A survey on L^2-invariants and their applications to algebra, geometry, and group theory

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Preface

- This is an advertisement talk for L^2-invariants.
- We want emphasize the overwhelming variety of applications of L^2-invariants to other areas of mathematics such as algebra, geometry, group theory, operator theory, and so on.
- We will avoid any technical statements, i.e., there will be no proofs and among applications we often have chosen the easy to state ones.
- We will give no information about the history of L^2-invariants. The pioneers were Atiyah and Gromov.

- We will not go through all the slides during the talk and simply skip some of them.
- The slides shall give valuable information for further reading.
- Hopefully the audience will understand after the talk why Daniel Wise called L^2-invariants Voodoo on a conference on topology and geometry conference in Bonn in 2015.

Theorem (Euler characteristic of amenable groups, Cheeger-Gromov)

Let G be a group which contains a normal infinite amenable subgroup. Suppose that there is a finite model for BG.

Then its Euler characteristic

$$\chi(BG) := \sum_{n\geq 0} (-1)^n \dim_{\mathbb{C}}(H_n(BG;\mathbb{C}))$$

vanishes.

Definition (Deficiency)

Let G be a finitely presented group. Define its deficiency

$$\operatorname{defi}(G) := \max\{g(P) - r(P)\}$$

where P runs over all presentations P of G and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation P.

- The deficiency is an important invariant in group theory and low-dimensional topology.
- Lower bounds can be obtained by investigating specific presentations. The hard part is to find upper bounds.
- Often the deficiency is not realized by the “obvious” presentation.

Example

- The group
$$(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$$
has the obvious presentation
$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = t_0^2 = [s_0, t_0] = s_1^3 = t_1^3 = [s_1, t_1] = 1 \rangle.$$

- One may think that its deficiency is -2.

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- One may think that its deficiency is -2.

- However, it turns out that its deficiency is -1 realized by the following presentation
$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = 1, [s_0, t_0] = t_0^2, s_1^3 = 1, [s_1, t_1] = t_1^3, t_0^2 = t_1^3\rangle.$$

Theorem (Deficiency and group extensions, L.)

Let $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ *be an exact sequence of infinite groups. Suppose that* G *is finitely presented and* H *is finitely generated. Then:*

$$\operatorname{defi}(G) \leq 1.$$

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$$\operatorname{defi}(G) \leq 1.$$

Theorem (Signatures of 4-manifolds and group extensions, L.)

Let M be a closed oriented 4*-manifold. Suppose that there is an exact sequence of infinite groups* $1 \to H \xrightarrow{i} \pi_1(X) \xrightarrow{q} K \to 1$ *such that H is finitely generated.*

Then

$$|\operatorname{sign}(M)| \leq \chi(M).$$

Conjecture (Idempotent Conjecture, Kaplansky)

Let G be a torsionfree group. Then all idempotents of $\mathbb{C}G$ are trivial, i.e., equal to 0 or 1.

Conjecture (Zero-divisor Conjecture, Kaplansky)

Let G be a torsionfree group. Then $\mathbb{C}G$ has no zero-divisors.

Conjecture (Embedding Conjecture, Malcev)

Let G be a torsionfree group. Then $\mathbb{C}G$ embeds into a skew-field.

- Embedding Conjecture $\Longrightarrow$ Zero-divisor Conjecture $\Longrightarrow$ Idempotent Conjecture.

Theorem (L.-Rœrdam)

Let G be a group and $H \subseteq G$ be a normal finite subgroup. Then the canonical map

$$\mathbb{Z} \otimes_{\mathbb{Z}G} \mathrm{Wh}(H) \to \mathrm{Wh}(G)$$

is rationally injective.

- The theorem above is a consequence of the K-theoretic Farrell-Jones Conjecture.
- Note that it holds for all groups.

Conjecture (Euler characteristic and sectional curvature, Hopf)

Let M be a closed Riemannian manifold of even dimension $2n$. Then:

- *If its sectional curvature satisfied $\sec(M) \leq 0$, then $(-1)^n \cdot \chi(M) \geq 0$;*
- *If its sectional curvature satisfied $\sec(M) < 0$, then $(-1)^n \cdot \chi(M) > 0$.*

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Theorem (S^1-actions and hyperbolic manifolds)

Any S^1-action on a hyperbolic closed manifold is trivial.

