

Some problems and conjectures about L^2 -invariants

Dominik Kirstein, Christian Kremer, and Wolfgang Lück

ABSTRACT. In this paper we discuss open problems and conjectures concerning L^2 -invariants.

1. Introduction

In this article we give a survey on open problems and conjectures concerning L^2 -invariants. We cover the whole portfolio and not only certain aspects as they are considered in the previous more specialized (and within their scope more detailed) survey articles [88, 90]. Moreover, we include some new results and problems, which have occurred after these two survey articles were written. The reader may select a specific topic by looking at the table of contents below.

Acknowledgements. This paper is funded by the ERC Advanced Grant “KL2MG-interactions” (no. 662400) of the third author granted by the European Research Council and by the Deutsche Forschungsgemeinschaft under Germany’s Excellence Strategy - GZ 2047/1, Projekt-ID 390685813, Hausdorff Center for Mathematics at Bonn. All three authors are supported by the Max Planck Institute for Mathematics in Bonn.

We thank Grigori Avramidi for comments on an earlier version of this article and Emma Brink for helpful discussions concerning Section 14.

The paper is organized as follows:

CONTENTS

1. Introduction	1
2. Basics about L^2 -invariants	2
3. The Atiyah Conjecture	8
4. The Singer Conjecture	12
5. The Determinant Conjecture	15
6. Approximation for finite index normal chains	16
7. Approximation for normal chains	23
8. L^2 -invariants and simplicial volume	26
9. L^2 -invariants of groups	27
10. Measure equivalence	30

2020 *Mathematics Subject Classification.* Primary 57Q99, Secondary 46L99, 20F99.
Key words and phrases. L^2 -invariants, condensed sets.

11. Zero-in-the-Spectrum-Conjecture	31
12. Twisting with finite-dimensional representations	31
13. Group von Neumann algebras over \mathbb{F}_p	32
14. L^2 -invariants and condensed mathematics	32
References	36

2. Basics about L^2 -invariants

We present some basic definitions, constructions, and properties of L^2 -invariants for the reader's convenience. The technical aspects of this section will not be needed in the other sections.

2.1. Group von Neumann algebras and their dimension function. The *group von Neumann algebra*

$$(2.1) \quad \mathcal{N}(G) := \mathcal{B}(L^2(G))^G$$

of the (discrete) group G is defined to be the \mathbb{C} -algebra of G -equivariant bounded operators from $L^2(G)$ to $L^2(G)$, where $L^2(G)$ is the complex Hilbert space of square integrable functions from G to \mathbb{C} . It can also be viewed as a completion of the complex group ring $\mathbb{C}G$ with respect to the weak (or equivalently strong) operator topology. We will view it in the sequel just as a ring with unit and essentially forget all the functional analytic aspects. It has much nicer properties than $\mathbb{C}G$, for example because it is *semi-hereditary*, i.e., any finitely generated submodule of a projective $\mathcal{N}(G)$ -module is projective again. More information about von Neumann algebras and their basic properties can be found for instance in [58, Section 2.2] and [85, Section 1.1 and Section 9.1].

The main feature is the *dimension function* that assigns to every $\mathcal{N}(G)$ -module M an element

$$(2.2) \quad \dim_{\mathcal{N}(G)}(M) \in \mathbb{R}^{\geq 0} \amalg \{\infty\}.$$

for $\mathbb{R}^{\geq 0} = \{r \in \mathbb{R} \mid r \geq 0\}$. We summarize its main properties:

- (1) It is *additive* on short exact sequences of $\mathcal{N}(G)$ -modules, also in the extreme cases following the convention that for $x \in \mathbb{R}^{\geq 0} \amalg \{\infty\}$ we set $x + \infty = \infty$.
- (2) The dimension function extends the classical Murray-von Neumann dimension for finitely generated projective $\mathcal{N}(G)$ -modules defined by the Hattori-Stalling rank associated to the standard trace of $\mathcal{N}(G)$. In particular, we have $\dim_{\mathcal{N}(G)}(\mathcal{N}(G)) = 1$, which together with additivity implies that the dimension of a finitely generated $\mathcal{N}(G)$ -module is finite.
- (3) It is *cofinal* in the sense that for any directed union $M = \bigcup_{i \in I} M_i$ of $\mathcal{N}(G)$ -modules the equality $\dim_{\mathcal{N}(G)}(M) = \sup\{\dim_{\mathcal{N}(G)}(M_i) \mid i \in I\}$ holds.
- (4) It satisfies the *continuity property*. A special case says that for any finitely generated $\mathcal{N}(G)$ -module M one can find a finitely generated projective $\mathcal{N}(G)$ -module PM together with two exact sequences $0 \rightarrow TM \rightarrow M \rightarrow PM \rightarrow 0$ and $0 \rightarrow \mathcal{N}(G)^n \rightarrow \mathcal{N}(G)^n \rightarrow TM \rightarrow 0$ for which $\dim_{\mathcal{N}(G)}(TM) = 0$ and $\dim_{\mathcal{N}(G)}(M) = \dim_{\mathcal{N}(G)}(PM)$ hold. The general version can be found in [85, Theorem 6.7].

- (5) We have $\dim_{\mathcal{N}(G)}(M) = 0$ if and only if every projective submodule of M is trivial. Moreover, a finitely generated projective $\mathcal{N}(G)$ -module M is trivial if and only if $\dim_{\mathcal{N}(G)}(M) = 0$ holds.

It turns out that the dimension function is uniquely determined by the properties (1) - (4). In the special case where G is finite and M is an $\mathcal{N}(G)$ -module one has $\mathcal{N}(G) = \mathbb{C}G$, and $|G| \cdot \dim_{\mathcal{N}(G)}(M)$ is the (classical) dimension of the underlying complex vector space of M . If G contains an element of infinite order, then every element $r \in \mathbb{R}^{\geq 0}$ can be realized as $r = \dim_{\mathcal{N}(G)}(M)$ for some finitely generated projective $\mathcal{N}(G)$ -module M . More information about the dimension function can be found for instance in [58, Sections 2.3 and 4.2], [83], [84], [85, Chapter 6].

2.2. L^2 -Betti numbers. Let Y be a G -space. Its singular chain complex $C_*^s(Y)$ is acted upon by G , so it acquires the structure of a chain complex of $\mathbb{Z}G$ -modules. Note that $\mathcal{N}(G)$ can be viewed as an $\mathcal{N}(G)$ - $\mathbb{Z}G$ -bimodule. Then we obtain an $\mathcal{N}(G)$ -chain complex $\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^s(Y)$. Its n th homology $H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^s(Y))$ is a $\mathcal{N}(G)$ -module. We define the n th- L^2 -Betti number of the G -space Y to be

$$(2.3) \quad b_n^{(2)}(Y; \mathcal{N}(G)) := \dim_{\mathcal{N}(G)}(H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^s(Y))) \in \mathbb{R}^{\geq 0} \amalg \{\infty\}.$$

If Y is a G -CW-complex, we can replace $C_*^s(Y)$ by the cellular $\mathbb{Z}G$ -chain complex $C_*(Y)$ in (2.3). If Y is a G -CW-complex such that Y/G is a finite d -dimensional CW-complex, then $b_n^{(2)}(Y; \mathcal{N}(G))$ is finite for every $n \geq 0$ and vanishes for $n > d$.

We will mainly be interested in the case where X is a path connected space possessing a universal covering $p: \tilde{X} \rightarrow X$. Recall that \tilde{X} then comes with a free π -action for π the fundamental group of X , and we abbreviate

$$(2.4) \quad b_n^{(2)}(\tilde{X}) = b_n^{(2)}(\tilde{X}; \mathcal{N}(\pi)) \in \mathbb{R}^{\geq 0} \amalg \{\infty\}.$$

If X is not path connected, one defines $b_n^{(2)}(\tilde{X})$ to be $\sum_{C \in \pi_0(X)} b_n^{(2)}(\tilde{C})$. If X is a CW-complex of finite type, then $b_n^{(2)}(\tilde{X})$ is finite. If M is a closed Riemannian manifold, then $b_n^{(2)}(\tilde{M})$ agrees with the original definition of the L^2 -Betti number due to Atiyah [9] in terms of heat kernel on the universal covering, namely, we have

$$(2.5) \quad b_n^{(2)}(\tilde{M}) = \lim_{t \rightarrow \infty} \int_{\mathcal{F}} \text{tr}(e^{-t\tilde{\Delta}_n}(\tilde{x}, \tilde{x})) \, \text{dvol}$$

where $\mathcal{F} \subseteq \tilde{X}$ is a fundamental domain for the π -action on \tilde{M} and $\text{tr}(e^{-t\tilde{\Delta}_n}(\tilde{x}, \tilde{x}))$ is the trace of the heat kernel on \tilde{M} at (\tilde{x}, \tilde{x}) for $\tilde{x} \in \tilde{X}$ at the time t in degree n .

If G is a (discrete) group, we define its n th- L^2 -Betti number

$$(2.6) \quad b_n^{(2)}(G) := b_n^{(2)}(EG; \mathcal{N}(G)) \in \mathbb{R}^{\geq 0} \amalg \{\infty\}.$$

where EG is the total space of the universal principal G -bundle $EG \rightarrow BG$. If \underline{EG} is the classifying space for proper G -actions, see for instance [86], then we conclude $b_n^{(2)}(G) := b_n^{(2)}(\underline{EG}; \mathcal{N}(G))$ from [85, Theorem 6.54 (2) on page 265]. This is useful since there are often small models of \underline{EG} available when EG may have no such nice models. For example, when G has torsion every CW-model for EG has cells in infinitely many dimension, whereas one might construct a finite dimensional model for \underline{EG} to conclude vanishing of L^2 -Betti numbers in high degrees.

Roughly speaking, the usefulness of the L^2 -Betti numbers stems often from the fact that they vanish quite often and pose interesting obstructions to the solution of geometric problems if they don't. For instance, for a connected CW-complex X

we have $b_0(X) = 1$, whereas $b_0^{(2)}(\tilde{X})$ vanishes if π is infinite and is equal to $|\pi|^{-1}$ if π is finite. We mention as an appetizer two rather surprising examples.

EXAMPLE 2.1. If the group G contains an amenable infinite normal subgroup $H \subseteq G$, then $b_n^{(2)}(G)$ vanishes for every $n \geq 0$. This implies that the Euler characteristic $\chi(BG)$ vanishes, if we additionally assume that BG has a finite CW -model. Note that we do not require anything about BH . This is a special case of [85, Theorem 7.2].

EXAMPLE 2.2. If $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ is an extension of infinite groups such that K is finitely generated and G is finitely presented, then $b_1^{(2)}(G)$ vanishes and hence the deficiency of G is bounded from above by 1. Recall that the deficiency is the maximum of the set of integers $g(P) - r(P)$, where P runs through all finite presentations P of G and $g(P)$ is the number of generators and $r(P)$ the number of relations. [81, Theorem 6.1]

2.3. Fuglede-Kadison determinants. Given an $\mathcal{N}(G)$ -map $f: P \rightarrow Q$ of finitely generated projective $\mathcal{N}(G)$ -modules, one can define its *Fuglede-Kadison determinant*

$$(2.7) \quad \det_{\mathcal{N}(G)}(f) \in \mathbb{R}^{\geq 0} := \{r \in \mathbb{R} \mid r \geq 0\}.$$

Let us briefly explain its definition, even though it will be of no concern to us later. One can interpret f as a bounded G -equivariant operator $f: V \rightarrow W$ of *finitely generated Hilbert $\mathcal{N}(G)$ -modules*, i.e., V is a Hilbert space V together with a linear isometric G -action such that there exists an isometric linear G -embedding of V into $L^2(G)^r$ for some natural number r , and analogously for W . Denote by $\{E_\lambda^{f^*f} \mid \lambda \in \mathbb{R}\}$ the (right-continuous) family of spectral projections of the positive operator $f^*f: V \rightarrow V$. Define the *spectral density function* of f by

$$F_f: \mathbb{R} \rightarrow \mathbb{R}^{\geq 0} \quad \lambda \mapsto \dim_{\mathcal{N}(G)}(\text{im}(E_{\lambda^2}^{f^*f})).$$

The spectral density function is monotone, non-decreasing, right-continuous and satisfies $F(0) = \dim_{\mathcal{N}(G)}(\ker(f))$. Let dF be the unique measure on the Borel σ -algebra on \mathbb{R} that satisfies $dF((a, b]) = F(b) - F(a)$ for $a < b$. Define Fuglede-Kadison determinant $\det_{\mathcal{N}(G)}(f) \in \mathbb{R}^{\geq 0}$ to be

$$\det_{\mathcal{N}(G)}(f) = \exp\left(\int_{0+}^{\infty} \ln(\lambda) dF\right)$$

if the Lebesgue integral $\int_{0+}^{\infty} \ln(\lambda) dF$ is a real number, and to be 0 if we have $\int_{0+}^{\infty} \ln(\lambda) dF = -\infty$. We say that f is of *determinant class* if $\det(f) \neq 0$. With our conventions we have $\det(0) = 1$ for the zero map $0: P \rightarrow Q$

EXAMPLE 2.3 (Fuglede-Kadison determinant for finite groups). To illustrate this definition, we look at the example where G is finite. Then $\mathcal{N}(G)$ is the same as $\mathbb{C}G$ and we can think of f as a $\mathbb{C}G$ -linear map of finite-dimensional unitary G -representations. The spectral density function F_f is the right-continuous step function, whose value at λ is the sum of the complex dimensions of the eigenspaces of f^*f for eigenvalues $\mu \leq \lambda^2$ divided by the order of G . Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the non-zero eigenvalues of f^*f with multiplicity μ_i . One computes

$$\det_{\mathcal{N}(G)}(f) = \exp\left(\sum_{i=1}^r \frac{\mu_i}{|G|} \cdot \ln(\sqrt{\lambda_i})\right) = \prod_{i=1}^r \lambda_i^{\frac{\mu_i}{2|G|}} = \det_{\mathbb{C}}(\overline{f^*f})^{\frac{1}{2|G|}}.$$

Here, $\overline{f^*f}$ denotes the automorphism of the orthogonal complement of the kernel of f^*f induced by f^*f and $\det_{\mathbb{C}}(\overline{f^*f})$ is put to be 1 if f is the zero operator. If $f: \mathbb{C}G^m \rightarrow \mathbb{C}G^m$ is an automorphism, we get

$$\det_{\mathcal{N}(G)}(f) = |\det_{\mathbb{C}}(f)|^{\frac{1}{|G|}}.$$

More information about the Fuglede-Kadison determinant can be found for instance in [85, Section 3.2].

2.4. L^2 -torsion. Let X be a connected finite CW-complex. Let $C_*(\tilde{X})$ be the cellular $\mathbb{Z}\pi$ -chain complex of its universal covering for π the fundamental group of X . Then $\mathcal{N}(\pi) \otimes_{\mathbb{Z}\pi} C_*(\tilde{X})$ is an $\mathcal{N}(\pi)$ -chain complex, which is finite dimensional and consists of finitely generated free $\mathcal{N}(\pi)$ -modules. If $c_n: C_n(\tilde{X}) \rightarrow C_{n-1}(\tilde{X})$ is the n th differential of $C_*(\tilde{X})$, then

$$\text{id}_{\mathcal{N}(\pi)} \otimes_{\mathbb{Z}\pi} c_n: \mathcal{N}(\pi) \otimes_{\mathbb{Z}\pi} C_n(\tilde{X}) \rightarrow \mathcal{N}(\pi) \otimes_{\mathbb{Z}\pi} C_{n-1}(\tilde{X})$$

is an $\mathcal{N}(\pi)$ -map of finitely generated free $\mathcal{N}(\pi)$ -modules and its Fuglede-Kadison determinant $\det_{\mathcal{N}(\pi)}(\text{id}_{\mathcal{N}(\pi)} \otimes_{\mathbb{Z}\pi} c_n) \in \mathbb{R}^{\geq 0}$ is defined. We say that X is of *determinant class* if $\det_{\mathcal{N}(\pi)}(\text{id}_{\mathcal{N}(\pi)} \otimes_{\mathbb{Z}\pi} c_n) \neq 0$ holds for every $n \geq 1$, and in this case we define the L^2 -torsion of X to be

$$(2.8) \quad \rho^{(2)}(\tilde{X}) = - \sum_{n \geq 1} (-1)^n \cdot \ln(\det_{\mathcal{N}(\pi)}(\text{id}_{\mathcal{N}(\pi)} \otimes_{\mathbb{Z}\pi} c_n)) \in \mathbb{R}.$$

We say that X is L^2 -acyclic if $b_n^{(2)}(\tilde{X}) = 0$ for $n \geq 0$. We say that X is \det - L^2 -acyclic if it is L^2 -acyclic and of determinant class. If X is not path connected, we define $\rho^{(2)}(\tilde{X}) = \sum_{C \in \pi_0(X)} \rho^{(2)}(\tilde{C})$.

The definition of L^2 -torsion in the analytic setting goes back to Lott [78] and Mathai [98], and in the topological setting to Lück-Rothenberg [93]. The equality of these two versions has been proved in [20].

2.5. Basic properties of L^2 -Betti numbers and L^2 -torsion.

Here is a list of basic properties of L^2 -Betti numbers and L^2 -torsion.

THEOREM 2.4.

(1) (Simple) Homotopy invariance, see [85, Theorem 1.35 (1) on page 37 and Theorem 3.96 (i) on page 163], [89, Theorem 6.7 (2)].

(a) If X and Y are homotopy equivalent spaces, then we get for $n \geq 0$

$$b_n^{(2)}(\tilde{X}) = b_n^{(2)}(\tilde{Y});$$

(b) Suppose that X and Y are homotopy equivalent finite CW-complexes and \tilde{X} or \tilde{Y} is \det - L^2 -acyclic. Then both \tilde{X} and \tilde{Y} are \det - L^2 -acyclic;

(c) Let $f: X \rightarrow Y$ be a homotopy equivalence of finite CW-complexes. Assume that \tilde{X} and \tilde{Y} are \det - L^2 -acyclic. Suppose that f is simple, or that $\pi_1(X)$ satisfies the Determinant Conjecture 5.1, or that the K -theoretic Farrell-Jones Conjecture for $\mathbb{Z}\pi_1(X)$ holds. Then

$$\rho^{(2)}(\tilde{Y}) = \rho^{(2)}(\tilde{X});$$

- (2) Euler-Poincaré formula, see [85, Theorem 1.35 (2) on page 37].

