ALGEBRAIC K-THEORY OF REDUCTIVE p-ADIC GROUPS

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Abstract. Motivated by the Farrell–Jones Conjecture for group rings, we formulate the $\mathcal{C}$op-Farrell–Jones Conjecture for the K-theory of Hecke algebras of td-groups. We prove this conjecture for (closed subgroups of) reductive p-adic groups G. In particular, the projective class group $\mathrm{K}_0(\mathcal{H}(G))$ for a (closed subgroup) of a reductive p-adic group G can be computed as a colimit of projective class groups $\mathrm{K}_0(\mathcal{H}(U))$ where U varies over the compact open subgroups of G. This implies that all finitely generated smooth complex representations of a reductive p-adic G admit finite projective resolutions by compactly induced representations. For $\mathrm{SL}_n(F)$ we translate the colimit formula for $\mathrm{K}_0(\mathcal{H}(G))$ to a more concrete cokernel description in terms of stabilizers for the action on the Bruhat-Tits building.

For negative K-theory we obtain vanishing results, while we identify the higher K-groups $\mathrm{K}_n(\mathcal{H}(G))$ with the value of G-homology theory on the extended Bruhat-Tits building. Our considerations apply to general Hecke algebras of the form $\mathcal{H}(G;R,\rho,\omega)$, where we allow a central character ω and a twist by an action ρ of G on R. For the $\mathcal{C}$op-Farrell–Jones Conjecture we need to assume $\mathbb{Q} \subseteq R$ and a regularity assumption. As a key intermediate step we introduce the $\mathcal{C}$vcy-Farrell–Jones conjecture. For the latter no regularity assumptions on R are needed.

1. Introduction

The Farrell–Jones conjecture [27] originated in surgery theory and has applications to the classification of manifolds, notably it implies (in dimension ≥ 5) Borel's conjecture on the topological rigidity of aspherical manifolds. The conjecture concerns the K- and L-groups of group rings and expresses these in terms of an equivariant homology theory. It can be viewed as reducing computations to the case of group rings for virtually cyclic groups. Under regularity assumptions there are often further reductions, typically to group rings of finite groups. Further information on the conjecture can be found for instance in [39, 40]. Farrell and Jones used the geodesic flow on non-positively curved manifolds as a tool to confirm their conjecture for fundamental groups of such manifolds [27].

In this paper we study the K-theory of Hecke algebras of td-groups and transfer the Farrell–Jones conjecture and the geodesic flow method to smooth representation theory. We obtain formulas for the K-theory of Hecke algebras $\mathcal{H}(G;R)$ where G is a closed subgroup of a reductive p-adic group and R is a field of characteristic 0[1]. These express the K-theory of $\mathcal{H}(G;R)$ as G-homology groups of the associated Bruhat-Tits building, see Corollary 1.8. On the level of K_0 this yields isomorphisms

$$\operatorname*{colim}_{U \in \mathtt{Sub}_{\mathcal{C}\mathrm{op}}(G)} \mathrm{K}_0(\mathcal{H}(U;R)) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{H}(G;R))$$

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[1]It suffices for R to contain $\mathbb{Q}$ and to satisfy a regularity assumption.

where the colimit is taken over a category of compact open subgroups of G, see Corollary 1.2. This confirms in particular a conjecture of Dat [21, Conj. 1.11]. For finitely generated smooth representations it implies the existence of finite length resolutions by compactly induced representations, generalizing a result of Schneider–Stuhler for admissable representations, see Subsection 1.D. A long standing conjecture in smooth representation theory asks whether all irreducible cuspidal representations of reductive p-adic groups G are compactly induced[2]. Our results imply that the K_0-classes of finitely generated representations can be expressed in terms of compact induction.

We proceed to explain our results in more detail.

1.A. **Hecke algebras and p-adic groups.** Let G be a *td-group*, i.e., a locally compact second countable totally disconnected topological Hausdorff group. In such a group the neutral element e has a countable neighborhood basis consisting of compact open subgroups. Let R be a not necessarily commutative ring with unit containing $\mathbb{Q}$. The *Hecke algebra* $\mathcal{H}(G;R)$ of G over R is the algebra of locally constant compactly supported R-valued functions on G. Its multiplication is given by convolution[3] relative to a $\mathbb{Q}$-valued left-invariant Haar measure on G[4]. There are more general Hecke algebras $\mathcal{H}(G;R,\rho,\omega)$ allowing for twists ρ by an action of G on R and a central character ω. Hecke algebras are in general not unital. A module M over the Hecke algebra is *non-degenerate*, if $\mathcal{H}(G;R)\cdot M = M$. A representation of G on an R-module V is said to be *smooth*, if all isotropy groups of the action of G on V are open. The category of non-degenerated $\mathcal{H}(G;R)$-modules is equivalent to the category of smooth representations on R-modules, see [29, Sec. 9]. By a *reductive p-adic group* we will mean the F-points of an algebraic group over F, whose component of the identity is reductive, where F is a non-Archimedian local field, i.e., a finite extension of the field of p-adic numbers or the field of formal Laurent series $k((t))$ over a finite field k. Reductive p-adic groups are td-groups.

Associated to a reductive p-adic group is its extended *Bruhat-Tits building* X [15, 16, 51]. This is a CAT(0)-space with a cocompact proper isometric G-action. The building can also be given the structure of a simplicial complex such that the action of X is simplicial and smooth. For a short review of the Bruhat-Tits building, emphasizing the aspects we need, see [6, Appendix A].

1.B. **Compact induction.** The *compact induction* of a smooth representation V of a compact open subgroup U of G is the G-representation consisting of compactly supported U-equivariant maps $G \to V$[5]. On the level of Hecke algebras compact induction is induced by the inclusion $\mathcal{H}(U;R) \subseteq \mathcal{H}(G;R)$. This inclusion exists for open subgroups U of G; locally constant functions on open subgroups can be extended by zero. Smooth representations of a reductive p-adic group G are often studied through compact induction. For example, type theory, introduced by Bushnell-Kutzko [17], aims at describing Bernstein blocks in the representation category as modules over endomorphism rings of representations that are compactly induced. Conjecturally, all irreducible cuspidal representations are induced from compact modulo center open subgroups. See Fintzen [28] for recent far reaching results concerning these conjectures.

[2] If G has non-trivial center, then one needs to consider open subgroups that are compact modulo center. There are versions of our results in this situation as well, see Corollary 1.18.

[3] $\varphi * \varphi'(g) = \int_G \varphi(gx)\varphi'(x^{-1})dx$.

[4] If μ an $\mathbb{R}$-valued Haar measure and K is compact open in G, then $\frac{\mu}{\mu(K)}$ is $\mathbb{Q}$-valued; the choice of Haar measure changes the Hecke algebra only by canonical isomorphism.

[5] The formula for the G-action is $(gf)(x) := f(xg)$ for $f\colon G \to V$, $g \in G$.

Following Dat [20] we study the K-theory of Hecke algebras (equivalently, of smooth representations) via compact induction. While this leads to less explicit results about smooth representations, it allows for very general results. Ultimately we will describe the K-theory of Hecke algebras of reductive p-adic groups in terms of the K-theory of Hecke algebras of compact open subgroups. We hope that the connection to the above mentioned conjectures can be explored in the future.

1.C. K_0 **of Hecke algebras.** Let $\mathcal{H}(G;R)$ be the Hecke algebra of a td-group G with coefficients in R. The projective class group $\mathrm{K}_0(\mathcal{H}(G;R))$ is the abelian group with a generator $[P]$ for each finitely generated projective $\mathcal{H}(G;R)$-module subject to the relation $[P \oplus P'] = [P] \oplus [P']$. Compact induction preserves finitely generated projective modules and induces a map on K_0. Combining these maps for all compact open subgroups of G we obtain

$$\operatorname*{colim}_{U \in \mathsf{Sub}_{\mathcal{C}\mathrm{op}}(G)} \mathrm{K}_0(\mathcal{H}(U;R)) \to \mathrm{K}_0(\mathcal{H}(G;R)) \tag{1.1}$$

where $\mathsf{Sub}_{\mathcal{C}\mathrm{op}}(G)$ is the following category. Objects are compact open subgroups of G. Morphisms $U \to U'$ are equivalence classes of group homomorphisms of the form $x \mapsto gxg^{-1}$ with $g \in G$. Two such group homomorphisms are identified if they differ by an inner automorphism of U'[6]. To study surjectivity the colimit in (1.1) can of course be replaced with the sum of the groups $\mathrm{K}_0(\mathcal{H}(U;R))$. Dat [20] has shown that (1.1) is rational surjective for G a reductive p-adic group and $R = \mathbb{C}$. In particular, the cokernel of (1.1) is a torsion group. Dat [21, Conj. 1.11] conjectured that this cokernel is $\widetilde{w}_G$-torsion. Here $\widetilde{w}_G$ is a certain multiple of the order of the Weyl group of G. Dat proved this conjecture for $G = \mathrm{GL}_n(F)$ [21, Prop. 1.13] and asked about integral surjectivity, see the comment following [21, Prop. 1.10]. The following will be a consequence of our main result.

Corollary 1.2. *Assume that G is a modulo a compact subgroup isomorphic to a closed subgroup of a reductive p-adic group. Let R be a ring containing $\mathbb{Q}$. Assume that R is uniformly regular, i.e., R is noetherian and there is l such that every R-module admits a projective resolution of length at most l. Then* (1.1) *is an isomorphism.*

This is a special case of Corollary 1.18 (ii), where we consider more general Hecke algebras, allowing for twists by actions of G on R and central characters.

1.D. **Resolutions of smooth representations.** Let G be a reductive p-adic group G. Bernstein [12] showed that the category of smooth complex representations is noetherian and has finite cohomological dimension. Consequently, any finitely generated smooth complex representation has a finite resolution

$$P_n \to P_{n-1} \to \cdots \to P_0 \to V \tag{1.3}$$

where the P_i are finitely generated projective. A smooth G-representation is said to be *admissible*, if for every compact open subgroup U of G the subspace V^U of U-fixed vectors is finite dimensional. It is called *compactly induced*, if it is for some compact open subgroup $U \subseteq G$ the compact induction of a finitely generated projective U-representation.

Schneider and Stuhler [49] showed that for finitely generated admissible V the P_i in the above resolution can be chosen to be finite direct sums of compactly induced representations. From Corollary 1.2 we obtain a generalization to arbitrary finitely generated V.

[6] In other words, $\mathrm{mor}_{\mathsf{Sub}_{\mathcal{C}\mathrm{op}}}(U,U')$ is the double coset $U' \backslash \{g \in G \mid gUg^{-1} \subseteq U'\}/C_G(U)$ where $C_G(U)$ is the centralizer of U in G.

Corollary 1.4. *Every finitely generated smooth complex representation V of G admits a finite resolution* (1.3) *where the P_i are direct sums of compactly induced representations.*

Proof. Under the equivalence of categories between smooth representations and (non-degenerated) Hecke modules the compactly induced representations correspond to the modules in the image of the induction map

$$\mathcal{H}(U;\mathbb{C})\text{-}\mathsf{Mod} \to \mathcal{H}(G;\mathbb{C})\text{-}\mathsf{Mod}, \quad M \mapsto \mathcal{H}(G;\mathbb{C}) \otimes_{\mathcal{H}(U;\mathbb{C})} M$$

for some compact open subgroup $U \subseteq G$. Let P be a finitely generated projective $\mathcal{H}(G;\mathbb{C})$-module. Corollary 1.2 implies that in K_0 we have $[P] = [W] - [W']$ where both W and W' are sums of compactly induced modules. This means that there is an isomorphism $P \oplus W \oplus Q \cong W' \oplus Q$ for some finitely generated projective $\mathcal{H}(G;\mathbb{C})$-module Q. As any finitely generated projective module is a direct summand of a compactly induced modules,[7] we can stabilize further and then absorb Q into W and W', i.e., we obtain $P \oplus W \cong W'$ with W and W' finite direct sums of compactly induced modules.

We can applying this to the P_i in (1.3). Thus by adding appropriate elementary chain complexes on compactly induced modules to (1.3) we obtain the desired resolution of V. □

1.E. **Smooth G-homology theories.** The *orbit category* has as objects homogeneous G-sets G/V with V closed in G and as morphisms G-maps. The *smooth orbit category* $\mathsf{Or}_{\mathcal{O}\mathrm{p}}(G)$ is the full subcategory on all G/U with U open in G. Let $\mathsf{Spectra}$ be the category of (not necessarily connective) spectra. Associated to a covariant functor $\mathbf{E}\colon \mathsf{Or}_{\mathcal{O}\mathrm{p}}(G) \to \mathsf{Spectra}$ there is a smooth G-homology theory

$$H_*^G(-;\mathbf{E}) \tag{1.5}$$

such that $H_n^G(G/H;\mathbf{E}) = \pi_n(\mathbf{E}(G/H))$ for $n \in \mathbb{Z}$. Here a *smooth G-homology theory* is to be understood in the obvious way: It digests (pairs of) smooth G-CW-complexes, yields an abelian group $H_n(X;\mathbf{E})$ for every $n \in \mathbb{Z}$, and satisfies the expected axioms, namely, functoriality in G-maps, G-homotopy invariance, the long exact sequence of a smooth G-CW-pair, and G-excision. All this is explained in [22]. The point here is that smooth G-CW-complexes are contravariant free $\mathcal{C}$-CW-complexes in the sense of [22, Def. 3.2] for $\mathcal{C} = \mathsf{Or}_{\mathcal{O}\mathrm{p}}(G)$. See also the discussions in [7, Sec. 2.C].

1.F. **The K-theory spectrum of Hecke algebras.** To generalize (1.1) to the K-theory spectrum, we introduce some notation. A *category with G-support* is a $\mathbb{Z}$-linear category $\mathcal{B}$ together with maps supp_G that associate to objects and morphisms compact subsets of G subject to a natural list of axioms, see Definition 3.1. Given a G-set X and such a $\mathcal{B}$, we naturally obtain a $\mathbb{Z}$-linear category $\mathcal{B}[X]$, see Definition 3.5. The key example associated to the Hecke algebra $\mathcal{H}(G;R)$ is the category $\mathcal{B}(G;R)$, see Example 3.3. Its objects are compact open subgroups $U \subseteq G$. Morphisms $U \to U'$ are elements f of $\mathcal{H}(G;R)$ satisfying $f = e_{U'} f e_U$, where e_U is the idempotent in $\mathcal{H}(G;R)$ associated to the compact open subgroup U. Here $\mathrm{supp}_G(f) = \{g \in G \mid f(g) \neq 0\}$, which is automatically compact as f is compactly supported and locally constant. We define

$$\mathbf{K}_R\colon \mathsf{Or}_{\mathcal{O}\mathrm{p}}(G) \to \mathsf{Spectra}, \quad G/U \mapsto \mathbf{K}\big(\mathcal{B}(G;R)[G/U]\big), \tag{1.6}$$

where $\mathbf{K}$ is the K-theory functor for $\mathbb{Z}$-linear categories, see Subsection 2.D. The homotopy groups of $\mathbf{K}_R(G/U)$ are the K-groups of the Hecke algebra $\mathcal{H}(U;R)$,

[7] If $v_1, \dots, v_n$ generates P and U fixes the v_i, then P is a direct summand of $\mathcal{H}(G;\mathbb{C}) \otimes_{\mathcal{H}(U;\mathbb{C})} \mathbb{C}^n$.

see [7, (6.8)]. As discussed in Subsection 1.E we can apply [22] and obtain a smooth G-homology theory $H_n^G(-;\mathbf{K}_R)$ with $H_n^G(G/U;\mathbf{K}_R) \cong \mathrm{K}_n(\mathcal{H}(U;R))$.

Associated to the family $\mathcal{C}$op of compact open subgroups, there is a G-CW-complex $E_{\mathcal{C}\mathrm{op}}(G)$ that is uniquely determined up to G-homotopy by the property that all its isotropy groups belong to $\mathcal{C}$op and $E_{\mathcal{C}\mathrm{op}}(G)^H$ is weakly contractible for $H \in \mathcal{C}$op. In particular $E_{\mathcal{C}\mathrm{op}}(G)$ is a proper smooth G-CW-complex. Every G-CW-complex X, whose isotropy belongs to $\mathcal{C}$op, has up to G-homotopy precisely one G-map to $E_{\mathcal{C}\mathrm{op}}(G)$, see [38, Subsec. 1.2]. Analogously one can define for the family $\mathcal{C}$om of all compact subgroups its classifying space $E_{\mathcal{C}\mathrm{om}}(G)$. It turns out that the canonical G-map $E_{\mathcal{C}\mathrm{op}}(G) \to E_{\mathcal{C}\mathrm{om}}(G)$ is a G-homotopy equivalence for a td-group G, see [38, Lemma 3.5].

The projection $E_{\mathcal{C}\mathrm{op}}(G) \to G/G$ induces a map

$$H_n^G(E_{\mathcal{C}\mathrm{op}}(G);\mathbf{K}_R) \to H_n^G(G/G;\mathbf{K}_R) = K_n\big(\mathcal{H}(G;R)\big). \tag{1.7}$$

If R is a regular ring containing $\mathbb{Q}$, then there is an isomorphism $H_0^G(E_{\mathcal{C}\mathrm{op}}(G);\mathbf{K}_R) \cong \operatorname{colim}_{U \in \mathsf{Sub}_{\mathcal{C}\mathrm{op}}(G)} \mathrm{K}_0(\mathcal{H}(U;R))$. Using this isomorphism (1.1) can be identified with (1.7) for $n = 0$, see [8, Thm 1.1 (iii))].

We note that if G is a reductive p-adic group, then we can take for $E_{\mathcal{C}\mathrm{op}}(G)$ the extended Bruhat-Tits building associated to G [38, Thm. 4.13][8]. The following will be a consequence of our main result.

Corollary 1.8. *Assume that G is a modulo a compact subgroup isomorphic to a closed subgroup of a reductive p-adic group. Let R be a ring containing $\mathbb{Q}$. Assume that R is uniformly[9] regular, i.e., R is noetherian and there is l such that every R-module admits a projective resolution of length at most l. Then* (1.7) *is an isomorphism.*

This is a special case of Corollary 1.18 (i), where we consider more general Hecke algebras, allowing for twists by actions of G on R and central characters. Conjecture 1.8 was stated in [40, Conjecture 119 on page 773] for $R = \mathbb{C}$.

1.G. **Vanishing of negative K-theory.** Bernstein's results from [12] which we briefly recalled in Subsection 1.D, also imply for a reductive p-adic group G that $\mathrm{K}_n(\mathcal{H}(G,\mathbb{C})) = 0$ holds for $n \leq -1$. Under the more general assumptions on G and R from Corollary 1.8 we get $\mathrm{K}_n(\mathcal{H}(G,R)) = 0$ for $n \leq -1$, see Corollary 1.18 (iii).

1.H. **The $\mathcal{C}$op-Farrell–Jones Conjecture.** To formulate our main result we generalize coefficients. For a category $\mathcal{B}$ with G-support we obtain

$$\mathbf{K}_{\mathcal{B}} \colon \mathsf{Or}_{\mathcal{O}\mathrm{p}}(G) \to \mathsf{Spectra}, \quad G/U \mapsto \mathbf{K}\big(\mathcal{B}[G/U]\big).$$

As discussed in Subsection 1.E we can apply [22] and obtain a smooth G-homology theory $H_n^G(-;\mathbf{K}_{\mathcal{B}})$. The projection $E_{\mathcal{C}\mathrm{op}}(G) \to G/G$ induces the $\mathcal{C}$op-assembly map

$$H_n^G(E_{\mathcal{C}\mathrm{op}}(G);\mathbf{K}_{\mathcal{B}}) \to H_n^G(G/G;\mathbf{K}_{\mathcal{B}}) = \mathrm{K}_n(\mathcal{B}). \tag{1.9}$$

We define *Hecke categories with G-support* in Definition 3.2. Essentially, these are categories with G-support satisfying axioms that are modeled on $\mathcal{B}(G;R)$, i.e., on Hecke algebras. In particular, $\mathcal{B}[G/U]$ is then equivalent to the subcategory $\operatorname{res}_G^U \mathcal{B}$ of $\mathcal{B}$ on objects and morphisms with support in U.

Conjecture 1.10 ($\mathcal{C}$op-Farrell–Jones Conjecture)**.** *Let G be a td-group and let $\mathcal{B}$ be a Hecke category with G-support. Assume that $\mathcal{B}$ satisfies (Reg) from Definition 3.11. Then* (1.9) *is an isomorphism for all n.*

[8]More general, if G is a closed subgroup of a reductive p-adic group $\widehat{G}$, then we can use the extended Bruhat-Tits building associated to $\widehat{G}$ with the restricted action.

[9]It is plausible that the result is also true if R is only assumed to be regular, but our proof certainly uses uniform regularity.

The following is our main result.

Theorem 1.11. *Conjecture 1.10 holds for reductive p-adic groups.*

Theorem 1.11 is a direct consequence of the $\mathcal{C}$vcy-Farrell–Jones Conjecture 5.12 for reductive p-adic groups from Theorem 5.15 and the Reduction Theorem 14.1 that reduces the $\mathcal{C}$op-Farrell–Jones Conjecture 1.10 to the $\mathcal{C}$vcy-Farrell–Jones Conjecture 5.12. The proof of Theorem 1.11 seems not to simplify if we only consider K_0; it uses for example localization sequences that combine all K_n.

Remark 1.12. The proof of the $\mathcal{C}$vcy-Farrell–Jones Conjecture 5.12 for reductive p-adic groups uses only their action on its extended Bruhat-Tits building.

Let M be a Coxeter matrix over a finite set I. Let C be a building of type M, in the sense of [23, §3]. Its realization $|C|$ is a CAT(0)-space, see [23, Thm. 11.1]. Let G be a td-group with a cofinite smooth proper action on C. We obtain an induced cocompact smooth proper isometric action on $|C|$. It seems to be reasonable to expect that the $\mathcal{C}$vcy-Farrell–Jones Conjecture 5.12 (and therefore the $\mathcal{C}$op-Farrell–Jones Conjecture 1.10) holds also in this situation. The only input to the proof of the $\mathcal{C}$vcy-Farrell–Jones Conjecture 5.12 for reductive p-adic groups that does not directly generalize to this situation is Theorem D.1. This result relies on a technical assumption for the action of G on the flow space associated to $|C|$, this is [6, Assumption 2.7]. Under this assumption the proof of the $\mathcal{C}$vcy-Farrell–Jones Conjecture 5.12 (and therefore also of the $\mathcal{C}$op-Farrell–Jones Conjecture 1.10) for reductive p-adic groups generalizes directly to G.

Remark 1.13 (Novikov Conjecture)**.** The Novikov Conjecture about the homotopy invariance of higher signatures of closed oriented manifolds with fundamental group Γ is equivalent to the rational injectivity of the L-theoretic assembly map $H_n(B\Gamma; \mathbf{L}_{\mathbb{Z}}) \to L_n(\mathbb{Z}\Gamma)$. Bökstedt-Hsiang-Madsen [14] proved using cyclotomic traces that the K-theoretic analogue $H_n(B\Gamma; \mathbf{K}_{\mathbb{Z}}) \to K_n(\mathbb{Z}\Gamma)$ is rationally split injective, if Γ satisfies some homological finiteness conditions, which are automatically satisfied, provided that $B\Gamma$ has a model of finite type. Mostad shows in his PhD-thesis [44] using the descent method of Carlsson-Pedersen [18] that the assembly map $H_n(B\Gamma; \mathbf{K}_R) \to K_n(R\Gamma)$ is split injective, if R is a ring and Γ is a torsion-free cocompact discrete subgroup of $\mathrm{SL}_n(\mathbb{Q}_p)$. Moreover, the descent method of Carlsson-Pedersen [18] has been used to show the split injectivity of the assembly map $H_n^{\Gamma}(E_{\mathcal{F}\mathrm{in}}(\Gamma); \mathbf{K}_R) \to K_n(R\Gamma)$ for a large class of groups and any ring R, see for instance [32, 33, 47], and also [39, Section 15.6], whereas the rational injectivity of the assembly map $H_n^{\Gamma}(E_{\mathcal{F}\mathrm{in}}(\Gamma); \mathbf{K}_{\mathbb{Z}}) \to K_n(\mathbb{Z}\Gamma)$ has been studied using cyclotomic traces in [41]. It would be interesting to see whether the descent method of Carlsson-Pedersen [18] leads to proofs of the split injectivity of the assembly (1.7) for classes of td-groups.

Remark 1.14. The Baum–Connes Conjecture for reductive p-adic groups has been proven by Lafforgue [37]. See also Baum–Higson–Plymen [11] for p-adic GL_n. This yields an Atiyah-Hirzebruch spectral sequence that computes the topological K-theory of the reduced group C*-algebra, compare Subsection 1.J below. We note that the spectral sequence in this case is not a first quadrant spectral sequence (because the negative topological K-theory does in general not vanish by Bott periodicity). Hence for topological K-theory one does not get formulas such as (1.17) or Corollary 1.2, where the relevant K_0-group is expressed in terms of K_0-group of compact open subgroups.

1.I. **Inheritance.** An advantage of the generalization from $\mathcal{B}(G; R)$ to Hecke categories with G-support is the following result, proven in [7, Theorem 1.5].

Theorem 1.15. *If Conjecture 1.10 holds for a td-group G, then it also holds for all td-groups G' which are modulo a normal compact subgroup isomorph to a closed subgroup of G.*

Thus Conjecture 1.10 holds for groups that are modulo a normal compact subgroup isomorphic to a closed subgroup of a reductive p-adic group. This applies in particular to parabolic subgroups that appear for example in parabolic induction and restriction.

1.J. **The Atiyah-Hirzebruch spectral sequence.** Given any smooth G-homology theory there is a (strongly convergent) *equivariant Atiyah-Hirzebruch spectral sequence*, see [22, Thm 4.7 and Sec. 7] and [8, Thm 2.1]. For $H_n^G(-;\mathbf{K}_R)$ it takes the form

$$E^2_{p,q} = BH^G_p(X; H^G_q(G/-;\mathbf{K}_R)) \implies H^G_{p+q}(X;\mathbf{K}_R). \tag{1.16}$$

The E^2-page is given by Borel homology. If R is regular and contains $\mathbb{Q}$, then the spectral sequence (1.16) is a first quadrant spectral sequence. In particular, $H^G_0(X;\mathbf{K}_R) = BH^G_0(X; H^G_0(G/-;\mathbf{K}_R))$. Thus, if R is uniformly regular and if G satisfies the $\mathcal{C}$op-Farrell–Jones conjecture, then we obtain

$$K_0(\mathcal{H}(G;R)) = BH^G_0(E_{\mathcal{C}\mathrm{op}}(G); H^G_0(G/-;\mathbf{K}_R)). \tag{1.17}$$

This homology group can then be described as the cokernel of a map between sums of K_0 of Hecke algebras of compact open subgroups of G. For a non-Archimedian local field F and $G = \mathrm{SL}_n(F)$, $\mathrm{PGL}_n(F)$ or $\mathrm{GL}_n(F)$ this is worked out in [8, Section 6].

1.K. **Central characters and actions on the coefficients.** Consider an exact sequence of td-groups $1 \to N \to G \xrightarrow{p} Q \to 1$, a unital ring R with $\mathbb{Q} \subseteq R$, a locally constant group homomorphism $\rho\colon Q \to \mathrm{aut}(R)$ to the group of ring automorphisms of R and a so called *normal character* $\omega\colon N \to \mathrm{cent}(R)^\times$, which is a locally constant group homomorphism to the multiplicative group of units of the center of R. We assume that that N is locally central[10] and that ω is G-conjugation invariant[11]. For example, N could be a closed subgroup of the center of G. We also assume that the Q-action on R fixes the image of ω. For example, Q could fix the center of R. In this situation we obtain a Hecke algebra $\mathcal{H}(G;R,\rho,\omega)$, see [5, Sec. 2.B]. Its elements are locally constant functions $s\colon G \to R$ with support compact modulo N satisfying $s(ng) = \omega(n)\cdot s(g)$ for all $n \in N$ and $g \in G$. In the special case that ρ is trivial and $\omega\colon N \to R^\times$ is a central character, i.e., $N \subseteq G$ is central and $\omega\colon N \to \mathrm{cent}(R)^\times$ is a locally constant homomorphism, this is the usual Hecke algebra of G with coefficients in R associated to the central character ω.

Similar to $\mathcal{B}(G;R)$ we obtain a category $\mathcal{B}(G;R,\rho,\omega)$, see [7, Section 6.C.]. The support of elements of $\mathcal{H}(G;R,\rho,\omega)$ in G is compact modulo N. Projecting we obtain compact subsets of Q. In this way $\mathcal{B}(G;R,\rho,\omega)$ can be viewed as a category with Q-support and we obtain

$$\mathbf{K}_{R,\rho,\omega}\colon \mathsf{Or}_{\mathcal{O}\mathrm{p}}(Q) \to \mathsf{Spectra}, \quad Q/U \mapsto \mathbf{K}\big(\mathcal{B}(G;R,\rho,\omega)[Q/U]\big).$$

For U open in Q the homotopy groups of $\mathbf{K}_{R,\rho,\omega}(Q/U)$ are the K-groups of the Hecke algebra associated to $p^{-1}(U)$, i.e., of $\mathcal{H}(p^{-1}(U);R,\rho,\omega)$[12]. We can again apply [22], see Subsection 1.E, and obtain a smooth Q-homology theory H^Q_n(-$-;\mathbf{K}_{R,\rho,\omega})$ with $H^G_n(Q/U;\mathbf{K}_{R,\rho,\omega}) \cong \mathrm{K}_n(\mathcal{H}(p^{-1}(U);R,\rho,\omega))$.

[10] I.e., the centralizer of N in G is an open subgroup of G.

[11] I.e., $\omega(gng^{-1}) = \omega(n)$ for all $g \in G$, $n \in N$.

[12] Strictly speaking we should write $\mathcal{H}(p^{-1}(U);R,\rho_{p^{-1}(U)},\omega)$.

The formulation of the $\mathcal{C}$op-Farrell–Jones conjecture can be applied in this situation as well and we have the following corollary to Theorem 1.11.

Corollary 1.18. *Assume that Q is a modulo a compact subgroup isomorphic to a closed subgroup of a reductive p-adic group and that R is a uniformly regular ring containing $\mathbb{Q}$. Then*

(i) The assembly map induced by the projection $E_{\mathcal{C}\mathrm{op}}(Q) \to Q/Q$

$$H_n^Q(E_{\mathcal{C}\mathrm{op}}(Q); \mathbf{K}_{R,\rho,\omega}) \to H_n^Q(Q/Q; \mathbf{K}_{R,\rho,\omega}) = K_n(\mathcal{H}(G; R, \rho, \omega))$$

is an isomorphism for all n;

(ii) The various inclusions $U \subseteq Q$ induce an isomorphism

$$\operatorname*{colim}_{U \in \mathtt{Sub}_{\mathcal{C}\mathrm{op}}(Q)} \mathrm{K}_0(\mathcal{H}(p^{-1}(U); R, \rho, \omega)) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{H}(G; R, \rho, \omega));$$

(iii) We have $K_n(\mathcal{H}(G; R, \rho, \omega)) = 0$ for $n \leq -1$.

Proof. See [8, Theorem 1.1]. $\square$

1.L. **Homotopy colimits.** We write $\mathtt{Or}_{\mathcal{C}\mathrm{op}}(G)$ for the full subcategory of $\mathtt{Or}(G)$ on the G/U with U compact open. The projections $G/U \to G/G$ for U compact open in G induce a map

$$\operatorname*{hocolim}_{G/U \in \mathtt{Or}_{\mathcal{C}\mathrm{op}}(G)} \mathbf{K}_{\mathcal{B}}(G/U) \to \mathbf{K}_{\mathcal{B}}(G/G) \simeq \mathbf{K}(\mathcal{B}). \tag{1.19}$$

This map can be identified with the map $\mathbf{H}^G(E_{\mathcal{C}\mathrm{op}}(G); \mathbf{K}_{\mathcal{B}}) \to \mathbf{H}^G(G/G; \mathbf{K}_{\mathcal{B}})$, see [22, Section 6]. Applying π_n to (1.19) therefore recovers (1.9).

Often the homotopy colimit in (1.19) can be replaced with a homotopy colimit over a smaller category than $\mathtt{Or}_{\mathcal{C}\mathrm{op}}(G)$. Let X be a simplicial complex with a smooth proper cellular simplicial action of G. *Cellular* means that, if $g \in G$ sends a simplex to itself, then g fixes the simplex pointwise. We also assume that X is a model for $E_{\mathcal{C}\mathrm{op}}(G)$, i.e., for $U \subseteq G$ compact open X^U is contractible. For example, if G is a p-adic group, then we can take for X (a subdivision of) the associated extended Bruhat-Tits building.

