$ is L^2-acyclic.

Then we get for any normal chain normal chain $\{G_i\}$

$$\rho^{(2)}(\overline{M}; \mathcal{N}(G)) = \lim_{i \to \infty} \rho^{(2)}(M[i]; \mathcal{N}(G/G_i)).$$

Conjecture (Approximation Conjecture for L^2-torsion for normal chains of finite index)

Let $\overline{M} \to M$ be a G-covering of the closed Riemannian manifold M. Then we get for any normal chain of finite index $\{G_i\}$

$$\rho^{(2)}(\overline{M}; \mathcal{N}(G)) = \lim_{i \to \infty} \frac{\rho_{RS}(M[i])}{[G : G_i]}$$

where $\rho_{RS}(M[i])$ is the Ray-Singer torsion.

- We are rather optimistic that the last conjecture holds in many interesting cases, although it is known only in very few instances, e.g., $G = \mathbb{Z}$ and $\overline{M}$ is L^2-acyclic.
- We are much less optimistic about the following conjectures concerning homological growth.

Homological growth and L^2-torsion

- The following conjecture is taken from Lück [25, Conjecture 1.12 (2)]. For locally symmetric spaces it reduces to the conjecture of Bergeron and Venkatesh [2, Conjecture 1.3].

Conjecture (Homological torsion growth and L^2-torsion)

Let M be an aspherical closed manifold and $\{G_i\}$ of $\pi_1(M)$ be a finite index normal chain of $G = \pi_1(M)$.

Then we get for any natural number n with $2n + 1 \neq \dim(M)$

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i];\mathbb{Z})\right)\right|\right)}{[G : G_i]} = 0.$$

If the dimension $\dim(M) = 2m + 1$ is odd, then $\widetilde{M}$ is $\det$-L^2-acyclic and we get

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_m(M[i];\mathbb{Z})\right)\right|\right)}{[G : G_i]} = (-1)^m \cdot \rho^{(2)}(\widetilde{M}).$$

Theorem (Lück)

Let M be an aspherical closed manifold with fundamental group $G = \pi_1(M)$. Suppose that M carries a non-trivial S^1-action or suppose that G contains a non-trivial elementary amenable normal subgroup.

Then M is L^2-acyclic and we get for all $n \geq 0$ and any finite index normal chain $\{G_i\}$ of $G = \pi_1(M)$

$$\begin{aligned} \lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i])\right)\right|\right)}{[G : G_i]} &= 0; \\ \rho^{(2)}(\widetilde{M}) &= 0. \end{aligned}$$

- Here is a weaker (and more realistic) version of the conjecture about homological torsion growth and L^2-torsion.

Conjecture (Modified Conjecture about homological torsion growth and L^2-torsion)

Let M be an aspherical closed manifold of odd dimension and $\{G_i\}$ of $G = \pi_1(M)$ be a finite index normal chain.

Then $\widetilde{M}$ is det-L^2-acyclic and we get

$$\lim_{i \to \infty} \left(\sum_{k \geq 0} (-1)^k \cdot \frac{\ln\left(\left|\operatorname{tors}\left(H_k(M[i]; \mathbb{Z})\right)\right|\right)}{[G : G_i]} \right) = \rho^{(2)}(\widetilde{M}).$$

- We mention the following special case of the conjectures above.

Conjecture (Hyperbolic 3-manifolds)

Let M be hyperbolic 3-manifold and $\{G_i\}$ be a finite index normal chain of $G = \pi_1(M)$
Then $\widetilde{M}$ is $\det$-L^2-acyclic and we get

$$\lim_{i\to\infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_1(M[i];\mathbb{Z})\right)\right|\right)}{[G:G_i]} = \lim_{i\to\infty} \frac{-\rho_{RS}(M[i])}{[G:G_i]} = -\rho^{(2)}(\widetilde{M}) = \frac{\operatorname{Vol}(M)}{6\pi}.$$

- In particular this would allow to read off the volume from the profinite completion of $\pi_1(M)$, see Kammeyer [17].
- Next we discuss the relation between the Approximation Conjecture for L^2-torsion for normal chains of finite index and the Modified Conjecture about homological torsion growth and L^2-torsion.