We get for the Euler characteristic $\chi(X)$ of a finite CW-complex X

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\tilde{X});$$

- (3) Sum formula, see [85, Theorem 3.96 (2) on page 164].

Consider a pushout of finite CW-complexes such that j_1 is an inclusion of CW-complexes, j_2 is cellular, and X inherits its CW-complex structure from X_0 , X_1 and X_2

$$\begin{array}{ccc} X_0 & \xrightarrow{j_1} & X_1 \\ j_2 \downarrow & & \downarrow i_1 \\ X_2 & \xrightarrow{i_2} & X. \end{array}$$

Assume that for $k = 0, 1, 2$ the map $\pi_1(i_k, x): \pi_1(X_k, x) \rightarrow \pi_1(X, j_k(x))$ induced by the obvious map $i_k: X_k \rightarrow X$ is injective for all base points x in X_k .

- (a) If \tilde{X}_0 , \tilde{X}_1 , and \tilde{X}_2 are L^2 -acyclic, then \tilde{X} is L^2 -acyclic;
(b) If \tilde{X}_0 , \tilde{X}_1 , and \tilde{X}_2 are \det - L^2 -acyclic, then \tilde{X} is \det - L^2 -acyclic and we get

$$\rho^{(2)}(\tilde{X}) = \rho^{(2)}(\tilde{X}_1) + \rho^{(2)}(\tilde{X}_2) - \rho^{(2)}(\tilde{X}_0);$$

- (4) Poincaré duality, see [85, Theorem 1.35 (3) on page 37 and Theorem 3.96 (3) on page 164].

Let M be a closed manifold of dimension d

- (a) Then

$$b_n^{(2)}(\tilde{M}) = b_{d-n}^{(2)}(\tilde{M});$$

- (b) Suppose that n is even and \tilde{M} is \det - L^2 -acyclic. Then

$$\rho^{(2)}(\tilde{M}) = 0;$$

- (5) Product formula, see [85, Theorem 1.35 (4) on page 37 and Theorem 3.96 (4) on page 164].

Let X and Y be finite CW-complexes.

- (a) Then

$$b_n^{(2)}(\widetilde{X \times Y}) = \sum_{\substack{i, j \in \mathbb{N} \\ n=i+j}} b_i^{(2)}(\tilde{X}) \cdot b_j^{(2)}(\tilde{Y});$$

- (b) Suppose that \tilde{X} is \det - L^2 -acyclic. Then $\widetilde{X \times Y}$ is \det - L^2 -acyclic and

$$\rho^{(2)}(\widetilde{X \times Y}) = \chi(Y) \cdot \rho^{(2)}(\tilde{X});$$

- (6) Multiplicativity, see [85, Theorem 1.35 (9) on page 38 and Theorem 3.96 (5) on page 164].

Let $X \rightarrow Y$ be a finite covering of finite CW-complexes with d sheets.

- (a) Then we get for $n \geq 0$

$$b_n^{(2)}(\tilde{X}) = d \cdot b_n^{(2)}(\tilde{Y});$$

(b) Then \tilde{X} is \det - L^2 -acyclic if and only if \tilde{Y} is \det - L^2 -acyclic, and in this case

$$\rho^{(2)}(\tilde{X}) = d \cdot \rho^{(2)}(\tilde{Y});$$

(7) Determinant class.

If $\pi_1(C)$ satisfies the Determinant Conjecture 5.1 for each component C of the finite CW-complex X , then \tilde{X} is of determinant class;

(8) 0th L^2 -Betti number, see [85, Theorem 1.35 (8) on page 38].

If X is a connected finite CW-complex with fundamental group π , then

$$b_0^{(2)}(\tilde{X}) = \begin{cases} \frac{1}{|\pi|} & \text{if } \pi \text{ is finite;} \\ 0 & \text{otherwise;} \end{cases}$$

(9) Fibration formula, see [85, Lemma 1.41 on page 45 and Corollary 3.103 on page 166].

(a) Let $p: E \rightarrow B$ a fibration whose base space B is a connected finite CW-complex and whose fiber is homotopy equivalent to a finite CW-complex Z . Suppose that for every $b \in B$ and $x \in F_b := p^{-1}(b)$ the inclusion $p^{-1}(b) \rightarrow E$ induces an injection on the fundamental groups $\pi_1(F_b, x) \rightarrow \pi_1(E, x)$, and that Z is L^2 -acyclic.

Then E is homotopy equivalent to a finite CW-complex X which is L^2 -acyclic;

(b) Let $F \xrightarrow{i} E \xrightarrow{p} B$ be locally trivial fiber bundle of finite CW-complexes. Suppose that B is connected, the map $\pi_1(F, x) \rightarrow \pi_1(E, i(x))$ is bijective for every base point $x \in F$, and \tilde{F} is \det - L^2 -acyclic.

Then \tilde{E} is \det - L^2 -acyclic and

$$\rho^{(2)}(\tilde{E}) = \chi(B) \cdot \rho^{(2)}(\tilde{F});$$

(10) S^1 -actions, see [85, Theorem 1.40 on page 43 and Theorem 3.105 on page 168].

Let X be a connected compact S^1 -CW-complex, for instance a closed smooth manifold with smooth S^1 -action. Suppose that for one orbit S^1/H (and hence for all orbits) the inclusion into X induces a map on π_1 with infinite image (so in particular the S^1 -action has no fixed points). Then \tilde{X} is \det - L^2 -acyclic and $\rho^{(2)}(\tilde{M})$ vanishes;

(11) Aspherical spaces, see [85, Theorem 3.111 on page 171 and Theorem 3.113 on page 172].

(a) Let M be an aspherical closed smooth manifold with a smooth S^1 -action. Then the conditions appearing in assertion (10) are satisfied and hence \tilde{M} is \det - L^2 -acyclic and $\rho^{(2)}(\tilde{X})$ vanishes;

(b) If X is an aspherical finite CW-complex whose fundamental group contains an elementary amenable infinite normal subgroup, then \tilde{X} is \det - L^2 -acyclic and $\rho^{(2)}(\tilde{X})$ vanishes;

(12) Mapping tori, see [85, Theorem 1.39 on page 42].

Let $f: X \rightarrow X$ be a self homotopy equivalence of a finite CW-complex. Denote by T_f its mapping torus.

(a) Then \tilde{T}_f is L^2 -acyclic;

(b) If \tilde{X} is \det - L^2 -acyclic, then $\rho^{(2)}(\tilde{T}_f)$ vanishes;

(13) Hyperbolic manifolds, see [50], [85, Theorem 1.39 on page 42].

Let M be a hyperbolic closed manifold of dimension d .

- (a) If d is odd, \widetilde{M} is \det - L^2 -acyclic;
- (b) Suppose that $d = 2m$ is even. Then $b_n^{(2)}(\widetilde{M})$ vanishes for $n \neq m$, and we have $(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M}) > 0$;
- (c) For every number m there exists an explicit constant $C_m > 0$ with the following property: If M is a hyperbolic closed manifold of dimension $(2m + 1)$ with volume $\text{vol}(M)$, then

$$\rho^{(2)}(\widetilde{M}) = (-1)^m \cdot C_m \cdot \text{vol}(M).$$

We have $C_1 = \frac{1}{6\pi}$. The number $\pi^m \cdot C_m$ is always rational;

(14) Approximation of L^2 -Betti numbers by classical Betti numbers, see [80], [85, Chapter 13].

Let X be a connected finite CW-complex with fundamental group $G = \pi_1(X)$. Suppose that G comes with a descending chain of subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$$

such that G_i is normal in G , the index $[G : G_i]$ is finite, and we have $\bigcap_{i \geq 0} G_i = \{1\}$.

Then $G_i \backslash \widetilde{X} \rightarrow X$ is a finite $[G : G_i]$ -sheeted covering, and we get for $n \geq 0$

$$b_n^{(2)}(\widetilde{X}) = \lim_{i \rightarrow \infty} \frac{b_n(G_i \backslash \widetilde{X})}{[G : G_i]},$$

where $b_n(G_i \backslash \widetilde{X})$ is the classical n th Betti number of the finite CW-complex $G_i \backslash \widetilde{X}$.

We refer for more information about L^2 -invariants and their applications to algebra, geometry, group theory, index theory, operator algebras, topology, and K -theory for instance to [58] and [85]. We will discuss some of them later in this article.

3. The Atiyah Conjecture

In this section we discuss one of the most prominent conjectures in the field of L^2 -invariants, the Atiyah Conjecture. It predicts the possible values of L^2 -Betti numbers and has many interesting implications. For example Kaplansky's Zero Divisor Conjecture for rational group rings follows from it.

3.1. Statement of the Atiyah Conjecture.

CONJECTURE 3.1 (Atiyah Conjecture). *Consider a field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Let G be a group possessing an upper bound on the orders of its finite subgroups. Let $\text{lcm}(G)$ be the natural number given by the least common multiple of the orders of its finite subgroups.*

We say that G satisfies the Atiyah Conjecture with coefficients in F , if for any finitely presented FG -module M the von Neumann dimension satisfies

$$\text{lcm}(G) \cdot \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{FG} M) \in \mathbb{Z}.$$

Note that G is torsionfree if and only if $\text{lcm}(G) = 1$, so the Atiyah conjecture in this case predicts integrality of L^2 -Betti numbers. For a field F , let us denote by \mathcal{A}_F the collection of all groups which admit a bound on the order of their finite subgroups and satisfy the Atiyah conjecture with coefficients in F . Obviously $\mathcal{A}_{\mathbb{C}} \subseteq \mathcal{A}_F \subseteq \mathcal{A}_{\mathbb{Q}}$.

REMARK 3.2 (Equivalent formulations of the Atiyah Conjecture 3.1). Consider a field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Let G be a group that possesses an upper bound on the orders of its finite subgroups. Then the following assertions are equivalent:

- (1) We have $G \in \mathcal{A}_F$;
- (2) For any matrix $A \in M_{m,n}(FG)$ the dimension $\dim_{\mathcal{N}(G)}(\ker(r_A))$ of the kernel of the $\mathcal{N}(G)$ -homomorphism $r_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n$ given by right multiplication with A satisfies $\text{lcm}(G) \cdot \dim_{\mathcal{N}(G)}(\ker(r_A)) \in \mathbb{Z}$;
- (3) For any FG -chain complex C_* of finitely generated free FG -modules and all $n \in \mathbb{Z}$ we have

$$\text{lcm}(G) \cdot \dim_{\mathcal{N}(G)}(H_n(\mathcal{N}(G) \otimes_{FG} C_*)) \in \mathbb{Z}.$$

If we additionally assume $F = \mathbb{Q}$, then the three assertions above and the following assertion are equivalent:

- (4) For any $\mathbb{Z}G$ -chain complex C_* of finitely generated free $\mathbb{Z}G$ -modules and all $n \in \mathbb{Z}$ we have

$$\text{lcm}(G) \cdot \dim_{\mathcal{N}(G)}(H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*)) \in \mathbb{Z}.$$

If we additionally assume $F = \mathbb{Q}$ and that G is finitely presented, then the four assertions above and the following assertion are equivalent:

- (5) For any closed manifold M with $\pi_1(M) \cong G$ and any $n \geq 0$ we have

$$\text{lcm}(G) \cdot b_n^{(2)}(\widetilde{M}) \in \mathbb{Z}.$$

All of these claims follow from [85, Lemma 10.5 on page 371 and Lemma 10.7 on 372], the equality $\dim_{\mathcal{N}(G)}(\mathcal{N}(G)) = 1$, and the Additivity of the dimension function $\dim_{\mathcal{N}(G)}$.

The Atiyah Conjecture 3.1 is rather surprising in view of (2.5) and assertion (5) appearing in Remark 3.2, since there seems to be no reason why the expression appearing on the right hand side of (2.5) should be an integer if $\pi_1(M)$ is torsionfree.

3.2. Status of the Atiyah Conjecture. The notions of elementary amenable groups and amenable groups are explained for instance in [85, Subsection 6.4.1].

DEFINITION 3.3 (Class of groups \mathcal{C}). Let \mathcal{C} be the smallest class of groups satisfying the following conditions:

- (1) \mathcal{C} contains all free groups;
- (2) If $\{G_i \mid i \in I\}$ is a directed system of subgroups directed by inclusion such that each G_i belongs to \mathcal{C} , then $G = \bigcup_{i \in I} G_i$ belongs to \mathcal{C} ;
- (3) Let $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ be an extension of groups such that H belongs to \mathcal{C} and Q is elementary amenable, then G belongs to \mathcal{C} .

DEFINITION 3.4 (Class of groups \mathcal{D}). Let \mathcal{D} be the smallest class of groups satisfying the following conditions:

- (1) The trivial group belongs to \mathcal{D} ;

- (2) If $\{G_i : i \in I\}$ is a filtered system of groups in \mathcal{D} (with arbitrary structure maps), then its colimit again belongs to \mathcal{D} ;
- (3) If $\{G_i : i \in I\}$ is a cofiltered system of groups in \mathcal{D} (with arbitrary structure maps), then its limit again belongs to \mathcal{D} ;
- (4) If G belongs to \mathcal{D} and $H \subseteq G$ is a subgroup, then $H \in \mathcal{D}$;
- (5) If $p: G \rightarrow A$ is an epimorphism of a torsionfree group G onto an elementary amenable group A and if $p^{-1}(B) \in \mathcal{A}_F$ for every finite group $B \subset A$, then $G \in \mathcal{A}_F$.

Note that each element in \mathcal{D} is a torsionfree group, and the class \mathcal{D} contains all residually (torsionfree elementary amenable) groups.

A group is called *locally indicable*, if every non-trivial finitely generated subgroup admits an epimorphism onto \mathbb{Z} . Locally indicable groups are torsionfree. Examples for locally indicable groups are one-relator groups.

THEOREM 3.5 (Status of the Atiyah Conjecture 3.1 with coefficients in F).
Consider a field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.

- (1) *If G belongs to \mathcal{C} and possesses an upper bound on the orders of its finite subgroups, then $G \in \mathcal{A}_F$;*
- (2) *If G belongs to \mathcal{D} , then G is torsionfree and $G \in \mathcal{A}_F$;*
- (3) *If $1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$ is an extension of groups, H is torsionfree and belongs to \mathcal{A}_F , and Q is locally indicable, then G is torsionfree and belongs to \mathcal{A}_F ;*
- (4) *If $G \in \mathcal{A}_F$ and $H \subseteq G$ is a subgroup, then $H \in \mathcal{A}_F$;*
- (5) *If G is the directed union $\bigcup_{i \in I} G_i$ of subgroups G_i directed by inclusion and each G_i belongs to \mathcal{A}_F , then G belongs to \mathcal{A}_F ;*
- (6) *The group G belongs to \mathcal{A}_F if and only if all its finitely generated subgroups belong to \mathcal{A}_F ;*
- (7) *If $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ is an extension of groups such that K is finite and G belongs to \mathcal{A}_F , then Q belongs to \mathcal{A}_F ;*
- (8) *Let M be a connected (not necessarily compact) d -dimensional manifold (possibly with non-empty boundary) such that $d \leq 3$ and its fundamental group $\pi_1(M)$ is torsionfree, then $\pi_1(M) \in \mathcal{C}$ and hence $\pi_1(M) \in \mathcal{A}_F$;*
- (9) *If the group G possesses an upper bound on the orders of its finite subgroups and belongs to one of the following classes below, then G belongs to \mathcal{A}_F :*
 - (a) *Residually {torsionfree elementary amenable} groups;*
 - (b) *Free by elementary amenable groups;*
 - (c) *Braid groups;*
 - (d) *Right-angled Artin and Coxeter groups;*
 - (e) *Torsionfree p -adic analytic pro- p -groups;*
 - (f) *Locally indicable groups;*
 - (g) *One-relator groups.*

PROOF. (1) This is due to Linnell, see for instance [75] or [85, Theorem 10.19 on page 378].

- (2) This follows from [53, Corollary 1.2], which is based on [26, Theorem 1.4].
- (3) This follows from [56, Proposition 6.5].
- (4) This follows from [85, Theorem 6.29 (2) on page 253].
- (5) See [85, Lemma 10.4 on page 371].

- (6) This follows from assertions (4) and (5).
 (7) This follows from [85, Lemma 13.45 on page 473].
 (8) This follows from [62, Theorem 1.1] for $d = 3$. The case $d = 2$ can be reduced to the case $d = 3$ by crossing with S^1 and assertion (4).
 (9) This follows from other assertions or from [53, Theorem 1.1 and Corollary 1.2] using [73, Theorem 2] and [32, Theorem 1.1]. \square

REMARK 3.6. The class \mathcal{A}_F is very large by aforementioned results. Nevertheless, we do not know whether the Atiyah Conjecture 3.1 holds for all hyperbolic groups or for all amenable groups.

There are partial results on the difficult question, whether the Atiyah Conjectures 3.1 holds for a group G if it holds for a subgroup of finite index, see for instance [74].

REMARK 3.7. Dropping the condition on an upper bound on the orders of finite subgroups, one might still ask if the L^2 -Betti numbers are always rational. This goes back to Atiyah's original question [9, page 72], who asked for rationality of L^2 -Betti numbers of closed manifolds. Austin [10, Corollary 1.2] gave the first example of a finitely generated group G , where for some matrix $A \in M_{m,n}(\mathbb{Q}G)$ the dimension $\dim_{\mathcal{N}(G)}(\ker(r_A))$ of the kernel of the $\mathcal{N}(G)$ -homomorphism $r_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n$ is irrational. Grabowski [41, Theorem 1.3] proved, using Turing machines, that any non-negative real number can occur in this way for some finitely generated group G and some matrix A . Löh and Uschold [77] investigate the computability degree of real numbers arising as L^2 -Betti numbers or L^2 -torsion of groups, parametrised over the Turing degree of the word problem. Roughly speaking, the complexity of the computation of L^2 -invariants of a group is the same as the complexity of the word problem. This is due to the combinatorial computation in terms of characteristic sequences mentioned in Remark 9.1.