Let C be a collection of simplices of X that contains at least one simplex from each orbit of the action of G on the set of simplices of X. Define a category $\mathcal{C}(C)$ as follows. Its objects are the simplices from C. A morphism $gG_\sigma \colon \sigma \to \tau$ is an element $gG_\sigma \in G/G_\sigma$ satisfying $g\sigma \subseteq \tau$, where we view a simplex of X as subspace of X in the obvious way. The composite of $gG_\sigma \colon \sigma \to \tau$ with $hG_\tau \colon \tau \to \rho$ is $hgG_\sigma \colon \sigma \to \rho$. Define a functor $\iota_C \colon \mathcal{C}(C)^{\mathrm{op}} \to \mathtt{Or}_{\mathcal{C}\mathrm{op}}(G)$ by sending an object σ to G/G_σ and a morphism $gG_\sigma \colon \sigma \to \tau$ to $G/G_\tau \to G/G_\sigma$, $g'G_\tau \mapsto g'gG_\sigma$.

Lemma 1.20. *The functor $\iota_C \colon \mathcal{C}(C) \to \mathtt{Or}_{\mathcal{C}\mathrm{op}}(G)$ is cofinal.*

Proof. For $C \subseteq C'$ it is not difficult to check that the inclusion $\mathcal{C}(C) \to \mathcal{C}(C')$ is an equivalence. Thus we can assume that C contains exactly one simplex from each orbit of the G-action. For $G/U \in \mathtt{Or}_{\mathcal{C}\mathrm{op}}(G)$ the category $G/U \downarrow \mathcal{C}$ can then be identified with the poset of simplices in X^U. By assumption X^U is contractible. $\square$

Lemma 1.20 in combination with the cofinality Lemma A.1 for homotopy colimits imply that the canonical map

$$\operatorname*{hocolim}_{c \in \mathcal{C}(C)} \mathbf{K}_{\mathcal{B}}(G/G_c) \xrightarrow{\sim} \operatorname*{hocolim}_{G/U \in \mathtt{Or}_{\mathcal{C}\mathrm{op}}(G)} \mathbf{K}_{\mathcal{B}}(G/U) \tag{1.21}$$

is an equivalence.

If X admits a strict fundamental domain X_0, i.e., a subcomplex X_0 that contains exactly one simplex from each orbit for the G-action on the set of simplices of X, then we can take for C the simplices from X_0. In this case $\mathcal{C}(C)$ can be identified with the poset (viewed as a category) $\mathrm{simp}(X_0)$ of simplices of X_0. If $\mathcal{B}$ is a Hecke category with G-support, then the inclusions $\operatorname{res}_G^{G_\sigma} \mathcal{B} \to \mathcal{B}[G/G_\sigma]$ are equivalences and induce an equivalence between $\mathbf{K}_{\mathcal{B}} \circ \iota_C$ and

$$\operatorname{simp}(X_0) \to \mathsf{Spectra}, \qquad \sigma \mapsto \mathbf{K}\big(\operatorname{res}_G^{G_\sigma} \mathcal{B}\big).$$

Thus, in this situation, (1.21) can be simplified further and (1.19) can be identified with the canonical map

$$\operatorname*{hocolim}_{\sigma \in \operatorname{simp}(X_0)} \mathbf{K}\big(\operatorname{res}_G^{G_\sigma} \mathcal{B}\big) \to \mathbf{K}(\mathcal{B}). \tag{1.22}$$

By Theorem 1.11, this map is an equivalence, if G is a reductive p-adic group and $\mathcal{B}$ satisfies (Reg) from Definition 3.11. In particular, for R uniformly regular with $\mathbb{Q} \subseteq R$, the canonical map

$$\operatorname*{hocolim}_{\sigma \in \operatorname{simp}(X_0)} \mathbf{K}\big(\mathcal{H}(G_\sigma; R)\big) \to \mathbf{K}\big(\mathcal{H}(G; R)\big) \tag{1.23}$$

is an equivalence, see Corollary 1.8.

Example 1.24 ($\mathrm{SL}_n(F)$)**.** Let R be a uniformly regular ring containing $\mathbb{Q}$. Any chamber of the Bruhat-Tits building for $\mathrm{SL}_n(F)$ is a strict fundamental domain and we obtain a homotopy pushout diagram from (1.23). We will illustrate this for $n = 2, 3$. Let $v\colon F \to \mathbb{Z} \cup \{\infty\}$ the valuation of F. Let $\mathcal{O} = \{v \geq 0\}$ be the ring of integers in F. Choose $\mu \in \mathcal{O}$ with $v(\mu) = 1$. Put

$$h := \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \mu & & & \end{pmatrix} \in \mathrm{GL}_n(F).$$

For $n = 2$ the homotopy pushout diagram is

$$\begin{tikzcd}
\mathbf{K}(\mathcal{H}(I;R)) \arrow[r] \arrow[d] & \mathbf{K}(\mathcal{H}(U_1;R)) \arrow[d] \\
\mathbf{K}(\mathcal{H}(U_0;R)) \arrow[r] & \mathbf{K}(\mathcal{H}(\mathrm{SL}_2(F);R)).
\end{tikzcd}$$

Here $U_0 = \mathrm{SL}_2(\mathcal{O})$, $U_1 = hU_0h^{-1}$, and $I = U_0 \cap U_1$ is the Iwahori subgroup. For the K-groups this yields a Mayer-Vietoris sequence, infinite to the left,

$$\begin{aligned}
\cdots \to K_n(\mathcal{H}(I;R)) \to K_n(\mathcal{H}(U_1;R)) \oplus K_n(\mathcal{H}(U_0;R)) \to K_n(\mathcal{H}(\mathrm{SL}_2(F);R)) \\
\to K_{n-1}(\mathcal{H}(I;R)) \to K_{n-1}(\mathcal{H}(U_1;R)) \oplus K_{n-1}(\mathcal{H}(U_0;R)) \to \cdots \\
\cdots \to K_0(\mathcal{H}(I;R)) \to K_0(\mathcal{H}(U_1;R)) \oplus K_0(\mathcal{H}(U_0;R)) \to K_0(\mathcal{H}(\mathrm{SL}_2(F);R)) \to 0,
\end{aligned}$$

and $K_n(\mathcal{H}(\mathrm{SL}_2(F);R)) = 0$ for $n \leq -1$.

For $n = 3$ we obtain the homotopy pushout diagram

$$\begin{tikzcd}
& \mathbf{K}(\mathcal{H}(U_{12};R)) \arrow[rr] \arrow[dd] & & \mathbf{K}(\mathcal{H}(U_2;R)) \arrow[dd] \\
\mathbf{K}(\mathcal{H}(I;R)) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbf{K}(\mathcal{H}(U_{02};R)) \arrow[ur] \arrow[dd] & \\
& \mathbf{K}(\mathcal{H}(U_1;R)) \arrow[rr] & & \mathbf{K}(\mathcal{H}(\mathrm{SL}_3(F);R)) \\
\mathbf{K}(\mathcal{H}(U_{01};R)) \arrow[ur] \arrow[rr] & & \mathbf{K}(\mathcal{H}(U_0;R)) \arrow[ur] &
\end{tikzcd}$$

where $U_0 = \mathrm{SL}_2(\mathcal{O})$, $U_1 = h\,\mathrm{SL}_2(\mathcal{O})h^{-1}$, $U_2 = h^2\,\mathrm{SL}_2(\mathcal{O})h^{-2}$, $U_{ij} = U_i \cap U_j$ and $I = U_0 \cap U_1 \cap U_2$ is the Iwahori subgroup.

In general, for $\mathrm{SL}_n(F)$ we obtain a homotopy pushout diagram whose shape is an n-cube.

1.M. **Comparison with the discrete case.** Let Γ be a discrete group. Let $\mathcal{F}\mathrm{in}$ be the collection of finite groups of Γ and $\mathcal{V}\mathrm{cyc}$ be the collection of the virtually cyclic subgroups of Γ[13]. We write $\mathsf{Or}_{\mathcal{F}\mathrm{in}}(\Gamma)$ and $\mathsf{Or}_{\mathcal{V}\mathrm{cyc}}(\Gamma)$ for the corresponding subcategories of the orbit category $\mathsf{Or}(\Gamma)$ of Γ. For a ring R there is a functor $\mathsf{Or}(\Gamma) \to \mathsf{Spectra}$ whose value on Γ/H is equivalent to the K-theory spectrum of $R[H]$, see [22, Sec. 2]. To distinguish it from (1.6) we will denote it here as $\mathbf{K}_R^{\mathrm{dis}}$. We obtain a commutative diagram

$$\begin{tikzcd} \operatorname*{hocolim}_{\Gamma/F \in \mathsf{Or}_{\mathcal{F}\mathrm{in}}(\Gamma)} \mathbf{K}_R^{\mathrm{dis}}(\Gamma/F) \arrow[rr, "\alpha^{\mathrm{dis}}_{\mathcal{F}\mathrm{in}}"] \arrow[dr, "\alpha^{\mathrm{dis}}_{\mathrm{rel}}"'] & & \mathbf{K}_R^{\mathrm{dis}}(\Gamma/\Gamma) \simeq \mathbf{K}(R[\Gamma]) \\ & \operatorname*{hocolim}_{\Gamma/V \in \mathsf{Or}_{\mathcal{V}\mathrm{cyc}}(\Gamma)} \mathbf{K}_R^{\mathrm{dis}}(\Gamma/V)\,. \arrow[ur, "\alpha^{\mathrm{dis}}_{\mathcal{V}\mathrm{cyc}}"'] & \end{tikzcd} \tag{1.25}$$

The map $\alpha^{\mathrm{dis}}_{\mathcal{F}\mathrm{in}}$ is the analog of (1.9) in the formulation (1.19). The (K-theoretic) Farrell–Jones Conjecture for discrete groups asserts that $\alpha^{\mathrm{dis}}_{\mathcal{V}\mathrm{cyc}}$ is an equivalence. If R is regular and contains $\mathbb{Q}$, then the relative assembly map $\alpha^{\mathrm{dis}}_{\mathrm{rel}}$ is an equivalence (for all Γ), see [40, Prop. 2.14]. Therefore, for such R, $\alpha^{\mathrm{dis}}_{\mathcal{F}\mathrm{in}}$ is an equivalence if and only if $\alpha^{\mathrm{dis}}_{\mathcal{V}\mathrm{cyc}}$ is an equivalence. The Farrell–Jones Conjecture holds for a large class of groups, including all groups Γ that admit a cocompact isometric action on a finite-dimensional CAT(0)-space X, see [2, Thm. B] and [53, Thm. 1.1]. The proof uses geodesic flows as pioneered by Farrell and Jones, e.g. [26, 27]. Here virtually cyclic subgroups appear as follows. For a bi-infinite geodesic $c\colon \mathbb{R} \to X$ let V_c be the subgroup of Γ consisting of all $g \in \Gamma$ for which there is $t_g \in \mathbb{R}$ such that $gc(t) = c(t + t_g)$ for all t. This is a virtually cyclic subgroup. More precisely, the homomorphism $V_c \to \mathbb{R}$, $g \mapsto t_g$ has discrete and therefore infinite cyclic or trivial image. The kernel of this homomorphism is finite as the action of Γ on X is proper. Thus V_c is either finite or admits a surjection onto $\mathbb{Z}$ with finite kernel.

To prove Theorem 1.11 we will use the action of a reductive p-adic group G on its associated extended Bruhat-Tits building X. The building X is a CAT(0)-space and our general strategy is to apply the geodesic flow method and argue along a variation of the diagram (1.25). For bi-infinite geodesics $c\colon : \mathbb{R} \to X$ we obtain the subgroups V_c of G as above. The V_c are now either compact (as pointwise stabilizers of bi-infinite geodesics) or admit a surjection onto $\mathbb{Z}$ with compact kernel.

Definition 1.26. For a td-group G we write $\mathcal{C}\mathrm{vcy}$ for the family of all closed subgroups V that are either compact or admit a surjection onto the infinite cyclic group with compact kernel[14]. We write $\mathsf{Or}_{\mathcal{C}\mathrm{vcy}}(G)$ for the full subcategory of $\mathsf{Or}(G)$ on all G/V with $V \in \mathcal{C}\mathrm{vcy}$.

Our general strategy will be to replace Γ with G, $\mathcal{F}\mathrm{in}$ with $\mathcal{C}\mathrm{op}$, and $\mathcal{V}\mathrm{cyc}$ with $\mathcal{C}\mathrm{vcy}$ in (1.25). However, two problems arise because the $V \in \mathcal{C}\mathrm{vcy}$ are in general not open in G. The first problem is that, if V not open in G, then $\mathcal{H}(V;R)$ is not a subalgebra of $\mathcal{H}(G;R)$; extending by zero does not produce locally compact functions on G from locally constant functions on V. Thus there is no induction map from $\mathbf{K}(\mathcal{H}(V;R))$ to $\mathbf{K}(\mathcal{H}(G;R))$ and it is not clear how $\mathbf{K}_R$ can be extended from $\mathsf{Or}_{\mathcal{C}\mathrm{op}}(G)$ to $\mathsf{Or}_{\mathcal{C}\mathrm{vcy}}(G)$. The second problem is less clear at this point, but it comes from the fact that (unlike the discrete case) a product of orbits $G/V \times G/V'$ cannot

[13]Alternatively one can work with the family of subgroups that are finite or admit a surjection onto $\mathbb{Z}$ with finite kernel.

[14]As $\mathbb{Z}$ is discrete the kernel is automatically open in V.

be written as a coproducts of orbits[15]. For this reason orbits are not necessarily the correct building blocks for topological groups and we will work with a category of formal products of orbits instead of the orbit category. As an added bonus this will allow us to disregard many morphisms in $\mathtt{Or}(G)$ and we arrive at the category $\mathsf{P}\mathcal{A}\mathrm{ll}(G)$, see Subsection 3.D. A technical point is that G/G is no longer terminal in $\mathsf{P}\mathcal{A}\mathrm{ll}(G)$, but this can be remedied by allowing the empty product $*$ and we obtain the category $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$ as our replacement for $\mathtt{Or}(G)$. For a *family*[16] $\mathcal{F}$ of closed subgroups of G we obtain the subcategory $\mathsf{P}\mathcal{F}(G)$ of $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$ on all (non-empty) formal products of the form $G/F_1 \times \cdots \times G/F_n$ with $F_i \in \mathcal{F}$. We will construct a functor $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G) \to \mathtt{Spectra}$, $P \mapsto \mathbf{K}(\mathsf{C}_G(P))$ in Subsection 5.E[17]. On orbits G/U with U open in G, the K-theory of $\mathsf{C}_G(P)$ will (up to a degree shift) be the K-theory of $\mathcal{H}(G;R)$, compare Proposition 5.16. Our replacement for (1.25) is then

$$\begin{tikzcd}[column sep=small]
\operatorname*{hocolim}_{P \in \mathsf{P}\mathcal{C}\mathrm{op}(G)} \mathbf{K}(\mathsf{C}_G(P)) \arrow[rr, "\alpha_{\mathcal{C}\mathrm{op}}"] \arrow[dr, "\alpha_{\mathrm{rel}}"'] & & \mathbf{K}(\mathsf{C}_G(*)) \simeq \Sigma\mathbf{K}(\mathcal{H}(G,R)) \\
& \operatorname*{hocolim}_{P \in \mathsf{P}\mathcal{C}\mathrm{vcy}(G)} \mathbf{K}(\mathsf{C}_G(P)) . \arrow[ur, "\alpha_{\mathcal{C}\mathrm{vcy}}"'] &
\end{tikzcd} \tag{1.27}$$

It is not difficult to identify $\alpha_{\mathcal{C}\mathrm{op}}$ with (1.19), compare Proposition 3.13. Thus the task is to show that $\alpha_{\mathcal{C}\mathrm{vcy}}$ and α_{rel} are equivalences. For $\alpha_{\mathcal{C}\mathrm{vcy}}$ this means that G satisfies the $\mathcal{C}$vcy-Farrell–Jones Conjecture 5.12, see Theorem 5.15. As in the discrete case Theorem 5.15 does not depend on any assumptions on the coefficients (here R). For α_{rel} it is the content of the Reduction Theorem 14.1; this depends on a regularity assumptions for the coefficients (here R).

The functor $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G) \to \mathtt{Spectra}$, $P \mapsto \mathbf{K}(\mathsf{C}_G(P))$ is not determined by its restriction to $\mathsf{P}_{+}\mathcal{O}\mathrm{p}(G)$. There are many variations of the category $\mathsf{C}_G(P)$ such that the K-theory is unchanged for $P \in \mathsf{P}_{+}\mathcal{O}\mathrm{p}(G)$. The specific choices from Subsection 5.E may seem overly complicated at first, but are made in order for the proof of Theorem 5.15 to work. For example, the foliated distance from Subsections 5.C and 5.D is modeled on foliated distance on flow spaces, see Subsection D.II. This in turn makes the proof of the reduction theorem more complicated than its discrete counterpart. In fact, we only know that α_{rel} is an equivalence under the assumption that $\alpha_{\mathcal{C}\mathrm{vcy}}$ is an equivalence. The reason is that we are not able to prove the reduction theorem directly for our functor $P \mapsto \mathbf{K}(\mathsf{C}_G(P))$, but only for a variation $P \mapsto \mathbf{K}(\widehat{\mathsf{C}}_G(P))$ thereof, see Subsection 14.B. There is a map $\mathbf{K}(\mathsf{C}_G(P)) \to \mathbf{K}(\widehat{\mathsf{C}}_G(P))$ and, under the assumption that $\alpha_{\mathcal{C}\mathrm{vcy}}$ is an equivalence, a simple diagram chase proves then the reduction theorem for $\mathbf{K}(\mathsf{C}_G(P))$.

While the construction of $\mathsf{C}_G(P)$ is in many ways the key ingredient to the proof of Theorem 1.11, a better understanding of it would still be desirable. The value of the functor $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G) \to \mathtt{Spectra}$, $P \mapsto \mathbf{K}(\mathsf{C}_G(P))$ does not only depend on the groups H_i occurring in $P = (G/H_1, \ldots, G/H_n)$ but also on how H_i sits in G unless each H_i is open. This is illustrated in Remark 14.18. For discrete groups the Farrell–Jones Conjecture (with appropriate coefficients) passes to subgroups. Similarly, the $\mathcal{C}$op-Farrell-Jones Conjecture 1.10 passes to closed subgroups. It is natural to expect the same for the $\mathcal{C}$vcy-Farrell-Jones Conjecture 5.12, but this remains open

[15] The precise place where this comes up is Theorem D.3. Locally there are maps on the flow space of the form $U \to G/V$, but if one patches them together over the flow space one ends up with maps to products of orbits.

[16] A family of closed subgroups is always assumed to be closed under conjugation and taking finite intersections.

[17] There is a functor for each category with G-support $\mathcal{B}$. To simplify the discussion here we tacitly assume $\mathcal{B} = \mathcal{B}(G;R)$ for a ring R containing $\mathbb{Q}$.

and seems to require a better understanding of $\mathrm{C}_G(P)$. This would imply the (K-theoretic) Farrell–Jones Conjecture for all discrete subgroups of reductive p-adic groups, because the $\mathcal{C}$vcy-Farrell–Jones Conjecture 5.12 reduces for discrete groups to the original K-theoretic Farrell–Jones Conjecture, see Remark 5.14.

1.N. **Open problems.** There is an interesting instance where the $\mathcal{C}$vcy-Farrell–Jones Conjecture 5.12 applies, but we do not know to what extend the $\mathcal{C}$op-Farrell–Jones Conjecture 1.10 applies. This arises as follows. Let G be a reductive p-adic group and R be a ring containing $1/p$. In this case G admits a pro-p-group U_0 as a compact open subgroup U_0. There is still a Hecke algebra $\mathcal{H}(G;R)$, see for example [52, Sec.I.3], which can be used to define a variant $\mathcal{B}'(G;R)$ of $\mathcal{B}(G;R)$, see Example 3.4. Now Theorem 5.15 applies and we obtain a homotopy colimit description of the K-theory of $\mathcal{H}(G;R)$ from (5.11). In general[18], for example if $R = \mathbb{Z}[1/p]$, we do not expect that $\mathcal{B}'(G;R)$ satisfies (Reg) from Definition 3.11. In particular, we do not expect that (1.7) is an isomorphism in this situation. Nevertheless, it seems interesting to evaluate this homotopy colimit from (5.11) in this situation further. For example, it is conceivable that (1.7) is an isomorphism modulo p-torsion.

For a reductive p-adic group G Bernstein decomposed the category of finitely generated non-degenerated $\mathcal{H}(G;\mathbb{C})$-modules as a direct sum of subcategories, now called Bernstein blocks, see [13, 12]. In particular, there is a corresponding direct sum decomposition of $K_n(\mathcal{H}(G;\mathbb{C}))$. By Corollary 1.8 there must then exist a corresponding decomposition of $H_n^G(E_{\mathcal{C}\mathrm{op}}(G);\mathbf{K}_\mathbb{C})$. It seems interesting to give a direct description of the summands in $H_n^G(E_{\mathcal{C}\mathrm{op}}(G);\mathbf{K}_\mathbb{C})$ corresponding to the Bernstein blocks in $K_n(\mathcal{H}(G;\mathbb{C}))$. Let $G = \mathrm{GL}_n(F)$ and let I be the Iwahori subgroup. The Iwahori-Hecke algebra $\mathcal{H}(G,I)$ is the (unital) subalgebra of I-bi-invariant functions of $\mathcal{H}(G;\mathbb{C})$. The Iwahori block in the category of finitely generated non-degenerated $\mathcal{H}(G;\mathbb{C})$-modules can be identified with the category of finitely generated $\mathcal{H}(G,I)$-modules. Even for this block it is not quite clear what the correct analog of the assembly map (1.7) should be.

1.O. **Overview.** Section 2 fixes some conventions and notations.

Section 3 contains the details of the formulation of the $\mathcal{C}$op-Farrell–Jones Conjecture and a reformulation using products of orbits as building blocks.

Controlled algebra is a key tool for proofs of the Farrell–Jones Conjecture for discrete groups sets up a variant of this theory also suitable for td-groups. In the usual theory controlled objects over a space X have as support a subset of X, while morphisms have as support subsets of $X \times X$. In our version of the theory objects also have a support in $X \times X$[19]. One can think of the controlled categories, that we introduce in Section 4, as generalizations of Hecke algebras. Thus it is quite natural that these categories also come with a notion of support in G.

Section 5 contains the formulation of the $\mathcal{C}$vcy-Farrell–Jones Conjecture. Central is the construction of the categories $\mathrm{C}_G(P)$ already discussed in Subsection 1.M. Their construction uses the language of controlled algebra. (This in contrast to the discrete case, where controlled algebra only enters proofs of the Farrell–Jones Conjecture but not its formulation.)

Section 6 contains the formal framework of the proof of the $\mathcal{C}$vcy-Farrell–Jones Conjecture of reductive p-adic groups. This framework is formally different from

[18]If $[U : U_0]$ is invertible for all compact open subgroups U of G containing the pro-p-group U_0, then we expect that $\mathcal{B}'(G;R)$ satisfies (Reg) from Definition 3.11. Thus, under this additional assumption the Reduction Theorem 14.1 should apply and lead for example to a version of Corollary 1.8 for the K-theory of $\mathcal{H}(G;R)$.

[19]If one thinks of objects as idempotents this is quite natural.

the one used for example in [2] for discrete groups, but also centers around a G-homology theory (here $\mathbf{D}_G$) and a transfer map, see Theorem 6.7. For technical reasons we also introduce a variant $\mathbf{D}^0_G$ of $\mathbf{D}_G$. The domain of the G-homology theory $\mathbf{D}_G$ is a category $\mathcal{R}$ of combinatorial G-space, whose building blocks are products of orbits and simplices. This category contains an analog $\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)$ of the numerable classifying space for $\mathcal{C}$vcy. The transfer map realizes the functor $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G) \to \mathsf{Spectra}$, $P \mapsto \mathbf{K}(\mathsf{C}_G(P))$ as a retract of $P \mapsto \mathbf{D}_G(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times P)$[20]. This allows then the application of excision and homotopy invariance results of $\mathbf{D}_G$ in the variable $\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)$.

Section 7 contains the construction of the functors $\mathbf{D}_G$ and $\mathbf{D}^0_G$ as the K-theory of certain categories. These categories are constructed using controlled algebra. Their construction builds on that of $\mathsf{C}_G(P)$ by adding a second control direction for what is called an ϵ-control condition. The precise formulation is tailored in order for $\mathbf{D}_G$ and $\mathbf{D}^0_G$ to satisfy the properties formulated in Section 6. With the exception of the transfer these properties are then verified in Section 8, following similar results in the discrete case.

The construction of the transfer is outlined in Section 9 and carried out in Sections 10, 11, 12 and 13. This depends on the construction of certain *almost equivariant* maps from the building X to a space $|\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge$ associated to $\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)$. This is the point where the dynamics of the geodesic flow on a flow space associated to X is exploited. The details of this construction is outsourced to [6], but we give an overview in Appendix D.

Section 14 contains the proof of the reduction theorem. The difficulty here is that it is not clear that the regularity of the coefficients induces a regularity property for $\mathsf{C}_G(P)$. Roughly, the controlled algebra nature of $\mathsf{C}_G(P)$ makes it too big to satisfy a regularity property. In a number of steps we reduce the problem to certain categories associated to infinite product categories (the limit category from Subsection 14.G) and use a K-theory computation from [4].

Appendix A reviews some results on homotopy colimits that are used throughout the paper. Appendix B reviews K-theory for dg-categories. This formalism is applied in Appendix C to homotopy coherent functors and ultimately used in the construction of the transfer in Section 11.

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The paper is organized as follows:

Contents

[20] Really, the transfer uses the close relative $\mathbf{D}^0_G$ of $\mathbf{D}_G$, but this is a technical point.

2. Preliminaries

2.a. **Convention on units.** Categories and rings will always be assumed to be unital, unless we explicitly allow non-unital categories or rings. Of course, Hecke algebras are typically not unital.

2.b. **Formally adding finite sums.** For a $\mathbb{Z}$-linear category $\mathcal{A}$ we obtain an additive category $\mathcal{A}_\oplus$ by formally adding finite sums. A concrete model for $\mathcal{A}_\oplus$ has as objects finite sequences $(A_1, \ldots, A_n)$ of objects in $\mathcal{A}$ and as morphisms $\varphi\colon (A_1, \ldots, A_n) \to (A'_1, \ldots, A'_{n'})$ matrices $\varphi = (\varphi_i^{i'} : A_i \to A'_{i'})_{i,i'}$ of morphisms in $\mathcal{A}$, see for example [22, p.214].

2.c. **Idempotent completion.** The idempotent completion $\operatorname{Idem} \mathcal{A}$ of a category $\mathcal{A}$ has as objects pairs (A, p), where p is an idempotent on A in $\mathcal{A}$. Morphisms $\varphi\colon (A, p) \to (A', p')$ are morphisms in $\mathcal{A}$ satisfying $\varphi = p' \circ \varphi \circ p$. For a category without units the idempotent completion makes still sense and produces a category with units: $\mathrm{id}_{(A,p)} = p$.

2.d. **K-theory.** A construction of the *non-connective K-theory spectrum* $\mathbf{K}^\infty(\mathcal{A})$ of a unital additive category $\mathcal{A}$ can be found for instance in [42] or [45]. The K-theory of a $\mathbb{Z}$-linear category $\mathcal{A}$ is defined as the K-theory of the additive category $\mathcal{A}_\oplus$. The canonical embedding $\mathcal{A} \to \operatorname{Idem} \mathcal{A}$ induces an equivalence in K-theory, see for instance [4, Lemma 3.3 (ii)].

A key tool for us will be a fiber sequence in K-theory that goes back to Karoubi [31] and Carlsson-Pedersen [18]. To state it we need a definition.

Definition 2.1. Let $\mathcal{U}$ be a full additive subcategory of an additive category $\mathcal{A}$.

(2.1a) The *quotient category* $\mathcal{A}/\mathcal{U}$ has the same objects as $\mathcal{A}$. Morphisms in $\mathcal{A}/\mathcal{U}$ are equivalence classes of morphisms in $\mathcal{A}$, where morphisms from $\mathcal{A}$ are identified in $\mathcal{A}/\mathcal{U}$ whenever their difference factors over an object from $\mathcal{U}$.

(2.1b) We say that $\mathcal{A}$ is *$\mathcal{U}$-filtered* if the following condition is satisfied. Let $A \in \mathcal{A}$, $U_-, U_+ \in \mathcal{U}$ and let $U_- \xrightarrow{\varphi_-} A \xrightarrow{\varphi_+} U_+$ be morphisms in $\mathcal{A}$. We require that there is a direct summand U of A with $U \in \mathcal{U}$ such that φ_- and φ_+ factor over U, i.e., if we write $p\colon A \to A$ for the projection associated to the direct summand U, then $\varphi_- = p \circ \varphi_-$ and $\varphi_+ = \varphi_+ \circ p$.

Definition 2.1 is originally due to Karoubi [31]. In (2.1b) we used Kasprowski's reformulation [32, Def. 5.4, Rem. 5.7 (1)].

Theorem 2.2 (Karoubi sequence)**.** *Let $\mathcal{U}$ be a Karoubi filtration of $\mathcal{A}$. Write $i\colon \mathcal{U} \to \mathcal{A}$ and $p\colon \mathcal{A} \to \mathcal{A}/\mathcal{U}$ for the associated inclusion and projection. Then*

$$\mathbf{K}(\mathcal{U}) \xrightarrow{i_*} \mathbf{K}(\mathcal{A}) \xrightarrow{p_*} \mathbf{K}(\mathcal{A}/\mathcal{U})$$

is a fibration sequence of spectra[21].

Proof. This is [18, Thm. 1.28]. $\square$

2.E. **Small compact open subgroups.**

Lemma 2.3. *Let G be a td-group, K be a compact subgroup of G, and W be an open neighborhood of K in G. Then there exists a compact open subgroup U with $K \subseteq U \subseteq W$.*

Proof. For each k there is an open subgroup V_k with $kV_k \subseteq W$. As K is compact there is $S \subset K$ finite with $K \subseteq \bigcup_{s\in S} sV_s$. Then $V := \bigcap_{s\in S} V_s$ is a compact open subgroup for which $kV \subseteq W$ for all $k \in K$. As $K \cap V$ has finite index in K, $N := \bigcap_{g\in K} gVg^{-1}$ is still compact open. Now K normalizes N and so $U := KN$ is a compact open subgroup containing K. Also $U = KN \subseteq KV \subseteq W$. $\square$

3. The $\mathcal{C}$op-Farrell–Jones Conjecture

3.A. **Categories with G-support.**

Definition 3.1. Let G be a td-group. A *category with G-support* is a $\mathbb{Z}$-linear category $\mathcal{B}$ together with a map that assigns to every morphism φ in $\mathcal{B}$ a compact subset $\operatorname{supp}\varphi$ of G. We require the following

(3.1a) $\operatorname{supp}\varphi = \emptyset \iff \varphi = 0$;

(3.1b) $\operatorname{supp}(\varphi' \circ \varphi) \subseteq \operatorname{supp}_G \varphi' \cdot \operatorname{supp}_G \varphi$;

(3.1c) $\operatorname{supp}(\varphi + \varphi') \subseteq \operatorname{supp}\varphi \cup \operatorname{supp}\varphi'$, $\operatorname{supp}(-\varphi) = \operatorname{supp}\varphi$.

We abbreviate $\operatorname{supp} B := \operatorname{supp}\operatorname{id}_B$.

Definition 3.2. A *Hecke category with G-support* is a category $\mathcal{B}$ with G-support such that the following holds.

(3.2a) *Subgroups*
$\operatorname{supp} B$ is a compact subgroup of G for all objects B. For morphisms $\varphi\colon B \to B'$ we have $\operatorname{supp}\varphi = \operatorname{supp} B' \cdot \operatorname{supp}\varphi \cdot \operatorname{supp} B'$. Moreover, the sets $\operatorname{supp} B' \backslash \operatorname{supp}\varphi$ and $\operatorname{supp}\varphi / \operatorname{supp} B$ are both finite;

(3.2b) *Translations*
For every $B \in \mathcal{B}$ and $g \in G$ there is an isomorphism $\varphi\colon B \xrightarrow{\cong} B'$ satisfying $\operatorname{supp} B' = g \operatorname{supp} B g^{-1}$, $\operatorname{supp}\varphi = g \operatorname{supp} B$, and $\operatorname{supp}\varphi^{-1} = g \operatorname{supp} B$;

(3.2c) *Morphism additivity*
Let $\varphi\colon B \to B'$ be a morphism. Suppose $\operatorname{supp}\varphi = L_1 \sqcup L_2$ is a disjoint union, where $\operatorname{supp}(B') \cdot L_i \cdot \operatorname{supp}(B) = L_i$. We require the existence of morphisms $\varphi_i\colon B \to B'$ for $i = 1, 2$ satisfying $\varphi = \varphi_1 + \varphi_2$ and $\operatorname{supp}\varphi_i = L_i$ for $i = 1, 2$;

(3.2d) *Support cofinality*
For every object $B \in \mathcal{B}$ and every subgroup L of finite index in $\operatorname{supp} B$ there are morphisms

$$B \xrightarrow{i_{B,L}} B|_L \xrightarrow{r_{B,L}} B$$

such that $\operatorname{supp} B|_L = L$, $\operatorname{supp} i_{B,L} = \operatorname{supp} r_{B,L} = \operatorname{supp} B$ and $r_{B,L} \circ i_{B,L} = \operatorname{id}_B$. Moreover, for L' a subgroup of finite index in L we require $B|_{L'} = (B|_L)|_{L'}$, $i_{B,L'} = i_{B|_L,L'} \circ i_{B,L}$ and $r_{B,L'} = r_{B,L} \circ r_{B|_L,L'}$.