- Consider the situation of the Modified Conjecture about homological torsion growth and L^2-torsion.
- Then one can define the *k-th regulator* of $M[i]$ to be the real number $R_k[i] > 0$ given by comparing the two basis on $H^k(M[i];\mathbb{R})$ coming from the Hodge-deRham isomorphism and the integral lattices in $H^k(M[i];\mathbb{R})$ coming from $H^k(M[i];\mathbb{Z})$.
- More precisely, there are canonical $\mathbb{R}$-isomorphisms

$$\begin{aligned}\hom_{\mathbb{Z}}(H_k(M[i];\mathbb{Z})/tors(H_k(M[i];\mathbb{Z})),\mathbb{R}) &\xrightarrow{\cong} H^k(M[i];\mathbb{R});\\ \mathcal{H}^k(M[i]) &\xrightarrow{\cong} H^k(M[i];\mathbb{R}),\end{aligned}$$

  where the first one comes from the Universal Coefficient Theorem and the second from the deRham-Hodge isomorphism and has the space of harmonic k-forms $\mathcal{H}^k(M[i])$ as source.
- So we get a canonical isomorphism

$$f_k\colon \hom_{\mathbb{Z}}(H_k(M[i];\mathbb{Z})/tors(H_k(M[i];\mathbb{Z})),\mathbb{R}) \xrightarrow{\cong} \mathcal{H}^k(M[i]).$$

- Choose any $\mathbb{Z}$-basis B_1 on the finitely generated $\mathbb{Z}$-module $H_k(M[i];\mathbb{Z})/tors(H_k(M[i];\mathbb{Z}))$ and let $B_1^{\mathbb{R}}$ be the induced $\mathbb{R}$-basis on $\hom_{\mathbb{Z}}(H_k(M[i];\mathbb{Z})/tors(H_k(M[i];\mathbb{Z})),\mathbb{R})$.
- The Riemannian metric on $M[i]$ induces a Hilbert space structure on $\mathcal{H}^k(M[i])$ and we can choose any orthogonal $\mathbb{R}$-bases B_2 on $\mathcal{H}^k(M[i])$. Let A_k be the matrix of f_k with respect to $B_1^{\mathbb{R}}$ and B_2.
- Define the *k-th regulator*

$$R_k[i] = |\det(A_k)| > 0.$$

- It is independent of the choices of B_1 and B_2.

- We have

$$\rho_{RS}(M[i]) - \sum_{k\geq 0}(-1)^k \cdot \ln\left(\left|\mathrm{tors}\left(H_k(M[i];\mathbb{Z})\right)\right|\right) = \sum_{k\geq 0}(-1)^k \cdot \ln(R_k[i]).$$

where $\rho_{RS}(M[i])$ is the Ray-Singer torsion.
- Hence the Approximation Conjecture for L^2-torsion for normal chains of finite index implies the Modified Conjecture about homological torsion growth and L^2-torsion if

$$\lim_{i\to\infty}\left(\sum_{k\geq 0}(-1)^k \cdot \frac{\ln(R_k[i])}{[G:G_i]}\right) = 0.$$

- For more information about approximation we refer to the survey article Lück [26].

Twisting with finite dimensional representations

- One can twist L^2-Betti numbers $b_n^{(2)}(\widetilde{X})$ with a finite-dimensional real representation V and obtains the V-twisted L^2-Betti numbers $b_n^{(2)}(\widetilde{X};V)$.
- If V is orthogonal, then it is easy to check

$$b_n^{(2)}(\widetilde{X};V) = \dim_{\mathbb{R}}(V) \cdot b_n^{(2)}(\widetilde{X}).$$

- There is the conjecture formulated as a question in Lück [27, Question 0.1] that this holds for all finite-dimensional real representations V.
- Boschheidgen-Jaikin-Zapirain [3, Theorem 1.1] have proved it if π is sofic.
- Therefore we will tacitly assume this conjecture to be true in the sequel.
- In particular $b_n^{(2)}(\widetilde{X};V)$ vanishes for all $n \geq 0$ if X is L^2-acyclic.

- This raises the question whether, for a connected finite CW-complex X which is L^2-acyclic, we can twist L^2-torsion $\rho^{(2)}(\widetilde{X})$ with a finite-dimensional real representation V and obtain the V-twisted L^2-torsion $\rho^{(2)}(\widetilde{X};V)$.

- This is easy if V is orthogonal but the result is not interesting since it will satisfy
$$\rho^{(2)}(\widetilde{X};V) = \dim_{\mathbb{R}}(V)\cdot\rho^{(2)}(\widetilde{X}).$$

- If V is any finite-dimensional real representation V, the proof that $\rho^{(2)}(\widetilde{X};V)$ is well-defined is much harder.