Theorem (Kähler manifolds and projective algebraic varieties, Gromov)

Let M be a closed Kähler manifold, i.e., a complex manifold which comes with a Kähler Hermitian metric and Kähler 2-form. Suppose that it admits some Riemannian metric with negative sectional curvature, or, more generally, that $\pi_1(M)$ is hyperbolic (in the sense of Gromov) and $\pi_2(M)$ is trivial.

Then M is a projective algebraic variety.

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Then M is a projective algebraic variety.

Theorem (Cochran-Orr-Teichner)

There are non-slice knots in 3-space whose Casson-Gordon invariants are all trivial.

Conjecture (Torsion growth in hyperbolic 3-manifolds, Bergeron-Venkatesh)

Let M be a closed hyperbolic 3-manifold and

$$\pi_1(M) = G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

be any descending chain of finite index normal subgroups with $\bigcap_{n \geq 0} G_n = \{1\}$. Let $M_i \to M$ be the finite cover associated to G_i.

Then

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M_i;\mathbb{Z})\right)\right|\right)}{[G : G_i]} = \begin{cases} \frac{1}{6\pi} \cdot \operatorname{Volume}(M) & n = 1; \\ 0 \text{ otherwise.} \end{cases}$$

- So far no L^2-invariants have occurred in the talk and the audience may wonder why the title contains the word L^2-invariants at all.
- The point is that the proofs of the results above or of the conjectures in certain special cases do rely on L^2-methods. The use of L^2-methods made a lot of progress possible although on the first glance they seem to be unrelated to the results and conjectures mentioned above.
- Next we give a very brief introduction to the L^2-setting.

Group von Neumann algebras

- Denote by $L^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$.

Definition (Group von Neumann algebra and its trace)

- Define the group von Neumann algebra

$$\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G))^G = \overline{\mathbb{C}G}^{\text{weak}}$$

to be the algebra of bounded G-equivariant operators $L^2(G) \to L^2(G)$.

- The von Neumann trace is defined by

$$\operatorname{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$

L^2-homology and L^2-Betti numbers

Definition (L^2-homology and L^2-Betti numbers)

- Let X be a connected CW-complex of finite type. Let $\widetilde{X}$ be its universal covering and $\pi = \pi_1(M)$. Denote by $C_*(\widetilde{X})$ its cellular $\mathbb{Z}\pi$-chain complex.
- Define its cellular L^2-chain complex to be the Hilbert $\mathcal{N}(\pi)$-chain complex

$$C_*^{(2)}(\widetilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) = \overline{C_*(\widetilde{X})}.$$

- Define its n-th L^2-homology to be the finitely generated Hilbert $\mathcal{N}(\pi)$-module

$$H_n^{(2)}(\widetilde{X}) := \ker(c_n^{(2)})/\overline{\operatorname{im}(c_{n+1}^{(2)})}.$$

- Define its n-th L^2-Betti number

$$b_n^{(2)}(\widetilde{X}) := \dim_{\mathcal{N}(\pi)} \left(H_n^{(2)}(\widetilde{X})\right) \quad \in \mathbb{R}^{\geq 0}.$$

Theorem (Main properties of Betti numbers)

Let X and Y be connected CW-complexes of finite type.

- *Homotopy invariance*

  If X and Y are homotopy equivalent, then

  $$b_n(X) = b_n(Y);$$

- *Euler-Poincaré formula*

  We have

  $$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n(X);$$

- *Poincaré duality*

  Let M be a closed manifold of dimension d. Then

  $$b_n(M) = b_{d-n}(M);$$

Theorem (Main properties of L^2-Betti numbers)

Let X and Y be connected CW-complexes of finite type.

- *Homotopy invariance*

  If X and Y are homotopy equivalent, then

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- *Poincaré duality*

  Let M be a closed manifold of dimension d. Then

  $$b_n^{(2)}(\widetilde{M}) = b_{d-n}^{(2)}(\widetilde{M});$$

Theorem (Continued)

- *Künneth formula*

$$b_n(X \times Y) = \sum_{p+q=n} b_p(X) \cdot b_q(Y);$$

- *Zero-th L^2-Betti number*

  We have

$$b_0(X) = 1;$$

Theorem (Continued)

- *Künneth formula*

$$b_n^{(2)}(\widetilde{X \times Y}) = \sum_{p+q=n} b_p^{(2)}(\widetilde{X}) \cdot b_q^{(2)}(\widetilde{Y});$$