3.3. Embedding the group ring of a torsionfree group into a skew-field. Associated to the von Neumann algebra $\mathcal{N}(G)$ is the algebra of affiliated operators $\mathcal{U}(G)$ which contains $\mathcal{N}(G)$. It can be defined analytically or just as the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicative subset of non-zero divisors. Now one can consider the so called division closure $\mathcal{D}(G)$ of $\mathbb{C}G$ in $\mathcal{U}(G)$, i.e., the smallest ring $\mathbb{C}G \subset \mathcal{D}(G) \subset \mathcal{U}(G)$ such that if $x \in \mathcal{D}(G)$ is invertible in $\mathcal{U}(G)$, its inverse is already contained in $\mathcal{D}(G)$.

The proof of the following is based on ideas of Peter Linnell from [75] which have been elaborated and expanded on in [85, Theorem 8.29 on page 330 and Lemma 10.39 on page 388] and [108], see also [35, Theorem 3.8 (1) and 2]. The proofs given in the references above for $F = \mathbb{C}$ and $F = \mathbb{Q}$ carry directly over to arbitrary F .

THEOREM 3.8 (Main properties of $\mathcal{D}(G)$). *Let G be a torsionfree group. Consider a field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.*

- (1) *The group G satisfies the Atiyah Conjecture with coefficients in F if and only if $\mathcal{D}(G)$ is a skew field;*
- (2) *Suppose that G satisfies the Atiyah Conjecture. Let C_* be a projective FG -chain complex. Then we get for all $n \geq 0$*

$$b_n^{(2)}(\mathcal{N}(G) \otimes_{FG} C_*) = \dim_{\mathcal{D}(G)}(H_n(\mathcal{D}(G) \otimes_{\mathbb{Q}G} C_*)).$$

In particular $b_n^{(2)}(\mathcal{N}(G) \otimes_{FG} C_*)$ is an integer or ∞ .

Theorem 3.8 shows that the Atiyah Conjecture 3.1 is related to the question whether for a torsionfree group G the group ring FG can be embedded into a skew field, see for instance [49]. Note that the existence of an embedding of FG into a skewfield implies that FG has no non-trivial zero-divisors which is predicted by the Zero-Divisor-Conjecture of Kaplansky.

3.4. The Algebraic Atiyah Conjecture. Recall that Linnell's program to approach the Atiyah Conjecture consists of a K -theoretic part and a ring theoretic part, namely that the composite

$$\operatorname{colim}_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG) \rightarrow K_0(\mathcal{D}(G))$$

is surjective and that $\mathcal{D}(G)$ is semisimple, see [85, Lemma 10.28 on page 382 and Theorem 10.38 on page 387]. There is the following so called algebraic Atiyah Conjecture, which is a purely K -theoretic statement and taken from [54, Conjecture 7.2], c.f. [85, Lemma 10.26 on page 382]. It implies the Atiyah Conjecture 3.1, see [85, Lemma 10.26 on page 382].

CONJECTURE 3.9 (Algebraic Atiyah Conjecture). *Let G be a group and let F be a field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$, which is closed under complex conjugation. Let \mathcal{R}_{FG} be the $*$ -regular closure of FG in $\mathcal{U}(G)$.*

We say that G satisfies the Algebraic Atiyah Conjecture with coefficients in F if the composite of the canonical maps

$$\operatorname{colim}_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG) \rightarrow K_0(\mathcal{R}_{FG})$$

is surjective.

Note that the Farrell-Jones Conjecture, which is known for a large class of groups, see for instance [91, Theorem 12.56 and Theorem 15.1], predicts the bijectivity of the map $\operatorname{colim}_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG)$. If the Farrell-Jones Conjecture holds, then the Algebraic Atiyah Conjecture 3.9 is equivalent to the claim that the map $K_0(FG) \rightarrow K_0(\mathcal{R}_{FG})$ is surjective. For a group G , for which there is a bound on its finite subgroups, the algebraic Atiyah Conjecture 3.9 is equivalent to the so called center-valued Atiyah Conjecture with coefficients in F , which implies the Atiyah Conjecture 3.1 with coefficients in F , see [48, Definition 2.5.1, Proposition 2.5.2, Theorem 3.1.4] and also [65, Definition 1.2, Proposition 1.3].

For more information about the Atiyah Conjecture we refer for instance to [85, Chapter 10].

4. The Singer Conjecture

We now turn attention to a series of conjectures about L^2 -invariants of aspherical manifolds. They deal with the phenomenon that L^2 -invariants often vanish outside of the middle dimension.

4.1. Statement of the Singer Conjecture.

CONJECTURE 4.1 (Singer Conjecture). *If M is an aspherical closed topological manifold, then we get for $n \geq 0$*

$$b_n^{(2)}(\widetilde{M}) = 0 \quad \text{if } 2n \neq \dim(M).$$

If M is an aspherical closed topological manifold of even dimension $2m$, then

$$(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M}) \geq 0.$$

If M is a closed connected smooth manifold of even dimension $2m$ admitting a Riemannian metric of negative sectional curvature, then

$$(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M}) > 0.$$

The equality $(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M})$ appearing in the Singer Conjecture 4.1 above follows from the the Euler-Poincaré formula $\chi(M) = \sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(\widetilde{M})$.

The Singer Conjecture 4.1 is consistent with the Atiyah Conjecture in the sense that it predicts that the L^2 -Betti numbers $b_n^{(2)}(\widetilde{M})$ for an aspherical closed manifold M are all integers.

In original versions of the Singer Conjecture 4.1 the condition aspherical closed manifolds was replaced by the condition closed Riemannian manifold with non-positive sectional curvature. Note that a closed Riemannian manifold with non-positive sectional curvature is aspherical by Hadamard's Theorem.

4.2. Hopf Conjectures. Obviously the Singer Conjecture 4.1 implies the following conjecture in the cases, where M is aspherical or has negative sectional curvature.

CONJECTURE 4.2 (Hopf Conjecture). *If M is an aspherical closed topological manifold of even dimension $\dim(M) = 2m$, then*

$$(-1)^m \cdot \chi(M) \geq 0.$$

If M is a closed smooth manifold of even dimension $\dim(M) = 2m$ with Riemannian metric and with sectional curvature $\sec(M)$, then

$$\begin{aligned} (-1)^m \cdot \chi(M) &> 0 && \text{if } \sec(M) < 0; \\ (-1)^m \cdot \chi(M) &\geq 0 && \text{if } \sec(M) \leq 0; \\ \chi(M) &= 0 && \text{if } \sec(M) = 0; \\ \chi(M) &\geq 0 && \text{if } \sec(M) \geq 0; \\ \chi(M) &> 0 && \text{if } \sec(M) > 0. \end{aligned}$$

The following version of the Hopf Conjecture for L^2 -torsion appears in [85, Conjecture 11.3 on page 418]

CONJECTURE 4.3 (Hopf Conjecture for L^2 -torsion). *If M is an aspherical closed topological manifold of odd dimension $\dim(M) = 2m + 1$, then M is \det - L^2 -acyclic, and its L^2 -torsion satisfies*

$$(-1)^m \cdot \rho^{(2)}(\widetilde{M}) \geq 0.$$

If M is a closed smooth manifold of odd dimension $\dim(M) = 2m + 1$ with Riemannian metric and with sectional curvature $\sec(M) < 0$, then M is \det - L^2 -acyclic, and its L^2 -torsion satisfies

$$(-1)^m \cdot \rho^{(2)}(\widetilde{M}) > 0.$$

If M is an aspherical closed topological manifold of odd dimension $\dim(M) = 2m + 1$ whose fundamental group contains an amenable infinite normal subgroup, then M is \det - L^2 -acyclic and its L^2 -torsion satisfies

$$\rho^{(2)}(\widetilde{M}) = 0.$$

The Hopf Conjecture 4.3 for L^2 -torsion is known to be true if one of the following conditions is satisfied:

- (1) $\dim(M) \leq 3$;
- (2) M is a locally symmetric space;
- (3) $\pi_1(M)$ contains an elementary amenable infinite normal subgroup;
- (4) M carries a non-trivial S^1 -action.

Statement (1) follows from combining [95, Theorem 0.7] with Thurston's Geometrization conjecture, which is known to be true by [64, 99] following the spectacular outline of Perelman. Statement (2) can be found in [85, Corollary 5.16 on page 231] and statements (3) and (4) are from [87, Corollary 1.13]. See also Remark 6.12.

4.3. Status of the Singer Conjecture. The Singer Conjecture 4.1 is known for an aspherical closed smooth manifold M in the following cases:

- (1) $\dim(M) \leq 3$;
- (2) M comes with a Riemannian metric whose sectional curvature is negative and satisfies certain pinching conditions;
- (3) M is a locally symmetric space;
- (4) M possesses a Riemannian metric, whose sectional curvature is negative and M carries some Kähler structure;
- (5) M is an aspherical closed Kähler manifold, whose fundamental group is word-hyperbolic in the sense of Gromov [44].
- (6) $\pi_1(M)$ contains an amenable infinite normal subgroup.
- (7) M carries a non-trivial S^1 -action;
- (8) M fibers over S^1 .

The precise statements and proofs and the relevant references, e.g., [14, 23, 45, 57, 81, 101], can be found in [85, Theorem 1.39 on page 42, Corollary 1.43 on page 48, Theorem 1.44 on page 48, Section 11.1], again using Thurston's Geometrization Conjecture for the three dimensional case. We also mention that proofs of (1) - (5) use geometric properties of the manifold whereas (6) - (8) use algebraic topological techniques and do not use the manifold structure.

Partial results about the Singer Conjecture 4.1 for right-angled Coxeter groups can be found in Davis-Okun [25].

The paper by Albanese, Di Cerbo and Lombardi [6] deals with the Singer Conjecture for aspherical complex surfaces and proves it for aspherical complex surfaces with residually finite fundamental groups in [6, Theorem 1.5].

In contrast to the Atiyah Conjecture, evidence for the Singer Conjecture 4.1 comes from computations only and no promising proof strategy is known. In some sense Poincaré duality together with L^2 -bounds on differential forms on \widetilde{M} seem to force the L^2 -Betti numbers $b_p^{(2)}(\widetilde{M})$ of an aspherical closed manifold to concentrate in the middle dimension. One may wonder what happens if we replace M by an aspherical finite Poincaré complex in the Singer Conjecture 4.1. There are counterexamples to the Singer Conjecture 4.1 if one weakens aspherical to rationally aspherical, see [11, Theorem 4]. The reader should also take a look at Remark 6.10 and Remark 6.12.

4.4. The proper Singer Conjecture and the action dimension. One may generalize the Singer Conjecture 4.1 to the following

CONJECTURE 4.4 (Proper Singer Conjecture). *A group G satisfies the proper Singer Conjecture, if for any contractible topological manifold M without boundary on which G acts properly and cocompactly, we have for all $n \geq 0$*

$$b_n^{(2)}(M; \mathcal{N}(G)) = 0 \quad \text{if } 2n \neq \dim(M).$$

Note that the proper Singer Conjecture is a statement about a group G , whereas the Singer Conjecture 4.1 is a statement about an aspherical closed manifold. Provided that G is torsionfree, the proper Singer Conjecture holds for G if and only if the Singer Conjecture 4.1 holds for one (and hence all) aspherical closed manifold M with $\pi_1(M) \cong G$.

The proper Singer Conjecture for a group G is equivalent to the cadim-Conjecture (cadim standing for compact action dimension) for G for manifolds with PL-boundary, see [100, Theorem 4.10]. The latter predicts that for a contractible topological manifold M (possibly with boundary) that admits a cocompact proper topological action of G such that the boundary ∂M of M carries a PL-structure for which the G -action on ∂M is through PL-automorphisms we have

$$2 \cdot \sup\{n \in \mathbb{Z}^{\geq 0} \mid b_n^{(2)}(G) \neq 0\} \leq \dim(M).$$

For more informations about the notion of the (compact) action dimension and its relationship to L^2 -Betti numbers we refer for instance to [12, 17, 100].

4.5. The \mathbb{F}_p -Singer Conjecture. One may also consider the \mathbb{F}_p -Singer Conjecture, which predicts that for an aspherical closed topological manifold M and any prime p we get for $n \geq 0$

$$b_n^{(2)}(\pi_1(M); \mathbb{F}_p) = 0 \quad \text{if } 2n \neq \dim(M).$$

Here $b_n^{(2)}(\pi_1(M); \mathbb{F}_p)$ is defined only if $\pi_1(M)$ is residually finite. Namely, for a chain of normal subgroups of finite index $\pi_1(M) = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots$ with $\bigcap_{n=0}^{\infty} \Gamma_n = \{1\}$, one puts $b_n^{(2)}(\pi_1(M); \mathbb{F}_p) = \limsup_{n \rightarrow \infty} \frac{b_n(\Gamma_n; \mathbb{F}_p)}{[\Gamma_n: \Gamma_k]}$. It is unknown in general whether this definition depends on the chain of subgroups Γ_k and the statement above is to be understood in the sense that it holds for any such chain. The definition of L^2 -Betti numbers over \mathbb{F}_p is motivated by the Approximation Theorem in characteristic zero 2.4 (14). The \mathbb{F}_p -Singer Conjecture is open for $\dim(M) = 3$. However, for any odd prime p , the \mathbb{F}_p -Singer Conjecture fails in all odd dimensions larger or equal than 7 and all even dimensions larger or equal than 14, see [13, Theorem 4].

5. The Determinant Conjecture

Next we explain the Determinant Conjecture [85, Conjecture 13.2 on page 454]. It is needed for the definition of L^2 -determinant class and implies homotopy invariance of L^2 -torsion, see Theorem 2.4 (1). Furthermore, it implies the Approximation Conjecture, see Remark 7.2.

CONJECTURE 5.1 (Determinant Conjecture for a group G).

For any matrix $A \in M_{r,s}(\mathbb{Z}G)$, the Fuglede-Kadison determinant of the $\mathcal{N}(G)$ -homomorphism $r_A: \mathcal{N}(G)^r \rightarrow \mathcal{N}(G)^s$ given by right multiplication with A satisfies

$$\det_{\mathcal{N}(G)}(r_A) \geq 1.$$

The class \mathcal{F} of groups for which it is true has the following properties:

- (1) *Trivial group.*
The trivial group belongs to \mathcal{F} ;
- (2) *Amenable quotient.*
Let $H \subset G$ be a normal subgroup. Suppose that $H \in \mathcal{F}$ and the quotient G/H is amenable. Then $G \in \mathcal{F}$;
- (3) *Filtered colimits.*
The class \mathcal{F} is closed under countable filtered colimits;
- (4) *Cofiltered limits.*
The class \mathcal{F} is closed under countable cofiltered limits;
- (5) *Subgroups.*
If H is isomorphic to a subgroup of a group G with $G \in \mathcal{F}$, then $H \in \mathcal{F}$.
- (6) *Quotients with finite kernel.*
Let $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ be an exact sequence of groups. If K is finite and G belongs to \mathcal{F} , then Q belongs to \mathcal{F} ;
- (7) *Sofic groups.*
Sofic groups belong to \mathcal{F} .

For the verification of the Determinant Conjecture for the groups above, see [28, Theorem 5], [85, Section 13.2 on pages 459 ff], [114, Theorem 1.21].

To sketch how large \mathcal{F} really is, let us mention that already the class of sofic groups is very large. It is closed under direct and free products, taking subgroups, taking inverse and direct limits over directed index sets, and is closed under extensions with amenable groups as quotients and a sofic group as kernel. In particular it contains all residually amenable groups. One expects that there exists non-sofic groups but no example is known. More information about sofic groups can be found for instance in [29] and [103]. More information about the Determinant Conjecture 5.1 can be found in [85, Chapter 13] and [114].

6. Approximation for finite index normal chains

Recall that the Approximation Theorem 14 predicts that L^2 -Betti numbers of spaces can be computed as limits of normalized ordinary Betti numbers of certain towers of coverings. In this section we discuss this phenomenon for more general invariants.

6.1. Basic setup for approximation for finite index normal chains. Let G be a (discrete) group. A *finite index normal chain* $\{G_i\}$ for G is a descending chain of subgroups

$$(6.1) \quad G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$$

such that G_i is normal in G , the index $[G : G_i]$ is finite and $\bigcap_{i \geq 0} G_i = \{1\}$.

Let $p: \overline{X} \rightarrow X$ be a G -covering. Put $X[i] := G_i \backslash \overline{X}$. We obtain a $[G : G_i]$ -sheeted covering $p[i]: X[i] \rightarrow X$. Its total space $X[i]$ inherits the structure of a finite CW -complex, a closed manifold, or a closed Riemannian manifold respectively if X has the structure of a finite CW -complex, a closed manifold, or a closed Riemannian manifold respectively.

Let α be a classical topological invariant such as the Euler characteristic, the signature, the n th Betti number with coefficients in the field \mathbb{Q} or \mathbb{F}_p , torsion in the sense of Reidemeister or Ray-Singer, or the logarithm of the cardinality of the

torsion subgroup of the n th homology group with integral coefficients. We want to study the sequence

$$\left(\frac{\alpha(X[i])}{[G : G_i]} \right)_{i \geq 0}.$$

PROBLEM 6.1 (Approximation Problem).

- (1) Does the sequence converge?
- (2) If yes, is the limit independent of the chain $\{G_i\}$?
- (3) If yes, what is the limit?

The hope is that the answer to the first two questions is yes and the limit turns out to be an L^2 -analogue $\alpha^{(2)}$ of α applied to the G -space \overline{X} , i.e., one can prove an equality of the type

$$(6.2) \quad \lim_{i \rightarrow \infty} \frac{\alpha(X[i])}{[G : G_i]} = \alpha^{(2)}(\overline{X}; \mathcal{N}(G)).$$

Here $\mathcal{N}(G)$ stands for the group von Neumann algebra and is a reminiscence of the fact that the G -action on \overline{X} plays a role. Equation (6.2) is often used to compute the L^2 -invariant $\alpha^{(2)}(\overline{X}; \mathcal{N}(G))$ by its finite-dimensional analogues $\alpha(X[i])$. On the other hand, it implies the existence of finite coverings with large $\alpha(X[i])$, if $\alpha^{(2)}(\overline{X}; \mathcal{N}(G))$ is known to be positive.