- It has been carried out by Lück [27, Theorem 7.7] provided that V is a $\mathbb{Q}\pi$-module which is finitely generated as $\mathbb{Q}$-module or if the representation V considered as a homomorphisms $\rho_V\colon \pi \to GL_d(\mathbb{R})$ factorizes through $\mathbb{Z}^k$ for $k \geq 0$.

- Let X be a finite connected CW-complex with fundamental group π which is L^2-acyclic. Let $\operatorname{Rep}_{\mathbb{R}}(\pi, d)$ be the real algebraic variety of d-dimensional real representations, i.e., of group homomorphisms $\pi \to GL_d(\mathbb{R})$.

Conjecture

The function

$$\rho_X^{(2)} \colon \operatorname{Rep}_{\mathbb{R}}(\pi, d) \to \mathbb{R}$$

is well-defined, continuous, and even smooth on manifold strata.

- We expect that $\rho_X^{(2)}$ carries interesting information, in particular when X is a compact connected irreducible 3-manifold M with infinite π whose boundary is empty or a union of incompressible tori.
- Question: Can we recover the Casson invariant of an integral homology 3-sphere N from $\rho_N^{(2)}$?
- Partial results show that $\rho_X^{(2)}$ seems to carry a lot of information.

- We know already that $\rho_M^{(2)}$ evaluated at the trivial d-dimensional representation is $-\frac{d}{6\pi} \cdot \mathrm{Vol}(M)$ for such M.

- If M is above, one can calculate $\rho_M^{(2)}(V)$ in terms of characteristic sequences as indicated above for group automorphisms, where the relevant matrices A can be read off from π and the representation $\pi \to GL_d(\mathbb{R})$.

- Next we explain the relation between $\rho_M^{(2)}$ and the Thurston norm, where M is a compact connected irreducible orientable 3-manifold M with infinite π whose boundary is empty or a union of incompressible tori. See [8, 9, 10, 11, 19, 20, 27].

The Thurston norm and the degree of the ϕ-twisted L^2-torsion function

- Consider an element $\phi \in H^1(M;\mathbb{Q}) = \hom(\pi,\mathbb{Q})$.
- We obtain for every $t \in (0,\infty)$ a 1-dimensional real representation $\mathbb{R}_{\phi,t}$ whose underlying real vector space is $\mathbb{R}$ and on which $w \in \pi$ acts by multiplication with $t^{\phi(w)}$.
- We obtain the ϕ-twisted L^2-torsion function

$$\rho^{(2)}(M;\phi)\colon (0,\infty) \to \mathbb{R}, \quad t \mapsto \rho^{(2)}(\widetilde{M};\mathbb{R}_{\phi,t}).$$

- Actually this function depends on a choice of a Spin^c-structure, but we will ignore this point since a different choice changes the $\rho^{(2)}_{\phi}$ by adding a function of the shape $E \cdot \ln(t)$.
- It turns out to be well-defined and continuous.

- There exist constants $C \geq 0$ and $D \geq 0$ such that we get for $0 < t \leq 1$

$$C \cdot \ln(t) - D \leq \rho^{(2)}(M;\phi)(t) \leq -C \cdot \ln(t) + D,$$

and for $t \geq 1$

$$-C \cdot \ln(t) - D \leq \rho^{(2)}(M;\phi)(t) \leq C \cdot \ln(t) + D.$$

- Define the degree of $\overline{\rho}^{(2)}(M;\phi)$ to be the non-negative real number

$$\deg\bigl(\rho^{(2)}(M;\phi)\bigr) := \limsup_{t\to\infty} \frac{\rho(t)}{\ln(t)} - \liminf_{t\to 0} \frac{\rho(t)}{\ln(t)}.$$

- Recall the definition of Thurston [35] of the so-called Thurston norm of $\phi \in H^1(M;\mathbb{Z})$

  $$x_M(\phi) := \min\{\chi_-(F) \mid F \subset M \text{ properly embedded surface dual to } \phi\},$$

  where, given a surface F with connected components $F_1, F_2, \ldots, F_k$, we define

  $$\chi_-(F) \; := \; \sum_{i=1}^{k} \max\{-\chi(F_i), 0\}.$$

- Thurston [35] showed that this defines a seminorm on $H^1(M;\mathbb{Z})$ which can be extended to a seminorm on $H^1(M;\mathbb{R})$.
- In particular we get for $r \in \mathbb{R}$ and $\phi \in H^1(M;\mathbb{R})$

  $$x_M(r \cdot \phi) = |r| \cdot x_M(\phi).$$

- If $K \subseteq S^3$ is a knot and we take M as its knot complement, then the Thurston norm of the element ϕ_K given by the knot is $2 \cdot \mathrm{genus}(K) - 1$.