We note that (3.2d) means that $\mathcal{B}$ is equipped with a choice of $B|_L, i_{B,L}, r_{B,L}$ for all B and L.

[21]The precise statement is as follows. The composition $p_* \circ i_0$ has a canonical null homotopy as all objects in $\mathcal{U}$ are isomorphic to 0 in $\mathcal{A}/\mathcal{U}$. The induced map from $\mathbf{K}(\mathcal{U})$ to the homotopy fiber of p_* is an equivalence.

Example 3.3. Let R be a ring containing $\mathbb{Q}$ and G be a td-group. Consider the Hecke algebra $\mathcal{H}(G;R)$ associated to a $\mathbb{Q}$-valued (left-invariant) Haar measure μ on G. Associated to $\mathcal{H}(G;R)$ is the category with G-supports $\mathcal{B}(G;R)$. Objects of $\mathcal{B}(G;R)$ are compact open subgroups of G. Morphisms $\varphi\colon U \to U'$ are elements of $\mathcal{H}(G;R)$ satisfying

$$\varphi(u'gu) = \varphi(g) \quad \text{for all } u' \in U',\ u \in U.$$

The support of φ is $\operatorname{supp}\varphi = \{g \in G \mid \varphi(g) \neq 0\}$[22]. The identity of U is the idempotent $\frac{\chi_U}{\mu(U)} \in \mathcal{H}(G;R)$ where χ_U is the characteristic function of U. The category $\operatorname{Idem}(\mathcal{B}(G;R)_\oplus)$ is equivalent to the category of finitely generated projective $\mathcal{H}(G;R)$-modules, compare [7, Lem. 6.6] In particular $\mathbf{K}\mathcal{B}(G;R) \simeq \mathbf{K}\mathcal{H}(G;R)$.

It is not difficult to check that $\mathcal{B}(G;R)$ is a Hecke category with G-support. The subgroup property and morphism additivity are clear from the definitions. For U compact open in G and $g \in G$ we have an isomorphism $\frac{\chi_{gU}}{\mu(gU)}\colon U \to gUg^{-1}$ in $\mathcal{B}(G;R)$; its inverse is $\frac{\chi_{Ug^{-1}}}{\mu(Ug^{-1})}$. This proves the translation property. For support cofinality we can set $U|_L := L$, $i_{U,L} = r_{U,L} := \frac{\chi_U}{\mu(U)}$. For more details see [7, Sec. 6.C], where also more general Hecke algebras are discussed.

Example 3.4. Let R be a ring and G be a td-group. Assume that G has at least one compact open subgroup U with the property[23] that

(3.4a) for all open subgroups V of U the index $[U : V]$ is invertible in R.

We can fix one such group U_0. If one normalizes a (left-invariant) Haar measure μ such that $\mu(U_0) = 1$, then μ it takes values in $\mathbb{Z}[1/n \mid 1/n \text{ is invertible in } R]$ and one obtains a Hecke algebra $\mathcal{H}(G;R)$. In this situation we obtain a variant $\mathcal{B}'(G;R)$ of the category from Example 3.3. Its objects are compact open subgroups satisfying (3.4a). Morphisms $U \to U'$ are elements of $\mathcal{H}(G;R)$ satisfying

$$\varphi(u'gu) = \varphi(g) \quad \text{for all } u' \in U',\ u \in U$$

as before. The point is that for such subgroups the measures $\mu(gU)$, $\mu(Ug)$ are invertible in R for all $g \in G$[24]. Thus formulas from Example 3.3 still work and $\mathcal{B}'(G;R)$ is a Hecke category with G-support.

Definition 3.5 (The category $\mathcal{B}[X]$)**.** Given a category $\mathcal{B}$ with G-support and a smooth G-set X we define the category $\mathcal{B}[X]$ as follows. Objects are pairs (B,x) with $B \in \mathcal{B}$ and $x \in X$. A morphism $(B,x) \to (B',x')$ is a morphism $\varphi\colon B \to B'$ in $\mathcal{B}$ such that $\operatorname{supp}\varphi \subseteq G_{x,x'} = \{g \in G \mid gx = x'\}$.

The construction of $\mathcal{B}[X]$ is natural in X and compatible with disjoint union, i.e., if $X = \coprod_{i\in I} X_i$ as G-sets, then the canonical functor

$$\coprod_{i\in I} \mathcal{B}[X_i] \xrightarrow{\sim} \mathcal{B}[X] \tag{3.6}$$

is an equivalence of $\mathbb{Z}$-linear categories. This reduces the computation of $\mathcal{B}[X]$ to the case of orbits G/U. Here U is open in G, since we are only allowing smooth G-sets. We write $\mathcal{B}|_U$ for the subcategory on objects and morphisms with support in U. If $\mathcal{B}$ is a Hecke category with G-support then, using (3.2b), there is an equivalence of $\mathbb{Z}$-linear categories

$$\mathcal{B}|_U \to \mathcal{B}[G/U], \tag{3.7}$$

see [7, Lem. 5.5].

[22]As φ is locally constant and compactly supported this is a compact subset of G.

[23]Such a subgroup exists for example if G is reductive p-adic and $1/p \in R$, see [43, Lemma 1.1].

[24]Indeed $\mu(gU) = \mu(U) = [U : U \cap U_0] \cdot \mu(U \cap U_0) = [U : U \cap U_0] \cdot [U_0 : U \cap U_0]^{-1}$ and $\mu(Ug) = \mu(g^{-1}Ug) = [g^{-1}Ug : g^{-1}Ug \cap U_0] \cdot \mu(g^{-1}Ug \cap U_0) = [U : U \cap gU_0g^{-1}] \cdot [U_0 : g^{-1}Ug \cap U_0]^{-1}$.

Definition 3.8 ($\mathcal{C}$op-assembly map)**.** Let G be a td-group and $\mathcal{B}$ be a category with G-support. The projections $G/U \to G/G$ induce a map

$$\operatorname*{hocolim}_{G/U \in \mathrm{Or}_{\mathcal{C}\mathrm{op}}(G)} \mathbf{K}\big(\mathcal{B}[G/U]\big) \to \mathbf{K}\big(\mathcal{B}[G/G]\big) \simeq \mathbf{K}\mathcal{B}. \tag{3.9}$$

We call this the $\mathcal{C}$op-*assembly map for* $\mathcal{B}$.

Remark 3.10. The notion of categories with G-support is very general and allows also for pathological examples. For this reason it is not sensible to conjecture that (3.9) is in general an equivalence. However, a significant part of our considerations work in the generality of categories with G-support.

3.B. **l-uniformly regular coherence and exactness.** Let $\mathcal{A}$ be an additive category. The Yoneda embedding $A \mapsto \mathrm{mor}_{\mathcal{A}}(-, A)$ embeds $\mathcal{A}$ into the abelian category of $\mathbb{Z}\mathcal{A}$-modules, i.e., the category of $\mathbb{Z}$-linear covariant functors from $\mathcal{A}$ to $\mathbb{Z}$-Mod. The $\mathbb{Z}\mathcal{A}$-modules in the image of this functor are called *finitely generate free*. A $\mathbb{Z}\mathcal{A}$-module is *finitely presented* if it is the cokernel of a map between finitely generated free modules. The additive category $\mathcal{A}$ is said to be *regular coherent* if any finitely presented $\mathbb{Z}\mathcal{A}$-module has a finite resolution by finitely generated projective $\mathbb{Z}\mathcal{A}$-modules[25]. It is *l-uniformly regular coherent* if in addition the resolution can be chosen to be of length at most l.

A sequence $A \to A' \to A''$ in $\mathcal{A}$ is *exact* at A', if its image is exact at $\mathrm{mor}_{\mathcal{A}}(-$-$, A')$ in $\mathbb{Z}\mathcal{A}$-modules. A functor $F\colon \mathcal{A} \to \mathcal{B}$ of additive categories is *exact*, if it sends sequences that are exact at A' to a sequence that is exact at $F(A')$. For a more detailed discussion see [4, Sec. 6].

3.C. **Formulation of the $\mathcal{C}$op-Farrell–Jones Conjecture.**

Definition 3.11 (Reg)**.** A Hecke category with G-support is said to satisfy condition (Reg) if for every natural number d there is a natural number $l(d)$ such that for every compact open subgroup $U \subseteq G$ the additive category $\mathcal{B}[G/U]_{\oplus}[\mathbb{Z}^d]$ is $l(d)$-uniformly regular coherent.

Conjecture 3.12 ($\mathcal{C}$op-Farrell–Jones Conjecture)**.** *Let G be a td-group and $\mathcal{B}$ be a Hecke category with G-support satisfying (Reg) from Definition 3.11. Then the* $\mathcal{C}$op-*assembly map* (3.9) *for $\mathcal{B}$ is an equivalence.*

As discussed in Subsection 1.L the $\mathcal{C}$op-assembly map (3.9) for $\mathcal{B}$ after applying π_n can be identified with (1.9). Thus Conjecture 3.12 is just a restatement of Conjecture 1.10 from the introduction.

3.D. **Product categories.** We digress briefly to introduce some notation for formal products, that will be useful later on. Let $\mathcal{C}$ be a category. We define the category $\mathsf{P}_{+}\mathcal{C}$ as follows. Objects of $\mathsf{P}_{+}\mathcal{C}$ are n-tuples of objects of $\mathcal{C}$, $(C_1, \dots, C_n)$. Here $n = 0$ is allowed; the empty tuple is the unique 0-tuple and will be written as $* \in \mathsf{P}_{+}\mathcal{C}$. Morphisms $f\colon (C_1, \dots, C_n) \to (C'_1, \dots, C'_{n'})$ are pairs $f = (u, \varphi)$, where $u\colon \{1, \dots, n'\} \to \{1, \dots, n\}$ and $\varphi\colon \{1, \dots, n'\} \to \mathrm{mor}_{\mathcal{C}}$ are maps such that for each $i' \in \{1, \dots, n'\}$ the morphism $\varphi(i')$ in $\mathcal{C}$ is of the shape $\varphi(i')\colon C_{u(i')} \to C'_{i'}$. The composition of

$$(C_1, \dots, C_n) \xrightarrow{(u,\varphi)} (C'_1, \dots, C'_{n'}) \xrightarrow{(u',\varphi')} (C''_1, \dots, C''_{n''})$$

is $\big(u \circ u', i'' \mapsto \varphi'(i'') \circ \varphi(u'(i''))\big)$.

As there is a unique map from the empty set to any other set, the empty tuple $*$ is a terminal object in $\mathsf{P}_{+}\mathcal{C}$. For objects $P = (C_1, \dots, C_n)$, $P' = (C'_1, \dots, C'_{n'})$ their product is given by $P \times P' = (C_1, \dots, C_n, C'_1, \dots, C'_{n'})$. For example the

[25] These are exactly the direct summands of finitely generated free $\mathbb{Z}\mathcal{A}$-modules.

projection $P \times P' \to P'$ is given by (u, φ) with $u(i) = i + n$ and $\varphi(i') = \mathrm{id}_{C'_{i'}}$ for $i' = 1, \dots, n'$. We write $\mathsf{P}\mathcal{C}$ for the full subcategory of $\mathsf{P}_{+}\mathcal{C}$ obtained by removing the empty product $*$.

One advantage of the product category in connection with homotopy colimits is that it often allows us to disregard all non-identity morphism in $\mathsf{Or}_{\mathcal{F}}(G)$. We write $\mathcal{F}(G)$ for the subcategory of $\mathsf{Or}_{\mathcal{F}}(G)$ that contains all objects, but only identity morphisms; for the corresponding subcategory of $\mathsf{Or}(G)$ containing all G/H with H a closed subgroup we write $\mathcal{A}\mathrm{ll}(G)$. Passing to product categories we obtain $\mathsf{P}_{+}\mathcal{F}(G)$ and $\mathsf{P}\mathcal{F}(G)$. Thus a morphism $u\colon (G/H_1, \dots, G/H_n) \to (G/H'_1, \dots, G/H'_{n'})$ in $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$ is a function $u\colon \{1, 2, \dots, n'\} \to \{1, 2, \dots, n\}$ satisfying $H_{u(i)} = H'_i$ for all i.

3.E. **A reformulation of the $\mathcal{C}$op-Farrell–Jones Conjecture.** Let G be a td-group and $\mathcal{B}$ be a category with G-support. In the following we write

$$\mathcal{B}[P] := \mathcal{B}[G/U_1 \times \cdots \times G/U_n]$$

for $P = (G/U_1, \dots, G/U_n) \in \mathsf{POr}_{\mathcal{O}\mathrm{p}}(G)$. We note that $\mathcal{B}[*]$ and $\mathcal{B}[(G/G, \dots, G/G)]$ are both just (the $\mathbb{Z}$-linear category underlying) $\mathcal{B}$. In terms of the notation introduced later $\mathcal{B}[P] = \mathcal{B}[|P|]$.

Proposition 3.13. *For a family $\mathcal{U}$ of open subgroups the canonical maps induced by the canoncial inclusions $\mathsf{Or}_{\mathcal{U}}(G) \to \mathsf{POr}_{\mathcal{U}}(G)$ and $\mathsf{P}\mathcal{U}(G) \to \mathsf{POr}_{\mathcal{U}}(G)$*

$$\begin{tikzcd}
\operatorname*{hocolim}_{G/U \in \mathsf{Or}_{\mathcal{U}}(G)} \mathbf{K}\big(\mathcal{B}[G/U]\big) \arrow[dr, "\alpha_1"', "\sim"] & & \operatorname*{hocolim}_{P \in \mathsf{P}\mathcal{U}(G)} \mathbf{K}\big(\mathcal{B}[P]\big) \arrow[dl, "\alpha_2", "\sim"'] \\
& \operatorname*{hocolim}_{P \in \mathsf{POr}_{\mathcal{U}}(G)} \mathbf{K}\big(\mathcal{B}[P]\big) &
\end{tikzcd}$$

are equivalences.

Proof. To show that α_1 is an equivalence we will use the transitivity Lemma A.2 for homotopy colimits. It thus suffices to show that for any $P = (G/U_1, \dots, G/U_n) \in \mathsf{POr}_{\mathcal{U}}(G)$ the canonical map

$$\operatorname*{hocolim}_{(G/U,f) \in \mathsf{Or}_{\mathcal{U}}(G)\downarrow P} \mathbf{K}\big(\mathcal{B}[G/U]\big) \xrightarrow{\sim} \mathbf{K}\big(\mathcal{B}[P]\big) \tag{3.14}$$

is an equivalence. As the U_i are open we have $G/U_1 \times \cdots \times G/U_n = \coprod_{j \in J} G/W_j$ with $W_j \in \mathcal{U}$. Then $\mathsf{Or}_{\mathcal{U}}(G) \downarrow P \simeq \coprod_j \big(\mathsf{Or}_{\mathcal{U}}(G) \downarrow G/W_j\big)$ and that id_{G/W_j} is a terminal object of $\mathsf{Or}_{\mathcal{U}}(G) \downarrow G/W_j$. Together with the compatibility of $\mathcal{B}[-]$ with coproducts (3.6) this implies that (3.14) is an equivalence.

Lemma A.6 implies directly that α_2 is an equivalence. □

Now we can reformulate Conjecture 3.12 by precomposing the assembly map (3.9) with α_1^{-1} or $\alpha_1^{-1} \circ \alpha_2$, thus changing the source $\operatorname{hocolim}_{G/U \in \mathsf{Or}_{\mathcal{C}\mathrm{op}}(G)} \mathbf{K}\big(\mathcal{B}[G/U]\big)$ of the assembly map (3.9) to $\operatorname{hocolim}_{P \in \mathsf{POr}_{\mathcal{U}}(G)} \mathbf{K}\big(\mathcal{B}[P]\big)$ or $\operatorname{hocolim}_{P \in \mathsf{P}\mathcal{U}(G)} \mathbf{K}\big(\mathcal{B}[P]\big)$.

4. Controlled algebra

4.A. **The category $\mathcal{B}_G(X)$.** Let X be a set. In the following we will often write 2-tupels in X as $\binom{x'}{x}$.

Definition 4.1. Let X be a G-set and $\mathcal{B}$ be a category with G-support. We define the category $\mathcal{B}_G(X)$ as follows. Objects are triples $\mathbf{B} = (S, \pi, B)$ where

(4.1a) S is a set,

(4.1b) $\pi\colon S \to X$ is a map,

(4.1c) $B\colon S \to \operatorname{ob}\mathcal{B}$ is a map.

Morphisms $\mathbf{B} = (S,\pi,B) \to \mathbf{B}' = (S',\pi',B')$ in $\mathcal{B}_G(X)$ are matrices $\varphi = (\varphi_s^{s'}\colon B(s) \to B'(s'))_{s\in S, s'\in S'}$ of morphisms in $\mathcal{B}$. Morphisms are required to be column finite: for each $s \in S$ there are only finitely many $s' \in S'$ with $\varphi_s^{s'} \neq 0$. Composition is matrix multiplication (using composition in $\mathcal{B}$)

$$(\varphi' \circ \varphi)_s^{s''} := \sum_{s'} {\varphi'}_{s'}^{s''} \circ \varphi_s^{s'}.$$

The formula for the identity $\mathrm{id}_{\mathbf{B}}$ of $\mathbf{B} = (S,\pi,B) \in \mathcal{B}_G(X)$ is

$$(\mathrm{id}_{\mathbf{B}})_s^{s'} = \begin{cases} \mathrm{id}_{B(s)} & s = s' \\ 0 & \text{else.} \end{cases}$$

Definition 4.2 (Support and finiteness for $\mathcal{B}_G(X)$)**.** The *support* of an object $\mathbf{B} = (S,\pi,B)$ in $\mathcal{B}(X)$ is defined to be

$$\operatorname{supp}_1 \mathbf{B} := \pi(S) \subseteq X.$$

The *support* of a morphism $\varphi\colon (S,\pi,B) \to (S',\pi',B')$ in $\mathcal{B}_G(X)$ is

$$\operatorname{supp}_2 \varphi := \left\{ \left(\begin{smallmatrix} \pi'(s') \\ g\pi(s) \end{smallmatrix} \right) \,\middle|\, s \in S, s' \in S', g \in \operatorname{supp}(\varphi_s^{s'}) \right\} \subseteq X \times X.$$

The G-support of a morphism φ in $\mathcal{B}_G(X)$ is

$$\operatorname{supp}_G \varphi := \bigcup_{s\in S, s'\in S'} \operatorname{supp} \varphi_s^{s'}.$$

We abbreviate

$$\operatorname{supp}_2 \mathbf{B} := \operatorname{supp}_2 \mathrm{id}_{\mathbf{B}} = \left\{ \left(\begin{smallmatrix} \pi(s) \\ g\pi(s) \end{smallmatrix} \right) \,\middle|\, s \in S, g \in \operatorname{supp}_G(\mathrm{id}_{B(s)}) \right\}$$

and $\operatorname{supp}_G \mathbf{B} := \operatorname{supp}_G \mathrm{id}_{\mathbf{B}}$.

For a subset A of X we will say that $\mathbf{B}$ is *finite over* A, if $\pi^{-1}(A)$ is finite. We will say that $\mathbf{B}$ is *finite*, if it is finite over X, i.e., if S is finite.

For $E, E' \subset X \times X$ we call

$$E' \circ E := \{ \left(\begin{smallmatrix} x'' \\ x \end{smallmatrix} \right) \mid \exists x' \text{with} \left(\begin{smallmatrix} x'' \\ x' \end{smallmatrix} \right) \in E', \left(\begin{smallmatrix} x' \\ x \end{smallmatrix} \right) \in E \}$$

the *composition* of E and E'. We call $E^{\mathrm{op}} := \{ \left(\begin{smallmatrix} x \\ x' \end{smallmatrix} \right) \mid \left(\begin{smallmatrix} x' \\ x \end{smallmatrix} \right) \in E \}$ the *opposite* of E. The product of two subsets M, M' of a group G is $M' \cdot M := \{ g'g \mid g' \in M', g \in M \}$. For a subset M of G we write $M^{-1} := \{ g^{-1} \mid g \in M \}$ for its elementwise inverse.

Note that for $\mathbf{B} \xrightarrow{\varphi} \mathbf{B}' \xrightarrow{\varphi'} \mathbf{B}''$ in $\mathbf{B}_G(X)$ we have

$$\operatorname{supp}_2(\varphi' \circ \varphi) \subseteq \operatorname{supp}_2 \varphi' \circ (\operatorname{supp}_G \varphi' \cdot \operatorname{supp}_2 \varphi). \tag{4.3}$$

Remark 4.4. We note that $\operatorname{supp}_2 \mathbf{B}$ is not necessarily contained in the diagonal of $X \times X$. The category $\mathcal{B}_G(X)$ is additive; the direct sum comes from disjoint unions, i.e.,

$$(S,\pi,B) \oplus (S',\pi',B') \cong (S \sqcup S', \pi \sqcup \pi', B \sqcup B').$$

Remark 4.5. Typically $\mathcal{B}_G(X)$ does not really encode information about X; any map $f\colon X \to Y$ between non-empty G-sets induces an equivalence $\mathcal{B}_G(X) \xrightarrow{\sim} \mathcal{B}_G(Y)$.

The use of $\mathcal{B}_G(X)$ will be as a home for interesting subcategories that we will exhibit using additional structure on X. The general framework to determine subcategories of $\mathcal{B}_G(X)$ uses the support notions from Definition 4.2 and the formalism of G-control structures that we discuss in Subsection 4.B.

Remark 4.6 (Functoriality)**.** The definition of $\mathcal{B}_G(X)$ does not really use a G-action on X; it is just the notion of support that makes use of the G-action. Any map $f\colon X \to Y$ induces a functor $f_*\colon \mathcal{B}_G(X) \to \mathcal{B}_G(X)$ with $f_*(S,\pi,B) = (S, f\circ \pi, B)$.

$$f_*(S,\pi,B) = (S, f\circ\pi, B), \qquad (f_*(\varphi))_s^{s'} = \varphi_s^{s'}.$$

We have $\operatorname{supp}_G f_*(\varphi) = \operatorname{supp}_G \varphi$ and $\operatorname{supp}_1 f_*(\mathbf{B}) = f(\operatorname{supp}_1 \mathbf{B})$. If f is G-equivariant, then $\operatorname{supp}_2 f_*(\varphi) = f^{\times 2}(\operatorname{supp}_2 \varphi)$. If f is not G-equivariant, then there is no general formula that expresses $\operatorname{supp}_2 f_*(\varphi)$ directly in terms of $\operatorname{supp}_2 \varphi$, but we have

$$\operatorname{supp}_2 f_*(\varphi) \subseteq f^{\times 2}(\operatorname{supp}_2 \varphi) \circ \Big\{ \left(\begin{smallmatrix} f(gx) \\ gf(x) \end{smallmatrix}\right) \;\Big|\; x \in \operatorname{supp}_1 \mathbf{B}, g \in \operatorname{supp}_G \varphi \Big\}. \tag{4.7}$$

Thus to estimate $\operatorname{supp}_2 f_*(\varphi)$ we need to estimate the failure of equivariance of f.

4.B. G-control structures.

Definition 4.8. Let G be a group and X be a G-set. A G-control structure on X is a triple $\mathfrak{E} = (\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_G)$ where

- $\mathfrak{E}_1$ is a collection of subsets of X that is closed under finite unions and taking subsets;
- $\mathfrak{E}_2$ is a collection of subsets of $X \times X$ that is closed under finite unions, taking subsets, opposites, and composition;
- $\mathfrak{E}_G$ is a collection of subsets of G that is closed under finite unions, taking subsets, elementwise inverses, and products.

We require in addition that for $M \in \mathfrak{E}_G$, $E \in \mathfrak{E}_2$ the product $M \cdot E := \big\{ \left(\begin{smallmatrix} gx' \\ gx \end{smallmatrix}\right) \bigm| g \in M, \left(\begin{smallmatrix} x' \\ x \end{smallmatrix}\right) \in E \big\}$ belongs to $\mathfrak{E}_2$.

One might wonder if the condition that $\mathfrak{E}_G$ is closed under elementwise inverses is really necessary. We use this condition in the proof of Lemma 4.21.

Remark 4.9. In our examples X will always be a topological space and the elements of $\mathfrak{E}_1$ will always have finite intersections with compact subsets of X.

Remark 4.10. In almost all our examples $\mathfrak{E}_G$ will be the collection of relatively compact subsets of G. The only other example for $\mathfrak{E}_G$ that we use is the collection of all subsets of G. It will only be used in Section 14 for the proof of the Reduction Theorem 14.1.

Example 4.11 (Trivial control structure)**.** Let X be a G-set. We obtain a G-control structure $\mathfrak{E}$ on X, where $\mathfrak{E}_1$ is the collection of all finite subsets, $\mathfrak{E}_2$ is the collection of all subsets of the diagonal in $X \times X$ and $\mathfrak{E}_G$ is the collection of all relatively compact subsets of G.

4.C. The category $\mathcal{B}_G(\mathfrak{E})$.

Definition 4.12. Let X be a G-set, $\mathfrak{E} = (\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_G)$ be a G-control structure on X, and $\mathcal{B}$ be a category with G-support. The additive category $\mathcal{B}_G(\mathfrak{E})$ is the following subcategory of $\mathcal{B}_G(X)$.

(4.12a) An object $\mathbf{B} = (S,\pi,B)$ from $\mathcal{B}_G(X)$ belongs to $\mathcal{B}_G(\mathfrak{E})$, iff it is finite over each point of X[26], $\operatorname{supp}_1 \mathbf{B} \in \mathfrak{E}_1$, $\operatorname{supp}_2 \mathbf{B} \in \mathfrak{E}_2$ and $\operatorname{supp}_G \mathbf{B} \in \mathfrak{E}_G$;

(4.12b) A morphism φ in $\mathcal{B}_G(X)$ between objects from $\mathcal{B}_G(\mathfrak{E})$ belongs to $\mathcal{B}_G(\mathfrak{E})$ iff $\operatorname{supp}_2 \varphi \in \mathfrak{E}_2$, $\operatorname{supp}_G \varphi \in \mathfrak{E}_G$ and φ is row finite[27].

[26] I.e., $\pi\colon S \to X$ is finite-to-one

[27] As morphisms in $\mathcal{B}_G(X)$ are already required to be column finite, this means that φ is column and row finite, i.e., for fixed s there are only finitely many s' with $\varphi_s^{s'} \neq 0$ and for fixed s' there are only finitely many s with $\varphi_s^{s'} \neq 0$.

Remark 4.13. For a smooth G-set X and the G-control structure $\mathfrak{E}$ from Example 4.11 the category $\mathcal{B}_G(\mathfrak{E})$ is equivalent to $(\mathcal{B}[X])_\oplus$.

Remark 4.14 (Summands)**.** Let $\mathbf{B} = (S, \pi, B) \in \mathcal{B}_G(\mathfrak{E})$. For $S_0 \subseteq S$ set $\mathbf{B}|_{S_0} := (S_0, \pi|_{S_0}, B|_{S_0})$. Consider

$$\mathbf{B}|_{S_0} \xrightarrow{i} \mathbf{B} \xrightarrow{r} \mathbf{B}|_{S_0} \quad \text{with} \quad i^s_{s_0} = r^{s_0}_s = \begin{cases} \mathrm{id}_{B(s)} & s = s_0; \\ 0 & s \neq s_0. \end{cases}$$

Then $\mathrm{id}_{\mathbf{B}|_{S_0}} = r \circ i$, $\mathrm{supp}_1 \mathbf{B}|_{S_0} \subseteq \mathrm{supp}_1 \mathbf{B}$, $\mathrm{supp}_2\, i = \mathrm{supp}_2\, r \subseteq \mathrm{supp}_2 \mathbf{B}$. Altogether $\mathbf{B}|_{S_0}$ is a direct summand of $\mathbf{B}$ in $\mathcal{B}_G(\mathfrak{E})$.

For $Y \subseteq X$ we abbreviate $\mathbf{B}|_Y := \mathbf{B}|_{\pi^{-1}(Y)}$.

Remark 4.15 (Corners)**.** Let $\varphi\colon \mathbf{B} = (S, \pi, B) \to \mathbf{B}' = (S', \pi', B')$. For $Y, Y' \subseteq X$ we obtain summands $\mathbf{B}|_Y \xrightarrow{i_Y} \mathbf{B} \xrightarrow{r^Y} \mathbf{B}|_Y$ and $\mathbf{B}'|_{Y'} \xrightarrow{i_{Y'}} \mathbf{B}' \xrightarrow{r^{Y'}} \mathbf{B}|_{Y'}$ as in Remark 4.14. We define $\varphi_Y^{Y'}$ as the composition $\mathbf{B}|_Y \xrightarrow{i_Y} \mathbf{B} \xrightarrow{\varphi} \mathbf{B}' \xrightarrow{r^{Y'}} \mathbf{B}'|_{Y'}$. Then

$$(\varphi_Y^{Y'})_s^{s'} = \begin{cases} \varphi_s^{s'} & \pi(s) \in Y, \pi'(s') \in Y'; \\ 0 & \text{else.} \end{cases}$$

If $\binom{x}{x'} \in \mathrm{supp}_2\, \varphi_Y^{Y'}$, then $x = \pi(s)$, $x' = g\pi'(s')$ for $\pi(s) \in Y$, $\pi'(s') \in Y'$, $g \in \mathrm{supp}_G\, \varphi$ and $g \in \mathrm{supp}\, \varphi_s^{s'}$. Thus

$$\mathrm{supp}_2\, \varphi_Y^{Y'} \subseteq Y' \times (\mathrm{supp}_G\, \varphi) \cdot Y. \tag{4.16}$$

Lemma 4.17. *Consider the situation of of Remark 4.15. Suppose that for all* $\binom{x'}{x} \in \mathrm{supp}_2\, \varphi$ *with* $x \in (\mathrm{supp}_G\, \varphi) \cdot Y$ *we have* $x' \in Y'$*. Then*

$$i_{Y'} \circ \varphi_Y^{Y'} = \varphi \circ i_Y \colon \mathbf{B}|_Y \to \mathbf{B}'.$$

Proof. We need to check that $\varphi_s^{s'} \neq 0$ with $s \in \pi^{-1}(Y)$ implies $\pi'(s') \in Y'$. If $\varphi_s^{s'} \neq 0$, then there is $g \in \mathrm{supp}_G\, \varphi_s^{s'} \subseteq \mathrm{supp}_G\, \varphi$ and so $\binom{\pi'(s')}{g\pi(s)} \in \mathrm{supp}_2\, \varphi$. Hence $\pi'(s') \in Y'$ by assumption. $\square$

Remark 4.18 (Shifted copy)**.** Let $\mathbf{B} = (S, \pi, B) \in \mathcal{B}_G(\mathfrak{E})$. Let $\sigma\colon S \to X$ be a finite-to-one map. Assume that $\sigma(S) \in \mathfrak{E}_1$ and that π and σ are $\mathfrak{E}_2$-equivalent in the sense that

$$E := \left\{ \binom{\pi(s)}{\sigma(s)} \;\middle|\; s \in S \right\} \in \mathfrak{E}_2.$$

Consider $\mathbf{B}_\sigma := (S, \sigma, B)$ and

$$\mathbf{B}_\sigma \xrightarrow{\varphi} \mathbf{B} \xrightarrow{\psi} \mathbf{B}_\sigma \quad \text{with} \quad \varphi_s^{s'} = \psi_s^{s'} = \begin{cases} \mathrm{id}_{B(s)} & s = s'; \\ 0 & s \neq s'. \end{cases}$$

Then $\mathrm{supp}_1 \mathbf{B}_\sigma = \sigma(S_0) \in \mathfrak{E}_1$, $\mathrm{id}_{\mathbf{B}_\sigma} = \psi \circ \varphi$, $\mathrm{id}_{\mathbf{B}} = \varphi \circ \psi$, and

$$\begin{aligned} \mathrm{supp}_2\, \varphi &= \left\{ \binom{\pi(s)}{g\sigma(s)} \;\middle|\; \mid s \in S, g \in \mathrm{supp}_G\, B(s) \right\} \subseteq \mathrm{supp}_2 \mathbf{B} \circ (\mathrm{supp}_G \mathbf{B}) \cdot E \in \mathfrak{E}_2; \\ \mathrm{supp}_2\, \psi &= \left\{ \binom{\sigma(s)}{g\pi(s)} \;\middle|\; \mid s \in S, g \in \mathrm{supp}_G\, B(s) \right\} \subseteq E^{\mathrm{op}} \circ \mathrm{supp}_2 \mathbf{B} \in \mathfrak{E}_2. \end{aligned}$$

Altogether, $\mathbf{B}_\sigma$ and $\mathbf{B}$ are canonically isomorphic in $\mathcal{B}_G(\mathfrak{E})$. We call $\mathbf{B}_\sigma$ *a shifted copy of* $\mathbf{B}$.