6.2. The Euler characteristic. The *Euler characteristic* $\chi(X)$ of a finite CW-complex is multiplicative under finite coverings. Because this implies $\chi(X) = \frac{\chi(X[i])}{[G : G_i]}$, the answer in this case is yes to the questions appearing in Problem 6.1, and the limit is

$$(6.3) \quad \lim_{i \rightarrow \infty} \frac{\chi(X[i])}{[G : G_i]} = \chi(X).$$

6.3. The Signature. Next we consider the *signature* $\text{sign}(M)$ of a closed oriented topological $4k$ -dimensional manifold M . It is known that it is multiplicative under finite coverings, however, the proof is more involved than the one for the Euler characteristic. It follows for instance from Hirzebruch's Signature Theorem, see [51], or Atiyah's L^2 -index theorem [9, (1.1)] in the smooth case; for closed topological manifolds see Schafer [113, Theorem 8]. Since this implies $\text{sign}(X) = \frac{\text{sign}(X[i])}{[G : G_i]}$, each of the questions appearing in Problem 6.1 has a positive answer, and the limit is

$$(6.4) \quad \lim_{i \rightarrow \infty} \frac{\text{sign}(X[i])}{[G : G_i]} = \text{sign}(X).$$

The next level of generality is to pass from a oriented closed topological manifold to a oriented finite Poincaré complex, whose definition is due to Wall [123]. For them the signature is still defined if the dimension is divisible by 4. There are Poincaré complexes X for which the signature is not multiplicative under finite coverings, see [106, Example 22.28], [123, Corollary 5.4.1]. Hence the situation is more complicated here. Nevertheless, it turns out in this case each of the questions appearing in Problem 6.1 has a positive answer, and the limit is

$$(6.5) \quad \lim_{i \rightarrow \infty} \frac{\text{sign}(X[i])}{[G : G_i]} = \text{sign}^{(2)}(\overline{X}; \mathcal{N}(G)),$$

where $\text{sign}^{(2)}(\overline{X}; \mathcal{N}(G))$ denotes the L^2 -signature, which is in general different from $\text{sign}(X)$ for a finite Poincaré complex X .

For more information and details we refer to [96, 97].

6.4. Approximation of L^2 -Betti numbers in characteristic zero. Fix a field F of characteristic zero. We consider the n th Betti number with F -coefficients $b_n(X; F) := \dim_F(H_n(X; F))$. Note that $b_n(X; F) = b_n(X; \mathbb{Q}) = \text{rk}_{\mathbb{Z}}(H_n(X; \mathbb{Z}))$ holds, where $\text{rk}_{\mathbb{Z}}$ denotes the rank of a finitely generated abelian group. In this case each of the questions appearing in Problem 6.1 has a positive answer by the main result of Lück's article [80].

THEOREM 6.2. *Let F be a field of characteristic zero, and let X be a finite CW-complex. Then for each finite index normal chain $\{G_i\}$ we have*

$$\lim_{i \rightarrow \infty} \frac{b_n(X[i]; F)}{[G : G_i]} = b_n^{(2)}(\overline{X}; \mathcal{N}(G)),$$

where $b_n^{(2)}(\overline{X}; \mathcal{N}(G))$ denotes the n th L^2 -Betti number.

Löh and Uschold [77, Proposition 6.6] prove a quantitative version of Theorem 6.2. Essentially they show

$$\left| b_n^{(2)}(\overline{X}; \mathcal{N}(G)) - \frac{b_n(X[i]; F)}{[G : G_i]} \right| \leq a \cdot \left(1 - \frac{1}{b \cdot i}\right)^{i^2} + \frac{a \cdot \log(b)}{\log(i)}$$

for certain constants a and b depending on X .

6.5. Approximation of L^2 -Betti numbers in prime characteristic. The situation is more complicated and unclear in prime characteristic. Fix a prime p . Let F be a field of characteristic p . We consider the n th Betti number with F -coefficients $b_n(X; F) := \dim_F(H_n(X; F))$. Note that $b_n(X; F) = b_n(X; \mathbb{F}_p)$ holds where \mathbb{F}_p is the field of p -elements. In this setting a general answer to Problem 6.1 is only known in special cases. The main problem is that one does not have an analogue of the von Neumann algebra in characteristic p and the construction of an appropriate extended dimension function, see [83], is not known in general.

If G is torsionfree elementary amenable, one gets the full positive answer by Linnell-Lück-Sauer [72, Theorem 0.2], where more explanations, e.g., about Ore localizations, are given and actually virtually torsionfree elementary amenable groups are considered.

THEOREM 6.3. *Let F be a field (of arbitrary characteristic) and X be a connected finite CW-complex. Let G be a torsionfree elementary amenable group. Then for each finite index normal chain $\{G_i\}$:*

$$\dim_{FG}^{\text{Ore}}(H_n(\overline{X}; F)) = \lim_{n \rightarrow \infty} \frac{b_n(X[i]; F)}{[G : G_n]}.$$

Note that Theorem 6.3 is consistent with Theorem 6.2 since for a field F of characteristic zero and a torsionfree elementary amenable group G we have $b_n^{(2)}(\overline{X}; \mathcal{N}(G)) = \dim_{FG}^{\text{Ore}}(H_n(\overline{X}; F))$. The latter equality follows from [85, Theorem 6.37 on page 259, Theorem 8.29 on page 330, Lemma 10.16 on page 376, and Lemma 10.39 on page 388].

Here is another special case taken from Bergeron-Lück-Linnell-Sauer [15], (see also Calegari-Emerton [21, 22]), where we know the answer only for special chains.

Let p be a prime, let n be a positive integer, and let $\phi: G \rightarrow \mathrm{GL}_n(\mathbb{Z}_p)$ be a homomorphism, where \mathbb{Z}_p denotes the p -adic integers. The closure of the image of ϕ , which is denoted by Λ , is a p -adic analytic group admitting an exhausting filtration by open normal subgroups $\Lambda_i = \ker(\Lambda \rightarrow \mathrm{GL}_n(\mathbb{Z}/p^i\mathbb{Z}))$. Put $G_i = \phi^{-1}(\Lambda_i)$.

THEOREM 6.4. *Let F be a field (of arbitrary characteristic). Put $d = \dim(\Lambda)$. Let X be a finite CW-complex. Then for any integer n and as i tends to infinity, we have:*

$$b_n(X[i]; F) = b_n^{(2)}(\overline{X}; F) \cdot [G : G_i] + O\left([G : G_i]^{1-1/d}\right),$$

where $b_n^{(2)}(\overline{X}; F)$ is the n th mod p L^2 -Betti numbers occurring in [15, Definition 1.3] which is defined using homology coefficients in the Iwasawa algebra of G . In particular

$$\lim_{i \rightarrow \infty} \frac{b_n(X[i]; F)}{[G : G_i]} = b_n^{(2)}(\overline{X}; F).$$

Returning to the setting of arbitrary finite index normal chain $(G_i)_{i \geq 0}$, we get by the universal coefficient theorem $b_n(X[i]; \mathbb{Q}) \leq b_n(X[i]; F)$ for any field F and hence by Theorem 6.2 the inequality

$$\liminf_{i \rightarrow \infty} \frac{b_n(X[i]; F)}{[G : G_i]} \geq b_n^{(2)}(\overline{X}; \mathcal{N}(G)).$$

If p is a prime and we additionally assume that each index $[G : G_i]$ is a p -power, then the sequence $\frac{b_n(X[i]; F)}{[G : G_i]}$ is monotone decreasing and in particular $\lim_{i \rightarrow \infty} \frac{b_n(X[i]; F)}{[G : G_i]}$ exists, see [15, Theorem 1.6].

QUESTION 6.5 (Approximation in prime characteristic). *Does the sequence $\frac{b_n(X[i]; F)}{[G : G_i]}$ converge and is the limit $\lim_{i \rightarrow \infty} \frac{b_n(X[i]; F)}{[G : G_i]}$ independent of the chain $\{G_i\}$ for all fields F and $n \geq 0$, provided that X is finite and \overline{X} is contractible?*

REMARK 6.6. The third author conjectured in the situation of Question 6.5 that the $\lim_{i \rightarrow \infty} \frac{b_n(X[i]; F)}{[G : G_i]}$ is equal to $b_n^{(2)}(X; \mathcal{N}(G))$ for every field F and for all finite index normal chains $\{G_i\}$, see [88, Conjecture 3.4 on page 275]. This is true in characteristic zero by Theorem 6.2 and if G is torsionfree elementary amenable by Theorem 6.3, but not in prime characteristic by Avramidi-Okun-Schreve [13, Corollary 2]. Namely, Avramidi-Okun-Schreve [13, Theorem 1] prove that for a right-angled Artin group A_L with defining flag complex L and any field F the limit $\lim_{i \rightarrow \infty} \frac{b_n(X[i]; F)}{[G : G_i]}$ exists, is independent of the chain $\{G_i\}$, and actually agrees with the reduced Betti number $\overline{b}_n(L; F)$ of L with coefficients in F . Furthermore, they construct examples where $\overline{b}_3(L; \mathbb{Q}) = 0$ and $\overline{b}_3(L; \mathbb{F}_2) = 1$ hold. These counterexamples do not exist in degree $n = 1$.

REMARK 6.7 (L^2 -Betti numbers in finite characteristic via skew fields). Recall from Theorem 3.8 that if a torsionfree group G satisfies the Atiyah conjecture, then the group ring $\mathbb{C}G$ embeds into a skew field $\mathcal{D}(G)$ and L^2 -Betti numbers are the Betti numbers with coefficients in this skew field. Motivated by this, one might hope that for a torsionfree group G and any field F there exists a skew field $\mathcal{D}_F(G)$ together with an embedding $FG \hookrightarrow \mathcal{D}_F(G)$ which can be used to define L^2 -Betti numbers with coefficients in F as $b_n^{(2)}(\overline{X}; F) := \dim_{\mathcal{D}_F(G)} H_n^G(\overline{X}; \mathcal{D}_F(G))$.

An approach to Question 6.5 would then be to show that the sequence $\frac{b_n(X[i]; F)}{[G : G_i]}$ converges to $b_n^{(2)}(\overline{X}; F)$ for $i \rightarrow \infty$.

Jaikin-Zapirain constructs such embeddings in [55, Corollary 1.3] for large classes of groups. He shows that such an embedding of FG into a skew field with very nice properties (it is a Hughes free division ring and the universal division ring of fractions of FG) exists if G is residually (locally indicable amenable). Combining the approximation results [55, Theorem 1.2] and [72, Theorem 0.2] it is easy to check that in this situation the equality

$$(6.6) \quad b_n^{(2)}(\overline{X}; F) = \inf \left\{ \frac{b_n(\overline{X}/H; F)}{[G : H]} : H \leq G, [G : H] < \infty \right\}$$

holds. In the case $F = \mathbb{Q}$ of the classical approximation theorem the easier to prove Kazhdan's inequality predicts that L^2 -Betti numbers are bounded by

$$b_n^{(2)}(\overline{X}; \mathcal{N}(G)) \geq \limsup_{i \rightarrow \infty} \frac{b_n(X[i]; \mathbb{Q})}{[G : G_i]}$$

for any finite index normal chain (G_i) for G . It is not clear, however, whether an analogue of this holds in the present setting. This would imply that (6.6) still holds when one replaces inf by the limit over a finite index normal chain.

6.6. Torsion. For a detailed discussion of approximation for finite index normal chains for torsion invariants such as the Ray-Singer torsion or the integral torsion we refer for instance to [88, Sections 8 – 10] and also to Subsection 7.3.

6.7. Homological torsion growth and L^2 -torsion. The following conjecture is taken from [87, Conjecture 1.12 (2)]. For locally symmetric spaces it reduces to the conjecture of Bergeron and Venkatesh [16, Conjecture 1.3].

CONJECTURE 6.8 (Homological torsion growth and L^2 -torsion). *Let M be an aspherical closed manifold.*

Then we get for any natural number n with $2n + 1 \neq \dim(M)$

$$\lim_{i \rightarrow \infty} \frac{\ln(|\text{tors}(H_n(M[i]; \mathbb{Z}))|)}{[G : G_i]} = 0.$$

If the dimension $\dim(M) = 2m + 1$ is odd, then \widetilde{M} is \det - L^2 -acyclic and we get

$$\lim_{i \rightarrow \infty} \frac{\ln(|\text{tors}(H_m(M[i]; \mathbb{Z}))|)}{[G : G_i]} = (-1)^m \cdot \rho^{(2)}(\widetilde{M}).$$

Conjecture 6.8 is true, if $\pi_1(M)$ contains an elementary amenable infinite normal subgroup or M carries a non-trivial S^1 -action, see [87, Corollary 1.13].

REMARK 6.9 (Criteria for vanishing homology torsion growth). Recently, there has been some progress in proving vanishing of homology growth and torsion homology growth for certain groups. These results are based on the general vanishing result of Abert-Bergeron-Fraczyk-Gaboriau [2, Theorem 10.20] and have been applied to show vanishing for special linear groups and mapping class groups [2, Theorem A, D]. Further applications of this theorem can be found in [7, Theorem A], [8, Theorem A], and [122, Proposition 1.6], where vanishing of homology growth and torsion homology growth of mapping tori of finitely generated free groups with respect to a polynomially growing automorphism and of right-angled Artin groups which are inner amenable is shown. Combining the first result with [24, Theorem

5.1] confirms Conjecture 6.8 for mapping tori of aspherical closed manifolds which have finitely generated free fundamental group and the monodromy is polynomially growing.

REMARK 6.10 (The Singer Conjecture 4.1 and Conjecture 6.8 are not compatible). The Singer Conjecture 4.1 and Conjecture 6.8 about homological torsion growth and L^2 -torsion cannot both be true in general. Namely, if both are true, then the \mathbb{F}_p -Singer Conjecture of Subsection 4.5 would be true as pointed out by Avramidi-Okun-Schreive before Theorem 4 appearing in [13]. The argument is as follows.

Suppose we have an aspherical closed d -dimensional manifold M and both the Singer Conjecture 4.1 and Conjecture 6.8 are true. Consider n with $2n \neq d$. We have to show

$$(6.7) \quad \lim_{i \rightarrow \infty} \frac{b_n(M[i]; \mathbb{F}_p)}{[G : G_i]} = 0$$

for $G = \pi_1(M)$ and a finite index normal chain $\{G_i\}$. Because of Poincaré duality it suffices to consider the case $2n > d$. Put $M^3 = M \times M \times M$, $G^3 = G \times G \times G$ and $G_i^3 = G_i \times G_i \times G_i$. Then $\widetilde{M^3} = \widetilde{M} \times \widetilde{M} \times \widetilde{M}$, $G^3 = \pi_1(M^3)$, and $M^3[i] = M[i]^3$. By the Singer Conjecture 4.1 applied to M^3 we get $b_{3n}^{(2)}(\widetilde{M^3}) = 0$. This implies by Theorem 6.3

$$\lim_{i \rightarrow \infty} \frac{\text{rk}_{\mathbb{Z}}(H_{3n}(M[i]^3; \mathbb{Z}))}{[G^3 : G_i^3]} = \lim_{i \rightarrow \infty} \frac{b_{3n}(M[i]^3; \mathbb{Q})}{[G^3 : G_i^3]} = b_{3n}^{(2)}(\widetilde{M^3}; \mathbb{Q}) = 0.$$

Since $2(3n) > 2(3n - 1) > 3d$ holds, we conclude from Conjecture 6.8

$$\begin{aligned} \lim_{i \rightarrow \infty} \frac{\ln(|\text{tors}(H_{3n}(M^3[i]; \mathbb{Z}))|)}{[G^3 : G_i^3]} &= 0, \\ \lim_{i \rightarrow \infty} \frac{\ln(|\text{tors}(H_{3n-1}(M^3[i]; \mathbb{Z}))|)}{[G^3 : G_i^3]} &= 0. \end{aligned}$$

We have for any $k \geq 0$

$$\begin{aligned} \dim_{\mathbb{F}_p}(\mathbb{Z}^k \otimes_{\mathbb{Z}} \mathbb{F}_p) &= \dim_{\mathbb{Z}}(\mathbb{Z}^k); \\ \dim_{\mathbb{F}_p}(\mathbb{Z}/p^k \otimes_{\mathbb{Z}} \mathbb{F}_p) &= 1; \\ \dim_{\mathbb{F}_p}(\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p^k, \mathbb{F}_p)) &= 1; \\ \ln(|\text{tors}(\mathbb{Z}/p^k)|) &= k \cdot \ln(p). \end{aligned}$$

If q is a prime different from p , we get $\mathbb{Z}/q^k \otimes_{\mathbb{Z}} \mathbb{F}_p = 0$ and $\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/q^k, \mathbb{F}_p) = 0$. Hence we get for any finitely generated abelian group M

$$\begin{aligned} \dim_{\mathbb{F}_p}(M \otimes_{\mathbb{Z}} \mathbb{F}_p) &\leq \dim_{\mathbb{Z}}(M) + \frac{\ln(|\text{tors}(M)|)}{\ln(2)}; \\ \dim_{\mathbb{F}_p}(\text{Tor}_1^{\mathbb{Z}}(M, \mathbb{F}_p)) &\leq \frac{\ln(|\text{tors}(M)|)}{\ln(2)}. \end{aligned}$$

We conclude from the Künneth Formula and the Universal Coefficient Theorem

$$\begin{aligned}
\frac{b_n(M[i]; \mathbb{F}_p)^3}{[G : G_i]^3} &= \frac{b_n(M[i]; \mathbb{F}_p)^3}{[G^3 : G_i^3]} \\
&\leq \frac{b_{3n}(M[i]^3; \mathbb{F}_p)}{[G^3 : G_i^3]} \\
&= \frac{\dim_{\mathbb{F}_p}(H_{3n}(M[i]^3; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p) + \dim_{\mathbb{F}_p}(\mathrm{Tor}_1^{\mathbb{Z}}(H_{3n-1}(M[i]; \mathbb{Z}), \mathbb{F}_p))}{[G^3 : G_i^3]} \\
&\leq \frac{\dim_{\mathbb{Z}}(H_{3n}(M[i]^3; \mathbb{Z}))}{[G^3 : G_i^3]} + \frac{\ln(|\mathrm{tors}(H_{3n}(M^3[i]; \mathbb{Z}))|)}{[G^3 : G_i^3] \cdot \ln(2)} \\
&\quad + \frac{\ln(|\mathrm{tors}(H_{3n-1}(M^3[i]; \mathbb{Z}))|)}{[G^3 : G_i^3] \cdot \ln(2)}.
\end{aligned}$$

Hence (6.7) is true.