- If $p\colon \overline{M} \to M$ is a finite covering with n sheets, then Gabai [12, Corollary 6.13] showed that

$$x_{\overline{M}}(p^*\phi) = n \cdot x_M(\phi).$$

- If $F \to M \xrightarrow{p} S^1$ is a fiber bundle for a 3-manifold M and compact surface F, and $\phi \in H^1(M;\mathbb{Z})$ is given by the homomorphism $H_1(p)\colon H_1(M) \to H_1(S^1) = \mathbb{Z}$, then by Thurston [35, Section 3] we have

$$x_M(\phi) = \begin{cases} -\chi(F), & \text{if } \chi(F) \leq 0; \\ 0, & \text{if } \chi(F) \geq 0. \end{cases}$$

Theorem (The Thurston norm and the degree of the ϕ-twisted L^2-torsion function)

We have

$$x_M(\phi) = \deg\bigl(\rho^{(2)}(M;\phi)\rho^{(2)}(M;\phi)\bigr).$$

- Actually, Thurston defines the so-called Thurston polytope which is essentially the unit ball with respect to the Thurston norm and carries information about the question which ϕ in $H^1(M;\mathbb{Z})$ are fibered.
- The Thurston polytope can be read of the universal L^2-torsion defined by Friedl-Lück [8] using [19] which actually determines also $\rho_X^{(2)}$ and hence $\rho^{(2)}(M;\phi)$.

An invariant of group automorphisms

Definition

Let G be a group with a finite model for BG. Let $f\colon G \xrightarrow{\cong} G$ be a group automorphism. Let T_{Bf} be the mapping torus of $Bf\colon BG \to BG$. Then T_{Bf} is L^2-acyclic and we can define the L^2-torsion of f

$$\rho^{(2)}(f) := \rho^{(2)}(\widetilde{T_f}) \quad \in \mathbb{R}$$

- One can generalize the construction above to the case where there is a finite model for $\underline{E}G$.
- Next we collect the main properties of $\rho^{(2)}(f)$.

- $\rho^{(2)}(f)$ depends only on the class of f in $\mathrm{Out}(G)$.
- Amalgamation formula

$$\rho^{(2)}(f_1 *_{f_0} f_2) = \rho^{(2)}(f_1) + \rho^{(2)}(f_2) - \rho^{(2)}(f_0).$$

- Trace property

  Let $u \colon G \xrightarrow{\cong} H$ and $v \colon H \xrightarrow{\cong} G$ group automorphisms. Then

$$\rho^{(2)}(u \circ v) = \rho^{(2)}(v \circ u).$$

  In particular $\rho^{(2)}(f)$ depends only on the conjugacy class of f in $\mathrm{Out}(G)$.

- **Additivity**

  If the following diagram commutes and has exact sequences as rows and automorphisms as vertical arrows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_0 & \xrightarrow{i} & G_1 & \xrightarrow{p} & G_2 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & G_0 & \xrightarrow{i} & G_1 & \xrightarrow{p} & G_2 & \longrightarrow & 1 \end{array}$$

  then

$$\rho^{(2)}(f_1) = \chi(BG_2) \cdot \rho^{(2)}(f_0).$$

- **Multiplicativity under finite index subgroups**

  If $f\colon G \xrightarrow{\cong} G$ is an automorphism of G and $H \subseteq G$ is a subgroup of finite index with $f(H) = H$, then

$$\rho^{(2)}(f|_H) = [G:H] \cdot \rho^{(2)}(f).$$

- **Multiplicativity under composition**

  For $m \geq 1$ we get

$$\rho^{(2)}(f^m) = m \cdot \rho^{(2)}(f)$$

  and we have

$$\rho^{(2)}(f^{-1}) = \rho^{(2)}(f).$$

- If BG is L^2-acyclic, then $\rho^{(2)}(f) = 0$.