4.D. **Quotients.**

Definition 4.19. Let $\mathfrak{C} = (\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_G)$ be a G-control structure on X. Let $\mathcal{Y}$ be a collection of subsets Y of X that is closed under finite unions and taking subsets. Suppose also that for $M \in \mathfrak{C}_G$ and $Y \in \mathcal{Y}$ we have $M \cdot Y \in \mathcal{Y}$. We obtain a G-control structure $\mathfrak{C}|_{\mathcal{Y}} := (\mathfrak{C}_1|_{\mathcal{Y}}, \mathfrak{C}_2, \mathfrak{C}_G)$ on X where $\mathfrak{C}_1|_{\mathcal{Y}} := \{F \cap Y \mid F \in \mathfrak{C}_1, Y \in \mathcal{Y}\}$.

Definition 4.20. In the situation of Definition 4.19 the category $\mathcal{B}_G(\mathfrak{C}|_{\mathcal{Y}})$ is a full subcategory of $\mathcal{B}_G(\mathfrak{C})$ and we define

$$\mathcal{B}_G(\mathfrak{C}, \mathcal{Y}) := \mathcal{B}_G(\mathfrak{C}) \;/\; \mathcal{B}_G(\mathfrak{C}|_{\mathcal{Y}}).$$

Lemma 4.21. *In the situation of Definition 4.19 the category $\mathcal{B}_G(\mathfrak{C})$ is $\mathcal{B}_G(\mathfrak{C}|_{\mathcal{Y}})$-filtered, see Definition 2.1.*

Proof. Let $\mathbf{U}_- \xrightarrow{\varphi_-} \mathbf{B} \xrightarrow{\varphi_+} \mathbf{U}_+$ be morphisms in $\mathcal{B}_G(\mathfrak{C})$ with $\mathbf{U}_-, \mathbf{U}_+ \in \mathcal{B}_G(\mathfrak{C}|_{\mathcal{Y}})$. Write $\mathbf{U}_\pm = (S_\pm, \pi_\pm, U_\pm)$ and $\mathbf{B} = (T, \rho, B)$. Let $T_- \subset T$ consist of all t, for which there is $s \in S_-$ with $(\varphi_-)_s^t \neq 0$, and let $T_+ \subset T$ consist of all t, for which there is $s_+ \in S_+$ with $(\varphi_+)_t^s \neq 0$. Set $T_0 := T_- \cup T_+$. We obtain the summand $\mathbf{B}|_{T_0}$ of $\mathbf{B}$ as in Remark 4.14. Clearly $\varphi_\pm$ factors over $\mathbf{B}|_{T_0}$. It suffices now to check that $\mathbf{B}|_{T_0}$ is isomorphic to an object in $\mathcal{B}_G(\mathfrak{C}|_{\mathcal{Y}})$. We now use a shifted copy of $\mathbf{B}|_{T_0}$. Choose $\sigma \colon T_0 \to X$ such that for every $t \in T_0$ there are either $s \in S_-$, $g \in G$ with $(\varphi_-)_s^t(g) \neq 0$ and $\sigma(t) = g\pi_-(s)$ or there are $s \in S_+$, $g \in G$ with $(\varphi_+)_t^s(g) \neq 0$ and $\sigma(t) = (g)^{-1}\pi_+(s)$. Then σ is finite-to-one because φ is column finite and φ' is row finite. Our choice of σ implies

$$E := \left\{ \left(\begin{smallmatrix} \rho(t) \\ \sigma(t) \end{smallmatrix} \right) \;\middle|\; t \in T_0 \right\} \subseteq \operatorname{supp}_2 \varphi_- \cup (\operatorname{supp}_G \varphi_+)^{-1} \cdot (\operatorname{supp}_2 \varphi_+)^{\mathrm{op}}.$$

Thus[28] $E \in \mathfrak{C}_2$ and the shifted copy $(\mathbf{B}|_{T_0})_\sigma$ of $\mathbf{B}|_{T_0}$ is isomorphic to $\mathbf{B}$, see Remark 4.18. By construction

$$\begin{aligned} \operatorname{supp}_1(\mathbf{B}|_{T_0})_\sigma = \sigma(T_0) \subseteq \\ & (\operatorname{supp}_G \varphi_-) \cdot \operatorname{supp}_1(\varphi_-) \cup (\operatorname{supp}_G \varphi_+)^{-1} \cdot \operatorname{supp}_1(\varphi_+) \in \mathcal{Y} \end{aligned}$$

and $(\mathbf{B}|_{T_0})_\sigma \in \mathcal{B}_G(\mathfrak{C}|_{\mathcal{Y}})$ as required. $\square$

Combining Lemma 4.21 with Theorem 2.2 we obtain a fibration sequence

$$\mathbf{K}(\mathcal{B}_G(\mathfrak{C}|_{\mathcal{Y}})) \to \mathbf{K}(\mathcal{B}_G(\mathfrak{C})) \to \mathbf{K}(\mathcal{B}_G(\mathfrak{C}, \mathcal{Y})). \tag{4.22}$$

More general, if $\mathcal{Y}_0$ is another collection of subsets of X also satisfying the conditions from Definition 4.20, and if $\mathcal{Y}_0 \subseteq \mathcal{Y}$, then

$$\mathbf{K}(\mathcal{B}_G(\mathfrak{C}|_{\mathcal{Y}}, \mathcal{Y}_0)) \to \mathbf{K}(\mathcal{B}_G(\mathfrak{C}, \mathcal{Y}_0)) \to \mathbf{K}(\mathcal{B}_G(\mathfrak{C}, \mathcal{Y})) \tag{4.23}$$

is a fibration sequence[29]. We will refer to sequences of additive categories of the form

$$\mathcal{B}_G(\mathfrak{C}|_{\mathcal{Y}}, \mathcal{Y}_0) \to \mathcal{B}_G(\mathfrak{C}, \mathcal{Y}_0) \to \mathcal{B}_G(\mathfrak{C}, \mathcal{Y})$$

[28]Here we use in particular that $\mathfrak{C}_G$ is closed under pointwise inverses.

[29](4.22) applies to the two vertical and the upper horizontal sequence in

$$\begin{tikzcd}
\mathcal{B}_G((\mathfrak{C}|_{\mathcal{Y}})|_{\mathcal{Y}_0}) \arrow[r, "="] \arrow[d] & \mathcal{B}_G(\mathfrak{C}|_{\mathcal{Y}_0}) \arrow[d] & \\
\mathcal{B}_G(\mathfrak{C}|_{\mathcal{Y}}) \arrow[r] \arrow[d] & \mathcal{B}_G(\mathfrak{C}) \arrow[r] \arrow[d] & \mathcal{B}_G(\mathfrak{C}, \mathcal{Y}) \arrow[d, "="] \\
\mathcal{B}_G(\mathfrak{C}|_{\mathcal{Y}}, \mathcal{Y}_0) \arrow[r] & \mathcal{B}_G(\mathfrak{C}, \mathcal{Y}_0) \arrow[r] & \mathcal{B}_G(\mathfrak{C}, \mathcal{Y})
\end{tikzcd}$$

and the lower horizontal sequence is therefore a fibration sequence in K-theory.

as Karoubi sequences. K-theory takes Karoubi sequences to a fibration sequences of spectra.

4.E. **Excision.** Let $\mathfrak{E}$ be a G-control structure on X. Let $\mathcal{Y}_0$ and $\mathcal{Y}_1$ be two collections of subsets of X satisfying the assumptions in Definition 4.19. Then $\mathcal{Y}_0 \cap \mathcal{Y}_1$ also satisfies these assumptions. The union of $\mathcal{Y}_0$ and $\mathcal{Y}_1$ may not, but we abuse notation and define $\mathcal{Y}_0 \cup \mathcal{Y}_1$ as the collection of all sets $Y_0 \cup Y_1$ with $Y_i \in \mathcal{Y}_i$. There is a natural functor

$$\mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_1}, \mathcal{Y}_0 \cap \mathcal{Y}_1) \to \mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_0 \cup \mathcal{Y}_1}, \mathcal{Y}_0). \tag{4.24}$$

It is not difficult to check that (4.24) is surjective on morphism sets and on isomorphism classes of objects, but it may fail to be injective on morphism sets. There are different possible assumptions that guarantee that (4.24) is an equivalence. We later use the following.

Lemma 4.25. *Assume that for all $Y_1 \in \mathcal{Y}_1$, $E \in \mathfrak{E}_2$ there are $Y_0 \in \mathcal{Y}_0$, $Y_1' \in \mathcal{Y}_1$ such that $Y_1 \subseteq Y_1' \cup Y_0$ and*

$$(Y_1')^E := \{y' \in X \mid \exists y \in Y \text{ with } \left(\begin{smallmatrix} y' \\ y \end{smallmatrix}\right) \in E\} \in \mathcal{Y}_1.$$

Then (4.24) *is an equivalence.*

Proof. We only need to discuss faithfulness on morphisms. Let $\varphi \colon \mathbf{B} \to \mathbf{B}'$ be a morphism in $\mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_1})$. Assume that φ can be factored in $\mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_0 \cup \mathcal{Y}_1})$ as

$$\mathbf{B} \xrightarrow{\varphi_-} \mathbf{X} \xrightarrow{\varphi_+} \mathbf{B}$$

where $\mathbf{X} \in \mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_0})$, i.e., $\operatorname{supp}_1 \mathbf{X} \in \mathcal{Y}_0$. We need to produce such a factorization over an $\mathbf{X}' \in \mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_0 \cap \mathcal{Y}_1})$, i.e., $\operatorname{supp}_1 \mathbf{X}' \in \mathcal{Y}_0 \cap \mathcal{Y}_1$. We have $Y_1 := \operatorname{supp}_G \varphi \cdot \operatorname{supp}_1 \mathbf{B} \in \mathcal{Y}_1$ and $E := (\operatorname{supp}_G \varphi_-)^{-1} \cdot \operatorname{supp}_2 \in \mathbf{E}_2$. Applying the assumption we find $Y_0 \in \mathcal{Y}_0$, $Y_1' \in \mathcal{Y}_1$ such that $Y_1 \subseteq Y_1' \cup Y_0$ and $(Y_1')^E \in \mathcal{Y}_1$. Now $\mathbf{B} = \mathbf{B}|_{Y_1'} \oplus \mathbf{B}|_{Y_0}$. Write $\pi_{Y_1'}, \pi_{Y_0} \colon \mathbf{B} \to \mathbf{B}$ for the corresponding projections. Then $\varphi - \varphi \circ \pi_{Y_1'} = \varphi \circ \pi_{Y_0}$ factors over $\mathbf{B}|_{Y_0} \in \mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_0})$. This allows us to replace φ with $\varphi \circ \pi_{Y_1'}$ and φ_- with $\varphi_- \circ \pi_{Y_1'}$, or put differently, we may assume without loss of generality $\operatorname{supp}_1 \mathbf{B} \subseteq Y_1'$.

Let now $Y_0' \subseteq \operatorname{supp}_1 \mathbf{X}$ consist of all $y_0 \in \operatorname{supp}_1 \mathbf{X}$ for which there are $y_1 \in \operatorname{supp}_1 \mathbf{B}$ and $g \in \operatorname{supp}_G \varphi_-$ with $\left(\begin{smallmatrix} y_0 \\ gy_1 \end{smallmatrix}\right) \in \operatorname{supp}_2 \varphi_-$, i.e., the matrix entry of φ for $\left(\begin{smallmatrix} y_0 \\ y_1 \end{smallmatrix}\right)$ is non-trivial. Then φ_- factors canonically over the inclusion $\mathbf{X}|_{Y_0'} \to \mathbf{X}$. This allows us to replace $\mathbf{X}$ with $\mathbf{X}|_{Y_0'}$. It remains to check that $Y_0' \in \mathcal{Y}_0 \cap \mathcal{Y}_1$. As $Y_0' \subseteq \operatorname{supp}_1 \mathbf{X} \in \mathcal{Y}_0$ we have $Y_0' \in \mathcal{Y}_0$. For $y_0 \in Y_0'$ there are $y_1 \in \operatorname{supp}_1 \mathbf{B} \subseteq Y_1'$ and $g \in \operatorname{supp}_G \varphi_-$ with $\left(\begin{smallmatrix} y_0 \\ gy_1 \end{smallmatrix}\right) \in \operatorname{supp}_2 \varphi_-$. This implies $y_0 \in (Y_1')^E$. As $(Y_1')^E \in \mathcal{Y}_1$ we now also have $Y_0' \in \mathcal{Y}_1$. □

Lemma 4.26. *Let $\mathcal{Y}$ and $\mathcal{Y}_0$ be two collections of subsets of X satisfying the assumptions from Definition 4.19. Then the canonical functor*

$$\mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_0}, \mathcal{Y}) \to \mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_0 \cup \mathcal{Y}}, \mathcal{Y})$$

is an equivalence.

Proof. The only difference between the two categories is that the category on the right has more objects, i.e., objects with support in $\mathcal{Y}$. But, by Definition, these additional objects are trivial (isomorphic to zero). □

Lemma 4.27. *Let $\mathcal{Y}, \mathcal{Y}_0, \mathcal{Y}_1$ be collections of subsets of X satisfying the assumptions in Definition 4.19. Assume that the condition from Lemma 4.25 holds. Then*

$$\begin{array}{ccc}
\mathbf{K}(\mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_0 \cap \mathcal{Y}_1}, \mathcal{Y})) & \longrightarrow & \mathbf{K}(\mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_1}, \mathcal{Y})) \\
\downarrow & & \downarrow \\
\mathbf{K}(\mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_0}, \mathcal{Y})) & \longrightarrow & \mathbf{K}(\mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_0 \cup \mathcal{Y}_1}, \mathcal{Y}))
\end{array}$$

is a homotopy pushout square.

Proof. We first argue that we may assume that $\mathcal{Y} \subseteq \mathcal{Y}_0 \cap \mathcal{Y}_1$. Indeed, Lemma 4.26 allows us to replace $\mathcal{Y}_0$ with $\mathcal{Y}_0 \cup \mathcal{Y}$ and $\mathcal{Y}_1$ with $\mathcal{Y}_1 \cup \mathcal{Y}$. It is not difficult to check that the condition from Lemma 4.25 is preserved.

Now the horizontal homotopy cofibers of the above diagram are determined by (4.23) and are given by $\mathbf{K}(\mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_0}, \mathcal{Y}_0 \cap \mathcal{Y}_1))$ and $\mathbf{K}(\mathcal{B}_G(\mathfrak{E}|_{\mathcal{Y}_0 \cup \mathcal{Y}_1}, \mathcal{Y}_1))$. The excision result of Lemma 4.25 applies to show that the induced maps between the homotopy cofibers is an equivalence. This implies the assertion. $\square$

4.F. **Swindles.** Let $\mathfrak{E}$ be a G-control structure on X. Let $\mathcal{Y}$ be a collection of subsets of X satisfying the assumptions in Definition 4.19. Sometimes it is easy to produce Eilenberg swindles on $\mathcal{B}_G(\mathfrak{E}, \mathcal{Y})$. Often such a swindle either comes from some map $f\colon X \to X$ that pushes everything to ∞, as in (4.28a) below. We will use the following formal result later on.

Lemma 4.28. *Assume that there is a G-map $f\colon X \to X$ satisfying*

(4.28a) for all $x \in X$ there is n such that $(f^{\circ n})^{-1}(x) = \emptyset$;
(4.28b) for all $Y \in \mathfrak{E}_1$ we have $\bigcup_{n\in\mathbb{N}} f^n(Y) \in \mathfrak{E}_1$;
(4.28c) for all $Y \in \mathcal{Y}$ we have $\bigcup_{n\in\mathbb{N}} f^n(Y) \in \mathcal{Y}$;
(4.28d) for all $E \in \mathfrak{E}_2$ we have $\bigcup_{n\in\mathbb{N}} (f\times f)^{\circ n}(E) \in \mathfrak{E}_2$;
(4.28e) for all $M \in \mathfrak{E}_G$ we have $\{\left(\begin{smallmatrix} f(x) \\ gx \end{smallmatrix}\right) \mid x \in X; g \in M\} \in \mathfrak{E}_2$.

Then the K-theory of $\mathcal{B}_G(\mathfrak{E}, \mathcal{Y})$ is trivial.

Proof. For $\mathbf{B} = (S, \pi, B) \in \mathcal{B}_G(\mathfrak{E})$ we define $\mathbf{B}^\infty = (S^\infty, \pi^\infty, B^\infty)$ where $S^\infty = S \times \mathbb{N}$, $\pi^\infty(s,n) = f^{\circ n}(s)$, $B^\infty(s,t) = B(s)$. Assumption (4.28a) implies that π^∞ is finite-to-one. Assumption (4.28b) implies $\operatorname{supp}_1 \mathbf{B}^\infty \in \mathfrak{E}_1$. Assumption (4.28d) implies $\operatorname{supp}_2 \mathbf{B}^\infty \in \mathfrak{E}_2$. Thus $\mathbf{B}^\infty \in \mathcal{B}_G(\mathfrak{E})$.

For $\varphi\colon (S, \pi, B) \to (S', \pi', B') \in \mathcal{B}_G(\mathfrak{E})$ we define φ^∞ by $(\varphi^\infty)_{s,t}^{s',t'} := \varphi_s^{s'}$. Assumption (4.28d) implies $\operatorname{supp}_2 \varphi^\infty \in \mathfrak{E}_2$. As φ^∞ is also row and column finite (because φ is), we have $\varphi^\infty \in \mathcal{B}_G(\mathfrak{E})$. Compatibility with composition is straight forward and we obtain an endofunctor $(-)^\infty$ of $\mathcal{B}_G(\mathfrak{E})$. For $\mathbf{B} = (S, \pi, B) \in \mathcal{B}_G(\mathfrak{E})$ let $i_{\mathbf{B}}\colon \mathbf{B} \to \mathbf{B}^\infty$ and $\operatorname{sh}_{\mathbf{B}}\colon \mathbf{B}^\infty \to \mathbf{B}^\infty$ be induced by the inclusions

$$\begin{array}{rcll} S & \to & S \times \mathbb{N}, & s \mapsto (s,0); \\ S \times \mathbb{N} & \to & S \times \mathbb{N}, & (s,t) \mapsto (s,t+1). \end{array}$$

Clearly, $i_{\mathbf{B}} \in \mathcal{B}_G(\mathfrak{E})$. Assumption (4.28e) (for $M = \operatorname{supp}_G \mathbf{B}$) implies that $\operatorname{sh}_{\mathbf{B}} \in \mathcal{B}_G(\mathfrak{E})$. Now $i \oplus \operatorname{sh}$ is a natural isomorphism $(-)^\infty \oplus \operatorname{id}_{\mathcal{B}_G(\mathfrak{E})} \cong (-)^\infty$. Altogether we defined a swindle on $\mathcal{B}_G(\mathfrak{E})$. Assumption (4.28c) ensures that this swindle descends to $\mathcal{B}_G(\mathfrak{E}, \mathcal{Y})$. $\square$

We will need a variation of the swindle from Lemma 4.28, where we can swindle towards some $Z \in \mathcal{Y}$ instead of towards ∞[30].

Lemma 4.29. *Assume that there are a G-map $f\colon X \to X$ and $Z \in \mathcal{Y}$ satisfying*

(4.29a) for all $x \in X \setminus Z$ there is n such that $(f^{\circ n})^{-1}(x) = \emptyset$;
(4.29b) for all $Y \in \mathfrak{E}_1$ we have $\bigcup_{n\in\mathbb{N}} Y_n \in \mathfrak{E}_1$ where $Y_0 = Y$ and $Y_{n+1} = f(Y_n \setminus Z)$;
(4.29c) for all $Y \in \mathcal{Y}$ we have $\bigcup_{n\in\mathbb{N}} Y_n \in \mathcal{Y}$ where $Y_0 = Y$ and $Y_{n+1} = f(Y_n \setminus Z)$;
(4.29d) for all $E \in \mathfrak{E}_2$ we have $\bigcup_{n\in\mathbb{N}} (f\times f)^{\circ n}(E) \in \mathfrak{E}_2$;
(4.29e) for all $M \in \mathfrak{E}_G$ we have $\{\left(\begin{smallmatrix} gx \\ f(x) \end{smallmatrix}\right) \mid x \in X; g \in M\} \in \mathfrak{E}_2$.

Then the K-theory of $\mathcal{B}_G(\mathfrak{E}, \mathcal{Y})$ is trivial.

[30] Of course, Lemma 4.28 is implied by Lemma 4.29 by taking $Z = \emptyset$.

Proof. In $\mathcal{B}_G(\mathfrak{E},\mathcal{Y})$ we have $\mathbf{B}|_Z \cong 0$ and so $\mathbf{B} \cong \mathbf{B}|_{X\setminus Z}$. Thus we can systematically get ride of everything over Z. A swindle on $\mathcal{B}_G(\mathfrak{E},\mathcal{Y})$ can be constructed almost verbatim as in Lemma 4.28. The only difference is that we use a subset S^∞_- in place of S^∞. To define this subset set $S_0 := S$, $\pi_0 := \pi$ and inductively $S_{n+1} := (\pi_n)^{-1}(X \setminus Z)$, $\pi_{n+1} := (f \circ \pi_n)|_{S_{n+1}}$. Then $S^\infty_- := \bigcup_n S_n \times \{n\}$. We remark that π_n is just the restriction of π^∞ to $S_n \cong S_n \times \{n\}$.

After restricting everything from S^∞ to S^∞_-, we obtain a swindle on $\mathcal{B}_G(\mathfrak{E},\mathcal{Y})$ as before. □

5. The $\mathcal{C}$vcy-Farrell–Jones Conjecture

5.a. **Delooping.** In order to formulate the $\mathcal{C}$vcy-Farrell–Jones Conjecture we will need categories $\mathsf{C}_G(P;\mathcal{B})$ for $P \in \mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$. To prepare for their construction in Definition 5.9 later on we discuss the Pedersen-Weibel delooping of K-theory from [45] in our set-up. Let X be a G-set. Define the G-control structure $\mathfrak{E}^{\mathrm{pw}}(X) = (\mathfrak{E}^{\mathrm{pw}}_1(X), \mathfrak{E}^{\mathrm{pw}}_2(X), \mathfrak{E}^{\mathrm{pw}}_G(X))$ on $X \times \mathbb{N}$ as follows.

- $\mathfrak{E}^{\mathrm{pw}}_1(X)$ is the collection of all subsets F of $X \times \mathbb{N}$ for which $F \cap X \times \{t\}$ is finite for all t;
- $\mathfrak{E}^{\mathrm{pw}}_2(X)$ is the collection of all $E \subseteq \mathbb{N} \times \mathbb{N}$ with
$$\operatorname{supp}\left\{|t-t'| \;\middle|\; \left(\begin{smallmatrix} x',t' \\ x,t \end{smallmatrix}\right) \in E\right\} < \infty;$$
- $\mathfrak{E}^{\mathrm{pw}}_G(X)$ is the collection of all relatively compact subsets of G.

Let $\mathcal{Y}$ be the collection of subsets Y of $X \times \mathbb{N}$ that are contained in $X \times \{1,\ldots,N\}$ for some N (depending on Y). Let $\mathcal{B}$ be a category with G-support. Then
$$\mathcal{B}_G(\mathfrak{E}^{\mathrm{pw}}(X)|_{\mathcal{Y}}) \to \mathcal{B}_G(\mathfrak{E}^{\mathrm{pw}}(X)) \to \mathcal{B}_G(\mathfrak{E}^{\mathrm{pw}}(X),\mathcal{Y})$$
is a Karoubi sequence. Applying K-theory we obtain a fibration sequence, see (4.22). Combining this sequence with Lemma 5.1 below we obtain $\Omega\mathbf{K}\big(\mathcal{B}_G(\mathfrak{E}^{\mathrm{pw}}(X),\mathcal{Y})\big) \simeq \mathbf{K}(\mathcal{B})$. Lemma 5.1 is standard, but the proof is instructive as we will use variation thereof later on.

Lemma 5.1.

(5.1a) There is an equivalence $\mathcal{B}_G(\mathfrak{E}^{pw}(X)|_{\mathcal{Y}}) \xrightarrow{\sim} \mathcal{B}_\oplus$ defined by $(S,\pi,B) \mapsto \oplus_{s\in S} B(s)$;

(5.1b) The K-theory of $\mathcal{B}_G(\mathfrak{E}^{pw}(X))$ is trivial.

Proof. If $(S,\pi,B) \in \mathcal{B}_G(\mathfrak{E}^{\mathrm{pw}}(X))$, then by definition of $\mathfrak{E}^{\mathrm{pw}}_1(X)$, for each $t \in \mathbb{N}$, $\pi^{-1}(X \times \{t\})$ is finite. If $(S,\pi,B) \in \mathcal{B}_G(\mathfrak{E}^{\mathrm{pw}}(X)|_{\mathcal{Y}})$, then in addition $\pi^{-1}(X \times \{t\})$ is non-empty for only finitely many t. Thus S is finite and $(S,\pi,B) \mapsto \oplus_{s\in S} B(s)$ defines a functor $\mathcal{B}_G(\mathfrak{E}^{\mathrm{pw}}(X)|_{\mathcal{Y}}) \to \mathcal{B}_\oplus$. It is straight forward to check that this functor is an equivalence.

Let $f\colon X \times \mathbb{N} \to X \times \mathbb{N}$ be the shift $(x,t) \mapsto (x,t+1)$. It is not difficult to check that f induces an Eilenberg swindle on $\mathcal{B}_G(\mathfrak{E}^{\mathrm{pw}}(X))$. More precisely, Lemma 4.28 applies to f and $\mathcal{B}_G(\mathfrak{E}^{\mathrm{pw}}(X))$. □

Recall from Subsection 1.m that for the Farrell–Jones Conjecture for a discrete group Γ the group rings over virtually cyclic subgroups of Γ play a central role. For td-groups the K-theory of the categories $\mathsf{C}_G(P;\mathcal{B})$ will take this role. Let G be a td-group and $V \in \mathcal{C}$vcy. Our difficulty is that, if V is closed but not open in G, there is no inclusion of $\mathcal{H}(V;R)$ into $\mathcal{H}(G;R)$. For a category $\mathcal{B}$ with G-support we can restrict to V and only consider morphisms whose support is contained in V. However, as the G-support is typically open this is not sensible and it is not clear how one might exhibit a subcategory associated to V. But once we use the (K-theoretic) deloopings $\mathcal{B}_G(\mathfrak{E}^{\mathrm{pw}}(X),\mathcal{Y})$ of $\mathcal{B}$ this changes; $\mathcal{B}_G(\mathfrak{E}^{\mathrm{pw}}(X),\mathcal{Y})$ has many subcategories. For example for any G-control structure $\mathfrak{E}$ on $X \times \mathbb{N}$ that is contained in $\mathfrak{E}^{\mathrm{pw}}(X)$ we obtain a subcategory $\mathcal{B}_G(\mathfrak{E}, \mathcal{Y})$. The $\mathbb{N}$-factor in $X \times \mathbb{N}$ allows us create G-control structures that become more restrictive with $t \to \infty$. In our definition later on we use this to approximate V by smaller and smaller neighborhoods of V in G as $t \to \infty$. For the precise G-control structure we use see Definition 5.8 later on. We will only change $\mathfrak{E}_2^{\mathrm{pw}}(X)$ by adding what we call the foliated control condition. (There are many possible variations for this G-control structure; our choice is carefully tailored to enable us to prove both the $\mathcal{C}$vcy-Farrell–Jones Conjecture for p-adic groups and the Reduction Theorem 14.1.)

5.B. **Two functors to G-spaces.** We define two functors $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G) \to G\text{-}\mathsf{Spaces}$ as follows. We recall from Subsection 3.D that objects in $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$ are n-tuples $(G/V_1, \dots, G/V_n)$, where $n = 0$ is allowed and each V_i is a closed subgroup of G. Given two such objects $(G/V_1, \dots, G/V_n)$ and $(G/V_1', \dots, G/V_{n'}')$, a morphism $u\colon (G/V_1, \dots, G/V_n) \to (G/V_1', \dots, G/V_{n'})$ is given by a function $u\colon \{1, \dots, n'\} \to \{1, \dots, n\}$ for which $V_i' = V_{u(i)}$ holds[31]. The first functor[32], written as $P \mapsto |P|$, is defined on objects by

$$|(G/V_1, \dots, G/V_n)| := G/V_1 \times \dots \times G/V_n.$$

It sends a morphism $u\colon (G/V_1, \dots, G/V_n) \to (G/V_1', \dots, G/V_{n'}')$ to the map

$$(x_1V_1, \dots, x_nV_n) \mapsto (x_{u(1)}V_1', \dots, x_{u(n')}V_{n'}').$$

The second one, written as $P \mapsto |P|^\wedge$, is defined on objects by

$$|G/V_1, \dots, G/V_n|^\wedge := G \times \dots \times G = G^n.$$

It sends a morphism $u\colon (G/V_1, \dots, G/V_n) \to (G/V_1', \dots, G/V_{n'}')$ to the map

$$(x_1, \dots, x_n) \mapsto (x_{u(1)}, \dots, x_{u(n')}).$$

There is a canonical natural transformation $|\cdot|^\wedge \to |\cdot|$ given on $(G/V_1, \dots, G/V_n)$ by the map

$$G \times \dots \times G \to G/V_1 \times \dots \times G/V_n, \quad (x_1, \dots, x_n) \mapsto (x_1V_1, \dots, x_nV_n).$$

For $P \in \mathsf{P}\mathcal{A}\mathrm{ll}(G)$ the action of G on $|P|^\wedge$ is free, and we can think of $|P|^\wedge \to |P|$ as a resolution. For the empty tuple $*$, both, $|*|$ and $|*|^\wedge$ are the empty product, i.e., a point.

5.C. **V-foliated distance.** Let G be a td-group. We can equip G with a left invariant proper metric d_G that generates the topology of G, see [30, Thm. 4.5] or [1, Thm. 1.1]. Let V be a closed subgroup of G. For $g, g' \in G$, $\beta \geq 0, \eta > 0$ we write

$$d_{V\text{-fol}}(g, g') < (\beta, \eta),$$

iff there is $v \in V$ with $d_G(e, v) = d_G(g, gv) \leq \beta$ and $d_G(gv, g') < \eta$. Similarly, for $g, g' \in G$, $\beta, \eta \geq 0$ we write $d_{V\text{-fol}}(g, g') \leq (\beta, \eta)$, iff there is $v \in V$ with $d_G(e, v) = d_G(g, gv) \leq \beta$ and $d_G(gv, g') \leq \eta$. We will not consider $< (\beta, 0)$.

The general idea here is to treat traveling in cosets of V different from traveling in arbitrary directions in G. Typically, β will be a bounded number, whereas η will be a small number. This definition is motivated by similar constructions for flow spaces, see Subsection D.II.

Remark 5.2. We have $d_{V\text{-fol}}(g, g') \leq (\beta, 0)$, iff $g^{-1}g' \in V$ and $d_G(g, g') \leq \beta$.

[31] if $n' = 0$ there is precisely one such u, if $n' \geq 1$ and $n = 0$, then there is no such u.

[32] Unlike the second the first one factors over $\mathsf{P}_{+}\mathsf{Or}(G)$.

Remark 5.3. Let U be an open subgroup of G. Then there is $\eta_0 > 0$ such that the η_0-neighborhood of U is just U. Thus for $g, g' \in G$ and $\eta < \eta_0$ we have

$$d_{U\text{-fol}}(g, g') \leq (\beta, \eta) \quad \Longrightarrow \quad d_{U\text{-fol}}(g, g') \leq (\beta + \eta, 0).$$

Remark 5.4. In our constructions of controlled categories later on (see Definition 5.9) we would ideally like to work with a G-invariant metric on G/V. Typically we would be interested in small distances in G/V[33]. Often there are however no G-invariant metrics on G/V (and neither are there G-invariant uniform structures on G/V). The notion of V-foliated control on G is (left) G-invariant and will serve us as a replacement for G/V with (a non-existing) G-invariant metric. One way to think about this replacement is that we have to add to points in G/V choices of lifts to G, where the choice of lifts is only relevant up to bounded distance in the fibers for $G \to G/V$. On the level of flow spaces this corresponds to the difference between parametrized geodesics and their images.

5.D. **P-foliated distance.** There is a natural extension of V-foliated distance to products. For $P = (V_1, \ldots, V_n) \in \mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$, $g = (g_1, \ldots, g_n), g' = (g'_1, \ldots, g'_n) \in |P|^\wedge = G^n$ we write

$$d_{P\text{-fol}}(g, g') < (\beta, \eta),$$

iff $d_{V_i\text{-fol}}(g_i, g'_i) < (\beta, \eta)$ for $i = 1, \ldots, n$. Similarly, we write $d_{P\text{-fol}}(g, g') \leq (\beta, \eta)$, iff $d_{V_i\text{-fol}}(g_i, g'_i) \leq (\beta, \eta)$ for $i = 1, \ldots, n$. Note that if $P = *$ is the empty tuple, then $d_{P\text{-fol}}(g, g') < (\beta, \eta)$ and $d_{P\text{-fol}}(g, g') \leq (\beta, \eta)$ are empty conditions and thus always satisfied. However, $|*|^\wedge$ is just a point so this is sensible[34].

Remark 5.5. Remark 5.3 also applies to $P \in \mathsf{P}_{+}\mathcal{O}\mathrm{p}(G)$: if $d_{P\text{-fol}}(\lambda, \lambda') \leq (\beta, \eta)$ with sufficiently small η, then $d_{P\text{-fol}}(\lambda, \lambda') \leq (\beta + \eta, 0)$.