We have already mentioned in Subsection 4.5 that the \mathbb{F}_p -Singer Conjecture is not true in general, see [13, Theorem 4]. Hence the Singer Conjecture 4.1 or Conjecture 6.8 is not true.

In view of Remark 6.10 we suggest to keep the Singer Conjecture 4.1 and weaken Conjecture 6.8 to the following version.

CONJECTURE 6.11 (Modified Homological torsion growth and L^2 -torsion). *Let M be an aspherical closed manifold of odd dimension $\dim(M) = 2m + 1$.*

Then M is det- L^2 -acyclic, the limit $\lim_{i \rightarrow \infty} \left(\sum_{n=0}^{2m+1} (-1)^n \cdot \frac{\ln(|\mathrm{tors}(H_n(M[i]; \mathbb{Z}))|)}{[G : G_i]} \right)$ exists and is given by

$$\lim_{i \rightarrow \infty} \left(\sum_{n=0}^{2m+1} (-1)^n \cdot \frac{\ln(|\mathrm{tors}(H_n(M[i]; \mathbb{Z}))|)}{[G : G_i]} \right) = \rho^{(2)}(\widetilde{M}).$$

REMARK 6.12 (Discussion of Conjecture 6.11). It is conceivable that both the Singer Conjecture 4.1 and the modified Conjecture 6.11 about homological torsion growth and L^2 -torsion are true, since the argument in Remark 6.10 that the Singer Conjecture 4.1 or Conjecture 6.8 is not true does not apply anymore.

The difference between Conjecture 6.8 and Conjecture 6.11 is that we drop the claim $\lim_{i \rightarrow \infty} \frac{\ln(|\mathrm{tors}(H_n(M[i]; \mathbb{Z}))|)}{[G : G_i]} = 0$ in Conjecture 6.8 and allow in Conjecture 6.11 contributions from all $n \geq 0$ and not only from $n = m$ as in Conjecture 6.8. We conclude $\mathrm{tors}(H_n(M[i]; \mathbb{Z})) \cong \mathrm{tors}(H_{2m-n}(M[i]; \mathbb{Z}))$ for $n \geq 0$ from Poincaré duality and the universal coefficient theorem and the vanishing of $\mathrm{tors}(H_0(M[i]; \mathbb{Z}))$. We get the equality

$$\begin{aligned}
&\sum_{n=0}^{2m+1} (-1)^n \cdot \frac{\ln(|\mathrm{tors}(H_n(M[i]; \mathbb{Z}))|)}{[G : G_i]} \\
&= (-1)^m \cdot \frac{\ln(|\mathrm{tors}(H_m(M[i]; \mathbb{Z}))|)}{[G : G_i]} + 2 \cdot \sum_{n=1}^{m-1} (-1)^n \cdot \frac{\ln(|\mathrm{tors}(H_n(M[i]; \mathbb{Z}))|)}{[G : G_i]}.
\end{aligned}$$

Hence in dimension $\dim(M) = 3$ Conjecture 6.8 and Conjecture 6.11 agree. This does not lead to any contradiction since no 3-dimensional counterexample to the \mathbb{F}_p -Singer Conjecture known.

Note that Conjecture 6.11 does not imply the Hopf Conjecture for L^2 -torsion 4.3 for aspherical closed topological manifolds. One may wonder whether the Hopf Conjecture for L^2 -torsion 4.3 is not true in odd dimensions $\dim(M) \geq 5$ in general. It is known to be true in dimension $\dim(M) = 3$ by [95, Theorem 0.7], yet again since Thurston's Geometrization Conjecture is known to be true. The modified Conjecture 6.11 resolves the problem mentioned above, but may only be true in special cases.

One may replace in Conjecture 6.11 the aspherical closed manifold M of odd dimension $\dim(M) = 2m + 1$ by a connected finite CW -complex X , for which \tilde{X} is \det - L^2 -acyclic. This corresponds to [88, Conjecture 8.9 on page 290]. This version is true if $\pi_1(X) \cong \mathbb{Z}$ holds, see [16, Theorem 7.3], but at the time of writing nothing is known in general if $\pi_1(X)$ is infinite and not equal to \mathbb{Z} .

For applications of these approximation conjectures for L^2 -torsion to questions about profinite rigidity for fundamental groups of 3-manifolds and lattices in higher rank Lie groups we refer to [58, Section 6.7] and [59]. Estimates for the homological growth in terms of the volume for aspherical manifolds are established in [111].

6.8. Further examples. Other invariants, such as the rank gradient and the cost, truncated Euler characteristics, and minimal numbers of generators, are discussed in [88, Sections 4 and 5]. For consideration concerning the speed of convergence we refer to [88, Section 6].

7. Approximation for normal chains

A *normal chain* $\{G_i\}$ for the group G is a descending chain of subgroups

$$(7.1) \quad G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

such that G_i is normal in G and $\bigcap_{i \geq 0} G_i = \{1\}$. Note that a normal chain is a finite index normal chain, if and only if $[G : G_i]$ is finite for each i . Next we want to discuss approximation results for normal chains.

7.1. The Approximation Conjecture for L^2 -Betti numbers. Next we deal with the Approximation Conjecture for L^2 -Betti numbers (see [114, Conjecture 1.10], [85, Conjecture 13.1 on page 453]).

CONJECTURE 7.1 (Approximation Conjecture for L^2 -Betti numbers). *A group G satisfies the Approximation Conjecture for L^2 -Betti numbers if for any normal chain $\{G_i\}$ one of the following equivalent conditions holds*

(1) *Matrix version*

Let $A \in M_{r,s}(\mathbb{Q}G)$ be a matrix. Then

$$\begin{aligned} \dim_{\mathcal{N}(G)}(\ker(r_A^{(2)} : L^2(G)^r \rightarrow L^2(G)^s)) \\ = \lim_{i \rightarrow \infty} \dim_{\mathcal{N}(G/G_i)}(\ker(r_{A[i]}^{(2)} : L^2(G/G_i)^r \rightarrow L^2(G/G_i)^s)); \end{aligned}$$

(2) *CW -complex version*

Let X be a G - CW -complex of finite type. Then $X[i] := G_i \backslash X$ is a G/G_i - CW -complex of finite type and we get for $n \geq 0$

$$b_n^{(2)}(X; \mathcal{N}(G)) = \lim_{i \rightarrow \infty} b_n^{(2)}(X[i]; \mathcal{N}(G/G_i)).$$

The two conditions appearing in Conjecture 7.1 are equivalent by [85, Lemma 13.4 on page 455].

REMARK 7.2 (The Determinant Conjecture and the Approximation Conjecture). Let G be a group and $\{G_i\}$ be a normal chain. Suppose that G and each G/G_i satisfies the Determinant Conjecture 5.1. Then the Approximation Conjecture 7.1 for L^2 -Betti numbers holds for G for this normal chain $\{G_i\}$ by [85, Theorem 13.3 (1) on page 454]. We mention that in [85, Theorem 13.3 (1) on page 454] there is a misprint, G_i has to be replaced by G/G_i . Moreover, as pointed out to us by Bin Sun, [85, Theorem 13.3 (2) on page 454] is not correct as stated. It is true that G is of $\det \geq 1$ -class if the group G belongs to \mathcal{G} . But it is not correct that G satisfies the Approximation Conjecture for any normal chain $\{G_i\}$ if G belongs to \mathcal{G} , one additionally needs that each G/G_i belongs to \mathcal{G} .

Recall that the Determinant Conjecture 5.1 is known for a large class of groups, for instance it is true for all sofic groups. Suppose that each quotient G/G_i is finite. Then we recover Theorem 6.2 from Remark 7.2.

A typical application of the Approximation Conjecture 7.1 is the following. If G satisfies the Approximation Conjecture 7.1 and for a given normal chain $\{G_i\}$ each G/G_i is torsionfree and satisfies the Atiyah Conjecture 3.1, then G is torsionfree and satisfies the Atiyah Conjecture 3.1, since a limit of a convergent sequence of integers is an integer again.

For more information about the Approximation Conjecture and its applications we refer to [85, Chapter 13] and [114].

7.2. Approximation of Fuglede-Kadison determinants. The following conjecture is taken from [88, Conjecture 14.1 on page 308].

CONJECTURE 7.3 (Approximation Conjecture for Fuglede-Kadison determinants). *A group G satisfies the Approximation Conjecture for Fuglede-Kadison determinants if for any normal chain $\{G_i\}$ and any matrix $A \in M_{r,s}(\mathbb{Q}G)$ we get for the Fuglede-Kadison determinant*

$$\begin{aligned} \det_{\mathcal{N}(G)}(r_A^{(2)} : L^2(G)^r \rightarrow L^2(G)^s) \\ = \lim_{i \in I} \det_{\mathcal{N}(G/G_i)}(r_{A[i]}^{(2)} : L^2(G/G_i)^r \rightarrow L^2(G/G_i)^s). \end{aligned}$$

An equivalent chain complex version can be found in [88, Conjecture 15.3 on page 309].

7.3. Approximation of L^2 -torsion. Let \overline{M} be a Riemannian manifold without boundary that comes with a proper free cocompact isometric G -action. Denote by $M[i]$ the Riemannian manifold obtained from \overline{M} by dividing out the G_i -action. The Riemannian metric on $M[i]$ is induced by the one on \overline{M} . There is an obvious proper free cocompact isometric G/G_i -action on $M[i]$ induced by the given G -action on \overline{M} . Notice that $M = \overline{M}/G$ is a closed Riemannian manifold and we get a G -covering $\overline{M} \rightarrow M$ and a G/G_i -covering $M[i] \rightarrow M$ which are compatible with the Riemannian metrics. Denote by

$$(7.2) \quad \rho_{\text{an}}^{(2)}(\overline{M}; \mathcal{N}(G)) \in \mathbb{R};$$

$$(7.3) \quad \rho_{\text{an}}^{(2)}(M[i]; \mathcal{N}(G/G_i)) \in \mathbb{R},$$

their *analytic L^2 -torsion* over $\mathcal{N}(G)$ and $\mathcal{N}(G/G_i)$ respectively, see [85, Definition 3.128 on page 178]. Note that we do *not* require that \overline{M} or $M[i]$ is L^2 -acyclic. If $[G : G_i]$ is finite, $\rho_{\text{an}}^{(2)}(M[i]; \mathcal{N}(G/G_i))$ is the Ray-Singer torsion of the closed Riemannian manifold multiplied with $\frac{1}{[G:G_i]}$ defined in [107].

CONJECTURE 7.4 (Approximation Conjecture for analytic L^2 -torsion). *A group G satisfies the Approximation Conjecture for analytic L^2 -torsion if for any normal chain $\{G_i\}$ and Riemannian manifold \overline{M} without boundary and with a proper free cocompact isometric G -action*

$$\rho_{\text{an}}^{(2)}(\overline{M}; \mathcal{N}(G)) = \lim_{i \in I} \rho_{\text{an}}^{(2)}(M[i]; \mathcal{N}(G/G_i)).$$

There are topological counterparts which we will denote by $\rho_{\text{top}}^{(2)}(X[i]; \mathcal{N}(G/G_i))$ and $\rho_{\text{top}}^{(2)}(\overline{X}; \mathcal{N}(G))$ see [85, Definition 3.120 on page 176]. They agree with their analytic versions by [20]. So Conjecture 7.4 is equivalent to its topological counterpart.

CONJECTURE 7.5 (Approximation Conjecture for topological torsion). *A group G satisfies the Approximation Conjecture for topological L^2 -torsion if for any normal chain $\{G_i\}$ and Riemannian manifold \overline{M} without boundary and with a proper free cocompact isometric G -action*

$$\rho_{\text{top}}^{(2)}(\overline{M}; \mathcal{N}(G)) = \lim_{i \rightarrow \infty} \rho_{\text{top}}^{(2)}(M[i]; \mathcal{N}(G/G_i))$$

The next result is proved in [88, Theorem 15.6 on page 310].

THEOREM 7.6. *Suppose that G satisfies the Approximation Conjecture 7.3 for Fuglede-Kadison determinants.*

Let \overline{M} be a Riemannian manifold without boundary that comes with a proper free cocompact isometric G -action. Suppose that $b_n^{(2)}(\overline{M}; \mathcal{N}(G)) = 0$ holds for all $n \geq 0$.

Then

$$\rho_{\text{an}}^{(2)}(\overline{M}; \mathcal{N}(G)) = \lim_{i \in I} \rho_{\text{an}}^{(2)}(M[i]; \mathcal{N}(G/G_i)).$$

REMARK 7.7 (Relating the Approximation Conjectures for Fuglede-Kadison determinant and torsion invariants). It is conceivable that Theorem 7.6 is still true, if we drop the assumption that $b_n^{(2)}(\overline{M}; \mathcal{N}(G))$ vanishes for all $n \geq 0$, but our present proof works only under this assumption, see [88, Remark 16.2 on page 315]. If we can drop this assumption in Theorem 7.6, then Theorem 7.6 just boils down to the statement that G satisfies the equivalent Conjectures 7.4 and 7.5 about L^2 -torsion, provided that G satisfies the Approximation Conjecture 7.3 for Fuglede-Kadison determinants.

REMARK 7.8 (Strategy of proof of Conjecture 7.3). A strategy to prove the Approximation Conjecture 7.3 for Fuglede-Kadison determinants is described in [88, Theorem 17.1 on page 316]. The key problem is isolated in the *uniform integrability condition* appearing in [88, Theorem 17.1 (v)]. Why it does not follow from known facts about spectral density functions is explained in [88, Lemma 17.2 on page 318], whereas in [88, Remark 17.3 on page 321] it is discussed why there is some hope that it may hold in many situations. In [88, Theorem 17.6 on page 322] a sufficient condition for the validity of the uniform integrability condition is described.

REMARK 7.9 ((Very poor) status of Conjecture 7.3). Unfortunately, there is no infinite group G besides infinite virtually cyclic groups for which the Approximation Conjecture 7.3 for Fuglede-Kadison determinants is known to be true. It holds for finite groups G for trivial reasons. For $G = \mathbb{Z}$ (and hence for every infinite virtually cyclic group) Conjecture 7.3 has been proven by Schmidt [116], see also [85, Lemma 13.53 on page 478] and [89, Lemma 7.25]. Note that the proof of this very special case already requires some input from the theory of diophantine equations, namely the estimate [85, 13.65] taken from [121, Corollary B1 on page 30].

We conclude from [88, Theorem 15.7] that in the situation of the Approximation Conjecture for Fuglede-Kadison determinants 7.3 we always have the inequality

$$(7.4) \quad \dim_{\mathcal{N}(G)}(\ker(r_A^{(2)}: L^2(G)^r \rightarrow L^2(G)^s)) \\ \geq \limsup_{i \rightarrow \infty} \dim_{\mathcal{N}(G/G_i)}(\ker(r_{A[i]}^{(2)}: L^2(G/G_i)^r \rightarrow L^2(G/G_i)^s))$$

provided that each quotient G/G_i satisfies the Determinant Conjecture 5.1. If $G = \mathbb{Z}^n$ and each G/G_i is finite, then the inequality (7.4) for the limit superior is known to be an equality by Lê [69, Theorem 3], see also Raimbault [105].

REMARK 7.10. The relationship between Conjecture 6.11 about homological torsion growth and L^2 -torsion and the Approximation Conjecture 7.5 for finite index normal chains is described in terms of regulators in [88, Sections 8 – 10].

7.4. More general setups. Some of the approximation conjectures above can also be formulated in more generality or a slightly different context. One can replace the normal chain $\{G_i\}$ indexed by $i \in \{0, 1, 2, \dots\}$ by an inverse system $\{G_i \mid i \in I\}$ of normal subgroups of G directed by inclusion over the directed set I such that $\bigcap_{i \in I} G_i = \{1\}$, see [88, Section 13]. One may consider so called *Farber sequences*, where the subgroups G_i are not necessarily normal but normal in an asymptotic sense, see [2, Section 10]. There is the notion of Benjamini-Schramm convergence, see [1, Definition 1.1]. Or one can consider convergence in the space or marked groups, see [56, Section 1.3].

8. L^2 -invariants and simplicial volume

The simplicial volume of a manifold is a topological variant of the (Riemannian) volume which agrees with it for hyperbolic manifolds up to a dimension constant and was introduced by Gromov [42].

CONJECTURE 8.1 (Simplicial volume and L^2 -invariants). *Let M be an aspherical closed orientable manifold of dimension ≥ 1 . Suppose that its simplicial volume $\|M\|$ vanishes. Then \widetilde{M} is of determinant class and*

$$b_p^{(2)}(\widetilde{M}) = 0 \quad \text{for } p \geq 0; \\ \rho^{(2)}(\widetilde{M}) = 0.$$

Gromov first asked in [46, Section 8A on page 232] whether under the conditions in Conjecture 8.1 the Euler characteristic of M vanishes, and notes that in all available examples even the L^2 -Betti numbers of M vanish. The part about L^2 -torsion appears in [82, Conjecture 3.2].

Conjecture 8.1 is discussed in detail in [85, Chapter 14]. No essential breakthrough has been made since then, but there are some interesting papers connected to this problem, such as [36, 110, 111]. So far, all evidence for Conjecture 8.1 has

been computational, but there is no structural reason known for it. It is intriguing, since it relates rather different invariants to one another and is one of the typical conjectures, which only make sense for closed manifolds when they are aspherical.