- *Zero-th L^2-Betti number*

  We have

$$b_0^{(2)}(\widetilde{X}) = \frac{1}{|\pi|};$$

  In particular $b_0^{(2)}(\widetilde{X}) = 0$ *if* π *is infinite.*

Theorem (Continued)

- *Künneth formula*

$$b_n^{(2)}(\widetilde{X \times Y}) = \sum_{p+q=n} b_p^{(2)}(\widetilde{X}) \cdot b_q^{(2)}(\widetilde{Y});$$

- *Zero-th L^2-Betti number*

  We have

$$b_0^{(2)}(\widetilde{X}) = \frac{1}{|\pi|};$$

  In particular $b_0^{(2)}(\widetilde{X}) = 0$ if π is infinite.

- *Finite coverings*

  If $X \to Y$ is a finite covering with d sheets, then

$$b_n^{(2)}(\widetilde{X}) = d \cdot b_n^{(2)}(\widetilde{Y}).$$

Theorem (Hodge - de Rham Theorem)

Let M be a closed Riemannian manifold. Put

$$\mathcal{H}^n(M) = \{\omega \in \Omega^n(M) \mid \Delta_n(\omega) = 0\}.$$

Then integration defines an isomorphism of real vector spaces

$$\mathcal{H}^n(M) \xrightarrow{\cong} H^n(M;\mathbb{R}).$$

Corollary (Betti numbers and heat kernels)

$$b_n(M) = \lim_{t\to\infty} \int_M \operatorname{tr}_{\mathbb{R}}(e^{-t\Delta_n}(x,x))\, d\mathrm{vol}\,.$$

where $e^{-t\Delta_n}(x,y)$ is the heat kernel on M.

Theorem (L^2-Hodge - de Rham Theorem, Dodziuk)

Let M be a closed Riemannian manifold. Put

$$\mathcal{H}^n_{(2)}(\widetilde{M}) = \{\widetilde{\omega} \in \Omega^n(\widetilde{M}) \mid \widetilde{\Delta}_n(\widetilde{\omega}) = 0,\ ||\widetilde{\omega}||_{L^2} < \infty\}.$$

Then integration defines an isomorphism of finitely generated Hilbert $\mathcal{N}(\pi)$-modules

$$\mathcal{H}^n_{(2)}(\widetilde{M}) \xrightarrow{\cong} H^n_{(2)}(\widetilde{M}).$$

Corollary (L^2-Betti numbers and heat kernels, Atiyah)

$$b_n^{(2)}(\widetilde{M}) = \lim_{t\to\infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{R}}(e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{x}))\, d\mathrm{vol}\,.$$

where $e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{y})$ is the heat kernel on $\widetilde{M}$ and $\mathcal{F}$ is a fundamental domain for the π-action.

- One may think of $\mathcal{N}(G)$ just as a ring by forgetting the topology.
- Then it has nice properties, e.g., it is semihereditary.
- One can extend the definition of the L^2-Betti numbers to arbitrary G-CW-complexes using the generalized dimension function of L.. Namely put

$$b_n^{(2)}(X;\mathcal{N}(G)) = \dim_{\mathcal{N}(G)}(H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*(X))) \in \mathbb{R}^{\geq 0} \amalg \{\infty\}.$$

- Now standard tools such as spectral sequences and homological algebra are available.
- In particular one can define for any group G its nth L^2-Betti number

$$b_n^{(2)}(G) := b_n^{(2)}(EG;\mathcal{N}(G)) \in \mathbb{R}^{\geq 0} \amalg \{\infty\}.$$

Some computations and results

Theorem (L.)

Let G be a group which contains an infinite amenable normal subgroup. Then all its L^2-Betti numbers $b_n^{(2)}(G)$ vanish.

- If we additionally assume that there is a finite model for BG, then we conclude

$$\chi(BG) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(G) = 0.$$

Sketch of proof.

- The functor $- \otimes_{\mathbb{Z}G} \mathcal{N}(G)$ is dimension flat for amenable G, i.e.,

$$\dim_{\mathcal{N}(G)}(\mathrm{Tor}_n^{\mathbb{Z}G}(M, \mathcal{N}(G)) = 0$$

  holds for all $n \geq 1$ and $\mathbb{Z}G$-modules M.