We will need the following version of the triangle inequality for $d_{P\text{-fol}}$. Note that in the statement δ depends not on P.

Lemma 5.6 (Foliated triangle inequality)**.** *Let $\alpha \geq 0$. Then for any $\epsilon > 0$ there is $\delta > 0$ such that for $P \in \mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$ and $g, g', g'' \in |P|^\wedge = G^n$*

$$d_{P\text{-fol}}(g, g'),\ d_{P\text{-fol}}(g', g'') \leq (\alpha, \delta) \quad \Longrightarrow \quad d_{P\text{-fol}}(g, g'') \leq (2\alpha, \epsilon).$$

Proof. This is an easy consequence of (5.7b) below. ☐

Lemma 5.7.

(5.7a) Let $M \subseteq G$ be compact. For any $\epsilon > 0$ there is $\delta > 0$ such that for all $g, g' \in G$, $v \in M$ we have

$$d_G(g, g') < \delta \implies d_G(gv, g'v) < \epsilon;$$

(5.7b) Let $\alpha \geq 0$. Then for any $\epsilon > 0$ there is $\delta > 0$ such that for any closed subgroup V of G and $g, g', g'' \in G$ we haven

$$d_{V\text{-fol}}(g, g'), d_{V\text{-fol}}(g', g'') \leq (\alpha, \delta) \quad \Longrightarrow \quad d_{V\text{-fol}}(g, g'') \leq (2\alpha, \epsilon).$$

Proof. This is [6, Lem. 3.1]. ☐

[33] For V open in G we could simply work with any discrete metric on G/V, for example the metric that put different points at distance 1. The difficulty here arises only if G/V is not discrete.

[34] Recall that we do not allow $\eta = 0$ when considering $< (\beta, \eta)$

5.E. **The category $\mathsf{C}_G(P)$.**

Definition 5.8. Let $P \in \mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$. We define the G-control structure $\mathfrak{C}(P) = \big(\mathfrak{C}_1(P), \mathfrak{C}_2(P), \mathfrak{C}_G(P)\big)$ on $|P|^{\wedge} \times \mathbb{N}$ as follows:

- $\mathfrak{C}_1(P)$ consists of all subsets F of $|P|^{\wedge} \times \mathbb{N}$ for which $F \cap |P|^{\wedge} \times \{t\}$ is finite for all $t \in \mathbb{N}$;
- $\mathfrak{C}_2(P)$ consists of all subsets E of $\big(|P|^{\wedge} \times \mathbb{N}\big)^{\times 2}$ satisfying the following two conditions
  - *bounded control over* $\mathbb{N}$: there is $\alpha > 0$ such that for all $\left(\begin{smallmatrix} \lambda', t' \\ \lambda, t \end{smallmatrix}\right) \in E$ we have $|t - t'| \leq \alpha$;
  - *foliated control over* $|P|^{\wedge}$: there is $\beta \geq 0$ such that for any $\eta > 0$ there is t_0 such that for all $t \geq t_0$ and all λ, λ', t' we have
  $$\left(\begin{smallmatrix} \lambda', t' \\ \lambda, t \end{smallmatrix}\right) \in E \quad \Longrightarrow \quad d_{P\text{-fol}}(\lambda, \lambda') < (\beta, \eta);$$
- $\mathfrak{C}_G(P)$ consists of all relatively compact subsets of G.

We write $\mathcal{Y}(P)$ for the collection of all subsets of $|P|^{\wedge} \times \mathbb{N}$ that are contained in $|P|^{\wedge} \times \{0, \dots, N\}$ for some N.

It is an exercise to check that this is indeed a G-control structure. To check that $\mathfrak{C}_2(P)$ is closed under composition, the triangle inequality from Lemma 5.6 is used.

Definition 5.9. Let G be a td-group and $\mathcal{B}$ be a category with G-support. We set
$$\mathsf{C}_G(P; \mathcal{B}) := \mathcal{B}_G(\mathfrak{C}(P), \mathcal{Y}(P)).$$
We will often drop $\mathcal{B}$ from the notation and abbreviate $\mathsf{C}_G(P) = \mathsf{C}_G(P; \mathcal{B})$.

The assignment $P \mapsto \mathsf{C}_G(P)$ is functorial in $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$[35]. We obtain an $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$-spectrum
$$P \mapsto \mathbf{K}\big(\mathsf{C}_G(P)\big).$$

5.F. **The $\mathcal{C}$vcy-Farrell–Jones Conjecture.**

Definition 5.10 ($\mathcal{C}$vcy-assembly map)**.** Let G be a td-group and $\mathcal{B}$ be a category with G-support. The maps $P \to *$ for $P \in \mathsf{P}\mathcal{C}\mathrm{vcy}(G)$ induce a map
$$\operatorname*{hocolim}_{P \in \mathsf{P}\mathcal{C}\mathrm{vcy}(G)} \mathbf{K}\big(\mathsf{C}_G(P; \mathcal{B})\big) \;\to\; \mathbf{K}\big(\mathsf{C}_G(*; \mathcal{B})\big). \tag{5.11}$$
This is the $\mathcal{C}$vcy-*assembly map for* $\mathcal{B}$.

In light of the Farrell–Jones Conjecture for discrete groups one might expect that the homotopy colimit in (5.11) should be taken over $\mathsf{Or}_{\mathcal{C}\mathrm{vcy}}(G)$ instead of $\mathsf{P}\mathcal{C}\mathrm{vcy}(G)$. However, as discussed in Subsection 1.M it is an important point that we allow products here. On the other hand with slightly different definitions we could use $\mathsf{POr}_{\mathcal{C}\mathrm{vcy}}(G)$ in place of $\mathsf{P}\mathcal{C}\mathrm{vcy}(G)$, see Subsection 14.A.

Conjecture 5.12 ($\mathcal{C}$vcy-Farrell–Jones Conjecture)**.**
Let G be a td-group and $\mathcal{B}$ be a Hecke category with G-support. Then the $\mathcal{C}$vcy-assembly map (5.11) *for $\mathcal{B}$ is an equivalence.*

Remark 5.13. By Proposition 5.16 below (for $P = *$) we have
$$\Omega \mathbf{K}(\mathsf{C}_G(*; \mathcal{B})) \simeq \mathbf{K}(\mathcal{B}_G[*]) = \mathbf{K}(\mathcal{B}).$$
If $\mathcal{B} = \mathcal{B}(G; R)$, then $\mathbf{K}(\mathcal{B}) = \mathbf{K}(\mathcal{H}(G; R))$ and so in this case the $\mathcal{C}$vcy-Farrell–Jones Conjecture 5.12 is about the K-theory of the Hecke algebra $\mathcal{H}(G; R)$.

[35] $P \mapsto \mathsf{C}_G(P)$ is not strictly functorial in $\mathsf{P}_{+}\mathsf{Or}(G)$. This can be fixed, using a construction that will appear later in Subsection 14.A. But for now we will ignore this.

Remark 5.14. Let Γ be a discrete group and $\mathcal{A}$ be an additive category with a Γ-action. One obtains a category $\mathcal{A}[\Gamma]$ whose objects are the objects of $\mathcal{A}$. Morphisms $A \to A'$ in $\mathcal{A}[\Gamma]$ are finite formal sums $\sum_\gamma \varphi_\gamma \cdot \gamma$ where $\varphi_\gamma \colon \gamma A \to A'$ is a morphism in $\mathcal{A}$. The K-theoretic Farrell–Jones conjecture with coefficients for Γ concerns the K-theory of $\mathcal{A}[\Gamma]$. As $\mathcal{A}[\Gamma]$ is a Hecke category with Γ-support in an obvious way, one can use Proposition 3.13 and Proposition 5.16 below to check that for discrete groups the $\mathcal{C}$vcy-Farrell–Jones Conjecture 5.12 implies the usual K-theoretic Farrell–Jones Conjecture with coefficients. In fact, for discrete groups the two conjectures are equivalent see [7, Remark 5.7].

Theorem 5.15 ($\mathcal{C}$vcy-Farrell–Jones Conjecture for reductive p-adic groups)**.** *Let G be a reductive p-adic group and $\mathcal{B}$ be a Hecke category with G-support. Then the $\mathcal{C}$vcy-assembly map* (5.11) *for $\mathcal{B}$ is an equivalence.*

The formal framework of the proof of Theorem 5.15 is discussed in Section 6. The proof is then carried out in Sections 7 to 13.

5.G. **Relating $\mathsf{C}_G(P;\mathcal{B})$ to $\mathcal{B}[P]$.**

Proposition 5.16. *Let $\mathcal{B}$ be a category with G-support. There is a zig-zag of weak equivalences between the $\mathsf{P}_{+}\mathcal{O}\mathrm{p}(G)$-$\mathsf{Spectra}$*

$$P \mapsto \Omega\mathbf{K}\big(\mathsf{C}_G(P;\mathcal{B})\big) \quad \textit{and} \quad P \mapsto \mathbf{K}\big(\mathcal{B}[P]\big).$$

We start the proof of Proposition 5.16 with the following observation. Let $P \in \mathsf{P}_{+}\mathcal{O}\mathrm{p}(G)$. As noted in Remark 5.5, if $d_{P\text{-fol}}(\lambda,\lambda') \leq (\beta,\eta)$ with small η, then $d_{P\text{-fol}}(\lambda,\lambda') \leq (\beta+\eta,0)$. This implies that (using the foliated control condition) for $E \in \mathfrak{C}_2(P)$ there are $\beta > 0$ and $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$ and all λ,λ',t' we have

$$\left(\begin{smallmatrix} \lambda',t' \\ \lambda,t \end{smallmatrix}\right) \in E \quad \Longrightarrow \quad d_{P\text{-fol}}(\lambda,\lambda') < (\beta,0).$$

In the following definition we strengthen this to all t, not just sufficiently large t. This produce a control structure that is discrete over $|P|$ (with respect to the projection $|P|^\wedge \to |P|$).

Definition 5.17. Let $P \in \mathsf{P}_{+}\mathcal{O}\mathrm{p}(G)$. We define the G-control structure $\mathfrak{C}^{\mathrm{dis}}(P) = \big(\mathfrak{C}_1^{\mathrm{dis}}(P),\mathfrak{C}_2^{\mathrm{dis}}(P),\mathfrak{C}_G^{\mathrm{dis}}(P)\big)$ as follows. We set $\mathfrak{C}_1^{\mathrm{dis}}(P) := \mathfrak{C}_1(P)$, $\mathfrak{C}_G^{\mathrm{dis}}(P) := \mathfrak{C}_G(P)$ and define $\mathfrak{C}_2^{\mathrm{dis}}(P)$ to consist of all $E \in \mathfrak{C}_2(P)$ satisfying in addition the following: there is $\beta > 0$ such that

$$\left(\begin{smallmatrix} \lambda',t' \\ \lambda,t \end{smallmatrix}\right) \in E \quad \Longrightarrow \quad d_{P\text{-fol}}(\lambda,\lambda') \leq (\beta,0).$$

We define $\mathsf{C}_G^{\mathrm{dis}}(P) := \mathcal{B}_G(\mathfrak{C}^{\mathrm{dis}}(P),\mathcal{Y}(P))$.

Lemma 5.18. *Let $P \in \mathsf{P}_{+}\mathcal{O}\mathrm{p}(G)$.*

(5.18a) The inclusion $\mathsf{C}_G^{\mathrm{dis}}(P) \to \mathsf{C}_G(P)$ is an equivalence.

(5.18b) The projection $|P|^\wedge \times \mathbb{N} \to |P|$ induces an equivalence $\mathcal{B}_G(\mathfrak{C}^{\mathrm{dis}}(P)|_{\mathcal{Y}(P)}) \to (\mathcal{B}[P])_\oplus$.

(5.18c) The K-theory of the category $\mathcal{B}_G(\mathfrak{C}^{\mathrm{dis}}(P))$ vanishes.

Proof. The first two are easy exercises in the Definitions. The third comes from the standard Eilenberg swindle on $\mathcal{B}_G(\mathfrak{C}^{\mathrm{dis}}(P))$ using the shift $(\lambda,t) \mapsto (\lambda,t+1)$, i.e., Lemma 4.28 applies[36]. $\square$

[36] It is instructive to note that this swindle does not work on $\mathcal{B}_G(\mathfrak{C}(P))$: For $\varphi \in \mathcal{B}_G(\mathfrak{C}(P))$ there can be $\left(\begin{smallmatrix} \lambda',t' \\ \lambda,t \end{smallmatrix}\right) \in \operatorname{supp}_2 \varphi$ where λ and λ' have different images in $|P|$, i.e., there is η such that $d_{P\text{-fol}}(\lambda,\lambda') < (\beta,\eta)$ fails regardless of β. Then $\left(\begin{smallmatrix} \lambda',t'+n \\ \lambda,t+n \end{smallmatrix}\right) \in \operatorname{supp}_2 \varphi^\infty$ for all n and this violates the foliated control condition over $|P|^\wedge$, i.e., $\varphi^\infty \notin \mathcal{B}_G(\mathfrak{C}(P))$. In other words (4.29d) fails.

Proof of Proposition 5.16. The Karoubi sequence

$$\mathcal{B}_G(\mathfrak{C}^{\mathrm{dis}}(P)|_{\mathcal{Y}(P)}) \to \mathcal{B}_G(\mathfrak{C}^{\mathrm{dis}}(P)) \to \mathcal{B}_G(\mathfrak{C}^{\mathrm{dis}}(P), \mathcal{Y}(P)) = \mathrm{C}^{\mathrm{dis}}_G(P)$$

induces a fibration sequence in K-theory, see (4.22). Using (5.18c) we obtain a weak equivalence $\Omega\mathbf{K}\big(\mathrm{C}^{\mathrm{dis}}_G(P)\big) \xrightarrow{\sim} \mathbf{K}\big(\mathcal{B}_G(\mathfrak{C}^{\mathrm{dis}}(P)|_{\mathcal{Y}(P)})\big)$. Now (5.18a) and (5.18b) give the result. □

6. Formal framework of proof of the $\mathcal{C}$vcy-Farrell–Jones Conjecture for reductive p-adic groups

The proof of the $\mathcal{C}$vcy-Farrell–Jones Conjecture for reductive p-adic groups (Theorem 5.15) is organized around two functors

$$\mathbf{D}_G(-;\mathcal{B}),\ \mathbf{D}^0_G(-;\mathcal{B}) : \mathcal{R} \to \mathsf{Spectra}.$$

We will define the source category below and then discuss some properties of these functors. Theorem 5.15 is then an easy consequence of these properties. The functors $\mathbf{D}_G(-;\mathcal{B}) = \mathbf{K}(\mathrm{D}_G(-;\mathcal{B}))$ and $\mathbf{D}^0_G(-;\mathcal{B}) = \mathbf{K}(\mathrm{D}^0_G(-;\mathcal{B}))$ will be constructed in Section 7 as the K-theory of certain additive categories. The verification of their properties will occupy Sections 7 to 13. For most of these properties we can work with any category with G-support $\mathcal{B}$. The exception is the transfer from Theorem 6.7, for which we need $\mathcal{B}$ to be a Hecke category with G-support. Often we will drop $\mathcal{B}$ from the notation and write $\mathbf{D}_G(-) = \mathbf{D}_G(-;\mathcal{B})$ and $\mathbf{D}^0_G(-) = \mathbf{D}^0_G(-;\mathcal{B})$.

6.A. $\mathcal{C}$-simplicial complexes.

Definition 6.1 ($\mathcal{C}$-simplicial complexes)**.** Let $\mathcal{C}$ be a category. A *$\mathcal{C}$-simplicial complex* is a pair $\mathbf{\Sigma} = (\Sigma, C)$, where Σ is a simplicial complex and $P\colon \mathrm{simp}(\Sigma)^{\mathrm{op}} \to \mathcal{C}$ is a contravariant functor from the poset of simplices of Σ, ordered by inclusion, to $\mathcal{C}$. A map of $\mathcal{C}$-complexes $(\Sigma, C) \to (\Sigma', C')$ is a pair $\mathbf{f} = (f, \kappa)$, where $f\colon \Sigma \to \Sigma'$ is a simplicial map and $\kappa\colon C \to C' \circ f_*$ is a natural transformation. Here we write $f_*\colon \mathrm{simp}(\Sigma) \to \mathrm{simp}(\Sigma')$ for the map induced by f.

The *dimension* of $\mathbf{\Sigma} = (\Sigma, C)$ is the dimension of Σ; its *d-skeleton* is $\mathbf{\Sigma}^d := (\Sigma^d, C|_{\mathrm{simp}(\Sigma^d)})$, where Σ^d is the d-skeleton of Σ.

Definition 6.2. We define $\mathcal{R}$ as the category of $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$-simplicial complexes and write $\mathcal{R}^0$ for the full subcategory of 0-dimensional $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$-simplicial complexes.

We can think about $\mathcal{R}^0$ as being obtained from $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$ by adding arbitrary coproducts to $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$. There is a product

$$\begin{array}{ccc} \mathcal{R} \times \mathcal{R}^0 & \to & \mathcal{R} \\ \big((\Sigma, P), (M, Q)\big) & \mapsto & \big(\Sigma \times M, (\sigma, m) \mapsto P(\sigma) \times Q(m)\big). \end{array} \tag{6.3}$$

We write $\mathcal{R}^0_{\mathcal{F}}$ for the full subcategory of $\mathcal{R}^0$ on all (M, P) where P takes values in $\mathsf{P}\mathcal{F}(G)$. A drawback of our notation is that $\mathcal{R}^0_{\mathcal{A}\mathrm{ll}(G)} \subsetneq \mathcal{R}^0$, because the empty product $*$ is not contained in $\mathsf{P}\mathcal{A}\mathrm{ll}(G)$. However, typically $\mathcal{F}$ will be a proper collection of subgroups, so this should not lead to serious confusion.

Example 6.4. Let $\mathcal{F}$ be a collection of closed subgroups of G. We write $\Sigma_{\mathcal{F}}(G)$ for the following simplicial complex. Vertices of $\Sigma_{\mathcal{F}}(G)$ are pairs (n, V) with $n \in \mathbb{N}$ and $V \in \mathcal{F}$. Vertices $(n_0, V_0), \ldots, (n_k, V_k)$ form a simplex of $\Sigma_{\mathcal{F}}(G)$, if and only if the n_i are pairwise distinct[37]. There is an evident functor $P_{\mathcal{F}}(G)\colon \mathrm{simp}(\Sigma_{\mathcal{F}}(G))^{\mathrm{op}} \to \mathsf{P}\mathcal{F}(G)$ that sends a simplex $\sigma = \big\{(n_0, V_0), \ldots, (n_k, V_k)\big\}$ to $(G/V_0, \ldots, G/V_k)$ with $n_0 < \cdots < n_k$, where we choose the numbering such that $n_0 < \cdots < n_k$. We obtain $\mathbf{J}_{\mathcal{F}}(G) := \big(\Sigma_{\mathcal{F}}(G), P_{\mathcal{F}}(G)\big) \in \mathcal{R}$.

[37] Alternatively, $\Sigma_{\mathcal{F}}(G)$ is the infinite join $*_{n\in\mathbb{N}}(\coprod_{F\in\mathcal{F}} G/F)$.

We write $\Sigma^N_{\mathcal{F}}(G)$ for the subcomplex of $\Sigma_{\mathcal{F}}(G)$ spanned by all vertices (n,V) with $n \le N$[38]. Then $\Sigma^N_{\mathcal{F}}(G)$ is a proper subcomplex of the N-skeleton $\big(\Sigma_{\mathcal{F}}(G)\big)^N$. For finite $\mathcal{F}$, $\Sigma^N_{\mathcal{F}}(G)$ is a finite complex, while the N-skeleton $\big(\Sigma_{\mathcal{F}}(G)\big)^N$ is never finite. We set $\mathbf{J}^N_{\mathcal{F}}(G) := \big(\Sigma^N_{\mathcal{F}}(G), P_{\mathcal{F}}(G)|_{\mathrm{simp}(\Sigma^N_{\mathcal{F}}(G))}\big)$.

We will discuss in Subsection 7.B realization functors from $\mathcal{R}$ to G-spaces. The realization $|\mathbf{J}_{\mathcal{F}}(G)|$ of $\mathbf{J}_{\mathcal{F}}(G)$ is the numerable classifying spaces for $\mathcal{F}$ [10, A1], see Example 7.2. This motivated the definition of $\mathbf{J}_{\mathcal{F}}(G)$.

6.B. **Coefficients of $\mathbf{D}_G$.** Write $I\colon \mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G) \to \mathcal{R}^0$ for the inclusion. The underlying simplicial complex of $I(P)$ consist of one vertex which is sent to P. We will show in Proposition 8.1 that there exists a zig-zag of equivalences of $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$-spectra between $I^*\Omega\mathbf{D}_G(-)$ and $\mathbf{K}\big(\mathsf{C}_G(-)\big)$. To ease notation we will often abbreviate $P = I(P)$ and omit I^* from the notation.

6.C. **Computation of $\mathbf{D}_G$ on $\mathcal{R}^0$.** Let $(M,P) \in \mathcal{R}^0$. We will show in Proposition 8.5 that the canonical map

$$\bigvee_{m\in M} \mathbf{D}_G(P(m)) \xrightarrow{\sim} \mathbf{D}_G((M,P)) \tag{6.5}$$

is an equivalence.

6.D. **$\mathbf{D}^0_G$ determines $\mathbf{D}_G$.** We will construct in Proposition 8.9 a diagram in $\mathcal{R}$-$\mathsf{Spectra}$

$$\begin{tikzcd}
\mathbf{D}^0_G(-) & \mathbf{D}^0_G(-) \arrow[l] \arrow[r] & \mathbf{D}^0_G(-) \\
\mathbf{D}^0_G(-) \arrow[u] \arrow[d] & \mathbf{D}^0_G(-) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathbf{D}^0_G(-) \arrow[u] \arrow[d] \\
\mathbf{D}^0_G(-) & \mathbf{D}^0_G(-) \arrow[l] \arrow[r] & \mathbf{D}^0_G(-)
\end{tikzcd}$$

whose homotopy colimit is equivalent to $\mathbf{D}_G(-)$.

6.E. **Homotopy invariance for $\mathbf{D}^0_G$.** Let $\mathbf{M} = (M,P) \in \mathcal{R}^0$. Let $\pi\colon M \times \Delta^d \to M$ be the projection. We obtain

$$\mathbf{\Delta}^d_{\mathbf{M}} := (M \times \Delta^d, P \circ \pi_*) \in \mathcal{R}.$$

A choice of a point $x_0 \in |\Delta^d|$ determines an inclusion $\mathbf{i}\colon \mathbf{M} \to \mathbf{\Delta}^d_{\mathbf{M}}$. We show in Proposition 8.16 that $\mathbf{i}$ induces an equivalence

$$\mathbf{D}^0_G(\mathbf{M}) \xrightarrow{\sim} \mathbf{D}^0_G(\mathbf{\Delta}^d_{\mathbf{M}}).$$

6.F. **Excision for $\mathbf{D}^0_G$.** Let $\mathbf{\Sigma} = (\Sigma, P)$ in $\mathcal{R}$ be d-dimensional. Assume that the vertices of Σ are locally ordered; then any simplex of Σ is canonically isomorphic to a standard simplex Δ^k. Let B be the set of d-simplices of Σ. We obtain a canonical map $f\colon B \times \Delta^d \to \Sigma$. Let $\pi\colon B \times \Delta^d \to B$ be the projection. Set $\widehat{\Sigma} := B \times \Delta^d$, $\widehat{P} := p|_B \circ \pi_*$ and $\widehat{\mathbf{\Sigma}} := (\widehat{\Sigma}, \widehat{P})$. Let $\mathbf{f} := (f,\kappa)\colon \widehat{\mathbf{\Sigma}} \to \mathbf{\Sigma}$ where κ is defined as follows. Let τ be a simplex of $\widehat{\Sigma}$. Then τ is contained in $\{\sigma\} \times \Delta^d$ for some d-simplex σ of Σ and we have $\widehat{P}(\tau) = P(\sigma)$ and $f(\tau) \subseteq \sigma$. We define $\kappa_\tau\colon \widehat{P}(\tau) \to P(f(\tau))$ as the evaluation of P on the inclusion $f(\tau) \subseteq \sigma$. Let $\mathbf{\Sigma}' = (\mathbf{\Sigma}', \mathbf{P}')$ be the $(d$-1)-skeleton

[38] Alternatively, $\Sigma^N_{\mathcal{F}}(G) = *_{n\le N}(\coprod_{F\in\mathcal{F}} G/F)$.

of $\mathbf{\Sigma}$ and $\widehat{\mathbf{\Sigma}}' = (\Sigma', P')$ be the $(d\text{-}1)$-skeleton of $\widehat{\mathbf{\Sigma}}$. Then $\mathbf{f}$ restricts to $\mathbf{f}'\colon \widehat{\mathbf{\Sigma}}' \to \mathbf{\Sigma}'$. We write $\widehat{\iota}\colon \widehat{\mathbf{\Sigma}}' \to \widehat{\mathbf{\Sigma}}$ and $\iota\colon \mathbf{\Sigma}' \to \mathbf{\Sigma}$ for the canonical inclusions and obtain

(6.6)
$$\begin{tikzcd} \widehat{\mathbf{\Sigma}}' \arrow[r, "\mathbf{f}'"] \arrow[d, "\widehat{\iota}"] & \mathbf{\Sigma}' \arrow[d, "\iota"] \\ \widehat{\mathbf{\Sigma}} \arrow[r, "\mathbf{f}"] & \mathbf{\Sigma}. \end{tikzcd}$$

We show in Proposition 8.20 that $\mathbf{D}^0_G(-)$ takes this diagram to a homotopy pushout diagram of spectra.

6.G. **Skeleton continuity of $\mathbf{D}^0_G$.** Let $\mathbf{\Sigma} \in \mathcal{R}$. We show in Proposition 8.22 that the canonical map
$$\operatorname*{hocolim}_{d\in\mathbb{N}} \mathbf{D}^0_G(\mathbf{\Sigma}^d) \xrightarrow{\sim} \mathbf{D}^0_G(\mathbf{\Sigma})$$
is an equivalence.

6.H. **Transfer.** We use $\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)$ from Example 6.4 and consider
$$\mathbf{D}^0_G\big(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times -;\mathcal{B}\big)\colon \mathcal{R} \to \mathsf{Spectra}.$$
The projections $\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times P \to P$ induce a projection
$$\mathbf{p}\colon \mathbf{D}^0_G\big(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times -;\mathcal{B}\big) \to \mathbf{D}^0_G(-;\mathcal{B})$$
in $\mathcal{R}$-spectra.

Theorem 6.7. *Assume that G is a reductive p-adic group and that $\mathcal{B}$ is a Hecke category with G-support. Then the projection $\mathbf{p}$ admits a section, i.e., there is $\mathbf{tr}\colon \mathbf{D}^0_G(-;\mathcal{B}) \to \mathbf{D}^0_G\big(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times -;\mathcal{B}\big)$ such that $\mathbf{p}\circ\mathbf{tr}$ is equivalent to the identity in $\mathcal{R}^0$-spectra.*

Proof of Theorem 5.15 modulo properties of $\mathbf{D}_G$ and $\mathbf{D}^0_G$. We need to show that the $\mathcal{C}$vcy-assembly map

(6.8)
$$\operatorname*{hocolim}_{P\in \mathsf{P}\mathcal{C}\mathrm{vcy}(G)} \mathbf{K}\big(\mathsf{C}_G(P)\big) \;\to\; \mathbf{K}\big(\mathsf{C}_G(*)\big)$$

is an equivalence. By the equivalence from Subsection 6.B we can equivalently show that

(6.9)
$$\operatorname*{hocolim}_{P\in \mathsf{P}\mathcal{C}\mathrm{vcy}(G)} \mathbf{D}_G(P) \;\to\; \mathbf{D}_G(*)$$

is an equivalence. We obtain the following factorization of (6.9)

(6.10)
$$\operatorname*{hocolim}_{P\in \mathsf{P}\mathcal{C}\mathrm{vcy}(G)} \mathbf{D}_G(P) \;\to\; \operatorname*{hocolim}_{(M,P)\in\mathcal{R}^0_{\mathcal{C}\mathrm{vcy}}} \mathbf{D}_G(M,P) \;\to\; \mathbf{D}_G(*).$$

For fixed $(M,P) \in \mathcal{R}^0_{\mathcal{C}\mathrm{vcy}}$ consider the canonical map

(6.11)
$$\operatorname*{hocolim}_{Q\in I\downarrow(M,P)} \mathbf{D}_G(Q) \to \mathbf{D}_G(M,P)$$

where I denotes the inclusion $\mathsf{P}\mathcal{C}\mathrm{vcy}(G) \to \mathcal{R}^0_{\mathcal{C}\mathrm{vcy}}$. It is not difficult to identify (6.11) with (6.5), which is an equivalence. The transitivity Lemma A.2 for homotopy colimits implies now that the first map in (6.10) is an equivalence. As $\mathbf{D}_G$ can be expressed as a homotopy colimit in $\mathbf{D}^0_G$, see Subsection 6.D, the second map in (6.10) is an equivalence if

(6.12)
$$\operatorname*{hocolim}_{(M,P)\in\mathcal{R}^0_{\mathcal{C}\mathrm{vcy}}} \mathbf{D}^0_G(M,P) \;\to\; \mathbf{D}^0_G(*)$$

is an equivalence. Theorem 6.7 implies that (6.12) is a retract of

(6.13)
$$\operatorname*{hocolim}_{(M,P)\in\mathcal{R}^0_{\mathcal{C}\mathrm{vcy}}} \mathbf{D}^0_G\big(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times (M,P)\big) \;\to\; \mathbf{D}^0_G(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)).$$

We now use that $\mathbf{D}^0_G$ is homotopy invariant (Subsection 6.E), satisfies an excision result (Subsection 6.F) and skeleta continuity (Subsection 6.G). These properties imply that $\mathbf{D}^0_G\big(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times -\big)$ can in $\mathcal{R}^0$-spectra be constructed as a homotopy colimit of functors of the form $\mathbf{D}^0_G\big((M_0, P_0) \times -\big)$ with $(M_0, P_0) \in \mathcal{R}^0_{\mathcal{C}\mathrm{vcy}}$. Lemma A.3 implies that for all $(M_0, P_0) \in \mathcal{R}^0_{\mathcal{C}\mathrm{vcy}}$

$$\operatorname*{hocolim}_{(M,P)\in\mathcal{R}^0_{\mathcal{C}\mathrm{vcy}}} \mathbf{D}^0_G\big((M_0,P_0)\times(M,P)\big) \xrightarrow{\sim} \mathbf{D}^0_G(M_0,P_0)$$

is an equivalence. Thus (6.13) is an equivalence and so is (6.8). □

Remark 6.14. It is possible to show that the functor $\mathbf{D}_G$, which we construct later on, also satisfies homotopy invariance, excision and continuity exactly as $\mathbf{D}^0_G$. Moreover, Theorem 6.7 holds also for $\mathbf{D}_G$. Thus it is possible to prove the Farrell–Jones Conjecture for reductive p-adic groups using only $\mathbf{D}_G$. In fact, homotopy invariance, excision and continuity for $\mathbf{D}_G$ can be proven in exactly the same way as for $\mathbf{D}^0_G$. However, the construction of the transfer map $\mathbf{tr}$ in Theorem 6.7 is technically easier for $\mathbf{D}^0_G$ than for $\mathbf{D}_G$; this is the reason for our small detour through $\mathbf{D}^0_G$. The other way round, we cannot replace $\mathbf{D}_G$ with $\mathbf{D}^0_G$ throughout; the equivalences from Subsections 6.B and 6.C do not hold for $\mathbf{D}^0_G$ in place of $\mathbf{D}_G$.

7. The categories $\mathrm{D}_G(\mathbf{\Sigma})$ and $\mathrm{D}^0_G(\mathbf{\Sigma})$

In this section we construct the two functors $\mathbf{D}_G$ and $\mathbf{D}^0_G$ promised in Section 6 as the K-theory of functors $\mathrm{D}_G(-)$ and $\mathrm{D}^0_G(-)$ to additive categories. We will need some preparations.

7.A. **Some notation for simplicial complexes.** Let Σ be an (abstract) simplicial complex. We write $\mathrm{vert}(\Sigma)$ for the set of vertices of Σ. We will write $|\Sigma|$ for the realization of Σ to topological spaces. For a simplex σ of Σ we write Δ_σ for the simplicial subcomplex of Σ spanned by σ and $\partial\Delta_\sigma$ for its boundary. So $\partial\Delta_\sigma$ is obtained from Δ_σ by omitting σ. For a vertex $v \in \mathrm{vert}(\Sigma)$ we will not distinguish between the abstract vertex v and the corresponding point $v \in |\Sigma|$. Any point $x \in |\Sigma|$ has unique barycentric coordinates, $x = \sum_{v\in\mathrm{vert}(\Sigma)} x(v)\cdot v$ with $x(v) \in [0,1]$, $\sum_{v\in\mathrm{vert}(\Sigma)} x(v) = 1$, $x(v) \neq 0$ for only finitely many v. Of course, $\sigma := \{v \mid x(v) \neq 0\}$ forms a simplex with $x \in |\Delta_\sigma| \setminus |\partial\Delta_\sigma|$. The ℓ^∞-metric[39] on $|\Sigma|$ is

$$d^\infty(x,x') := \max_{v\in\mathrm{vert}(\Sigma)} |x(v) - x'(v)|. \tag{7.1}$$

For $\sigma \in \mathrm{simp}(\Sigma)$ we set

$$U_\sigma := \big\{x \in |\Sigma| \;\big|\; \forall v \in \sigma : x(v) > 0\big\},$$

this is the open star of σ, i.e., the union of the interiors of those simplices which contain σ as face. It is an open neighborhood of $|\Delta_\sigma| \setminus |\partial\Delta_\sigma|$. For $\epsilon > 0$ we set

$$K_{\sigma,\epsilon} := \{x \in |\Sigma| \mid \forall v \in \sigma : x(v) \geq \epsilon\}.$$

This is a closed subset of U_σ. We record that the $K_{\sigma,\epsilon}$ get larger with decreasing ϵ and that $U_\sigma = \bigcup_{\epsilon>0} K_{\sigma,\epsilon}$. Moreover U_σ is the ϵ-neighborhood of $K_{\sigma,\epsilon}$ with respect to d^∞.