For an aspherical closed orientable manifold of dimension ≥ 1 its simplicial volume $\|M\|$ vanishes if $\pi_1(M)$ is amenable, see [42, page 40] and [52, Theorem 4.3]. However, it is not known whether $\|M\|$ vanishes if $\pi_1(M)$ contains a normal infinite amenable subgroup or at least an elementary amenable infinite normal subgroup. Recall that $\rho^{(2)}(\widetilde{M})$ vanishes if $\pi_1(M)$ contains a normal infinite elementary amenable subgroup, see (11) in Subsection 2.5, but it is not known whether $\rho^{(2)}(\widetilde{M})$ vanishes if $\pi_1(M)$ contains a normal infinite amenable subgroup.

As an approach to Conjecture 8.1, Gromov suggested in [47] to use a new invariant, the integral foliated simplicial volume, to estimate L^2 -Betti numbers from above. This was carried out in [117, Corollary 5.28]. However, it is still open whether vanishing of simplicial volume implies vanishing of integral foliated simplicial volume (of oriented closed aspherical manifolds).

9. L^2 -invariants of groups

Recall that L^2 -Betti numbers $b_n^{(2)}(G)$ of a group G were defined in (2.6). We call a group G *admissible*, if there exists a finite CW -model BG for its classifying space, we have $b_n^{(2)}(EG; \mathcal{N}(G)) = 0$ for $n \geq 0$, and G satisfies the Determinant Conjecture 5.1. We define the L^2 -torsion of an admissible group G

$$(9.1) \quad \rho^{(2)}(G) = \rho^{(2)}(EG; \mathcal{N}(G)) \in \mathbb{R}$$

by the L^2 -torsion of $EG = \widetilde{BG}$. Since two models for EG are G -homotopy equivalent and G satisfies the Determinant Conjecture 5.1, the real number $\rho^{(2)}(EG; \mathcal{N}(G))$ is well-defined and is independent of the choice of the model for EG . Hence there notion of the L^2 -torsion of an admissible group in (9.1) makes sense.

9.1. Vanishing results. Some vanishing criterions for the L^2 -Betti numbers $b_n^{(2)}(G)$ of a group can be found in [85, Theorem 7.2 on page 294 and Theorem 7.4 on page 295]. For instance, if G contains a normal subgroup H such that $b_n^{(2)}(H) = 0$ for all $n \leq d$ for some fixed natural number d , then $b_n^{(2)}(G) = 0$ for all $n \leq d$. If G contains an amenable infinite normal subgroup, then $b_n^{(2)}(G) = 0$ holds for all $n \geq 0$. If G occurs as an extension $1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$ of infinite groups such that H is finitely generated or, more generally, satisfies $b_1^{(2)}(H) < \infty$, then $b_1^{(2)}(G) = 0$. See [109] and [112] for generalizations of the last assertion. If G is admissible and contains an elementary amenable infinite normal subgroup, then $\rho^{(2)}(G) = 0$ holds, see (11) in Subsection 2.5. A very interesting interaction between the notions of L^2 -torsion and entropy is developed in [70] and used to show that for any admissible group which is amenable, $\rho^{(2)}(G) = 0$ holds, see [70, Theorem 1.3]. It is unknown whether $\rho^{(2)}(G) = 0$ vanishes if G contains an amenable infinite normal subgroup.

REMARK 9.1. A combinatorial computation of L^2 -invariants is described in [85, Section 3.7] Given a matrix $A \in M_{m,n}(\mathbb{C}G)$, one can assign to it its *characteristic sequence* $\{c(A, K)_n\}$ for some large enough number K . The sequence $\{c(A, K)_n\}$ is monotone decreasing sequence of non-negative real numbers that converges to the von Neumann dimension $\dim_{\mathcal{N}(G)}(\ker(R_A))$ of the kernel of the induced $\mathcal{N}(G)$ -homomorphism $R_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n$. There is some control over the speed of

convergence, which is often very fast, actually exponentially in n . Analogously there is a way of computing the Fuglede-Kadison determinant of R_A in terms of the characteristic sequence. Given an algorithm to decide the word problem in G , one obtains an algorithm to compute the characteristic sequence. Unfortunately this algorithm seems to be exponentially running. But at least each $c(A, K)_n$ gives an upper bound for $\dim_{\mathcal{N}(G)}(R_A)$. This may be useful in view of the Atiyah Conjecture 3.1 to prove the vanishing of L^2 -Betti numbers $b_n^{(2)}(\tilde{X})$ for a finite CW -complex X for which there is an upper bound on the orders of the finite subgroups of $\pi_1(X)$. See also Remark 3.7.

9.2. The first L^2 -Betti number and applications to group theory. The vanishing of $b_1^{(2)}(G)$ has some interesting consequences, provided that G is finitely presented. Namely, it implies that the deficiency of G is bounded from above by 1 and that for any oriented closed manifold M of dimension M the inequality $|\text{sign}(M)| \leq \chi(M)$ holds for its signature $\text{sign}(M)$ and its Euler characteristic $\chi(M)$, see [81, Theorems 5.1 and 6.1]. The following result is due to Kielak [61, Theorem 5.3] and generalizes the work of Agol [5]

THEOREM 9.2. *Let G be an infinite finitely generated group which is virtually RFRS, where RFRS stands for residually finite rationally solvable.*

Then G is virtually fibered, in the sense that it admits a finite index subgroup mapping onto Z with a finitely generated kernel, if and only if $b_1^{(2)}(G) = 0$ holds.

There are relations between the non-vanishing of $b_1^{(2)}(G)$ and the question whether the finitely presented group G is *large*, i.e. has a subgroup of finite index which maps surjectively to a non-abelian free group, see for instance [67, Theorem 1.4], [68, Theorem 1.6].

The following questions are taken from [88, Section 4], where also more explanations and references to the literature, such as [3, 4, 19, 30, 38, 39, 40, 66, 92, 102, 115], are given.

QUESTION 9.3 (Rank gradient, cost, and first L^2 Betti number). *Let G be an infinite finitely generated residually finite group. Let $(G_i)_{i \geq 0}$ be a descending chain of normal subgroups of finite index of G with $\bigcap_{i \geq 0} G_i = \{1\}$.*

Do we have

$$b_1^{(2)}(G) = \text{cost}(G) - 1 = \text{RG}(G; (G_i)_{i \geq 0})?$$

QUESTION 9.4 (Rank gradient, cost, first L^2 -Betti number, and approximation). *Let G be a finitely presented infinite residually finite group. Let (G_i) be a descending chain of normal subgroups of finite index of G with $\bigcap_{i \geq 0} G_i = \{1\}$. Let F be any field.*

Do we have

$$\lim_{i \rightarrow \infty} \frac{b_1(G_i; F) - 1}{[G : G_i]} = b_1^{(2)}(G) - b_0^{(2)}(G) = \text{cost}(G) - 1 = \text{RG}(G; (G_i)_{i \geq 0})?$$

Note that a positive answer to the questions above also includes the statement, that $\lim_{i \rightarrow \infty} \frac{b_1(X[i; F])}{[G : G_i]}$ and $\text{RG}(G; (G_i)_{i \geq 0})$ are independent of the chain and the characteristic of F . It is possible that the answer is positive also in the case, where the characteristic of F is not zero, since in Remark 6.6 the counterexamples of Avramidi-Okun-Schreive [13, Theorem 1] do not work in degree $n = 1$.

QUESTION 9.5. *Let G be a finitely generated group. Do we have*

$$b_1^{(2)}(G) = \text{cost}(G) - 1$$

and is the Fixed Price Conjecture true? (The Fixed Price Conjecture predicts that the cost of every standard action of G , i.e., an essentially free G -action on a standard Borel space with G -invariant probability measure, is equal to the cost of G .)

Higher rank versions of the rank gradient are discussed in [88, Section 5].

9.3. The L^2 -torsion and applications to group theory. As we have explained above, L^2 -Betti numbers have been exploited for group theory. We think that there is a lot of potential for the L^2 -torsion of a group G to have striking applications to group theory, and we encourage group theorists to work on these types of questions.

A typical question is the following. If M is a closed hyperbolic manifold of odd dimension, then $\rho^{(2)}(\pi_1(M))$ is a up to a constant depending only on the dimension the volume of M , see [50, 95]. Since $\pi_1(M)$ is a word-hyperbolic group, one may ask what $\rho^{(2)}(G)$ measures for a det- L^2 -acyclic torsionfree word-hyperbolic group.

We want to mention the following invariant of an automorphism $f: BG \rightarrow BG$ of a group G for which there exists a finite model for BG and which satisfies the Determinant Conjecture 5.1. Then $G \rtimes_f \mathbb{Z}$ is admissible, and hence we can define the L^2 -torsion of f by

$$(9.2) \quad \rho^{(2)}(f) := \rho^{(2)}(G \rtimes_f \mathbb{Z}).$$

This invariant has the following properties, see [85, Theorem 7.27 on page 305].

THEOREM 9.6. *Suppose that all groups appearing below have finite CW-models for their classifying spaces and satisfy the Determinant Conjecture 5.1.*

- (1) *Suppose that G is the amalgamated product $G_1 *_{G_0} G_2$ for subgroups $G_i \subset G$ and the automorphism $f: G \rightarrow G$ is the amalgamated product $f_1 *_{f_0} f_2$ for automorphisms $f_i: G_i \rightarrow G_i$. Then*

$$\rho^{(2)}(f) = \rho^{(2)}(f_1) + \rho^{(2)}(f_2) - \rho^{(2)}(f_0);$$

- (2) *Let $f: G \rightarrow H$ and $g: H \rightarrow G$ be isomorphisms of groups. Then*

$$\rho^{(2)}(f \circ g) = \rho^{(2)}(g \circ f).$$

In particular $\rho^{(2)}(f)$ is invariant under conjugation with automorphisms;

- (3) *Suppose that the following diagram of groups*

$$\begin{array}{ccccccc} 1 & \longrightarrow & G_1 & \xrightarrow{i} & G_2 & \xrightarrow{p} & G_3 & \longrightarrow & 1 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow \text{id} & & \\ 1 & \longrightarrow & G_1 & \xrightarrow{i} & G_2 & \xrightarrow{p} & G_3 & \longrightarrow & 1 \end{array}$$

commutes, has exact rows and its vertical arrows are automorphisms. Then

$$\rho^{(2)}(f_2) = \chi(BG_3) \cdot \rho^{(2)}(f_1);$$

- (4) *Let $f: G \rightarrow G$ be an automorphism of a group. Then for all integers $n \geq 1$*

$$\rho^{(2)}(f^n) = n \cdot \rho^{(2)}(f);$$

- (5) *We have $\rho^{(2)}(f) = 0$, if G satisfies one of the following conditions:*

- (a) We have $b_n^{(2)}(G) = 0$ for every $n \geq 0$;
- (b) G contains an amenable infinite normal subgroup.

Let S be a compact connected orientable 2-dimensional manifold, possibly with boundary. Let $f: S \rightarrow S$ be an orientation preserving homeomorphism. The mapping torus T_f is a compact connected orientable 3-manifold, whose boundary is empty or a disjoint union of 2-dimensional tori. Then there is a maximal family of embedded incompressible tori, which are pairwise not isotopic and not boundary parallel, such that it decomposes T_f into pieces that are Seifert or hyperbolic. Let M_1, M_2, \dots, M_r be the hyperbolic pieces. They all have finite volume $\text{vol}(M_i)$. The following result is taken from [85, Theorem 7.28 on page 307]

THEOREM 9.7. *If S is S^2 , D^2 , or T^2 , then $\rho^{(2)}(f) = 0$. Otherwise we get*

$$\rho^{(2)}(\pi_1(f): \pi_1(S) \rightarrow \pi_1(S)) = \frac{-1}{6\pi} \cdot \sum_{i=1}^r \text{vol}(M_i).$$

The combinatorial approach for the computation of $\rho^{(2)}(f)$ in terms of characteristic sequences of Remark 9.1 is described in detail in [85, Subsection 7.34]. The favourite and so far unexplored case is, when G is a finitely generated free group.

QUESTION 9.8. *Let G be finitely generated non-abelian free group.*

- (1) Is $\rho^{(2)}(f) \leq 0$ for any automorphism f of G ;
- (2) What is the structure of the countable set $\{\rho^{(2)}(f)\}$, where f runs through the automorphisms of G ?
- (3) Given a real number $r < 0$, is the set of conjugacy classes of fully irreducible automorphisms f of G with $\rho^{(2)}(f) = r$ finite?

We will discuss measure equivalence in Section 10 and twisted L^2 -torsion in Section 12.

10. Measure equivalence

Gaboriau [39] introduced L^2 -Betti numbers of measured equivalence relations and proved that two measure equivalent countable groups have proportional L^2 -Betti numbers. This notion turned out to have many important applications in recent years, most notably through the work of Popa [104].

The notion of *measure equivalence* was introduced by Gromov [46, 0.5.E].

DEFINITION 10.1. Two countable groups G and H are called *measure equivalent with index $c = I(G, H) > 0$* if there exists a non-trivial standard measure space (Ω, μ) on which $G \times H$ acts such that the restricted actions of $G = G \times \{1\}$ and $H = \{1\} \times H$ have measurable fundamental domains $X \subset \Omega$ and $Y \subset \Omega$, with $\mu(X) < \infty$, $\mu(Y) < \infty$, and $c = \mu(X)/\mu(Y)$. The space (Ω, μ) is called a *measure coupling* between G and H (of index c).

The following conjecture is taken from [94, Conjecture 1.2].

CONJECTURE 10.2. *Let G and H be two admissible groups, which are measure equivalent with index $I(G, H) > 0$. Then*

$$\rho^{(2)}(G) = I(G, H) \cdot \rho^{(2)}(H).$$

Due to Gaboriau [39], the vanishing of the n th L^2 -Betti number $b_n^{(2)}(G)$ is a invariant of the measure equivalence class of a countable group G . If all L^2 -Betti numbers vanish and G is an admissible group, then the vanishing of the L^2 -torsion is a secondary invariant of the measure equivalence class of a countable group G provided that Conjecture 10.2 holds.

Evidence for Conjecture 10.2 comes from [94, Conjecture 1.10] which says that Conjecture 10.2 is true if we replace measure equivalence by the stronger notion of uniform measure equivalence, see [94, Definition 1.3], and assume that G satisfies the Measure Theoretic Determinant Conjecture, see [94, Conjecture 1.7].

11. Zero-in-the-Spectrum-Conjecture

The next conjecture appears for the first time in Gromov's article [43, page 120].

CONJECTURE 11.1 (Zero-in-the-Spectrum Conjecture). *Suppose that \widetilde{M} is the universal covering of the aspherical closed Riemannian manifold M (with the Riemannian metric coming from M). Then for some $p \geq 0$ zero is in the spectrum of the minimal closure*

$$(\Delta_p)_{\min}: \text{dom}((\Delta_p)_{\min}) \subset L^2\Omega^p(\widetilde{M}) \rightarrow L^2\Omega^p(\widetilde{M})$$

of the Laplacian acting on smooth p -forms on \widetilde{M} .

The Zero-in-the-Spectrum Conjecture 11.1 is known to be true if one of the following conditions is satisfied:

- (1) $\dim(M) \leq 3$;
- (2) M is a locally symmetric space;
- (3) M possesses a Riemannian metric whose sectional curvature is non-positive;
- (4) M is an aspherical closed Kähler manifold whose fundamental group is word-hyperbolic in the sense of [44];
- (5) $\pi_1(M)$ satisfies the strong Novikov Conjecture.

If one drops the conditions "aspherical" in the Zero-in-the-Spectrum Conjecture 11.1, then there are counterexamples.

For the proofs of the claim above and the relevant reference, such as [31, 43, 79], and for more information about the Zero-in-the-Spectrum Conjecture 11.1, we refer to [85, Chapter 12].

12. Twisting with finite-dimensional representations

A prominent open problem is whether one can twist L^2 -invariants with (not necessarily unitary) finite-dimensional representations. A basic and systematical treatment of this problem can be found in [89].

For L^2 -Betti numbers there is the conjecture that this just boils down to multiplying the untwisted L^2 -Betti number with the dimension of the representation, see [18, Conjecture 2] and [89, Question 0.1]. This conjecture is proved for sofic groups by Boschheidgen-Jaikin-Zapirian[18, Theorem 1.1] and for locally indicable groups by Kielak-Sun [63, Theorem 4.5].

For L^2 -torsion the effect of the twisting is much more interesting and this leads to new invariants. Especially in dimension 3 there has been made a lot of progress, and interesting open problems occur. For instance, there are interesting relations between the Alexander-torsion function, which is given by twisting the L^2 -torsion with a family of one-dimensional representations associated to an element

in $H^1(M; \mathbb{Z})$, and the Thurston polytope. A prominent open problem is whether the regular Fuglede-Kadison determinant is continuous, see [89, Question 9.11]. We refer for more information and the relevant references, such as [27, 33, 34, 35, 37, 60, 71, 76], to [89, Section 10] and the survey article [90].

13. Group von Neumann algebras over \mathbb{F}_p

Some of the problems discussed above are of the shape that we understand the case of a field of characteristic zero well using the group von Neumann algebra $\mathcal{N}(G)$, but do not know what happens in prime characteristic p . A prominent example are the questions about approximation of L^2 -Betti numbers in prime characteristic, see Subsection 6.5. It seems to be conceivable that the relevant limits exists and are independent of the chains, but the value of the limits depend on whether we work in characteristic zero or in prime characteristic. So the original hope that one always gets as a limit the L^2 -invariants defined in terms of the von Neumann algebra also in prime characteristic turns out not to be fulfilled. So we face the new problem what the limit could be in the prime characteristic case.