- Hence for an amenable group G we get for $n \geq 1$

$$\begin{aligned} b_n^{(2)}(G) &= \dim_{\mathcal{N}(G)}\big(H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*(EG))\big) \\ &= \dim_{\mathcal{N}(G)}\big(\mathcal{N}(G) \otimes_{\mathbb{Z}G} H_n(C_*(EG))\big) \\ &= \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{Z}G} \{0\}) = 0. \end{aligned}$$

- Now the general case follows from the Lyndon-Hochschild-Serre spectral sequence.

□

Theorem (Hyperbolic manifolds, Dodziuk)

Let M be a hyperbolic closed Riemannian manifold of dimension d. Then:

$$b_n^{(2)}(\widetilde{M}) = \begin{cases} = 0 & , \text{if } 2n \neq d; \\ > 0 & , \text{if } 2n = d. \end{cases}$$

Theorem (S^1-actions on aspherical manifolds, L.)

Let M be an aspherical closed manifold with non-trivial S^1-action. Then

1. *The action has no fixed points;*
2. $b_n^{(2)}(\widetilde{M}) = 0$ *for* $n \geq 0$ *and* $\chi(M) = 0$.

Theorem

Let M be a hyperbolic closed manifold of dimension d. Then

1. *If $d = 2m$ is even, then*

$$(-1)^m \cdot \chi(M) > 0;$$

2. *M carries no non-trivial S^1-action.*

Theorem (3-manifolds, Lott-L.)

Let the 3-manifold M be the connected sum $M_1 \sharp \ldots \sharp M_r$ of (compact connected orientable) prime 3-manifolds M_j. Assume that $\pi_1(M)$ is infinite. Then

$$\begin{aligned}
b_1^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} - \chi(M) \\
&\quad + \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right|; \\
b_2^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} \\
&\quad + \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right|; \\
b_n^{(2)}(\widetilde{M}) &= 0 \quad \text{for } n \neq 1, 2.
\end{aligned}$$

Corollary

Let M be a compact n-manifold such that $n \leq 3$ and its fundamental group is torsionfree.

Then all its L^2-Betti numbers are integers.

- L^2-Betti numbers are obstruction against fibering.

Theorem (Mapping tori, L.)

Let $f\colon X \to X$ be a cellular self-homotopy equivalence of a connected CW-complex X of finite type. Let T_f be the mapping torus. (This is up to homotopy just the total space of a fibration $X \to E \to S^1$.)

Then

$$b_n^{(2)}(\widetilde{T_f}) = 0 \quad \text{for } n \geq 0.$$

Proof.

- As $T_{fd} \to T_f$ is up to homotopy a d-sheeted covering, we get
$$b_n^{(2)}(\widetilde{T_f}) = \frac{b_n^{(2)}(\widetilde{T_{fd}})}{d}.$$

- There is up to homotopy equivalence a CW-structure on T_{fd} with $\beta_n(T_{fd}) = \beta_n(X) + \beta_{n-1}(X)$, where $\beta_n(X)$ is the number of n-cells. We have
$$b_n^{(2)}(\widetilde{T_{fd}}) \leq \beta_n(T_{fd}).$$

□

Proof continued.

- This implies for all $d \geq 1$

$$b_n^{(2)}(\widetilde{T}_f) \leq \frac{\beta_n(X) + \beta_{n-1}(X)}{d}.$$

- Taking the limit for $d \to \infty$ yields the claim.

$\square$

Theorem (Agol)

Every compact irreducible orientable 3-manifold with trivial Euler characteristic and non-trivial RFRS *(= residually finite rational solvable) fundamental group virtually fibers, that is, admits a finite covering which fibers over S^1.*

- Since closed hyperbolic 3-manifolds are irreducible and have vanishing Euler characteristic, it follows that such manifolds are virtually fibered.

Theorem (Stallings)

Let M be a closed 3-manifold. Then M fibers over S^1 if and only if there exists a short exact sequence $1 \to K \to \pi_1(M) \to \mathbb{Z} \to 1$ with finitely generated K.

Theorem (Kielak)

Let G be an infinite finitely generated group which is virtually RFRS*.*

Then G is virtually fibered in the sense that it admits a finite index subgroup mapping onto $\mathbb{Z}$ with a finitely generated kernel, if and only if $b_1^{(2)}(G) = 0$.