- If there is an automorphism $a\colon S \to S$ of a compact orientable surface different from S^2 and D^2, then its mapping torus T_f is a connected compact irreducible manifold of dimension 3 whose boundary is empty or a union of incompressible tori, and we get

$$\rho^{(2)}(\pi_1(a)) = -\frac{1}{6\pi} \cdot \mathrm{Vol}(T_a).$$

- One should investigate $\rho^{(2)}(f)$ in particular for elements $f \in \mathtt{Out}(F_r)$ for the free group F_r of rank r.
- It is an interesting question whether $\rho(f)$ determines the conjugacy class of f in $\mathtt{Out}(F_r)$ up to finite ambiguity provided that f has exponential growth
- Next we describe a recipe how to compute $\rho^{(2)}(f)$ for $f \in \mathtt{Out}(F_r)$.

- Write $G = F_r \rtimes_f \mathbb{Z}$ for the semi-direct product associated to f. Let $t \in \mathbb{Z}$ be a generator and denote the corresponding element in G also by t.

- Define a (r, r)-matrix A over $\mathbb{Z}[F_r]$ by

$$A = \left(\frac{\partial}{\partial s_j} f(s_i)\right)_{1 \leq i,j \leq r}$$

  where $\frac{\partial}{\partial s_j}$ denotes the Fox derivative.

- Choose a large enough real number $K > 0$.

- Denote by

$$\mathrm{tr}_{\mathbb{Z}[G]} \colon \mathbb{Z}[G] \to \mathbb{Z}, \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \lambda_e$$

  the standard trace on $\mathbb{Z}[G]$.

- Define the so called characteristic sequence for $p \geq 0$

$$c(A,K)_p = \operatorname{tr}_{\mathbb{Z}[G]}\left(\left(1 - K^{-2} \cdot (1 - tA)(1 - A^*t^{-1})\right)^p\right).$$

- In the setting above the sequence $c(A,K)_p$ is a monotone decreasing sequence of non-negative real numbers, and the L^2-torsion of f satisfies

$$\rho^{(2)}(f) = -r \cdot \ln(K) + \frac{1}{2} \cdot \sum_{p=1}^{\infty} \frac{1}{p} \cdot c(A,K)_p \leq 0.$$

- The convergence of the infinite sum above is exponential.

- The complexity of the computation of $\rho^{(2)}(f)$ has been analyzed by Löh-Uschold [21].

Simplicial volume and L^2-invariants

- The simplicial volume of a manifold is a topological variant of the (Riemannian) volume which agrees with it for hyperbolic manifolds up to a dimension constant and was introduced by Gromov [15].

Definition (Simplicial volume)

Let M be a closed connected orientable manifold of dimension n. Define its simplicial volume to be the non-negative real number

$$||M|| := ||j([M])||_1 \quad \in [0, \infty)$$

for any choice of fundamental class $[M] \in H_n^{\text{sing}}(M)$ and $j \colon H_n^{\text{sing}}(M) \to H_n^{\text{sing}}(M; \mathbb{R})$ the change of coefficients map associated to the inclusion $\mathbb{Z} \to \mathbb{R}$, where $||j([M])||_1$ is the infimum over the L^1-norms of any cycle in the singular chain complex $C_*^{sing}(M; \mathbb{R})$ representing $j([M])$.

Conjecture (Simplicial volume and L^2-invariants)

Let M be an aspherical closed orientable manifold of dimension ≥ 1. Suppose that its simplicial volume $||M||$ vanishes. Then:

$$\begin{aligned} b_n^{(2)}(\widetilde{M}) &= 0 \quad \text{for } n \geq 0; \\ \rho^{(2)}(\widetilde{M}) &= 0. \end{aligned}$$

- Gromov first asked in [16, Section 8A on page 232] whether under the conditions in the conjecture above the Euler characteristic of M vanishes, and notes that in all available examples even the L^2-Betti numbers of M vanish. The part about L^2-torsion appears in Lück [23, Conjecture 3.2].

L^2-torsion and measure equivalence

- Gaboriau [14] introduced L^2-Betti numbers of measured equivalence relations and proved that two measure equivalent countable groups have proportional L^2-Betti numbers. This notion turned out to have many important applications in recent years, most notably through the work of Popa [33].
- The notion of *measure equivalence* was introduced by Gromov [16, 0.5.E].

Definition (Measure equivalence)

Two countable groups G and H are called measure equivalent with index $c = I(G,H) > 0$ if there exists a non-trivial standard measure space (Ω, μ) on which $G \times H$ acts such that the restricted actions of $G = G \times \{1\}$ and $H = \{1\} \times H$ have measurable fundamental domains $X \subset \Omega$ and $Y \subset \Omega$, with $\mu(X) < \infty$, $\mu(Y) < \infty$, and $c = \mu(X)/\mu(Y)$. The space (Ω, μ) is called a measure coupling between G and H (of index c).