This raises the question whether there is an \mathbb{F}_p -analogue of the group von Neumann algebra? In order to treat L^2 -Betti numbers in prime characteristic, one would hope for the existence of a \mathbb{F}_p -algebra $\mathcal{N}(G; p)$ which contains the group ring $\mathbb{F}_p G$ and comes with a dimension function for finitely generated projective $\mathcal{N}(G; p)$ -modules satisfying [85, Assumption 6.2 on page 238]. Then [85, Definition 6.6 and Theorem 6.7 on page 239] would apply, and one would get a dimension function for all $\mathcal{N}(G; p)$ -modules which has many useful features. Finally, one would define the n th L^2 -Betti number in characteristic p of a G -space X to be

$$(13.1) \quad b_n^{(2)}(Y; \mathcal{N}(G; p)) := \dim_{\mathcal{N}(G; p)}(H_n(\mathcal{N}(G; p) \otimes_{\mathbb{F}_p G} C_*^s(Y; \mathbb{F}_p))) \in \mathbb{R}^{\geq 0} \amalg \{\infty\},$$

where $C_*^s(Y; \mathbb{F}_p)$ is the $\mathbb{F}_p G$ -chain complex given by the singular chain complex with coefficient in \mathbb{F}_p . For a group G one would define its n th L^2 -Betti number in characteristic p

$$(13.2) \quad b_n^{(2)}(G; p) = b_n^{(2)}(EG; \mathcal{N}(G; p)).$$

The hope is that then the corresponding sequences of normalized Betti numbers with coefficients in \mathbb{F}_p converge for any normal chain to the L^2 -Betti numbers with coefficients in \mathbb{F}_p , see also Remark 6.7.

14. L^2 -invariants and condensed mathematics

Condensed mathematics, a theory recently developed by Clausen-Scholze [119, 118, 120], is a framework which aims to remedy ill behaviour of the category of topological spaces with respect to algebraic structure, for example the fact that the category of topological abelian groups is not an abelian category. It has proven useful in incorporating geometric and analytic structures that appear in arithmetic, or complex geometry and we are far from understanding its full capabilities. In the context of L^2 -invariants, condensed mathematics poses at least two interesting questions.

- (1) The theory of L^2 -invariants heavily relies on techniques mixing algebra and topology or functional analysis. Can condensed mathematics help to extend the current formalism of L^2 -invariants in a way which sheds light

on open problems in the area? For example, are there analogues of the group von Neumann algebra of a discrete group over \mathbb{F}_p ?

- (2) Can L^2 -invariants be applied to a wider class of geometric problems coming from condensed mathematics?

In this section, we attempt to make a first step towards the second question by defining L^2 -Betti numbers of condensed sets carrying an action of a discrete group in three different ways that extend the current definition for nicely behaved spaces, e.g., CW-complexes:

- (1) homology through solidification;
- (2) condensed cohomology;
- (3) condensed singular homology.

We would hope that more sophisticated setups to associate L^2 -Betti numbers to objects in analytic geometry lead to new interesting invariants. The reader should be aware that we are freely using the language of ∞ -categories.

Let us begin by giving a brief summary of the main definitions from condensed mathematics necessary for this purpose. For more details and proofs, we refer the reader to Peter Scholze's lecture notes [118, 119, 120]. We also ignore set theoretic size issues, which are treated with more care in aforementioned references.

Denote by \mathbf{edCH} the category of extremally disconnected compact Hausdorff spaces (extremally disconnected meaning that the closure of any of its open subsets is again open) and continuous maps. It can be made into a site with coverings given by finite families of jointly surjective maps. For \mathbf{C} a category with finite limits, the category $\mathbf{Cond}(\mathbf{C})$ of condensed objects in \mathbf{C} is defined as the category of sheaves on \mathbf{edCH} with values in \mathbf{C} . It is not hard to check that this identifies with the full subcategory of $\mathbf{Fun}(\mathbf{edCH}^{\text{op}}, \mathbf{C})$ of finite product preserving functors. Explicitly, a condensed object in \mathbf{C} is a functor

$$X : \mathbf{edCH}^{\text{op}} \rightarrow \mathbf{C}$$

which satisfies

- (1) $X(\emptyset)$ is the terminal object in \mathbf{C} ;
- (2) For any two objects T_1 and T_2 in \mathbf{edCH} the natural map

$$X(T_1 \amalg T_2) \rightarrow X(T_1) \times X(T_2)$$

is an isomorphism.

The category $\mathbf{Cond}(\mathbf{Set})$ of condensed sets is an enlargement of the category of compactly generated topological spaces in the following sense:

PROPOSITION 14.1 ([119, Proposition 1.7]). *The functor $\mathbf{Top} \rightarrow \mathbf{Cond}(\mathbf{Set})$, $X \mapsto \underline{X}$ given by the restricted Yoneda embedding admits a left adjoint and is fully faithful when restricted to compactly generated topological spaces.*

The nicely behaved enlargement of topological abelian groups is now given by the category $\mathbf{Cond}(\mathbf{Ab})$ of condensed abelian groups. It is a complete and cocomplete abelian category additionally satisfying Grothendieck's axioms (AB5) and (AB6). We now summarise some additional structure:

- (1) The forgetful functor $\mathbf{Cond}(\mathbf{Ab}) \rightarrow \mathbf{Cond}(\mathbf{Set})$ admits a left adjoint $X \mapsto \mathbb{Z}[X]$.
- (2) The category $\mathbf{Cond}(\mathbf{Ab})$ admits a closed symmetric monoidal structure with $M \otimes N$ given by the sheafification of the presheaf $T \mapsto M(T) \otimes_{\mathbb{Z}} N(T)$.

- (3) There is an equivalence $D(\text{Cond}(\text{Ab})) \simeq \text{Cond}(D(\mathbb{Z}))$ between the derived ∞ -category of condensed abelian groups and condensed objects in the derived ∞ -category of \mathbb{Z} -modules. The category $\text{Cond}(D(\mathbb{Z}))$ is a stable presentably symmetric monoidal ∞ -category with symmetric monoidal structure induced from the one on $D(\mathbb{Z})$. It can also be identified with the left derived tensor product on condensed abelian groups.

We now want to outline approach (1) homology through solidification to define L^2 -Betti numbers of condensed sets. First we recall some background on solid abelian groups from [119, Lecture 5 and 6].

DEFINITION 14.2. For a profinite set T define

$$\mathbb{Z}[T]^{\blacksquare} = \underline{\text{Hom}}_{\text{Cond}(\text{Ab})}(C(T, \mathbb{Z}), \mathbb{Z})$$

where $C(T, \mathbb{Z})$ denotes the space of continuous maps endowed with the compact open topology. It comes with a canonical map $\mathbb{Z}[T] \rightarrow \mathbb{Z}[T]^{\blacksquare}$. Equivalently, if $T = \lim_i T_i$ is given as the cofiltered limit of finite sets, one can identify $\mathbb{Z}[T]^{\blacksquare} \simeq \lim_i \mathbb{Z}[T_i]$ where the limit is formed in $\text{Cond}(\text{Ab})$.

A condensed abelian group M is then called *solid* if for all profinite sets T the induced map

$$\text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[T]^{\blacksquare}, M) \rightarrow \text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[T], M)$$

is an equivalence. Denote by $\text{Solid}(\text{Ab}) \subset \text{Cond}(\text{Ab})$ the full subcategory of solid abelian groups.

Similarly, a derived condensed abelian group $C \in D(\text{Cond}(\text{Ab})) \simeq \text{Cond}(D(\mathbb{Z}))$ is called *solid* if for all profinite sets T the induced map

$$\text{Hom}_{D(\text{Cond}(\text{Ab}))}(\mathbb{Z}[T]^{\blacksquare}, C) \rightarrow \text{Hom}_{D(\text{Cond}(\text{Ab}))}(\mathbb{Z}[T], C)$$

is an equivalence in $D(\mathbb{Z})$. Denote by $\text{Solid}(D(\mathbb{Z})) \subseteq \text{Cond}(D(\mathbb{Z}))$ the full subcategory of solid derived \mathbb{Z} -modules.

The following theorem is the main result from [119, Lecture 5]

THEOREM 14.3. *The full inclusion $\text{Solid}(D(\mathbb{Z})) \subseteq \text{Cond}(D(\mathbb{Z}))$ admits a left adjoint $(-)^{L\blacksquare}$ called the solidification. Furthermore, a condensed derived \mathbb{Z} -module is solid if and only if all of its homology groups are solid abelian groups.*

Now we come to the definition of L^2 -Betti numbers of condensed sets. Let X be a condensed set with G -action, i.e., an element in $\text{Cond}(\text{Set}^{BG}) \simeq \text{Cond}(\text{Set})^{BG}$. Consider the following composition

$$(14.1) \quad \text{Cond}(\text{Set})^{BG} \xrightarrow{\mathbb{Z}[-]} \text{Cond}(D(\mathbb{Z}))^{BG} \xrightarrow{(-)^{L\blacksquare}} \text{Cond}(D(\mathbb{Z}))^{BG} \\ \simeq \text{Cond}(D(\mathbb{Z}))^{BG} \simeq \text{Cond}(D(\mathbb{Z}[G]))$$

and denote the image of X under (14.1) by $\mathbb{Z}[X]^{L\blacksquare}$.

DEFINITION 14.4. Let R be a $\mathbb{Z}[G]$ -algebra and X a condensed set with G -action. We define solid equivariant homology of X with coefficients in R by

$$(14.2) \quad H_n^{G, \blacksquare}(X; R) := \Gamma(H_n(\mathbb{Z}[X]^{L\blacksquare} \otimes_{\mathbb{Z}[G]} R)).$$

Here, $\Gamma: \text{Cond}(\text{Mod}_R) \rightarrow \text{Mod}_R$ is the global sections functor and $- \otimes_{\mathbb{Z}[G]} R: D(\mathbb{Z}[G]) \rightarrow D(R)$ denotes the (derived) base change.

Furthermore, we define the L^2 -Betti numbers of X by

$$(14.3) \quad b_n^{(2)}(X; \mathcal{N}(G)) := \dim_{\mathcal{N}(G)}(H_n^{G, \blacksquare}(X; \mathcal{N}(G))).$$

For free G -CW-complexes, we now compare equivariant homology and L^2 -Betti numbers from Definition 14.4 with the classical ones, keeping Proposition 14.1 in mind. For this we need one further ingredient, giving a description of the singular chain complex of a CW-complex in the condensed world.

PROPOSITION 14.5 ([119, Example 6.5]). *There is an equivalence of the two functors*

$$\begin{aligned} \text{CW} &\xrightarrow{(-)} \text{Cond}(\text{Set}) \xrightarrow{\mathbb{Z}[-]} \text{Cond}(D(\mathbb{Z})) \xrightarrow{(-)^{L, \blacksquare}} \text{Cond}(D(\mathbb{Z})) \text{ and} \\ \text{CW} &\xrightarrow{C_\bullet(-)} D(\mathbb{Z}) \xrightarrow{(-)} \text{Cond}(D(\mathbb{Z})), \end{aligned}$$

where $\text{CW} \subseteq \text{Top}$ denotes the full subcategory of CW-complexes, $C_\bullet(X)$ denotes the singular chain complex of X , and the functor $(-): D(\mathbb{Z}) \rightarrow \text{Cond}(D(\mathbb{Z}))$ sends an object M to the sheafification of the constant presheaf with value M .

We now explain why this generalises the L^2 -Betti number of G -CW-complexes defined in the body of this article.

THEOREM 14.6. *Let X be a G -space which is G -homotopy equivalent to a G -CW-complex and R a $\mathbb{Z}[G]$ -algebra. Then Borel homology of X with coefficients in R and homology of X in the sense of Definition 14.4 are equivalent. In particular, if X is G -homotopy equivalent to a free G -CW-complex, the L^2 -Betti numbers of X in the sense of (2.3) and Definition 14.3 agree.*

PROOF. Consider the diagram

$$(14.4) \quad \begin{array}{ccccccc} \text{CW}^{BG} & \xrightarrow{C_\bullet(-)} & D(\mathbb{Z})^{BG} & \xrightarrow{\simeq} & D(\mathbb{Z}[G]) & \xrightarrow{-\otimes_{\mathbb{Z}[G]} R} & D(R) \\ \downarrow \mathbb{Z}[-] & & \downarrow (-) & & \downarrow (-) & & \downarrow (-) \\ \text{Cond}(D(\mathbb{Z}))^{BG} & \xrightarrow{(-)^{L, \blacksquare}} & \text{Cond}(D(\mathbb{Z}))^{BG} & \xrightarrow{\simeq} & \text{Cond}(D(\mathbb{Z}[G])) & \xrightarrow{-\otimes_{\mathbb{Z}[G]} R} & \text{Cond}(D(R)). \end{array}$$

The left square commutes by Proposition 14.5 and the rest of the diagram commutes for obvious reasons. The composite of the upper horizontal arrows recovers Borel homology of a G -CW-complex. It remains to note that taking global sections $\Gamma: \text{Cond}(D(R)) \rightarrow D(R)$ is left inverse to the functor $(-): D(R) \rightarrow \text{Cond}(D(R))$.

For a free G -CW-complex X , Borel homology of \overline{X} with coefficients in R can be computed as homology of the degreewise tensor product of the singular chain complex of X with R , as used in the definition of L^2 -Betti numbers in [85, Definition 6.50 on page 263].

To extend this result to all G -spaces G -homotopy equivalent to free G -CW-complexes, observe that the functor $\text{Top} \rightarrow \text{Cond}(D(\mathbb{Z})), X \mapsto \mathbb{Z}[\underline{X}]^{L, \blacksquare}$ sends homotopic maps to homotopic maps which formally follows from $\mathbb{Z}[[0, 1]]^{L, \blacksquare} \simeq \mathbb{Z}$. \square

REMARK 14.7 (General actions). Note that the above definition does not recover L^2 -Betti numbers for non-free G -CW-complexes. To remedy this, one can work parametrised over the orbit category $\text{Or}(G)$ instead of BG .

We now briefly explain the two alternative definitions of L^2 -Betti numbers of condensed sets with G -action mentioned in the beginning of Section 14.

REMARK 14.8 (Condensed singular homology). One alternative is to model the construction of singular homology inside condensed sets defined as homology of the simplicial set

$$\mathrm{Hom}_{\mathrm{Cond}(\mathrm{Set})}(\underline{\Delta}^\bullet, X).$$

This in fact agrees with singular homology for all topological spaces since the spaces Δ^n are compactly generated so that the realization of the condensed set $\underline{\Delta}^n$ is Δ^n .

REMARK 14.9 (A cohomological variant). One can also define equivariant cohomology of condensed G -sets analogous to Definition 14.4. For this, one assigns to a condensed set X with G -action the homology of

$$\mathrm{Hom}_{\mathrm{Cond}(D(\mathbb{Z}[G]))}(\mathbb{Z}[X]^{L^\blacksquare}, \underline{R}) \in D(R).$$

Then the same arguments as in Theorem 14.6 show that for free G -CW-complexes this recovers Borel cohomology.

Note that as for any abelian group the induced constant condensed abelian group is solid we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Cond}(D(\mathbb{Z}[G]))}(\mathbb{Z}[X]^{L^\blacksquare}, \underline{R}) &\simeq \mathrm{Hom}_{\mathrm{Cond}(D(\mathbb{Z}))}(\mathbb{Z}[X]^{L^\blacksquare}, \underline{R})^{hG} \\ &\simeq \mathrm{Hom}_{\mathrm{Cond}(D(\mathbb{Z}))}(\mathbb{Z}[X], \underline{R})^{hG} \\ &\simeq \mathrm{Hom}_{\mathrm{Cond}(D(\mathbb{Z}[G]))}(\mathbb{Z}[X], \underline{R}). \end{aligned}$$

Because of that, this version of equivariant condensed cohomology looks most natural to us. From the viewpoint of L^2 -invariants, working with homology instead of cohomology is often easier (for instance homology behaves better with respect to colimits) and chose to present the dual version in more detail.

References

- [1] M. Abert, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov, J. Raimbault, and I. Samet. On the growth of L^2 -invariants for sequences of lattices in Lie groups. *Ann. of Math. (2)*, 185, 2017.
- [2] M. Abert, N. Bergeron, M. Fraczyk, and D. Gaboriau. On homology torsion growth. Preprint, arXiv:2106.13051 [math.GT], 2021.
- [3] M. Abért, A. Jaikin-Zapirain, and N. Nikolov. The rank gradient from a combinatorial viewpoint. *Groups Geom. Dyn.*, 5(2):213–230, 2011.
- [4] M. Abért and N. Nikolov. Rank gradient, cost of groups and the rank versus Heegaard genus problem. *J. Eur. Math. Soc. (JEMS)*, 14(5):1657–1677, 2012.
- [5] I. Agol. Criteria for virtual fibering. *J. Topol.*, 1(2):269–284, 2008.
- [6] M. Albanese, L. Di Cerbo, and L. Lombardi. Aspherical complex surfaces, the Singer Conjecture, and Gromov’s inequality $\chi \geq |\sigma|$. Preprint, arXiv:2311.10226 [math.DG], 2023.
- [7] N. Andrew, Y. Guerch, S. Hughes, and M. Kudlinska. Homology growth of polynomially growing mapping tori. Preprint, arXiv:2305.10410 [math.GR], 2023.
- [8] N. Andrew, S. Hughes, and M. Kudlinska. Torsion homology growth of polynomially growing free-by-cyclic groups. Preprint, arXiv:2211.04389 [math.GR], 2022.
- [9] M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. *Astérisque*, 32-33:43–72, 1976.
- [10] T. Austin. Rational group ring elements with kernels having irrational dimension. *Proc. Lond. Math. Soc. (3)*, 107(6):1424–1448, 2013.
- [11] G. Avramidi. Rational manifold models for duality groups. *Geom. Funct. Anal.*, 28(4):965–994, 2018.
- [12] G. Avramidi, M. W. Davis, B. Okun, and K. Schreve. The action dimension of right-angled Artin groups. *Bull. Lond. Math. Soc.*, 48(1):115–126, 2016.
- [13] G. Avramidi, B. Okun, and K. Schreve. Mod p and torsion homology growth in nonpositive curvature. *Invent. Math.*, 226(3):711–723, 2021.