- Kielak et al. investigate more generally Bieri-Strebel invariants in terms of L^2-Betti numbers.

The fundamental square and the Atiyah Conjecture

Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)

Let G be a torsionfree finitely presented group. We say that G satisfies the Atiyah Conjecture if for any closed Riemannian manifold M with $\pi_1(M) \cong G$ we have for every $n \geq 0$

$$b_n^{(2)}(\widetilde{M}) \in \mathbb{Z}.$$

- All computations presented above support the Atiyah Conjecture.

- The fundamental square is given by the following inclusions of rings

$$\begin{array}{ccc} \mathbb{C}G & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G). \end{array}$$

- $\mathcal{U}(G)$ is the algebra of affiliated operators.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{C}G$ in $\mathcal{U}(G)$.

Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- If G is a finitely presented torsionfree group, the two versions of the Atiyah Conjecture above are equivalent.
- Obviously the Atiyah Conjecture implies the Embedding Conjecture and hence the Zero-divisor Conjecture and the Idempotent Conjecture due to Kaplansky.
- There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.
- However, there exist closed Riemannian manifolds whose universal coverings have an L^2-Betti number which is irrational, see Austin, Grabowski.

Theorem (Linnell-Schick)

If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.

- A group is called locally indicable if every non-trivial finitely generated subgroup admits an epimorphism onto $\mathbb{Z}$. Examples are one-relator-groups.

Theorem (Jaikin-Zapirain & Lopez-Alvarez)

If G is locally indicable, then it satisfies the Atiyah Conjecture.

Approximation

- In general there are no relations between the Betti numbers $b_n(X)$ and the L^2-Betti numbers $b_n^{(2)}(\widetilde{X})$ for a connected CW-complex X of finite type except for the Euler Poincaré formula

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{X}) = \sum_{n \geq 0} (-1)^n \cdot b_n(X).$$

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Theorem (Monotonicity, Avramidi-L.)

Let $f \colon G \to Q$ be an epimorphism of residually (locally indicable amenable) groups with kernel K. Let X be a free G-CW-complex of finite type. Then we get for any $n \geq 0$

$$b_n^{(2)}(X; \mathcal{N}(G)) \leq b_n^{(2)}(X/K; \mathcal{N}(Q)).$$

In particular we get $b_n^{(2)}(X; \mathcal{N}(G))) \leq b_n(X/G)$.

Theorem (Approximation Theorem, L.)

Let X be a connected CW-complex of finite type. Suppose that π is residually finite, i.e., there is a nested sequence

$$\pi = G_0 \supset G_1 \supset G_2 \supset \ldots$$

of normal subgroups of finite index with $\cap_{i \geq 1} G_i = \{1\}$. Let X_i be the finite $[\pi : G_i]$-sheeted covering of X associated to G_i.

Then for any such sequence $(G_i)_{i \geq 1}$

$$b_n^{(2)}(\widetilde{X}) = \lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}.$$

Applications to deficiency and signature

Lemma

Let G be a finitely presented group. Then

$$\operatorname{defi}(G) \;\leq\; 1 - |G|^{-1} + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Proof.

We have to show for any presentation P

$$g(P) - r(P) \ \leq \ 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Let X be a CW-complex realizing P. Then

$$\chi(X) = 1 - g(P) + r(P) = b_0^{(2)}(\widetilde{X}) + b_1^{(2)}(\widetilde{X}) - b_2^{(2)}(\widetilde{X}).$$

Since the classifying map $X \to BG$ is 2-connected, we get

$$\begin{array}{lcll} b_n^{(2)}(\widetilde{X}) & = & b_n^{(2)}(G) & \quad \text{for } n = 0, 1; \\ b_2^{(2)}(\widetilde{X}) & \geq & b_2^{(2)}(G). & \end{array}$$

□

Theorem (Deficiency and extensions, L.)

Let $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ *be an exact sequence of infinite groups. Suppose that G is finitely presented and H is finitely generated. Then:*

1. $b_1^{(2)}(G) = 0$;
2. $\operatorname{defi}(G) \leq 1$;
3. *Let M be a closed oriented 4-manifold with G as fundamental group. Then*
$$|\operatorname{sign}(M)| \leq \chi(M).$$

Conjecture (Singer Conjecture)

If M is an aspherical closed manifold, then

$$b_n^{(2)}(\widetilde{M}) = 0 \quad \text{if } 2n \neq \dim(M).$$

If M is a closed Riemannian manifold with negative sectional curvature, then

$$b_n^{(2)}(\widetilde{M}) \begin{cases} = 0 & \text{if } 2n \neq \dim(M); \\ > 0 & \text{if } 2n = \dim(M). \end{cases}$$

- The computations presented above do support the Singer Conjecture.