[39] In general, the topology of the ℓ^∞-metric is coarser than the weak topology on $|\Sigma|$, but we will mostly only use it on finite subcomplexes of Σ, where both topologies coincide. Also, on finite dimensional subcomplexes the ℓ^∞-metric and the ℓ^1-metric (that we used for example in [9]) are Lipschitz equivalent. Using the ℓ^∞-metric is more convenient here, but there is no substantial difference.

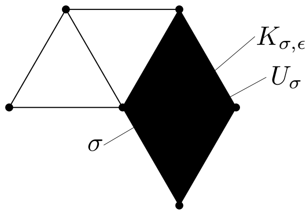

FIGURE 1. U_σ and $K_{\sigma,\epsilon}$

7.B. **The G-spaces $|\mathbf{\Sigma}|$ and $|\mathbf{\Sigma}|^\wedge$.** Write $\Delta\colon \operatorname{simp}(\Sigma) \to \mathsf{Spaces}$ for the functor $\sigma \mapsto |\Delta_\sigma|$ and define the realization functor

$$\begin{array}{rcl} |-| : \mathcal{R} & \to & G\text{-}\mathsf{Spaces} \\ \mathbf{\Sigma} = (\Sigma, P) & \mapsto & \Delta \times_{\operatorname{simp}(\Sigma)} |P(-)|. \end{array}$$

In the construction of $\mathsf{C}_G(P)$ for $P \in \mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$ we used the G-space $|P|^\wedge$ from Subsection 5.B, thought of as a resolution of $|P|$. We will need a similar resolution for $|\mathbf{\Sigma}|$. Define

$$\begin{array}{rcl} |-|^\wedge : \mathcal{R} & \to & G\text{-}\mathsf{Spaces} \\ \mathbf{\Sigma} = (\Sigma, P) & \mapsto & \Delta \times_{\operatorname{simp}(\Sigma)} |P(-)|^\wedge. \end{array}$$

The projections maps $|P|^\wedge \to *$ induce a map

$$p_{\mathbf{\Sigma}}\colon |\mathbf{\Sigma}|^\wedge \to |\Sigma| = \Delta \times_{\operatorname{simp}\Sigma} *.$$

Let σ be a simplex of Σ. It comes with a canonical map

$$|\Delta_\sigma| \times |P(\sigma)|^\wedge \to |\mathbf{\Sigma}|^\wedge.$$

We will write this map as $(x, \lambda) \mapsto [x, \lambda]_\sigma$. Of course $p_{\mathbf{\Sigma}}\big([x,\lambda]_\sigma\big) = x$. Altogether these canonical maps define the projection

$$\coprod_\tau |\Delta_\tau| \times |P(\tau)|^\wedge \xrightarrow{q_{\mathbf{\Sigma}}} |\mathbf{\Sigma}|^\wedge$$

which is an identification of topological spaces. For a simplex σ the preimage of $U_\sigma \subseteq |\Sigma|$ under $p_{\mathbf{\Sigma}} \circ q_{\mathbf{\Sigma}}$ is $\coprod_{\tau, \sigma \leq \tau} \big(|\Delta_\tau| \setminus \partial_\sigma |\Delta_\tau|\big) \times |P(\tau)|^\wedge$, where $\partial_\sigma|\Delta_\tau| = \bigcup_{\mu \subseteq \tau, \sigma \not\subseteq \tau} |\Delta_\mu|$. We define $\lambda_{\sigma,\tau}\colon \big(|\Delta_\tau| \setminus \partial_\sigma|\Delta_\tau|\big) \times |P(\tau)|^\wedge \to |P(\sigma)|^\wedge$ to be the composite of the projection $\big(|\Delta_\tau| \setminus \partial_\sigma|\Delta_\tau|\big) \times |P(\tau)|^\wedge \to |P(\tau)|^\wedge$ with the map $|P(\tau)|^\wedge \to |P(\sigma)|^\wedge$ induced by $\sigma \subseteq \tau$. One easily checks that the map

$$\coprod_{\tau, \sigma \leq \tau} \lambda_{\sigma,\tau} \;:\; \coprod_{\tau, \sigma \leq \tau} \big(|\Delta_\tau| \setminus \partial_\sigma|\Delta_\tau|\big) \times |P(\tau)|^\wedge \to |P(\sigma)|^\wedge$$

factorizes over the projection $\coprod_{\tau,\sigma \leq \tau} |\Delta_\tau| \setminus \partial_\sigma|\Delta_\tau| \times |P(\tau)|^\wedge \to (p_{\mathbf{\Sigma}})^{-1}(U_\sigma)$, to a map $\lambda_\sigma\colon (p_{\mathbf{\Sigma}})^{-1}(U_\sigma) \to |P(\sigma)|^\wedge$. We note that $\lambda_\sigma\big([x,\lambda]_\sigma\big) = \lambda$.

Example 7.2 (Realizations of $\mathbf{J}_{\mathcal{F}}(G)$ and $\mathbf{J}^N_{\mathcal{F}}(G)$)**.** Let $\mathbf{J}_{\mathcal{F}}(G) = (\Sigma_{\mathcal{F}}(G), P_{\mathcal{F}}(G))$ be as in Example 6.4. It is not hard to check that then $|\mathbf{J}_{\mathcal{F}}(G)| = *_{n \in \mathbb{N}}(\coprod_{F \in \mathcal{F}} G/F)$ and $|\mathbf{J}_{\mathcal{F}}(G)|^\wedge = *_{n\in\mathbb{N}}(\coprod_{F\in\mathcal{F}} G) = *_{n\in\mathbb{N}}(G \times \mathcal{F})$ hold. The canonical projection $(\coprod_{F\in\mathcal{F}} G) = G \times \mathcal{F} \to \mathcal{F}$ induces the projection $p_{\mathbf{J}_{\mathcal{F}}(G)}\colon |\mathbf{J}_{\mathcal{F}}(G)|^\wedge = *_{n\in\mathbb{N}}(G \times \mathcal{F}) \to *_{n\in\mathbb{N}}(\mathcal{F}) = |\Sigma_{\mathcal{F}}(G)|$.

Similarly, $|\mathbf{J}^N_{\mathcal{F}}(G)| = *_{n \leq N}(\coprod_{F\in\mathcal{F}} G/F)$ and $|\mathbf{J}_{\mathcal{F}}(G)|^\wedge = *_{n\leq N}(\coprod_{F\in\mathcal{F}} G) = *_{n \leq N}(G \times \mathcal{F})$. In this description of points in $|\mathbf{J}_{\mathcal{F}}(G)|^\wedge$ can be written as $z = [t_0 \cdot (g_0, V_0), \ldots, t_N \cdot (g_N, V_N)]$ with $t_i \in [0,1]$, $g_i \in G$, $V_i \in \mathcal{F}$ where $\sum t_i = 1$. In this notation $[t_0 \cdot (g_0, V_0), \ldots, t_N \cdot (g_N, V_N)] = [t'_0 \cdot (g'_0, H'_0), \ldots, t'_N \cdot (g'_N, H'_N)]$ if and only if $t_i = t'_i$ for $i = 0, \ldots, N$, and $(g_i, V_i) = (g'_i, H'_i)$ for all i with $t_i = t'_i \neq 0$.

7.C. **Foliated distance in $|\mathbf{\Sigma}|^\wedge$.** We extend the notion of foliated distance from Subsection 5.D to $\mathcal{R}$. Let $\mathbf{\Sigma} = (\Sigma, P) \in \mathcal{R}$ and $\beta, \eta, \epsilon > 0$. For $z, z' \in |\mathbf{\Sigma}|^\wedge$ we write

$$d_{\mathbf{\Sigma}\text{-fol}}(z, z') < (\beta, \eta, \epsilon),$$

iff the following two conditions are satisfied

(7.3a) $d^\infty\big(p_{\mathbf{\Sigma}}(z), p_{\mathbf{\Sigma}}(z')\big) < \epsilon$;

(7.3b) for all $\sigma \in \operatorname{simp}(\Sigma)$ with $p_\Sigma(z) \in K_{\sigma,\epsilon}$ or $p_\Sigma(z') \in K_{\sigma,\epsilon}$, we require

$$d_{P(\sigma)\text{-fol}}\big(\lambda_\sigma(z), \lambda_\sigma(z')\big) < (\beta, \eta).$$

Note that the first condition implies that if $p_\Sigma(z) \in K_{\sigma,\epsilon}$ or $p_\Sigma(z') \in K_{\sigma,\epsilon}$, then both, $p_\Sigma(z)$ and $p_\Sigma(z')$, belong to U_σ, and both, $\lambda_\sigma(z)$ and $\lambda_\sigma(z')$, are defined.

We note that this definition is compatible with restrictions to subcomplexes. More precisely, let $\Sigma' \subseteq \Sigma$ be a subcomplex and let $\mathbf{\Sigma}' := (\Sigma', P|_{\operatorname{simp}(\Sigma')})$. If σ is a simplex of Σ', then $K^{\Sigma'}_{\sigma,\epsilon} = \Sigma' \cap K^{\Sigma}_{\sigma,\epsilon}$, where the upper index indicates in which complex we form $K_{\sigma,\epsilon}$. Thus for $z, z' \in |\mathbf{\Sigma}'|^\wedge \subseteq |\mathbf{\Sigma}|^\wedge$ we have $d_{\mathbf{\Sigma}'\text{-fol}}(z, z') < (\beta, \eta, \epsilon)$ iff $d_{\mathbf{\Sigma}\text{-fol}}(z, z') < (\beta, \eta, \epsilon)$.

Remark 7.4. Recall that we think of V-foliated distance on G as a way to get around the problem that there may not exist G-invariant metrics on G/V, see Remark 5.4

Given $\mathbf{\Sigma} = (\Sigma, P) \in \mathcal{R}$, we would ideally like to equip the G-space $|\mathbf{\Sigma}|$ with a G-invariant metric (and as for G/V we would be interested in small distances in $|\mathbf{\Sigma}|$). However, this space can have isotropy groups for which the orbit G/V admits no G-invariant metric and then neither does $|\mathbf{\Sigma}|$. The notion of foliated distance in $|\mathbf{\Sigma}|^\wedge$ is our replacement for $|\mathbf{\Sigma}|$ with (a non-existing) G-invariant metric. A way to think about this replacement is that we add to points in $|\mathbf{\Sigma}|$ a choice of lift to $|\mathbf{\Sigma}|^\wedge$, where the choice of lift is only relevant up to bounded distance in the fibers for $|\mathbf{\Sigma}|^\wedge \to |\mathbf{\Sigma}|$. Alternatively, we can think of this as adding to points in $|\Sigma|$ a choice of lift to $|\mathbf{\Sigma}|^\wedge$, where the choice of lift is only relevant up to foliated distance in the fibers for $p_{\mathbf{\Sigma}}\colon |\mathbf{\Sigma}|^\wedge \to |\Sigma|$.

Remark 7.5. One should think of $d_{\mathbf{\Sigma}\text{-fol}}(z, z') < (\beta, \eta, \epsilon)$ as a two stage condition. The first stage just uses the images of z, z' in $|\Sigma|$ and requires their distance to be $< \epsilon$. For general z and z' we can not compare the two fibers for $p_{\mathbf{\Sigma}}\colon |\mathbf{\Sigma}|^\wedge \to |\Sigma|$ containing them. But whenever one of z and z' projects into $K_{\sigma,\epsilon}$, then $\lambda_\sigma(z), \lambda_\sigma(z') \in P(\sigma)$ are both defined and we require $d_{P(\sigma)\text{-fol}}(\lambda_\sigma(z), \lambda_\sigma(z')) < (\beta, \eta)$ in the second stage of the condition.

We point out the following detail about the second stage. Recall $K_{\sigma,\epsilon} \subseteq U_\sigma$. One might be tempted to require $d_{P(\sigma)\text{-fol}}(\lambda_\sigma(z), \lambda_\sigma(z')) < (\beta, \eta)$ whenever both z and z' project into U_σ, as this suffices for $\lambda_\sigma(z), \lambda_\sigma(z')$ to be defined. However, if neither z nor z' projects into $K_{\sigma,\epsilon}$ (but their images in $|\Sigma|$ are close), then both z and z' are close to $p_{\mathbf{\Sigma}}^{-1}\big(|\Delta_\tau|\big)$ for a simplex τ (of smaller dimension than σ) and in passing from z to z' one might take a short-cut through $p_{\mathbf{\Sigma}}^{-1}(|\Delta_\tau|)$ and mostly avoid $p_{\mathbf{\Sigma}}^{-1}(|\Delta_\sigma|)$. Thus in this situation $d_{P(\sigma)\text{-fol}}\big(\lambda_\sigma(z), \lambda_\sigma(z')\big)$ is not necessarily relevant in comparing z and z'. Requiring that that at least one of z and z' projects into $K_{\sigma,\epsilon}$ avoids this problem.

More formally, our formulation of (7.3a) and (7.3b) guarantees (using Lemma 5.6) the following version of the triangle inequality. Given $\beta > 0$, $\eta > 0$ there is $\rho > 0$ such that for $\epsilon > 0$, $z, z', z'' \in |\mathbf{\Sigma}|^\wedge$ we have

$$d_{\mathbf{\Sigma}\text{-fol}}(z, z') < (\beta, \rho, \epsilon), d_{\mathbf{\Sigma}\text{-fol}}(z', z'') < (\beta, \rho, \epsilon) \implies d_{\mathbf{\Sigma}\text{-fol}}(z, z'') < (2\beta, \eta, 2\epsilon).$$

Example 7.6 (Foliated distance for $\mathbf{J}^N_{\mathcal{F}}(G)$)**.** For $\mathbf{J}^N_{\mathcal{F}}(G) = (\Sigma^N_{\mathcal{F}}(G), P_{\mathcal{F}}(G))$ from Example 6.4 we can use the join description of $|\mathbf{J}^N_{\mathcal{F}}(G)|^\wedge = *_{n\leq N}(G \times \mathcal{F})$ from

Example 7.2 to unravel the definition of the foliated distance as follows. Let $z := [t_0 \cdot (g_0, V_0), \dots, t_N \cdot (g_N, V_N)]$, $z' := [t'_0 \cdot (g'_0, V'_0), \dots, t'_N \cdot (g'_N, V'_N)] \in |\mathbf{J}^N_{\mathcal{F}}(G)|^\wedge$. Then $d_{\mathbf{J}^N_{\mathcal{F}}(G)\text{-fol}}(z, z') < (\beta, \eta, \epsilon)$ if and only if

(7.6a) $|t_i - t'_i| < \epsilon$ for all i;
(7.6b) for all i with $\max\{t_i, t'_i\}$ we have $V_i = V'_i$ and $d_{V_i\text{-fol}}(g_i, g'_i) < (\beta, \eta)$.

7.D. **The G-control structures $\mathfrak{D}(\mathbf{\Sigma})$ and $\mathfrak{D}^0(\mathbf{\Sigma})$.** Let $\mathbf{\Sigma} = (\Sigma, P) \in \mathcal{R}$. In the following definition we will define a control structure $\mathfrak{D}(\mathbf{\Sigma})$ on $|\mathbf{\Sigma}|^\wedge \times \mathbb{N}^{\times 2}$. The two $\mathbb{N}$-directions will be used to encode two different control conditions. The first factor will be used to encode a foliated control conditions over the $P(\sigma)$, that is compatible with $\mathfrak{C}(P(\sigma))$. The second $\mathbb{N}$-factor will be used to encode an ϵ-control condition over $|\Sigma|$ with respect to d^∞. In particular Definition 7.7 is not symmetric in t_0 and t_1. The remarks following the definition provide some discussion and motivation.

We will use the ℓ^1-norm $|(t_0, t_1)| = t_0 + t_1$ on $\mathbb{N}^{\times 2}$.

Definition 7.7. Let $\mathbf{\Sigma} = (\Sigma, P) \in \mathcal{R}$. We define the G-control structure $\mathfrak{D}(\mathbf{\Sigma}) = \big(\mathfrak{D}_1(\mathbf{\Sigma}), \mathfrak{D}_2(\mathbf{\Sigma}), \mathfrak{D}_G(\mathbf{\Sigma})\big)$ on the G-space $|\mathbf{\Sigma}|^\wedge \times \mathbb{N}^{\times 2}$ as follows.

(7.7a) $\mathfrak{D}_1(\mathbf{\Sigma})$ consists of all subsets F of $|\mathbf{\Sigma}|^\wedge \times \mathbb{N}^{\times 2}$ satisfying the following conditions
- *Finiteness over* $\mathbb{N}^{\times 2}$: for all $\underline{t} \in \mathbb{N}^{\times 2}$ the set $F \cap |\mathbf{\Sigma}|^\wedge \times \{\underline{t}\}$ is finite;
- *Compact support in* $|\Sigma|$: for every $t_0 \in \mathbb{N}$ there exists a finite subcomplex Σ_0 of Σ such that $F \cap |\mathbf{\Sigma}|^\wedge \times \{t_0\} \times \mathbb{N} \subseteq p_{\mathbf{\Sigma}}^{-1}\big(|\Sigma_0|\big) \times \mathbb{N}^{\times 2}$;
- *Finite dimensional support*: $F \subseteq |\mathbf{\Sigma}^d|^\wedge \times \mathbb{N}^{\times 2}$;

(7.7b) $\mathfrak{D}_2(\mathbf{\Sigma})$ consists of all subsets E of $\big(|\mathbf{\Sigma}|^\wedge \times \mathbb{N}^{\times 2}\big)^{\times 2}$ satisfying the following conditions
- *Bounded control over* $\mathbb{N}^{\times 2}$: there is $\alpha > 0$ such that if $\left(\begin{smallmatrix} z', \underline{t}' \\ z, \underline{t} \end{smallmatrix}\right) \in E$, then $|\underline{t} - \underline{t}'| \leq \alpha$;
- *Foliated control over* $\mathbf{\Sigma}$: for any $\epsilon > 0$ there is $k_0 \in \mathbb{N}$ such that for all $t_0 \in \mathbb{N}_{\geq k_0}$ there is $\beta > 0$ such that for all $\eta > 0$ there is $k_1 \in \mathbb{N}$ such that for all $t_1 \geq \mathbb{N}_{\geq k_1}$ and all $z, z' \in |\mathbf{\Sigma}|^\wedge$, $\underline{t}' \in \mathbb{N}^{\times 2}$, with $\underline{t} := (t_0, t_1)$ we have
$$\left(\begin{smallmatrix} z', \underline{t}' \\ z, \underline{t} \end{smallmatrix}\right) \in E \implies d_{\mathbf{\Sigma}\text{-fol}}(z, z') < (\beta, \eta, \epsilon);$$

(7.7c) $\mathfrak{D}_G(\mathbf{\Sigma})$ consists of all relatively compact subsets of G.

It is an exercise to check that this is indeed a G-control structure. To check that $\mathfrak{D}_2(\mathbf{\Sigma})$ is closed under composition, the triangle inequality from Remark 7.5 is used.

We define the G-control structure $\mathfrak{D}^0(\mathbf{\Sigma}) = \big(\mathfrak{D}^0_1(\mathbf{\Sigma}), \mathfrak{D}^0_2(\mathbf{\Sigma}), \mathfrak{D}^0_G(\mathbf{\Sigma})\big)$ as follows. Set $\mathfrak{D}^0_1(\mathbf{\Sigma}) := \mathfrak{D}_1(\mathbf{\Sigma})$, $\mathfrak{D}^0_G(\mathbf{\Sigma}) := \mathfrak{D}_G(\mathbf{\Sigma})$. We define $\mathfrak{D}^0_2(\mathbf{\Sigma})$ to consist of all $E \in \mathfrak{D}_2(\mathbf{\Sigma})$ satisfying
$$\left(\begin{smallmatrix} z', \underline{t}' \\ z, \underline{t} \end{smallmatrix}\right) \in E \implies \underline{t}' = \underline{t}.$$

Remark 7.8. Using quantifiers the foliated control in Definition 7.7 reads as
$$\forall \epsilon > 0 \; \exists k_0 \; \forall t_0 \geq k_0 \; \exists \beta > 0 \; \forall \eta > 0 \; \exists k_1 \; \forall \big(t_1 \geq k_1, z, z', \underline{t}'\big) \text{ we have}$$
$$\left(\begin{smallmatrix} z', \underline{t}' \\ z, \underline{t} \end{smallmatrix}\right) \in E \implies d_{\mathbf{\Sigma}\text{-fol}}(z, z') < (\beta, \eta, \epsilon).$$

Remark 7.9 (ϵ-control over $|\Sigma|$)**.** The foliated control condition in Definition 7.7 implies that for any $\epsilon > 0$ there is k_0 such that for all $t_0 \geq k_0$ there is k_1 such that for all $t_1 \geq k_1$ and all $z, z', \underline{t}'$ with $\left(\begin{smallmatrix} z', \underline{t}' \\ z, \underline{t} \end{smallmatrix}\right) \in E$ for $\underline{t} = (t_0, t_1)$ we have

$d^\infty\big(p_{\mathbf{\Sigma}}(z), p_{\mathbf{\Sigma}}(z')\big) < \epsilon$. A possible shape of a region in the $\mathbb{N}^{\times 2}$-plane where ϵ-control holds for a fixed $\epsilon > 0$ is illustrated in Figure 2.

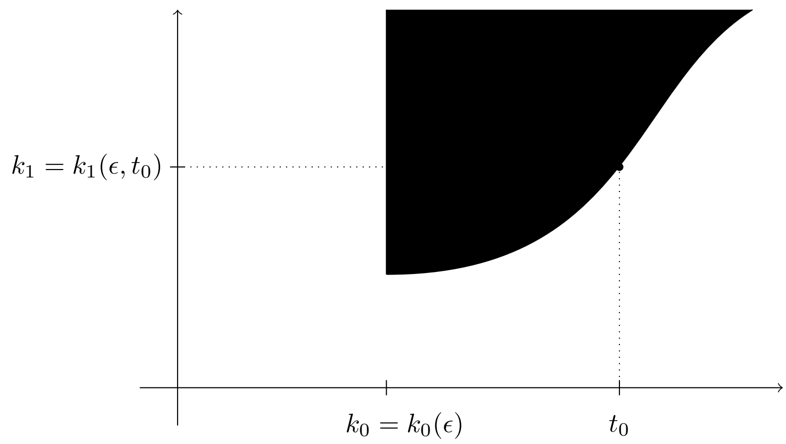

FIGURE 2. Where we have ϵ-control.

Remark 7.10 (ϵ-control and excision)**.** It is mostly the ϵ-control from Remark 7.9 that guarantees that $\mathbf{K}\big(\mathsf{D}^0_G(-)\big)$ as defined below is excisive as required in Proposition 8.20. This is analogous to many other similar results in controlled topology/algebra, see for example the construction of the homology theory associated to the K-theory spectrum of a ring by Pedersen-Weibel in [46].

Remark 7.11 (Foliated control over $P(\sigma)$)**.** The foliated control condition in Definition 7.7 includes a second stage[40] that implies that for certain $z, z' \in |\mathbf{\Sigma}|^\wedge$ we have $d_{P(\sigma)\text{-fol}}\big(\lambda_\sigma(z), \lambda_\sigma(z')\big) < (\beta, \eta)$. See also Remark 7.5 where $d_{\mathbf{\Sigma}\text{-fol}}(z, z') < (\beta, \eta, \epsilon)$ is explained as a two stage condition. Figure 3 illustrates where this (β, η)-control applies along a vertical ray in the $\mathbb{N}^{\times 2}$ plane for fixed $\epsilon > 0$ and t_0. Here $\beta(t_0)$ is fixed along the ray and $\eta(t_0, t_1) \to 0$ with $t_1 \to \infty$.

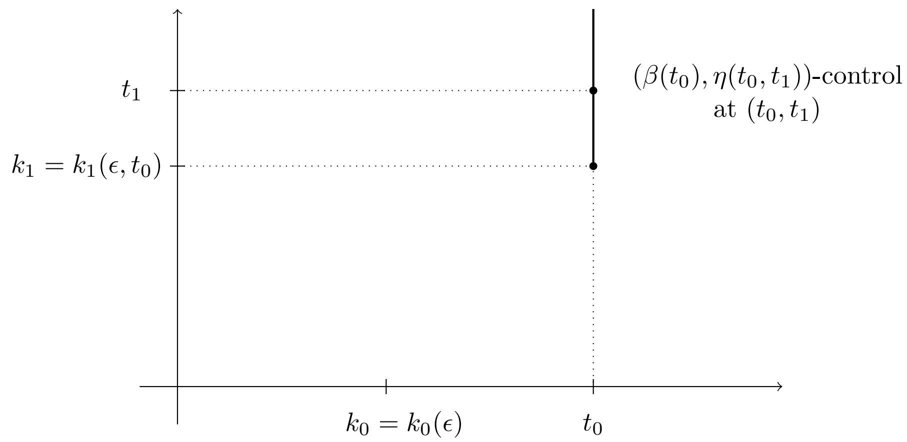

FIGURE 3. Where we have (β, η)-control.

Remark 7.12 (On non-uniform compact support)**.** An important aspect of Definition 7.7 is that the compact support condition in (7.7a) is not uniform over all

[40]The first stage is the ϵ-control condition discussed in Remark 7.9

$\underline{t} \in \mathbb{N}^{\times 2}$. This creates some difficulties in the computation of $\mathbf{D}_G(\mathbf{M})$ for $\mathbf{M} \in \mathcal{R}^0$ in Proposition 8.5 (which would be easier using a uniform compact support condition). But the non-uniformness will be crucial for the construction of the required transfer in Theorem 6.7[41]. The construction of the transfer depends on certain maps $X \to |\mathbf{J}_{\mathcal{C}\mathrm{vcy}}|^\wedge$ where X is the extended Bruhat-Tits building associated to G, see Theorem D.1. The construction of these maps uses an intermediate step maps $X \to FS(X)$, where $FS(X)$ is the flow space associated to X, see Subsection D.III. In the construction of these latter maps the geodesic flow on $FS(X)$ is used for arbitrary long times. Roughly, this has the effect that the images of X in $FS(X)$ are spread out over large parts of $FS(X)$ and ultimately we do not have uniform control over the images of the maps $X \to |\mathbf{J}_{\mathcal{C}\mathrm{vcy}}|^\wedge$. This forces us to work with the non-uniform compact support condition over $|\Sigma|$. This is also the reason for the non-uniform nature of $\mathfrak{C}_1(P)$ in Definition 5.8. The non-uniformness of this compact support condition in turn force us to work with P-foliated control over $|P|^\wedge$ instead of continuous control of $|P|$, compare Remark 5.4.

Remark 7.13. In contrast to the compact support conditions, the finite dimensional support condition is uniform in $\mathbb{N}^{\times 2}$. This is crucial for (and directly implies) the skeleton continuity of $\mathbf{D}^0_G$ in Proposition 8.22.

Definition 7.14. For $\mathbf{\Sigma} \in \mathcal{R}$ we define $\mathcal{Y}(\mathbf{\Sigma})$ as the collection of all subsets Y of $|\mathbf{\Sigma}|^\wedge \times \mathbb{N}^{\times 2}$ satisfying the following condition: there is k_0 such that for each $t_0 \geq k_0$ there is k_1 with

$$Y \cap |\mathbf{\Sigma}|^\wedge \times \{t_0\} \times \mathbb{N}_{\geq k_1} = \emptyset.$$

Definition 7.15. Let $\mathcal{B}$ be a category with G-support. For $\mathbf{\Sigma} \in \mathcal{R}$ we apply Definition 4.20 and define

$$\begin{aligned} \mathrm{D}_G(\mathbf{\Sigma};\mathcal{B}) &:= \mathcal{B}_G(\mathfrak{D}(\mathbf{\Sigma}), \mathcal{Y}(\mathbf{\Sigma})); \\ \mathrm{D}^0_G(\mathbf{\Sigma};\mathcal{B}) &:= \mathcal{B}_G(\mathfrak{D}^0(\mathbf{\Sigma}), \mathcal{Y}(\mathbf{\Sigma})). \end{aligned}$$

Often we drop $\mathcal{B}$ from the notation and write $\mathrm{D}_G(\mathbf{\Sigma}) = \mathrm{D}_G(\mathbf{\Sigma};\mathcal{B})$ and $\mathrm{D}^0_G(\mathbf{\Sigma}) = \mathrm{D}^0_G(\mathbf{\Sigma};\mathcal{B})$.

Remark 7.16. The category $\mathrm{D}_G(\mathbf{\Sigma})$ can be described slightly more explicit as follows. Objects of $\mathrm{D}_G(\mathbf{\Sigma})$ are objects of $\mathcal{B}_G(\mathfrak{D}(\mathbf{\Sigma}))$. Morphisms in $\mathrm{D}_G(\mathbf{\Sigma})$ are equivalence classes of morphisms in $\mathcal{B}_G(\mathfrak{D}(\mathbf{\Sigma}))$, where $\varphi, \psi \colon (S,\pi,B) \to (S',\pi',B')$ are identified, if and on if there is $Y \in \mathcal{Y}(\mathbf{\Sigma})$ such that

$$\varphi_s^{s'} = \psi_s^{s'}$$

whenever $s \in S$, $s' \in S'$ satisfy $\pi(s), \pi'(s') \notin Y$.

Remark 7.17 ($\mathrm{D}^0_G(\mathbf{\Sigma})$ as sequences)**.** An advantage of $\mathbf{D}^0_G(\mathbf{\Sigma})$ over $\mathbf{D}_G(\mathbf{\Sigma})$ is that it admits the following description. For $\underline{t} \in \mathbb{N}^{\times 2}$ we can restrict to $|\mathbf{\Sigma}|^\wedge \times \{\underline{t}\}$ and obtain a functor

$$\mathrm{res}_{\underline{t}} \colon \mathcal{B}_G(\mathfrak{D}^0(\mathbf{\Sigma})) \to \mathcal{B}_G(|\mathbf{\Sigma}|^\wedge).$$

Write $\prod'_{\mathbb{N}^{\times 2}} \mathcal{B}_G(|\mathbf{\Sigma}|^\wedge)$ for the following category. Objects are sequences $(\mathbf{B}_{\underline{t}})_{\underline{t} \in \mathbb{N}^{\times 2}}$ of objects in $\mathcal{B}_G(|\mathbf{\Sigma}|^\wedge)$. Morphisms are equivalence classes of sequences $(\varphi_{\underline{t}})_{\underline{t} \in \mathbb{N}^{\times 2}}$ of morphisms in $\mathcal{B}_G(|\mathbf{\Sigma}|^\wedge)$, where two sequences $(\varphi_{\underline{t}})_{\underline{t} \in \mathbb{N}^{\times 2}}$, $(\varphi'_{\underline{t}})_{\underline{t} \in \mathbb{N}^{\times 2}}$ are equivalent

[41] This issue comes also up in proofs of the Farrell–Jones Conjecture for certain discrete groups, but somewhat less visible. For example the category $\mathcal{D}^G(Y;\mathcal{A})$ in [9, Sec. 3.3] does use a uniform compact support condition, but the proof later also uses the category $\mathcal{O}^G(Y,(Z_n,d_n)_{n \in \mathbb{N}})$ where the compact support condition is not uniform in $n \in \mathbb{N}$.

if there is k_0 such that for any $t_0 \geq k_0$ there is k_1 such that for all $t_1 \geq k_1$ we have $\varphi_{t_0,t_1} = \psi_{t_0,t_1}$. The above restrictions combine to a faithful functor

$$\mathsf{D}^0_G(\mathbf{\Sigma}) \;\to\; \prod_{\mathbb{N}^{\times 2}}{}' \mathcal{B}_G(|\mathbf{\Sigma}|^\wedge). \tag{7.18}$$

A sequence $\mathbf{B} = (\mathbf{B}_{\underline{t}})_{\underline{t}\in\mathbb{N}^{\times 2}}$ of objects in $\mathcal{B}_G(|\mathbf{\Sigma}|^\wedge)$ is in the image of (7.18), if and only if the following four conditions are satisfied:

(7.18a) $\operatorname{supp}_1 \mathbf{B} = \big\{(z,\underline{t}) \bigm| z \in \operatorname{supp}_1 \mathbf{B}_{\underline{t}}\big\} \in \mathfrak{D}^0_1(\mathbf{\Sigma})$;

(7.18b) $\operatorname{supp}_2 \mathbf{B} = \Big\{\Big(\begin{smallmatrix} z',\underline{t} \\ z,\underline{t}\end{smallmatrix}\Big) \Bigm| \big(\begin{smallmatrix} z' \\ z\end{smallmatrix}\big) \in \operatorname{supp}_2 \mathbf{B}_{\underline{t}}\Big\} \in \mathfrak{D}^0_2(\mathbf{\Sigma})$;

(7.18c) $\operatorname{supp}_G \mathbf{B} = \bigcup_{\underline{t}\in\mathbb{N}^{\times 2}} \operatorname{supp}_G \mathbf{B}_{\underline{t}} \in \mathfrak{D}^0_G(\mathbf{\Sigma})$;

(7.18d) $\mathbf{B}_{\underline{t}}$ is finite for all $\underline{t}$[42].