- [14] W. Ballmann and J. Brüning. On the spectral theory of manifolds with cusps. *J. Math. Pures Appl. (9)*, 80(6):593–625, 2001.
- [15] N. Bergeron, P. Linnell, W. Lück, and R. Sauer. On the growth of Betti numbers in p -adic analytic towers. *Groups Geom. Dyn.*, 8(2):311–329, 2014.
- [16] N. Bergeron and A. Venkatesh. The asymptotic growth of torsion homology for arithmetic groups. *J. Inst. Math. Jussieu*, 12(2):391–447, 2013.
- [17] M. Bestvina, M. Kapovich, and B. Kleiner. Van Kampen’s embedding obstruction for discrete groups. *Invent. Math.*, 150(2):219–235, 2002.
- [18] J. Boschheidgen and A. Jaikin-Zapirain. Twisted L^2 -Betti numbers of sofic groups. Preprint, arXiv:2201.03268 [math.GR], 2022.
- [19] M. R. Bridson and D. H. Kochloukova. Volume gradients and homology in towers of residually-free groups. *Math. Ann.*, 367(3-4):1007–1045, 2017.
- [20] D. Burghelea, L. Friedlander, T. Kappeler, and P. McDonald. Analytic and Reidemeister torsion for representations in finite type Hilbert modules. *Geom. Funct. Anal.*, 6(5):751–859, 1996.
- [21] F. Calegari and M. Emerton. Bounds for multiplicities of unitary representations of cohomological type in spaces of cusp forms. *Ann. of Math. (2)*, 170(3):1437–1446, 2009.
- [22] F. Calegari and M. Emerton. Mod- p cohomology growth in p -adic analytic towers of 3-manifolds. *Groups Geom. Dyn.*, 5(2):355–366, 2011.
- [23] J. Cheeger and M. Gromov. L_2 -cohomology and group cohomology. *Topology*, 25(2):189–215, 1986.
- [24] M. Clay. ℓ^2 -torsion of free-by-cyclic groups. *Q. J. Math.*, 68(2):617–634, 2017.
- [25] M. W. Davis and B. Okun. Vanishing theorems and conjectures for the ℓ^2 -homology of right-angled Coxeter groups. *Geom. Topol.*, 5:7–74 (electronic), 2001.
- [26] J. Dodziuk, P. Linnell, V. Mathai, T. Schick, and S. Yates. Approximating L^2 -invariants and the Atiyah conjecture. *Comm. Pure Appl. Math.*, 56(7):839–873, 2003. Dedicated to the memory of Jürgen K. Moser.
- [27] J. Dubois, S. Friedl, and W. Lück. The L^2 -Alexander torsion of 3-manifolds. *J. Topol.*, 9(3):889–926, 2016.
- [28] G. Elek and E. Szabó. Hyperlinearity, essentially free actions and L^2 -invariants. The sofic property. *Math. Ann.*, 332(2):421–441, 2005.
- [29] G. Elek and E. Szabó. On sofic groups. *J. Group Theory*, 9(2):161–171, 2006.
- [30] M. Ershov and W. Lück. The first l^2 -Betti number and approximation in arbitrary characteristics. *Documenta*, 19:313–331, 2014.
- [31] M. Farber and S. Weinberger. On the zero-in-the-spectrum conjecture. *Ann. of Math. (2)*, 154(1):139–154, 2001.
- [32] D. R. Farkas and P. A. Linnell. Congruence subgroups and the Atiyah conjecture. In *Groups, rings and algebras*, volume 420 of *Contemp. Math.*, pages 89–102. Amer. Math. Soc., Providence, RI, 2006.
- [33] S. Friedl and W. Lück. Universal L^2 -torsion, polytopes and applications to 3-manifolds. *Proc. Lond. Math. Soc. (3)*, 114(6):1114–1151, 2017.
- [34] S. Friedl and W. Lück. L^2 -Euler characteristics and the Thurston norm. *Proc. Lond. Math. Soc. (3)*, 118(4):857–900, 2019.
- [35] S. Friedl and W. Lück. The L^2 -torsion function and the Thurston norm of 3-manifolds. *Comment. Math. Helv.*, 94(1):21–52, 2019.
- [36] R. Frigerio, C. Löh, C. Pagliantini, and R. Sauer. Integral foliated simplicial volume of aspherical manifolds. *Israel J. Math.*, 216(2):707–751, 2016.
- [37] F. Funke and D. Kielak. Alexander and Thurston norms, and the Bieri-Neumann-Strebel invariants for free-by-cyclic groups. *Geom. Topol.*, 22(5):2647–2696, 2018.
- [38] D. Gaboriau. Coût des relations d’équivalence et des groupes. *Invent. Math.*, 139(1):41–98, 2000.
- [39] D. Gaboriau. Invariants l^2 de relations d’équivalence et de groupes. *Publ. Math. Inst. Hautes Études Sci.*, 95:93–150, 2002.
- [40] D. Gaboriau. On orbit equivalence of measure preserving actions. In *Rigidity in dynamics and geometry (Cambridge, 2000)*, pages 167–186. Springer, Berlin, 2002.
- [41] L. Grabowski. On Turing dynamical systems and the Atiyah problem. *Invent. Math.*, 198(1):27–69, 2014.

- [42] M. Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, 56:5–99 (1983), 1982.
- [43] M. Gromov. Large Riemannian manifolds. In *Curvature and topology of Riemannian manifolds (Katata, 1985)*, pages 108–121. Springer-Verlag, Berlin, 1986.
- [44] M. Gromov. Hyperbolic groups. In *Essays in group theory*, pages 75–263. Springer-Verlag, New York, 1987.
- [45] M. Gromov. Kähler hyperbolicity and L_2 -Hodge theory. *J. Differential Geom.*, 33(1):263–292, 1991.
- [46] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.
- [47] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. Birkhäuser Boston Inc., Boston, MA, 1999. Based on the 1981 French original [MR 85e:53051], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by S. M. Bates.
- [48] F. Henneke. Embeddings of group rings and L^2 -invariants. PhD-thesis, Bonn, 2021.
- [49] F. Henneke and D. Kielak. Agrarian and L^2 -invariants. *Fund. Math.*, 255(3):255–287, 2021.
- [50] E. Hess and T. Schick. L^2 -torsion of hyperbolic manifolds. *Manuscripta Math.*, 97(3):329–334, 1998.
- [51] F. Hirzebruch. The signature theorem: reminiscences and recreation. In *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970)*, pages 3–31. Ann. of Math. Studies, No. 70. Princeton Univ. Press, Princeton, N.J., 1971.
- [52] N. Ivanov. Foundations of the theory of bounded cohomology. *J. Soviet Math.*, 37:1090–1114, 1987.
- [53] A. Jaikin-Zapirain. The base change in the Atiyah and the Lück approximation conjectures. *Geom. Funct. Anal.*, 29(2):464–538, 2019.
- [54] A. Jaikin-Zapirain. L^2 -Betti numbers and their analogues in positive characteristic. In *Groups St Andrews 2017 in Birmingham*, volume 455 of *London Math. Soc. Lecture Note Ser.*, pages 346–405. Cambridge Univ. Press, Cambridge, 2019.
- [55] A. Jaikin-Zapirain. The universality of Hughes-free division rings. *Selecta Math. (N.S.)*, 27(4):Paper No. 74, 33, 2021.
- [56] A. Jaikin-Zapirain and D. López-Álvarez. The strong Atiyah and Lück approximation conjectures for one-relator groups. *Math. Ann.*, 376(3-4):1741–1793, 2020.
- [57] J. Jost and Y. L. Xin. Vanishing theorems for L^2 -cohomology groups. *J. Reine Angew. Math.*, 525:95–112, 2000.
- [58] H. Kammeyer. *Introduction to ℓ^2 -invariants*, volume 2247 of *Lecture Notes in Mathematics*. Springer, Cham, 2019.
- [59] H. Kammeyer and S. Kionke. On the profinite rigidity of lattices in higher rank Lie groups. *Math. Proc. Cambridge Philos. Soc.*, 174(2):369–384, 2023.
- [60] D. Kielak. The Bieri-Neumann-Strebel invariants via Newton polytopes. *Invent. Math.*, 219(3):1009–1068, 2020.
- [61] D. Kielak. Residually finite rationally solvable groups and virtual fibring. *J. Amer. Math. Soc.*, 33(2):451–486, 2020.
- [62] D. Kielak and M. Linton. The Atiyah conjecture for three-manifold groups. Preprint, arXiv:2303.15907 [math.GT], 2023.
- [63] D. Kielak and B. Sun. Agrarian and L^2 -Betti numbers of locally indicable groups, with a twist. Preprint, arXiv:2112.07394 [math.GT], 2021.
- [64] B. Kleiner and J. Lott. Notes on Perelman’s papers. *Geom. Topol.*, 12(5):2587–2855, 2008.
- [65] A. Knebusch, P. Linnell, and T. Schick. On the center-valued Atiyah conjecture for L^2 -Betti numbers. *Doc. Math.*, 22:659–677, 2017.
- [66] M. Lackenby. Expanders, rank and graphs of groups. *Israel J. Math.*, 146:357–370, 2005.
- [67] M. Lackenby. Large groups, property (τ) and the homology growth of subgroups. *Math. Proc. Cambridge Philos. Soc.*, 146(3):625–648, 2009.
- [68] M. Lackenby. Detecting large groups. *J. Algebra*, 324(10):2636–2657, 2010.
- [69] T. Le. Homology torsion growth and Mahler measure. *Comment. Math. Helv.*, 89(3):719–757, 2014.
- [70] H. Li and A. Thom. Entropy, determinants, and L^2 -torsion. *J. Amer. Math. Soc.*, 27(1):239–292, 2014.
- [71] P. Linnell and W. Lück. Localization, Whitehead groups and the Atiyah conjecture. *Ann. K-Theory*, 3(1):33–53, 2018.

- [72] P. Linnell, W. Lück, and R. Sauer. The limit of \mathbb{F}_p -Betti numbers of a tower of finite covers with amenable fundamental groups. *Proc. Amer. Math. Soc.*, 139(2):421–434, 2011.
- [73] P. Linnell, B. Okun, and T. Schick. The strong Atiyah conjecture for right-angled Artin and Coxeter groups. *Geom. Dedicata*, 158:261–266, 2012.
- [74] P. Linnell and T. Schick. Finite group extensions and the Atiyah conjecture. *J. Amer. Math. Soc.*, 20(4):1003–1051 (electronic), 2007.
- [75] P. A. Linnell. Division rings and group von Neumann algebras. *Forum Math.*, 5(6):561–576, 1993.
- [76] Y. Liu. Degree of L^2 -Alexander torsion for 3-manifolds. *Invent. Math.*, 207(3):981–1030, 2017.
- [77] C. Löh and M. Uschold. L^2 -Betti numbers and computability of reals. Preprint, arXiv:2202.03159 [math.GR], 2022.
- [78] J. Lott. Heat kernels on covering spaces and topological invariants. *J. Differential Geom.*, 35(2):471–510, 1992.
- [79] J. Lott. The zero-in-the-spectrum question. *Enseign. Math. (2)*, 42(3-4):341–376, 1996.
- [80] W. Lück. Approximating L^2 -invariants by their finite-dimensional analogues. *Geom. Funct. Anal.*, 4(4):455–481, 1994.
- [81] W. Lück. L^2 -Betti numbers of mapping tori and groups. *Topology*, 33(2):203–214, 1994.
- [82] W. Lück. L^2 -torsion and 3-manifolds. In *Low-dimensional topology (Knoxville, TN, 1992)*, pages 75–107. Internat. Press, Cambridge, MA, 1994.
- [83] W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and L^2 -Betti numbers. I. Foundations. *J. Reine Angew. Math.*, 495:135–162, 1998.
- [84] W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and L^2 -Betti numbers. II. Applications to Grothendieck groups, L^2 -Euler characteristics and Burnside groups. *J. Reine Angew. Math.*, 496:213–236, 1998.
- [85] W. Lück. *L^2 -Invariants: Theory and Applications to Geometry and K -Theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.
- [86] W. Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005.
- [87] W. Lück. Approximating L^2 -invariants and homology growth. *Geom. Funct. Anal.*, 23(2):622–663, 2013.
- [88] W. Lück. Approximating L^2 -invariants by their classical counterparts. *EMS Surv. Math. Sci.*, 3(2):269–344, 2016.
- [89] W. Lück. Twisting l^2 -invariants with finite-dimensional representations. *J. Topol. Anal.*, 10(4):723–816, 2018.
- [90] W. Lück. Survey on L^2 -invariants and 3-manifolds. *Bull. Lond. Math. Soc.*, 53(6):1583–1620, 2021.
- [91] W. Lück. Isomorphism Conjectures in K - and L -theory. in preparation, see <http://www.him.uni-bonn.de/lueck/data/ic.pdf>, 2024.
- [92] W. Lück and D. Osin. Approximating the first L^2 -Betti number of residually finite groups. *J. Topol. Anal.*, 3(2):153–160, 2011.
- [93] W. Lück and M. Rothenberg. Reidemeister torsion and the K -theory of von Neumann algebras. *K-Theory*, 5(3):213–264, 1991.
- [94] W. Lück, R. Sauer, and C. Wegner. L^2 -torsion, the measure-theoretic determinant conjecture, and uniform measure equivalence. *Journal of Topology and Analysis*, 2 (2):145–171, 2010.
- [95] W. Lück and T. Schick. L^2 -torsion of hyperbolic manifolds of finite volume. *GAF*, 9:518–567, 1999.
- [96] W. Lück and T. Schick. Various L^2 -signatures and a topological L^2 -signature theorem. In *High-dimensional manifold topology*, pages 362–399. World Sci. Publishing, River Edge, NJ, 2003.
- [97] W. Lück and T. Schick. Approximating L^2 -signatures by their compact analogues. *Forum Math.*, 17(1):31–65, 2005.
- [98] V. Mathai. L^2 -analytic torsion. *J. Funct. Anal.*, 107(2):369–386, 1992.

- [99] J. Morgan and G. Tian. *The geometrization conjecture*, volume 5 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2014.
- [100] B. Okun and K. Schreve. The L^2 -(co)homology of groups with hierarchies. *Algebr. Geom. Topol.*, 16(5):2549–2569, 2016.
- [101] M. Olbrich. L^2 -invariants of locally symmetric spaces. *Doc. Math.*, 7:219–237, 2002.
- [102] D. Osin. Rank gradient and torsion groups. *Bull. Lond. Math. Soc.*, 43(1):10–16, 2011.
- [103] V. G. Pestov. Hyperlinear and sofic groups: a brief guide. *Bull. Symbolic Logic*, 14(4):449–480, 2008.
- [104] S. Popa. Deformation and rigidity for group actions and von Neumann algebras. In *International Congress of Mathematicians. Vol. I*, pages 445–477. Eur. Math. Soc., Zürich, 2007.
- [105] J. Raimbault. Exponential growth of torsion in abelian coverings. *Algebr. Geom. Topol.*, 12(3):1331–1372, 2012.
- [106] A. A. Ranicki. *Algebraic L-theory and topological manifolds*. Cambridge University Press, Cambridge, 1992.
- [107] D. B. Ray and I. M. Singer. R -torsion and the Laplacian on Riemannian manifolds. *Advances in Math.*, 7:145–210, 1971.
- [108] H. Reich. L^2 -Betti numbers, isomorphism conjectures and noncommutative localization. In *Non-commutative localization in algebra and topology*, volume 330 of *London Math. Soc. Lecture Note Ser.*, pages 103–142. Cambridge Univ. Press, Cambridge, 2006.
- [109] P. Sánchez-Peralta. On vanishing criteria of L^2 -Betti numbers of groups. Preprint, arXiv:2307.07031 [math.GR], 2023.
- [110] R. Sauer. Amenable covers, volume and L^2 -Betti numbers of aspherical manifolds. *J. Reine Angew. Math.*, 636:47–92, 2009.
- [111] R. Sauer. Volume and homology growth of aspherical manifolds. *Geom. Topol.*, 20(2):1035–1059, 2016.
- [112] R. Sauer and A. Thom. A spectral sequence to compute L^2 -Betti numbers of groups and groupoids. *J. Lond. Math. Soc. (2)*, 81(3):747–773, 2010.
- [113] J. A. Schafer. Topological Pontrjagin classes. *Comment. Math. Helv.*, 45:315–332, 1970.
- [114] T. Schick. L^2 -determinant class and approximation of L^2 -Betti numbers. *Trans. Amer. Math. Soc.*, 353(8):3247–3265 (electronic), 2001.
- [115] J.-C. Schlage-Puchta. A p -group with positive rank gradient. *J. Group Theory*, 15(2):261–270, 2012.
- [116] K. Schmidt. *Dynamical systems of algebraic origin*, volume 128 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1995.
- [117] M. Schmidt. *L^2 -Betti Numbers of \mathcal{R} -Spaces and the Integral Foliated Simplicial Volume*. PhD thesis, WWU Münster, 2005. Available online at <http://nbn-resolving.de/urn:nbn:de:hbz:6-05699458563>.
- [118] P. Scholze and D. Clausen. Lectures on analytic geometry. lecture notes URL: <https://people.mpim-bonn.mpg.de/scholze/Analytic.pdf>, 2019.
- [119] P. Scholze and D. Clausen. Lectures on condensed mathematics. lecture notes URL: <https://people.mpim-bonn.mpg.de/scholze/Condensed.pdf>, 2019.
- [120] P. Scholze and D. Clausen. Condensed mathematics and complex geometry. lecture notes URL: <https://people.mpim-bonn.mpg.de/scholze/Complex.pdf>, 2022.
- [121] T. N. Shorey and R. Tijdeman. *Exponential Diophantine equations*. Cambridge University Press, Cambridge, 1986.
- [122] M. Uschold. Torsion homology growth and cheap rebuilding of inner-amenable groups. Preprint, arXiv:2212.07916 [math.GR], 2022.
- [123] C. T. C. Wall. Poincaré complexes. I. *Ann. of Math. (2)*, 86:213–245, 1967.

Email address: kirstein@math.uni-bonn.de

URL: <https://dkirstein.github.io>

Email address: kremer@mpim-bonn.mpg.de

URL: <https://sites.google.com/view/christian-kremer-math>

Email address: wolfgang.lueck@him.uni-bonn.de

URL: <http://www.him.uni-bonn.de/lueck>

MATHEMATICAL INSTITUTE OF THE UNIVERSITY OF BONN, ENDENICHER ALLEE 60, 53115
BONN, GERMANY