- The following conjecture is taken from Lueck-Sauer-Wegner [31, Conjecture 1.2].

Conjecture (L^2-torsion and measure equivalence)

Let G and H be two admissible groups, which are measure equivalent with index $I(G, H) > 0$. Then

$$\rho^{(2)}(G) = I(G, H) \cdot \rho^{(2)}(H).$$

- Due to Gaboriau [14], the vanishing of the nth L^2-Betti number $b_n^{(2)}(G)$ is an invariant of the measure equivalence class of a countable group G. If all L^2-Betti numbers vanish and G is an admissible group, then the vanishing of the L^2-torsion is a secondary invariant of the measure equivalence class of a countable group G provided that the conjecture above holds.

- Evidence for the conjecture above comes from Lueck-Sauer-Wegner [31, Conjecture 1.10] which says that the conjecture above is true if we replace measure equivalence by the stronger notion of uniform measure equivalence, see [31, Definition 1.3], and assume that G satisfies the Measure Theoretic Determinant Conjecture, see [31, Conjecture 1.7].

- Here is a very interesting aside concerning Fuglede-Kadison determinants and Mahler measures.

Definition (Mahler measure)

Let $p(z) \in \mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$ be a non-trivial element. Write it as $p(z) = c \cdot z^k \cdot \prod_{i=1}^{r}(z - a_i)$ for an integer $r \geq 0$, non-zero complex numbers c, a_1, ..., a_r and an integer k. Define its Mahler measure

$$M(p) = |c| \cdot \prod_{\substack{i=1,2,\ldots,r \\ |a_i|>1}} |a_i|.$$

- The following famous and open problem goes back to a question of Lehmer [18].

Problem (Lehmer's Problem)

Does there exist a constant $\Lambda > 1$ such that for all non-trivial elements $p(z) \in \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[z, z^{-1}]$ with $M(p) \neq 1$ we have

$$M(p) \geq \Lambda?$$

- There is even a candidate for which the minimal Mahler measure is attained, namely, Lehmer's polynomial

$$L(z) := z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.$$

- It is actual $-z^5 \cdot \Delta(z)$ for the Alexander polynomial $\Delta(z)$ of the bretzel knot given by $(2, 3, 7)$.

- It is conceivable that for any non-trivial element $p \in \mathbb{Z}[\mathbb{Z}]$ with $M(p) > 1$

$$M(p) \geq M(L) = 1.17628\ldots$$

holds.

- For a survey on Lehmer's problem we refer for instance to [4, 5, 7, 34].

Lemma

The Mahler measure $m(p)$ is the square root of the Fuglede-Kadison determinant of the operator $L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ given by multiplication with $p(z) \cdot \overline{p(z)}$.

Definition (Lehmer's constant of a group)

Define Lehmer's constant of a group G

$$\Lambda^w(G) \in [1, \infty)$$

to be the infimum of the set of Fuglede-Kadison determinants

$$\det{}^{(2)}_{\mathcal{N}(G)}\bigl(r_A^{(2)} \colon L^2(G)^r \to L^2(G)^r\bigr),$$

where A runs through all (r, r)-matrices with coefficients in $\mathbb{Z}[G]$ for all $r \geq 1$, for which $r_A^{(2)} \colon L^2(G)^r \to L^2(G)^r$ is a weak isomorphism and the Fuglede-Kadison determinant satisfies $\det{}^{(2)}_{\mathcal{N}(G)}(r_A^{(2)}) > 1$.

- We can show, see Lück [28]

$$\Lambda^w(\mathbb{Z}^n) \geq M(L)$$

for all $n \geq 1$, provided that Lehmer's problem has a positive answer.

- We know $1 \leq \Lambda^w(G) \leq M(L)$ for torsionfree G.

Problem (Generalized Lehmer's Problem)

For which torsionfree groups G do we have

$$1 < \Lambda^w(G)?$$

Example (Weeks manifold)

There is a closed hyperbolic 3-manifold W, the so called Weeks manifold, which is the unique closed hyperbolic 3-manifold with smallest volume, see Gabai-Meyerhoff-Milley [13, Corollary 1.3]. Its volume is between 0,942 and 0,943. Hence we get

$$\Lambda^w(\pi) \leq \exp\left(\frac{1}{6\pi} \cdot 0,943\right) \leq 1,06.$$

This implies $\Lambda^w(\pi) < M(L)$.

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