A sequence of morphisms is in the image of (7.18) if and only if it is equivalent to a sequence $(\varphi_{\underline{t}})_{\underline{t}\in\mathbb{N}^{\times 2}}$ of morphisms in $\mathcal{B}_G(|\mathbf{\Sigma}|^\wedge)$ satisfying

(7.18e) $\operatorname{supp}_2 \varphi = \big\{\Big(\begin{smallmatrix} z',\underline{t} \\ z,\underline{t}\end{smallmatrix}\Big) \mid \big(\begin{smallmatrix} z' \\ z\end{smallmatrix}\big) \in \operatorname{supp}_2 \varphi_{\underline{t}}\big\} \in \mathfrak{D}^0_2(\mathbf{\Sigma})$;

(7.18f) $\operatorname{supp}_G \varphi = \bigcup_{\underline{t}\in\mathbb{N}^{\times 2}} \operatorname{supp}_G \varphi_{\underline{t}} \in \mathfrak{D}^0_G(\mathbf{\Sigma})$.

For $\mathbf{\Sigma} = (\Sigma, P) \in \mathcal{R}$ and $Q \in \mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$ we have $\mathbf{\Sigma} \times Q = (\Sigma, \sigma \mapsto P(\sigma) \times Q)$ as a special case of (6.3). Note that $|\mathbf{\Sigma} \times Q|^\wedge = |\mathbf{\Sigma}|^\wedge \times |Q|^\wedge$.

Lemma 7.19. *Let $\mathbf{\Sigma} \in \mathcal{R}$ and $Q \in \mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$.*

(7.19a) For $F \in \mathfrak{D}^0_1(\mathbf{\Sigma})$, $F' \in \mathfrak{D}^0_1(\mathbf{M})$ we have

$$\big\{(z,\lambda,\underline{t}) \mid (z,\underline{t}) \in F, (\lambda,\underline{t}) \in F'\big\} \in \mathfrak{D}^0_1(\mathbf{\Sigma} \times Q);$$

(7.19b) For $E \in \mathfrak{D}^0_2(\mathbf{\Sigma})$, $E' \in \mathfrak{D}^0_2(\mathbf{M})$ we have

$$\Big\{\Big(\begin{smallmatrix} z',\lambda',\underline{t} \\ z,\lambda,\underline{t}\end{smallmatrix}\Big) \Bigm| \Big(\begin{smallmatrix} z',\underline{t} \\ z,\underline{t}\end{smallmatrix}\Big) \in E, \Big(\begin{smallmatrix} \lambda',\underline{t} \\ \lambda,\underline{t}\end{smallmatrix}\Big) \in E'\Big\} \in \mathfrak{D}^0_2(\mathbf{\Sigma} \times Q).$$

Proof. This is an easy exercise in the definitions. □

Definition 7.20. For a category $\mathcal{B}$ with G-support we define the two functors $\mathbf{D}_G(-;\mathcal{B})$, $\mathbf{D}^0_G(-;\mathcal{B}) : \mathcal{R} \to \mathtt{Spectra}$ by

$$\mathbf{D}_G(\mathbf{\Sigma};\mathcal{B}) := \mathbf{K}\big(\mathsf{D}_G(\mathbf{\Sigma};\mathcal{B})\big) \quad \text{and} \quad \mathbf{D}^0_G(\mathbf{\Sigma};\mathcal{B}) := \mathbf{K}\big(\mathsf{D}^0_G(\mathbf{\Sigma};\mathcal{B})\big).$$

We often abbreviate $\mathbf{D}_G(-) = \mathbf{D}_G(-;\mathcal{B})$ and $\mathbf{D}^0_G(-) = \mathbf{D}^0_G(-;\mathcal{B})$.

8. Properties of $\mathsf{D}_G(-)$ and $\mathsf{D}^0_G(-)$

8.A. Computation of $\mathsf{D}_G(P)$.

We write again $I \colon \mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G) \to \mathcal{R}$ for the inclusion.

Proposition 8.1. *There is a zig-zag of equivalences of $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$-spectra between $I^*\Omega\mathbf{K}(\mathsf{D}_G(-))$ and $\mathbf{K}(\mathsf{C}_G(-))$.*

The proof of Proposition 8.1 will need a preparation.

Definition 8.2. Let $P \in \mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$. Let $\mathcal{Y}_0(P)$ be the collection of all subsets Y of $|P|^\wedge \times \mathbb{N} \times \mathbb{N}$ that are contained in $|P|^\wedge \times \mathbb{N} \times \{0,\ldots,N\}$ for some N (depending on Y). We define

$$\begin{aligned} \mathsf{D}^{\mathrm{fin}}_G(P) &:= \mathcal{B}_G(\mathfrak{D}(P)|_{\mathcal{Y}(P)}, \mathcal{Y}_0(P)); \\ \mathsf{D}^{\mathrm{sw}}_G(P) &:= \mathcal{B}_G(\mathfrak{D}(P), \mathcal{Y}_0(P)). \end{aligned}$$

Here $\mathcal{Y}(P)$ is from Definition 7.14.

[42] This condition comes from the finite over points condition in (4.12a) and the finiteness over $\mathbb{N}^{\times 2}$ in (7.7a).

As $\mathsf{D}_G(P) = \mathcal{B}_G(\mathfrak{D}(P), \mathcal{Y}(P))$ and $\mathcal{Y}_0(P) \subseteq \mathcal{Y}(P)$ we obtain a Karoubi sequence

$$\mathsf{D}_G^{\mathrm{fin}}(P) \;\to\; \mathsf{D}_G^{\mathrm{sw}}(P) \;\to\; \mathsf{D}_G(P). \tag{8.3}$$

Lemma 8.4. *Let $P \in \mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$.*

(8.4a) The inclusion $|P|^\wedge \times \mathbb{N} \to |P|^\wedge \times \mathbb{N}^{\times 2}$, $(\lambda, t) \mapsto (\lambda, 0, t)$ induces an equivalence $\mathsf{C}_G(P) \to \mathsf{D}^{\mathrm{fin}}(P)$;

(8.4b) The K-theory of $\mathsf{D}_G^{\mathrm{sw}}(P)$ vanishes.

Proof. The first statement is an easy exercise in the definitions. The second comes from a standard Eilenberg swindle on $\mathsf{D}_G^{\mathrm{sw}}(P)$ using the shift $(\lambda, t_0, t_1) \mapsto (\lambda, t_0 + 1, t_1)$; formally we use Lemma 4.28. Let $P \in \mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$. Let $\mathbf{\Sigma} \in \mathcal{R}^0$ be the object whose underlying simplicial complex consist of one vertex which is sent to P, i.e., $\mathbf{\Sigma} = I(P)$. Now the point is that for $z, z' \in |\mathbf{\Sigma}|^\wedge = |P|^\wedge$ we have for any $\epsilon > 0$

$$d_{\mathbf{\Sigma}\text{-fol}}(z, z') < (\beta, \eta, \epsilon) \quad \iff \quad d_{P\text{-fol}}(z, z') < (\beta, \eta).$$

Thus the conditions in (7.7b) are constant in the first coordinate. This is used to verify (4.29d), the other assumptions of Lemma 4.28 are straight forward to check. ☐

Proof of Proposition 8.1. The Karoubi sequence (8.3) induces a fibration sequence in K-theory, see (4.23). Thus (8.4a) and (8.4b) give the result. ☐

8.B. Computation of $\mathsf{D}_G(-)$ on $\mathcal{R}^0$.

Proposition 8.5. *Let $\mathbf{M} = (M, P) \in \mathcal{R}^0$. The canonical map*

$$\bigvee_{m \in M} \mathbf{K}(\mathsf{D}_G(P(m))) \xrightarrow{\sim} \mathbf{K}(\mathsf{D}_G(\mathbf{M}))$$

is an equivalence.

The proof of Proposition 8.5 requires some preparations.

Definition 8.6. Let $\mathbf{M} = (M, P) \in \mathcal{R}^0$. We define the G-control structure $\mathfrak{D}^{\mathrm{dis}}(\mathbf{M}) = \big(\mathfrak{D}_1^{\mathrm{dis}}(\mathbf{M}), \mathfrak{D}_2^{\mathrm{dis}}(\mathbf{M}), \mathfrak{D}_G^{\mathrm{dis}}(\mathbf{M})\big)$ as follows. We set $\mathfrak{D}_1^{\mathrm{dis}}(\mathbf{M}) := \mathfrak{D}_1(\mathbf{M})$ and $\mathfrak{D}_G^{\mathrm{dis}}(\mathbf{M}) := \mathfrak{D}_G(\mathbf{M})$. We define $\mathfrak{D}_2^{\mathrm{dis}}(\mathbf{M})$ to consist of all $E \in \mathfrak{D}_2(\mathbf{M})$ that are 0-controlled over M, i.e., satisfy the following. Let $\left(\begin{smallmatrix} \lambda', \underline{t}' \\ \lambda, \underline{t} \end{smallmatrix}\right) \in E$. Write m and m' for the images of λ and λ' in M under the projection $|\mathbf{M}|^\wedge \to M$. We require $m = m'$.

Let $\mathcal{Y}_0(M)$ be the collection of all subsets Y of $|\mathbf{M}|^\wedge \times \mathbb{N} \times \mathbb{N}$ that are contained in $|\mathbf{M}|^\wedge \times \mathbb{N} \times \{0, \dots, N\}$ for some N (depending on Y). We define

$$\begin{aligned}
\mathsf{D}_G^{\mathrm{dis,fin}}(\mathbf{M}) &:= \mathcal{B}_G(\mathfrak{D}^{\mathrm{dis}}(\mathbf{M})|_{\mathcal{Y}(\mathbf{M})}, \mathcal{Y}_0(\mathbf{M}));\\
\mathsf{D}_G^{\mathrm{dis,sw}}(\mathbf{M}) &:= \mathcal{B}_G(\mathfrak{D}^{\mathrm{dis}}(\mathbf{M}), \mathcal{Y}_0(\mathbf{M}));\\
\mathsf{D}_G^{\mathrm{dis}}(\mathbf{M}) &:= \mathcal{B}_G(\mathfrak{D}^{\mathrm{dis}}(\mathbf{M}), \mathcal{Y}(\mathbf{M})).
\end{aligned}$$

Remark 8.7. The ϵ-control aspect of the foliated control condition in (7.7b) implies that for all $E \in \mathfrak{D}_2(\mathbf{M})$ there is $Y \in \mathcal{Y}(\mathbf{M})$ such that

$$E \cap Y \times Y \;\in\; \mathfrak{D}_2^{\mathrm{dis}}(\mathbf{M}).$$

Lemma 8.8. *Let $\mathbf{M} = (M, P) \in \mathcal{R}^0$.*

(8.8a) The canonical map

$$\bigoplus_{m \in M} \mathsf{D}_G^{\mathrm{dis,fin}}(P(m)) \xrightarrow{\sim} \mathsf{D}_G^{\mathrm{dis,fin}}(\mathbf{M})$$

is an equivalence

(8.8b) The K-theory of $\mathsf{D}_G^{\mathrm{dis,sw}}(\mathbf{M})$ vanishes.

(8.8c) The inclusion $\mathsf{D}_G^{\mathrm{dis}}(\mathbf{M}) \to \mathsf{D}_G(\mathbf{M})$ *is an equivalence.*

Proof. The first statement follows from the compact support condition over $|\Sigma| = M$ in (7.7a) and the fact that we have 0-control over over M. The second statement uses the Eilenberg swindle on $\mathsf{D}_G^{\mathrm{dis,sw}}(\mathbf{M})$ induced by the map $(\lambda, t_0, t_1) \mapsto (\lambda, t_0 + 1, t_1)$, see Lemma 4.28. To verify (4.28d) it is again important that 0-control over B is in $\mathfrak{D}^{\mathrm{dis}}(\mathbf{M})$ enforced for all $t_0 \in \mathbb{N}$[43]. The third statement is an easy exercise in the definitions and uses the observation from Remark 8.7. $\square$

Proof of Proposition 8.5. By (8.8c) it suffices to prove the assertion for $\mathsf{D}_G^{\mathrm{dis}}$ in place of D_G. The Karoubi sequence

$$\mathsf{D}_G^{\mathrm{dis,fin}}(\mathbf{M}) \;\to\; \mathsf{D}_G^{\mathrm{dis,sw}}(\mathbf{M}) \;\to\; \mathsf{D}_G^{\mathrm{dis}}(\mathbf{M})$$

induces a fibration sequence in K-theory, see (4.23). Using (8.8b) we obtain an equivalence $\Omega\mathbf{K}\big(\mathsf{D}_G^{\mathrm{dis}}(-)\big) \xrightarrow{\sim} \mathbf{K}\big(\mathsf{D}_G^{\mathrm{dis,fin}}(-)\big)$. Thus it suffices to prove the assertion with $\mathsf{D}_G^{\mathrm{dis,fin}}$ in place of $\mathsf{D}_G^{\mathrm{dis}}$. In this formulation the assertion follows from (8.8a). $\square$

8.c. The K-theory of $\mathsf{D}_G^0(\mathbf{\Sigma})$ determines the K-theory of $\mathsf{D}_G(\mathbf{\Sigma})$.

Proposition 8.9. *There exists a diagram in* $\mathcal{R}$-$\mathsf{Spectra}$

$$\begin{tikzcd}
\mathbf{K}\big(\mathsf{D}_G^0(-)\big) & \mathbf{K}\big(\mathsf{D}_G^0(-)\big) \arrow[l] \arrow[r] & \mathbf{K}\big(\mathsf{D}_G^0(-)\big) \\
\mathbf{K}\big(\mathsf{D}_G^0(-)\big) \arrow[u] \arrow[d] & \mathbf{K}\big(\mathsf{D}_G^0(-)\big) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathbf{K}\big(\mathsf{D}_G^0(-)\big) \arrow[u] \arrow[d] \\
\mathbf{K}\big(\mathsf{D}_G^0(-)\big) & \mathbf{K}\big(\mathsf{D}_G^0(-)\big) \arrow[l] \arrow[r] & \mathbf{K}\big(\mathsf{D}_G^0(-)\big)
\end{tikzcd}$$

whose homotopy colimit is equivalent to $\mathbf{K}\big(\mathsf{D}_G(-)\big)$.

The proof of Proposition 8.9 requires some preparations.

Definition 8.10. We define the G-control structure

$$\mathfrak{D}^{\cdot 5}(\mathbf{\Sigma}) = \big(\mathfrak{D}_1^{\cdot 5}(\mathbf{\Sigma}), \mathfrak{D}_2^{\cdot 5}(\mathbf{\Sigma}), \mathfrak{D}_G^{\cdot 5}(\mathbf{\Sigma})\big)$$

as follows. Set $\mathfrak{D}_1^{\cdot 5}(\mathbf{\Sigma}) := \mathfrak{D}_1(\mathbf{\Sigma})$, $\mathfrak{D}_G^{\cdot 5}(\mathbf{\Sigma}) := \mathfrak{D}_G(\mathbf{\Sigma})$. We define $\mathfrak{D}_2^{\cdot 5}(\mathbf{\Sigma})$ to consist of all $E \in \mathfrak{D}_2(\mathbf{\Sigma})$ satisfying

$$\left(\begin{smallmatrix} z', t_0', t_1' \\ z, t_0, t_1 \end{smallmatrix}\right) \in E \implies t_0 = t_0'.$$

We set

$$\mathsf{D}_G^{\cdot 5}(\mathbf{\Sigma}) := \mathcal{B}_G(\mathfrak{D}^{\cdot 5}(\mathbf{\Sigma}), \mathcal{Y}(\mathbf{\Sigma})).$$

Note that $\mathfrak{D}_2^0(\mathbf{\Sigma}) \subseteq \mathfrak{D}_2^{\cdot 5}(\mathbf{\Sigma}) \subseteq \mathfrak{D}_2(\mathbf{\Sigma})$ and so $\mathsf{D}_G^0(\mathbf{\Sigma}) \subseteq \mathsf{D}_G^{\cdot 5}(\mathbf{\Sigma}) \subseteq \mathsf{D}_G(\mathbf{\Sigma})$.

Lemma 8.11. *There are homotopy pushouts in* $\mathcal{R}$-$\mathsf{Spectra}$

$$\begin{tikzcd}
\mathbf{K}\big(\mathsf{D}_G^0(-)\big) \arrow[r] \arrow[d] & \mathbf{K}\big(\mathsf{D}_G^0(-)\big) \arrow[d] \\
\mathbf{K}\big(\mathsf{D}_G^0(-)\big) \arrow[r] & \mathbf{K}\big(\mathsf{D}_G^{\cdot 5}(-)\big)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{K}\big(\mathsf{D}_G^{\cdot 5}(-)\big) \arrow[r] \arrow[d] & \mathbf{K}\big(\mathsf{D}_G^{\cdot 5}(-)\big) \arrow[d] \\
\mathbf{K}\big(\mathsf{D}_G^{\cdot 5}(-)\big) \arrow[r] & \mathbf{K}\big(\mathsf{D}_G(-)\big).
\end{tikzcd}$$

[43]Neither (8.8a) nor (8.8b) hold if we use $\mathfrak{D}(\mathbf{M})$ instead of $\mathfrak{D}^{\mathrm{dis}}(\mathbf{M})$

We will only give the construction of the right homotopy pushout square in Lemma 8.11; the construction of the left homotopy pushout square is entirely analogous.

Definition 8.12. For $X \subseteq \mathbb{N}$ let $\mathcal{Y}^X(\mathbf{\Sigma})$ be the collection of all subsets of $|\mathbf{\Sigma}|^\wedge \times X \times \mathbb{N}$. We define

$$\mathsf{D}_G^X(\mathbf{\Sigma}) := \mathcal{B}_G(\mathfrak{D}(\mathbf{\Sigma})|_{\mathcal{Y}^X(\mathbf{\Sigma})}, \mathcal{Y}(\mathbf{\Sigma})).$$

The definition amounts to replacing $|\mathbf{\Sigma}|^\wedge \times \mathbb{N} \times \mathbb{N}$ with $|\mathbf{\Sigma}|^\wedge \times X \times \mathbb{N}$. Note that because of the bounded control requirement over $\mathbb{N}\times\mathbb{N}$ in (7.7b) the category $\mathsf{D}_G^X(\mathbf{\Sigma})$ depends on X with the metric it inherits from $\mathbb{N}$. In particular, the properties of $\mathsf{D}_G^X(\mathbf{\Sigma})$ depend on the coarse structure of X.

We now choose natural numbers $0 = a_0 < a_1 < a_2 < \ldots$ such that $a_{n+1} - a_n \to \infty$ as $n \to \infty$. We set

$$\begin{aligned} A &:= [a_0, a_3] \cup [a_4, a_7] \cup [a_8,\, a_{11}] \cup [a_{12}, a_{15}] \cup \ldots; \\ B &:= [a_2, a_5] \cup [a_6, a_9] \cup [a_{10}, a_{13}] \cup [a_{14}, a_{17}] \cup \ldots, \end{aligned}$$

where $[a, b]$ is the discrete interval $\{a, a+1, \ldots, b\}$. The point is that each of the three sets A, B and $A \cap B$ is the infinite union of intervals, where both the length and the distance between successive grow to ∞. We have

(8.13a) for any r there is R such that for $a \in A \setminus B$ and $b \in B$ we have either $a, b \le R$ or $|b - a| \le r$;

(8.13b) for any r there is R such that if $|a_i - a_j| \le r$ and $a_i \ne a_j$, then $a_i, a_j \le R$.

Lemma 8.14. *The square*

$$\begin{tikzcd} \mathbf{K}\big(\mathsf{D}_G^{A\cap B}(-)\big) \arrow[r] \arrow[d] & \mathbf{K}\big(\mathsf{D}_G^{A}(-)\big) \arrow[d] \\ \mathbf{K}\big(\mathsf{D}_G^{B}(-)\big) \arrow[r] & \mathbf{K}\big(\mathsf{D}_G(-)\big) \end{tikzcd}$$

is a homotopy pushout square in $\mathcal{R}$-$\mathsf{Spectra}$.

Proof. Let $\mathbf{\Sigma} \in \mathcal{R}$. We check that Lemma 4.27 applies to the above square evaluated on $\mathbf{\Sigma}$. Let $Y_A \in \mathcal{Y}^A(\mathbf{\Sigma})$, $E \in \mathfrak{D}_2(\mathbf{\Sigma})$ be given. Set $Y_B := Y_A \cap \big(|\mathbf{\Sigma}|^\wedge \times B \times \mathbb{N}\big)$ and $Y_A' := Y_A \setminus |\mathbf{\Sigma}|^\wedge \times B \times \mathbb{N}$. Then $Y_A \in \mathcal{Y}^A(\mathbf{\Sigma})$, $Y_B \in \mathcal{Y}^B(\mathbf{\Sigma})$ and $Y_A = Y_A' \cup Y_B$. The bounded control condition from (7.7b) together with (8.13a) implies that there is $R > 0$ such that for all $(z, t_0, t_1) \in (Y_A')^E$ we have $t_0 \in A$ or $t_0 \le R$. Thus $(Y_A')^E \in \mathcal{Y}^A(\mathbf{\Sigma})$ and Lemma 4.27 applies. □

Lemma 8.15. *In* $\mathcal{R}$-$\mathsf{Spectra}$*, the functors* $\mathbf{K}\big(\mathsf{D}_G^{A\cap B}(-)\big)$, $\mathbf{K}\big(\mathsf{D}_G^{A}(-)\big)$, $\mathbf{K}\big(\mathsf{D}_G^{B}(-)\big)$ *are all equivalent to* $\mathbf{K}\big(\mathsf{D}_G^{\cdot 5}(-)\big)$.

Proof. The argument is almost the same in all three cases. We treat $\mathbf{K}\big(\mathsf{D}_G^{A}(-)\big)$. Let $A_0 := \{a_0, a_4, a_8, a_{12}, \ldots\}$. We claim that for any $\mathbf{\Sigma} \in \mathcal{R}$

(8.15a) the bijection $\mathbb{N} \to A_0$, $i \mapsto a_{4i}$ induces an equivalence $\mathsf{D}_G^{\cdot 5}(\mathbf{\Sigma}) \to \mathsf{D}_G^{A_0}(\mathbf{\Sigma})$ of categories;

(8.15b) the inclusion $A_0 \to A$ induces an equivalence $\mathsf{D}_G^{A_0}(\mathbf{\Sigma}) \to \mathsf{D}_G^{A}(\mathbf{\Sigma})$ in K-theory.

Clearly, (8.15a) and (8.15b) together give us an equivalence $\mathbf{K}\big(\mathsf{D}_G^{\cdot 5}(-)\big) \to \mathbf{K}\big(\mathsf{D}_G^{A}(\text{-}-)\big)$.

The difference between $\mathfrak{D}^{\cdot 5}(\mathbf{\Sigma})$ and $\mathfrak{D}(\mathbf{\Sigma})$ is that for $\left(\begin{smallmatrix} z', t_0', t_1' \\ z, t_0, t_1' \end{smallmatrix}\right) \in E \in \mathfrak{D}_2(\mathbf{\Sigma})$ we can have $t_0' \ne t_0$, while this does not happen for $\mathfrak{D}_2^{\cdot 5}(\mathbf{\Sigma})$. However, if $t_0 = a_i$ and $t_0' = a_{i'}$ then (8.13b) implies that either $t_0 = t_0'$ or t_0, t_0' are bounded. More formally, for $E \in \mathfrak{D}_2(\mathbf{\Sigma})$ there is $Y \in \mathcal{Y}(\mathbf{\Sigma})$ such that $(E \backslash Y \times Y) \cap \big(|\mathbf{\Sigma}|^\wedge \times A_0 \times \mathbb{N}\big)^{\times 2} \in \mathfrak{D}_2^{.5}(\mathbf{\Sigma})$ and from this it is not difficult to verify (8.15a).

The Karoubi sequence

$$\mathsf{D}_G^{A_0}(\mathbf{\Sigma}) \to \mathsf{D}_G^{A}(\mathbf{\Sigma}) \to \mathcal{B}_G(\mathfrak{D}(\mathbf{\Sigma})|_{\mathcal{Y}^A(\mathbf{\Sigma})}, \mathcal{Y}^{A_0}(\mathbf{\Sigma}))$$

induce a fibration sequence in K-theory, see (4.23). To prove (8.15b) it suffice to show that $\mathcal{B}_G(\mathfrak{D}(\mathbf{\Sigma})|_{\mathcal{Y}^A(\mathbf{\Sigma})}, \mathcal{Y}^{A_0}(\mathbf{\Sigma}))$ admits an Eilenberg swindle. Set $Z := |\mathbf{\Sigma}|^\wedge \times A_0 \times \mathbb{N}$. Consider $f\colon |\mathbf{\Sigma}|^\wedge \times A \times \mathbb{N} \to |\mathbf{\Sigma}|^\wedge \times A \times \mathbb{N}$ with

$$f(z, t_0, t_1) := \begin{cases} (z, t_0, t_1) & t_0 \in A_0 \\ (z, t_0 - 1, t_1) & t_0 \notin A_0. \end{cases}$$

It is not difficult to check that Lemma 4.29 applies and we obtain a swindle on $\mathcal{B}_G^\wedge(\mathfrak{C}(\mathbf{\Sigma})|_{\mathcal{Y}^A(\mathbf{\Sigma})}, \mathcal{Y}^{A_0}(\mathbf{\Sigma}))$. $\mathcal{B}_G(\mathfrak{C}(\mathbf{\Sigma})|_{\mathcal{Y}^A(\mathbf{\Sigma})}, \mathcal{Y}^{A_0}(\mathbf{\Sigma}))$. ☐

Proof of Lemma 8.11. The existence of the right hand square follows directly from Lemma 8.14 and Lemma 8.15. The left hand square can be constructed by a similar argument. ☐

Proof of Proposition 8.9. This follows from Lemma 8.11. ☐

8.D. **Homotopy invariance for D_G^0.** Let $\mathbf{M} = (M, P) \in \mathcal{R}^0$. Let $\pi\colon M \times \Delta^n \to M$ be the projection. We obtain $\mathbf{\Delta}_{\mathbf{M}}^d = (M \times \Delta^d, P \circ \pi_*) \in \mathcal{R}$ as in Subsection 6.E. A choice of a point $x_0 \in |\Delta^d|$ determines an inclusion $\mathbf{i}\colon \mathbf{M} \to \mathbf{\Delta}_{\mathbf{M}}^d$.

Proposition 8.16. *The inclusion* $\mathbf{i}$ *induces an equivalence*

$$\mathbf{K}\big(\mathsf{D}_G^0(\mathbf{M})\big) \xrightarrow{\sim} \mathbf{K}\big(\mathsf{D}_G^0(\mathbf{\Delta}_{\mathbf{M}}^d)\big).$$

Proof. We have $|\mathbf{\Delta}_{\mathbf{M}}^d|^\wedge = |\mathbf{M}|^\wedge \times |\Delta^d|$. Let $\mathcal{Y}(x_0)$ be the collection of all subsets of $|\mathbf{M}|^\wedge \times \{x\} \times \mathbb{N} \times \mathbb{N} \subset |\mathbf{\Delta}_{\mathbf{M}}^d|^\wedge \times \mathbb{N} \times \mathbb{N}$. Then $\mathsf{D}_G^0(\mathbf{M}) = \mathcal{B}_G(\mathfrak{D}^0(\mathbf{M}), \mathcal{Y}(\mathbf{M}))$ is equivalent to $\mathcal{B}_G(\mathfrak{D}^0(\mathbf{\Delta}_{\mathbf{M}}^d)|_{\mathcal{Y}(x_0)}, \mathcal{Y}(\mathbf{\Delta}_{\mathbf{M}}^d))$. We obtain a Karoubi sequence

$$\begin{aligned}\mathcal{B}_G(\mathfrak{D}^0(\mathbf{\Delta}_{\mathbf{M}}^d)|_{\mathcal{Y}(x_0)}, \mathcal{Y}(\mathbf{\Delta}_{\mathbf{M}}^d)) &\to \mathcal{B}_G(\mathfrak{D}^0(\mathbf{\Delta}_{\mathbf{M}}^d), \mathcal{Y}(\mathbf{\Delta}_{\mathbf{M}}^d)) \\ &\to \mathcal{B}_G(\mathfrak{D}^0(\mathbf{\Delta}_{\mathbf{M}}^d), \mathcal{Y}(\mathbf{\Delta}_{\mathbf{M}}^d) \cup \mathcal{Y}(x_0))\end{aligned}$$

as in (4.23). It suffices to show that the K-theory of right most category of this sequence is trivial. To this end we produce an Eilenberg swindle and use Lemma 4.29 with $Z = |\mathbf{M}|^\wedge \times \{x_0\} \times \mathbb{N} \times \mathbb{N}$. The point of the swindle is that we can contract $|\Delta^d|$ linearly to $\{x_0\}$. The ϵ-control part (see (7.3a)) of the foliated control condition in (7.7b) requires a bit of care here: with increasing t_0 we need to push slower and slower. More precisely, we construct $f\colon |\mathbf{M}|^\wedge \times |\Delta^d| \times \mathbb{N} \times \mathbb{N} \to |\mathbf{M}|^\wedge \times |\Delta^d| \times \mathbb{N} \times \mathbb{N}$ as follows. Write $f_{t_0}\colon |\Delta^d| \to |\Delta^d|$ for the map that sends x to the point x' on the straight line from x to x_0 in $|\Delta^d|$, whose distance from x is $\min\{1/t_0, d(x, x_0)\}$. We can set

$$f(\lambda, x, t_0, t_1) := (\lambda, f_{t_0}(x), t_0, t_1).$$

The foliated control condition in (7.7b) also involves foliated control over $|(P \circ \pi_*)(\{m\} \times \sigma)|^\wedge = |P(m)|^\wedge$ for each $m \in M$, see (7.3b). The map f preserves this condition, because f acts as the identity on the $|\mathbf{M}|^\wedge$-coordinate and because $|(P \circ \pi_*)(\{m\} \times \sigma)$ depends on m but not on σ (otherwise the passage from a subsimplex to a larger simplex could create problems). With this observation it is not difficult to check that the assumptions of Lemma 4.29 are satisfied. ☐

8.E. **Excision for $\mathrm{D}^0_G(-)$.** Let $\mathbf{\Sigma} = (\Sigma, P) \in \mathcal{R}$ be d-dimensional. Let $\mathbf{\Sigma}' = (\Sigma', P')$ be its (d-1)-skeleton. For $W \subseteq |\Sigma|$ we let $|W|^\wedge$ be the preimage of W under $|\mathbf{\Sigma}|^\wedge \xrightarrow{p_{\mathbf{\Sigma}}} |\Sigma|$ and define $\mathcal{Y}_W$ as the collection of all subsets of $|W|^\wedge \times \mathbb{N}^{\times 2}$. We set

$$\mathrm{D}^{0,W}_G(\mathbf{\Sigma}) := \mathcal{B}_G(\mathfrak{D}^0(\mathbf{\Sigma})|_{\mathcal{Y}_W}, \mathcal{Y}(\mathbf{\Sigma})),$$

i.e., $\mathrm{D}^{0,W}_G(\mathbf{\Sigma})$ is the full subcategory of $\mathrm{D}^{0,W}_G(\mathbf{\Sigma})$ on all objects whose support is contained in $|W|^\wedge \times \mathbb{N}^{\times 2}$. Setting $\mathcal{Y}^+_W := \mathcal{Y}_W \cup \mathcal{Y}(\mathbf{\Sigma})$, we can apply Lemma 4.26 to conclude that $\mathrm{D}^{0,W}_G(\mathbf{\Sigma})$ is canonically equivalent to $\mathcal{B}_G(\mathfrak{D}^0(\mathbf{\Sigma})|_{\mathcal{Y}^+_W}, \mathcal{Y}(\mathbf{\Sigma}))$.

Fix $0 < \epsilon < 1/(d+1)$. Let N be the (open[44]) ϵ-neighborhood of $|\Sigma'|$ in $|\Sigma|$ (always with respect to the l^∞-metric). The choice of ϵ guarantees $|\Sigma'| \subseteq N \subsetneq |\Sigma|$. Let M be the complement of the $\epsilon/2$-neighborhood of $|\Sigma'|$ in $|\Sigma|$. Thus $|\Sigma| = N \cup M$.