- Under certain negative pinching conditions the Singer Conjecture has been proved by Ballmann-Brüning, Donnelly-Xavier, Jost-Xin.

- Because of the Euler-Poincaré formula

$$\chi(M) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{M})$$

the Singer Conjecture implies in the case $\dim(M) = 2n$

$$(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M})$$

and hence the Hopf Conjecture.

- The Singer Conjecture gives also evidence for the Atiyah Conjecture.

L^2-torsion

- There is the notion of L^2-torsion due to L.-Rothenberg in the topological and to Lott, Mathai in the analytic setting. These are the L^2-analogues of Reidemeister torsion and analytic Ray-Singer torsion. Burghelea-Friedlander-Kappeller-McDonald proved that these two notions agree.

- The L^2-torsion is a secondary invariant in the sense that it is defined if all L^2-Betti numbers vanish.

- There are many interesting results and open problems concerning L^2-torsion which can be found on the slides of my talk *Survey on L^2-torsion and its (future) applications* from May 2024 in Bonn.

- At least we mention the following conjecture which is completely open and encompasses the conjecture of Bergeron-Venkatesh since for a hyperbolic closed 3-manifold its L^2torsion satisfies, $\rho^{(2)}(M) = \frac{1}{6 \cdot \pi} \cdot \text{Volume(M)}$.

Conjecture (Homological torsion growth, L.)

Let M be an aspherical closed manifold. Suppose that there is a nested sequence

$$\pi = G_0 \supset G_1 \supset G_2 \supset \dots$$

of normal subgroups of finite index with $\cap_{i \geq 1} G_i = \{1\}$. Let M_i be the finite $[\pi : G_i]$-sheeted covering of M associated to G_i.

Then we get for any natural number n with $2n + 1 \neq \dim(M)$

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(M[i];\mathbb{Z})\right)\right|\right)}{[G : G_i]} = 0.$$

If the dimension $\dim(M) = 2m + 1$ is odd, then we get

$$\lim_{i \to \infty} \frac{\ln\left(\left|\operatorname{tors}\left(H_m(M[i];\mathbb{Z})\right)\right|\right)}{[G : G_i]} = (-1)^m \cdot \rho^{(2)}(\widetilde{M}).$$

Outlook

- Unfortunately, there are a lot of very interesting aspects and very deep results by many people, which we have not covered. At least we want to mention some highlights.

- Gaboriau showed that the L^2-Betti numbers are (up to scaling) invariants of the measure equivalences class.

- Using L^2-Betti numbers and Gaboriau's ideas Popa solved some prominent outstanding problems about von Neumann algebras.

- Connes-Shlyakhtenko have defined L^2-Betti numbers for finite von Neumann algebras using Hochschild homology and the generalized dimension function of L.. If one can show that their definition applied to $\mathcal{N}(G)$ agrees with the L^2-Betti numbers of G, this would lead to a positive solution to the outstanding Free Group Factor Isomorphism Problem whether two finitely generated free groups are isomorphic if and only if their group von Neumann algebras are isomorphic. This is important for free probability theory.

- There are certain attempts to define L^2-Betti number also in prime characteristic. This has been done in a satisfactory way for elementary amenable groups by Bergeron-Linnel-L.-Sauer based on work by Elek.

  The state of the art is the work of Jaikin-Zapirain which allows to consider residually (locally indicable amenable) groups.

  Interesting results about prime characteristic are due to Avramidi-Okun-Schreve inspired by computations for Artin groups by Davis-Leary.

- The definition of L^2-Betti numbers have recently be extended to the setting of condensed sets by Kirstein-Kremer-L..

- L^2-torsion and Fuglede-Kadison determinants have been linked to entropy by Deninger and Li-Thom.

- Universal L^2-torsion has been defined by Friedl-L. and related to the Thurston polytope for 3-manifolds.

- There is the conjecture that for an aspherical closed manifold with vanishing simplicial volume in the sense of Gromov & Thurston all its L^2-invariants vanish.