Lemma 8.17. *The functor $\mathrm{D}^0_G(\mathbf{\Sigma}') \to \mathrm{D}^{0,N}_G(\mathbf{\Sigma})$ induced by the inclusion $\mathbf{\Sigma}' \to \mathbf{\Sigma}$ yields an equivalence in K-theory.*

Proof. First note that the inclusion $\mathbf{\Sigma}' \to \mathbf{\Sigma}$ induces an equivalence $\mathrm{D}^0_G(\mathbf{\Sigma}') \to \mathrm{D}^{0,|\Sigma'|}_G(\mathbf{\Sigma})$. We have a Karoubi sequence

$$\begin{aligned}\mathcal{B}_G(\mathfrak{D}^0(\mathbf{\Sigma})|_{\mathcal{Y}_{|\Sigma'|}}, \mathcal{Y}(\mathbf{\Sigma})) \to \mathcal{B}_G(\mathfrak{D}^0(\mathbf{\Sigma})|_{\mathcal{Y}_N}, \mathcal{Y}(\mathbf{\Sigma}))&\\ \to \mathcal{B}_G(\mathfrak{D}^0(\mathbf{\Sigma}|_{\mathcal{Y}_N}, \mathcal{Y}(\mathbf{\Sigma}) \cup \mathcal{Y}_{|\Sigma'|}))&\end{aligned}$$

as in (4.23). The first map can be identified with $\mathrm{D}^{0,|\Sigma'|}_G(\mathbf{\Sigma}) \to \mathrm{D}^{0,N}_G(\mathbf{\Sigma})$. Thus we need to show that the K-theory of the third term is trivial. To this end we will use Lemma 4.28 to construct an Eilenberg swindle on $\mathcal{B}_G(\mathfrak{D}^0(\mathbf{\Sigma})|_{\mathcal{Y}(N)}, \mathcal{Y}(\mathbf{\Sigma}) \cup \mathcal{Y}_{|\Sigma'|})$. The swindle will be similar to the one constructed in the proof of homotopy invariance in Proposition 8.16. This time we swindle towards $|\mathbf{\Sigma}'|^\wedge$ in $|N|^\wedge$. Let $Z := |\mathbf{\Sigma}'|^\wedge \times \mathbb{N} \times \mathbb{N}$. Let $p\colon N \to |\Sigma'|$ be the radial projection, i.e., if σ is a d-simplex of Σ and $x \in N \cap |\Delta_\sigma|$, then $p(x)$ is the unique point in $|\partial\Delta_\sigma|$ such that x lies on the straight line between $p(x)$ and the barycenter of σ. Let $f_t(x)$ be the point on the straight line from x to $p(x)$ of distance $\min\{1/(t+1), d(x, p(x))$ from x. Let $f^\wedge_t\colon |N|^\wedge \to |N|^\wedge$ be the map that sends $[x, \lambda]_\sigma \in |N|^\wedge$ to $[f_t(x), \lambda]_\sigma$. We now define $f\colon |N|^\wedge \times \mathbb{N} \times \mathbb{N} \to |W|^\wedge \times \mathbb{N} \times \mathbb{N}$ by

$$f(z, t_0, t_1) = (f^\wedge_{t_0}, t_0, t_1).$$

It is not too difficult to check that the assumptions of Lemma 4.29 are satisfied. As in the proof of Proposition 8.16, it is important here that f_t pushes less with increasing t, in order to preserve the ϵ-control part (see (7.3a)) of the foliated control condition in (7.7b). As f_t pushes linearly towards the boundary of $|\Delta_\sigma|$, it does preserve the sets $K_{\tau,\epsilon}$ from (7.3b). With these observations it is easy to control the interaction of $f^\wedge_t$ with the foliated control condition in (7.7b). □

Proposition 8.18. *The diagram*

$$\begin{tikzcd} \mathrm{D}^{0,M\cap N}_G(\mathbf{\Sigma}) \arrow[r] \arrow[d] & \mathrm{D}^{0,M}_G(\mathbf{\Sigma}) \arrow[d] \\ \mathrm{D}^{0,N}_G(\mathbf{\Sigma}) \arrow[r] & \mathrm{D}^0_G(\mathbf{\Sigma}) \end{tikzcd}$$

is a homotopy pushout.

Proof. It is not difficult to check that Lemma 4.27 applies, where we use $\mathcal{Y}_0 := \mathcal{Y}^+_N$, $\mathcal{Y}_1 := \mathcal{Y}^+_M$ and observe that $\mathcal{Y}_0 \cap \mathcal{Y}_1 = \mathcal{Y}_{M\cap N}$ and that, as $|\Sigma| = M \cup N$, $\mathrm{D}^0_G(\mathbf{\Sigma})|_{\mathcal{Y}_{M\cup N}} = \mathrm{D}^0_G(\mathbf{\Sigma})$. To verify that the assumption from Lemma 4.25 (as

[44] We could equally well work with closed neighborhoods.

required in Lemma 4.27) is satisfied, observe that the distance between points in $M \setminus N$ and in $N \setminus M$ is uniformly bounded from below by some $\delta > 0$. Let $Y_1 \in \mathcal{Y}_M^+$ and $E \in \mathfrak{D}_2^0(\mathbf{\Sigma})$ be given. The ϵ-control requirement in (7.7b) for E gives us some $k_0 \in \mathbb{N}$ such that if $\left(\begin{smallmatrix} z', t_0', t_1' \\ z, t_0, t_1 \end{smallmatrix}\right) \in E$ with $t_0 = t_0' \geq k_0$, then the distance of $p_{\mathbf{\Sigma}}(z)$ and $p_{\mathbf{\Sigma}}(z')$ in $|\Sigma|$ is $< \delta$; in particular $p_{\mathbf{\Sigma}}(z)$ and $p_{\mathbf{\Sigma}}(z')$ are either both in M or in N. Then with $Y_0 := |\mathbf{\Sigma}|^\wedge \times \mathbb{N}_{\leq k_0} \in \mathcal{Y}(\mathbf{\Sigma}) \subseteq \mathcal{Y}_N^+$, $Y_1' := Y_1 \setminus Y_1 \in \mathcal{Y}_M^+$ we have $(Y_1')^E \in \mathcal{Y}_M^+$. $\square$

Let now $\widehat{\mathbf{\Sigma}} = (\widehat{\Sigma}, \widehat{P})$ be as in Subsection 6.F, i.e., $\widehat{\Sigma} = B \times \Delta^d$, where B is the set of d-simplices of Σ. We can also apply the previous definitions to $\widehat{\mathbf{\Sigma}}$ and obtain $\widehat{N}$, $\mathsf{D}_G^{0,\widehat{M}}(\widehat{\mathbf{\Sigma}})$ and so on.

Lemma 8.19. *The functors*

$$\mathsf{D}_G^{0,\widehat{M}\cap\widehat{N}}(\widehat{\mathbf{\Sigma}}) \to \mathsf{D}_G^{0,M\cap N}(\mathbf{\Sigma}) \quad \textit{and} \quad \mathsf{D}_G^{0,\widehat{M}}(\widehat{\mathbf{\Sigma}}) \to \mathsf{D}_G^{0,M}(\mathbf{\Sigma})$$

induced by the projection $\widehat{\mathbf{\Sigma}} \xrightarrow{\mathbf{f}} \mathbf{\Sigma}$ *are equivalences.*

Proof. The projection $|\widehat{\Sigma}| \xrightarrow{f} \Sigma|$ restricts to an bijection $\widehat{M} \to M$. The restrictions of the l^∞-metrics of $|\widehat{\Sigma}|$ and $|\Sigma|$ to $\widehat{M}$ and M are different, but agree on path components. Moreover, in both cases the distance between points in different path components is bounded from below by a universal constant (depending only on d). With these observations it is not difficult to verify the assertion. $\square$

Proposition 8.20. *The diagram*

$$\begin{tikzcd} \mathbf{K}\big(\mathsf{D}_G^0(\widehat{\mathbf{\Sigma}}')\big) \arrow[r, "\mathbf{f}'_*"] \arrow[d, "\widehat{\iota}_*"] & \mathbf{K}\big(\mathsf{D}_G^0(\mathbf{\Sigma}')\big) \arrow[d, "\iota_*"] \\ \mathbf{K}\big(\mathsf{D}_G^0(\widehat{\mathbf{\Sigma}})\big) \arrow[r, "\mathbf{f}_*"] & \mathbf{K}\big(\mathsf{D}_G^0(\mathbf{\Sigma})\big) \end{tikzcd} \tag{8.21}$$

obtained by applying K-theory to (6.6) *is a homotopy pushout diagram.*

Proof. Consider

$$\begin{tikzcd} \mathsf{D}_G^{0,\widehat{M}\cap\widehat{N}}(\widehat{\mathbf{\Sigma}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & \mathsf{D}_G^{0,\widehat{M}}(\widehat{\mathbf{\Sigma}}) \arrow[d] \\ \mathsf{D}_G^{0,\widehat{N}}(\widehat{\mathbf{\Sigma}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & \mathsf{D}_G^{0}(\widehat{\mathbf{\Sigma}}) \arrow[d] \\ \mathsf{D}_G^{0,N}(\mathbf{\Sigma}) \arrow[r] & \mathsf{D}_G^{0}(\mathbf{\Sigma}) \end{tikzcd} \qquad \begin{tikzcd} \mathsf{D}_G^{0,\widehat{M}\cap\widehat{N}}(\widehat{\mathbf{\Sigma}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "(3)"] & \mathsf{D}_G^{0,\widehat{M}}(\widehat{\mathbf{\Sigma}}) \arrow[d] \\ \mathsf{D}_G^{0,M\cap N}(\mathbf{\Sigma}) \arrow[r] \arrow[d] \arrow[dr, phantom, "(4)"] & \mathsf{D}_G^{0,M}(\mathbf{\Sigma}) \arrow[d] \\ \mathsf{D}_G^{0,N}(\mathbf{\Sigma}) \arrow[r] & \mathsf{D}_G^{0}(\mathbf{\Sigma}) \end{tikzcd}$$

We will argue that (2) is a homotopy pushout in K-theory. This will give the assertion of the proposition, as Lemma 8.17 (which also applies to $\widehat{\mathbf{\Sigma}}$) allows us to replace $\mathsf{D}_G^{0,\widehat{N}}(\widehat{\mathbf{\Sigma}})$ with $\mathsf{D}_G^{0}(\widehat{\mathbf{\Sigma}}')$ and $\mathsf{D}_G^{0,N}(\mathbf{\Sigma})$ with $\mathsf{D}_G^{0}(\mathbf{\Sigma}')$. (Note that the positions of the right/top and left/bottom corners in (8.21) and in (2) are switched.)

Proposition 8.18 (which also applies to $\widehat{\mathbf{\Sigma}}$) tells us that (1) and (4) are homotopy pushouts in K-theory. Lemma 8.19 implies that (3) is a homotopy pushouts in K-theory as well. As the combinations of (1) with (2) and (3) with (4) agree, this implies that (4) is homotopy pushouts in K-theory. $\square$

8.F. **Skeleton continuity of $\mathrm{D}^0_G(-)$.**

Proposition 8.22. *For any $\mathbf{\Sigma} \in \mathcal{R}$ the canonical map*

$$\operatorname*{hocolim}_{d\in\mathbb{N}} \mathbf{K}\big(\mathrm{D}^0_G(\mathbf{\Sigma}^d)\big) \;\to\; \mathbf{K}\big(\mathrm{D}^0_G(\mathbf{\Sigma})\big)$$

is an equivalence.

Proof. The finite dimensional support condition in Definition 7.7 directly implies that $\mathrm{D}^0_G(\mathbf{\Sigma})$ is the directed union of the $\mathrm{D}^0_G(\mathbf{\Sigma}^d)$ and this gives the result. □

9. Outline of the construction of the transfer

We now assume that $\mathcal{B}$ is a Hecke category with G-support. As before we abbreviate $\mathrm{D}^0_G(\mathbf{\Sigma}) = \mathrm{D}^0_G(\mathbf{\Sigma};\mathcal{B})$. Theorem 6.7 asserts that the maps

$$\mathbf{K}\mathrm{D}^0_G\big(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times \mathbf{M}\big) \xrightarrow{\mathbf{p}_M} \mathbf{K}\mathrm{D}^0_G(\mathbf{M}) \tag{9.1}$$

induced by the projections $\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times \mathbf{M} \to \mathbf{M}$ admit sections $\mathbf{tr}_{\mathbf{M}}$ that are natural in $\mathbf{M} \in \mathcal{R}^0$. In this outline we will concentrate on the case where $\mathbf{M} = *$ is the terminal object[45] in $\mathcal{R}$; the general case requires no real additional input.

9.A. **Sequences.** To construct a section to (9.1) for $\mathbf{M} = *$ we will work with finite chain complexes and construct a homotopy coherent[46] functor

$$\mathrm{D}^0_G(*) \to \mathrm{ch}_{\mathrm{fin}}\,\mathrm{Idem}\,\mathrm{D}^0_G\big(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\big). \tag{9.2}$$

We use the sequence description of $\mathrm{D}^0_G(\mathbf{\Sigma})$ from Remark 7.17 as a subcategory of $\prod'_{\mathbb{N}^{\times 2}}\mathcal{B}_G(|\mathbf{\Sigma}|^\wedge)$. We will for each $\underline{t} \in \mathbb{N}^{\times 2}$ construct a (homotopy coherent) functor

$$\tilde{F}_{\underline{t}} \,:\, \mathcal{B}_G(*) \;\to\; \mathrm{ch}_{\mathrm{fin}}\,\big(\mathrm{Idem}\,\mathcal{B}_G(|\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge)\big)$$

such that their product

$$\prod_{\mathbb{N}^{\times 2}}{}' \tilde{F}_{\underline{t}} \,:\, \mathcal{B}_G(*) \;\to\; \prod_{\mathbb{N}^{\times 2}}{}' \mathrm{ch}_{\mathrm{fin}}\,\big(\mathrm{Idem}\,\mathcal{B}_G(|\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge)\big)$$

restricts to the desired functor (9.2). This boils down to verifying the conditions (7.18a), (7.18b), (7.18c), (7.18d), (7.18e), (7.18c) spelled out in Remark 7.17.

Objects in $\mathrm{D}^0_G(*)$ are sequences $\mathbf{B} = (\mathbf{B}_{\underline{t}})_{\underline{t}\in\mathbb{N}^{\times 2}}$ of objects in $\mathcal{B}_G(*)$, such that $\mathrm{supp}_G\,\mathbf{B}_{\underline{t}} \subseteq K$ for all $\underline{t}$ for some compact subset K that does not depend on $\underline{t}$. (In fact, we will pass to a subcategory of $\mathcal{B}_G(*)$, and have a bit more control over the $\mathrm{supp}_G\,\mathbf{B}_{\underline{t}}$.) Write $(\tilde{F}_{\underline{t}}(\mathbf{B}_{\underline{t}}))_n$ for the n-th chain module of $\tilde{F}_{\underline{t}}(\mathbf{B}_{\underline{t}})$. In order for $\prod'_{\mathbb{N}^{\times 2}}\tilde{F}_t$ to restrict to the desired functor, we will, among other things, require that $\mathrm{supp}_1(\tilde{F}_{\underline{t}}(\mathbf{B}_{\underline{t}}))_n \subseteq |\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge$ is finite for all $\underline{t} \in \mathbb{N}^{\times 2}$. Morphisms in $\mathrm{D}^0_G(*)$ are (equivalence classes of) sequences of morphisms $\varphi = (\varphi_{\underline{t}})_{\underline{t}\in\mathbb{N}^{\times 2}}$ in $\mathcal{B}_G(*)$ such that $\mathrm{supp}_G\,\varphi_{\underline{t}} \subseteq K$ for all $\underline{t}$ for some compact subset K that does not depend on $\underline{t}$. In order for $\prod'_{\mathbb{N}^{\times 2}}\tilde{F}_{\underline{t}}$ to restrict to the desired functor, we will, among other things, need to verify the foliated control condition from (7.7b). This means roughly the following. Given a compact subset K of G we need $\beta > 0$, $\eta_{\underline{t}}, \epsilon_{\underline{t}} > 0$ for $\underline{t} = (t_0, t_1) \in \mathbb{N}^{\times 2}$ with $\eta_{\underline{t}} \to 0$ as $t_1 \to \infty$ (for fixed t_0) and $\epsilon_{\underline{t}} \to 0$ as $t_0 \to \infty$ (uniform in t_1), such that if φ is a morphism in $\mathcal{B}_G(*)$ with $\mathrm{supp}_G\,\varphi \subseteq K$, then

$$\big(\begin{smallmatrix} z' \\ z \end{smallmatrix}\big) \in \mathrm{supp}_2\,\tilde{F}_{\underline{t}}(\varphi) \;\implies\; d_{\mathbf{\Sigma}\text{-fol}}(z, z') < (\beta, \eta_{\underline{t}}, \epsilon_{\underline{t}}).$$

[45] i.e., $* = (\Delta^0, *_{\mathrm{P}_{\mathcal{A}\mathrm{ll}(G)}})$.

[46] See Appendix C.

These two required properties, finiteness for objects, and foliated control for morphisms, are in tension with each other[47]. Let X be the extended Bruhat-Tits building associated to G. The functors $\tilde{F}_{\underline{t}}$ are constructed as a composition

$$\mathcal{B}_G(*) \xrightarrow{F_{\underline{t}}} \mathrm{ch_{fin}}\,\mathrm{Idem}\,\mathcal{B}_G(X) \xrightarrow{(f_{\underline{t}})_*} \mathrm{ch_{fin}}\,\big(\mathrm{Idem}\,\mathcal{B}_G(|\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge)\big)$$

where the $F_{\underline{t}}$ are given by a tensor product with certain complexes over X and $f_{\underline{t}}\colon X \to |\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge$ are certain maps. We give a brief outline for both below.

9.B. **The diagonal tensor product.** Given two smooth G-representations V, W we can equip $V \otimes W$ with the diagonal action $g \cdot (v \otimes w) = gv \otimes gw$. We obtain a functor

$$V \otimes - \,:\, \mathsf{Rep}(G) \;\to\; \mathsf{Rep}(G)$$

on categories of smooth representations. If the underlying vector space of V is finite dimensional, then this functor preserves finitely generated and projective representations. If V is the permutation representation of a smooth G-set Σ, then for Hecke category $\mathcal{B}$ with G-suport this can be generalized[48] to

$$\Sigma \otimes - \,:\, \mathcal{B}_G(*) \;\to\; \mathrm{Idem}\,\mathcal{B}_G(*).$$

Moreover, a map $c\colon \Sigma \to X$ determines a lift of this functor to

$$(\Sigma, c) \otimes - \,:\, \mathcal{B}_G(*) \;\to\; \mathrm{Idem}\,\mathcal{B}_G(X).$$

In Subsection 10.A we define a $\mathbb{Z}$-linear category $\mathcal{S}^G(X)$, whose objects are pairs (Σ, c) as above, and in Subsection 11.C we obtain a functor

$$- \otimes - \,:\, \mathcal{S}^G(X) \times \mathcal{B}_G(*) \;\to\; \mathrm{Idem}\,\mathcal{B}_G(X).$$

The functor $(\Sigma, c) \otimes -$ will provide a splitting for $\mathcal{B}_G(X) \to \mathcal{B}_G(*)$ if $\Sigma = G/G$, but this does not give us enough flexibility to construct a sequence F_t that will satisfy the foliated control condition from (7.7b). We can view the singular chain complex $\mathbf{S}_*(X)$ of X as a chain complex over $\mathcal{S}^G(X)$[49], see Subsection 10.B, and obtain

$$\mathbf{S}_*(X) \otimes - \,:\, \mathcal{B}_G(*) \;\to\; \mathrm{ch}\,\mathrm{Idem}\,\mathcal{B}_G(X).$$

As X is contractible this functor is much closer to providing a splitting and is much better compatible with control conditions for morphisms, see (11.9c). The remaining problem is that the singular chain complex is very large and this will lead to conflict with the finiteness conditions (7.7a), see Remark 11.11. Of course, as X is contractible, $\mathbf{S}_*(X)$ is finite up to homotopy. But such a homotopy involves moving through X and incorporating it into our construction would again lead to conflict with the foliated control condition from (7.7b). The solution is a compromise between X and a point. We will use large balls $B_{\underline{t}}$ in X. Moreover, in place of the singular complex we will use the simplicial complex of a suitable (fine) triangulation of $B_{\underline{t}}$. The balls $B_{\underline{t}}$ are not G-invariant. To resolve this we use that $B_{\underline{t}} \subseteq X$ is a deformation retract via the radial projection $X \to B_{\underline{t}}$. This way $B_{\underline{t}}$ inherits a homotopy coherent G-action from the G-action on X. Altogether we will construct a homotopy coherent functor

$$F_{\underline{t}}\colon \mathcal{B}_{G,U_{\underline{t}}}(*) \;\to\; \mathrm{ch_{fin}}\,\mathrm{Idem}\,\mathcal{B}_G(X),$$

[47] For example, it is not difficult to construct a functor $F\colon \mathcal{B}_G(*) \to \mathrm{ch_{fin}}\,\big(\mathcal{B}_G(|\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge)\big)$ such that for $\binom{z}{z'} \in \mathrm{supp}_2\,\tilde{F}_t(\varphi_t)$ we even have $z = z'$, but such an F will fail the required finiteness property for objects, see Remark 11.11.

[48] The formula in Subsection 11.C looks a priori different. The translation between the two functors uses a shearing isomorphism.

[49] For technical reason we will have to replace X with the set $S(X)$ of singular simplices in X and view $\mathbf{S}_*(X)$ as a chain complex over $\mathcal{S}^G(S(X))$; applying the barycenter map $S(X) \to X$ we then obtain a chain complex over $\mathcal{S}^G(X)$.

see Subsection 11.G. Here $U_{\underline{t}}$ is the compact open subgroup of G that fixes the ball $B_{\underline{t}}$ pointwise and $\mathcal{B}_{G,U_{\underline{t}}}(*) \subseteq \mathcal{B}_G(*)$ is the full subcategory on objects (S,π,B) where $\operatorname{supp} B(s) \subseteq U_t$ for all s. Passing to this subcategory is not a serious restriction, by the support cofinality property (3.2d) for $\mathcal{B}$, the idempotent completions of $\mathcal{B}_{G,U_{\underline{t}}}(*)$ and $\mathcal{B}_G(*)$ coincide. The deformation of X onto $B_{\underline{t}}$ still moves through X, creating conflict with the foliated control condition from (7.7b). We will outline how this conflict is resolved in the next subsection.

9.C. **The maps $X \to |\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge$.** Recall that morphisms in $\mathrm{D}^0_G(*)$ have relatively compact G-support. Using the equivalence relation on morphisms in $\mathrm{D}^0_G(*)$ (or in $\prod'\mathcal{B}_G(*)$) this means that in the construction of the $\tilde{F}_{\underline{t}}$ for each fixed $\underline{t}$ we only have to control its interaction with a relatively compact set in G, specified later in Subsection 13.B and denoted $M_{\underline{t}}$. The important point is that $M_{\underline{t}} = M_{t_0,t_1} \to \infty$ as $t_0 \to \infty$. In a similar way we will for fixed $\underline{t}$ not need to worry about the deformation of all of X onto $B_{\underline{t}}$, but only about its restriction to the $L_{\underline{t}}$-neighborhood $B_{\underline{t}}^{(L_{\underline{t}})}$ of $B_{\underline{t}}$. Here it will be important that $L_{\underline{t}} = L_{t_0,t_1} \to \infty$ as $t_1 \to \infty$[50]. This leads to the following requirements for the maps $f_{\underline{t}}\colon X \to |\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge$.

(a) The restriction of $f_{\underline{t}}$ to $B_{\underline{t}}^{(L_{\underline{t}})}$ should be $M_{\underline{t}}$-equivariant up to a $\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)$-foliated error-term; i.e., we have control over
$$d_{\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\text{-fol}}(gf_{\underline{t}}(x), f_{\underline{t}}(gx))$$
for $g \in M_{\underline{t}}$ and $x \in B_{\underline{t}}^{(L_{\underline{t}})}$. The precise formulation is (13.1a);

(b) The restriction of $f_{\underline{t}}$ to the tracks of the radial deformation of $B_{\underline{t}}^{(L_{\underline{t}})}$ to $B_{\underline{t}}$ is constant up to a $\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)$-foliated error-term; i.e., we have control over
$$d_{\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\text{-fol}}(f_{\underline{t}}(x), f_{\underline{t}}(\pi_{R'}(x)))$$
for $x \in B_{\underline{t}}^{(L_{\underline{t}})}$ and $\pi_{R'}(x)$ on the geodesic between x and its image in $B_{\underline{t}}$ under the radial projection $B_{\underline{t}}^{(L_{\underline{t}})} \to B_{\underline{t}}$. The precise formulation is (13.1b).

There is third requirement (13.1c) for $f_{\underline{t}}$. This should be thought of as a substitute for continuity of $f_{\underline{t}}$. In fact, with a more careful choice for the resolution $|\mathbf{J}\mathcal{C}\mathrm{vcy}(G)|^\wedge \to |\mathbf{J}\mathcal{C}\mathrm{vcy}(G)|$ we could arrange for the $f_{\underline{t}}$ to be continuous, but we found it more convenient to allow some non-continuity for the $f_{\underline{t}}$. The construction of the $f_{\underline{t}}$ uses a geodesic flow on X and depends on the fact that X is CAT(0). It is outlined in Appendix D, details are worked out in [6].

10. The category $\mathcal{S}^G(\Omega)$

Throughout this section we fix a smooth G-space X. We assume that for $K \subseteq X$ compact, the pointwise isotropy group $G_K = \bigcap_{x\in K} G_x$ is an open subgroup of G. Later X will be the extended Bruhat-Tits building associated to a reductive p-adic group.

10.A. The category $\mathcal{S}^G(\Omega)$.

Definition 10.1. For a smooth G-set Ω we define the additive category $\mathcal{S}^G(\Omega)$ as follows. Objects are pairs $\mathbf{V} = (\Sigma, c)$ where Σ is a smooth G-set and $c\colon \Sigma \to \Omega$ is a G-map. A morphism $\rho\colon \mathbf{V} = (\Sigma, c) \to \mathbf{V}' = (\Sigma', c')$ is an $\Sigma \times \Sigma'$-matrix $(\rho_\sigma^{\sigma'})_{\sigma\in\Sigma,\sigma'\in\Sigma'}$ over $\mathbb{Z}$ satisfying the following two conditions

(10.1a) for all $\sigma \in \Sigma$ the set $\{\sigma' \in \Sigma' \mid \rho_\sigma^{\sigma'} \neq 0\}$ is finite;

(10.1b) for all $g \in G$, $\sigma \in \Sigma$, $\sigma' \in \Sigma'$ we have $\rho_{g\sigma}^{g\sigma'} = \rho_\sigma^{\sigma'}$.

[50] Typically the radii of the $B_{\underline{t}}$ will grow much quicker than the $L_{\underline{t}}$.

The support of ρ is

$$\operatorname{supp}_2 \rho := \left\{ \left(\begin{smallmatrix} c'(\sigma') \\ c(\sigma) \end{smallmatrix} \right) \,\middle|\, \rho_\sigma^{\sigma'} \neq 0 \right\} \subseteq \Omega \times \Omega.$$

The support of $\mathbf{V}$ is $\operatorname{supp}_1 \mathbf{V} := c(\Sigma)$. Composition is matrix multiplication $(\rho' \circ \rho)_\sigma^{\sigma''} := \sum_{\sigma'} \rho'^{\sigma''}_{\sigma'} \circ \rho_\sigma^{\sigma'}$. We will say that $\mathbf{V}$ is finite, if Σ is finite.

The identity $\mathrm{id}_{\mathbf{V}}$ of $\mathbf{V} = (\Sigma, c)$ is given by $(\mathrm{id}_{\mathbf{V}})_\sigma^{\sigma'} = 1$ for $\sigma = \sigma'$ and $(\mathrm{id}_{\mathbf{V}})_\sigma^{\sigma'} = 0$ for $\sigma \neq \sigma'$.

10.B. **The singular chain complex of X as a chain complex over $\mathcal{S}^G(S(X))$.** Let $S_n(X)$ be the set of singular n-simplices of X. Let $S(X)$ be the union of the $S_n(X)$. By our assumption on X this is a smooth G-set via the G-action on X. Let $c_n \colon S_n(X) \to S(X)$ be the inclusion. We obtain $(S_n(X), c_n) \in \mathcal{S}^G(S(X))$. We define $\partial_n \colon (S_n(X), c_n) \to (S_{n-1}(X), c_{n-1})$ by

$$(\partial_n)_\sigma^{\sigma'} := \begin{cases} (-1)^i & \text{if } \sigma' \text{ is the } i\text{-th face of } \sigma; \\ 0 & \text{else.} \end{cases} \tag{10.2}$$

We write $\mathbf{S}_*(X) \in \operatorname{ch} \mathcal{S}^G(S(X))$ for the chain complex in $\mathcal{S}^G(S(X))$ obtained this way.

Later we will use the G-map $\mathrm{bary} \colon S(X) \to X$ that sends $\sigma \colon |\Delta^n| \to X$ to the image of the barycenter of $|\Delta^n|$ under σ. Then $\mathrm{bary}_*(\mathbf{S}_*(X)) \in \operatorname{ch} \mathcal{S}^G(X)$[51]. Working in $\mathcal{S}^G(S(X))$ and not in $\mathcal{S}^G(X)$ will allow us to consider certain restrictions in Subsection 12.D.

10.C. **Subspace.** For some purposes the singular chain complex $\mathbf{S}_*(X)$ is too big. To replace it by smaller chain complexes we will pass from X to a subspace (and later to a subspace equipped with a triangulation). Typically, the subspace will not be G-invariant, and we will have to pass to an open subgroup as well.

Let $U \subseteq G$ be an open subgroup. Let $D \subseteq X$ be a U-invariant subspace[52]. We write $S_n(D)$ for the set of singular simplices of D. We obtain the chain complex $\mathbf{S}_*(D)$ over $\mathcal{S}^U(S(X))$ whose n-th chain module is $(S_n D, c_n|_{S_n(D)})$. The boundary map is still given by (10.2).

Let D' be a further U-invariant subspace. A U-equivariant map $f \colon D \to D'$ induces a chain map $f_* \colon \mathbf{S}_*(D) \to \mathbf{S}_*(D')$ in $\operatorname{ch} \mathcal{S}^U(S(X))$ with

$$(f_*)_\sigma^{\sigma'} := \begin{cases} 1 & \text{if } \sigma' = f \circ \sigma; \\ 0 & \text{else.} \end{cases}$$

A U-equivariant homotopy $H \colon D \times [0,1] \to D'$ with $H(-,0) = f_0$, $H(-,1) = f_1$ determines a chain homotopy $H_* \colon (f_0)_* \simeq (f_1)_*$ by the usual formula. In order to give the formula in detail, we write v_k for the k-th vertex of Δ^n and let $i_j \colon |\Delta^{n+1}| \to |\Delta^n| \times [0,1]$ be the affine map determined by

$$i_j(v_k) := \begin{cases} (v_k, 0) & j \leq k; \\ (v_{k-1}, 1) & j > k. \end{cases}$$

Then[53]

$$(H_*)_\sigma^{\sigma'} = \sum_{j \colon \sigma' = H \circ (\sigma \times \mathrm{id}_{[0,1]}) \circ i_j} (-1)^j.$$

[51] A G-map $f \colon \Omega \to \Omega'$ induces a map $f_* \colon \operatorname{ch} \mathcal{S}^G(\Omega) \to \mathcal{S}^G(\Omega')$ by composition.

[52] Later on D will be contained in the U-fixed point set X^U

[53] Usually H_* is not written as a matrix; for this reason the formula may not look familiar at a first glance.

Lemma 10.3. *For $H\colon D \times [0,1] \to D'$ we have*

$$\binom{\sigma'}{\sigma} \in \operatorname{supp}_2 H_* \quad \Longrightarrow \quad \operatorname{im}(\sigma') \subseteq \operatorname{im}(H \circ \sigma \times \operatorname{id}_{[0,1]}).$$

Proof. This is a direct consequence of the formula for H_* reviewed above. $\square$

Lemma 10.4. *Let $H\colon D \times [0,1] \to D'$ be a homotopy between f_0 and f_1. Let d_X be a metric on X. Assume that D is compact. Assume that for all $s \in [0,1]$ and all $x, x' \in B$ we have*

$$d_X\big(H(x,s), H(x',s)\big) \leq d_X(x,x').$$

Let $\epsilon > 0$. Then there is a chain homotopy $\widetilde{H}\colon \mathbf{S}_*(D) \to \mathbf{S}_{*+1}(D')$ between $(f_0)_*$ and $(f_1)_*$ with the following property. Let $\binom{\sigma'}{\sigma} \in \operatorname{supp}_2 \widetilde{H}$[54]*. Then*

(a) if the diameter of the image of σ is $< \kappa$, then the diameter of the image of σ' is $< \kappa + \epsilon$;

(b) $\operatorname{im}\sigma' \subseteq \operatorname{im}(H \circ \sigma \times \operatorname{id}_{[0,1]})$.

Proof. As D is compact we can find $\delta > 0$ such that $d_X(H(x,s), H(x,s')) < \epsilon/2$ for all $x \in D$ and all $s, s' \in [0,1]$ with $|s - s'| < \delta$. Now we choose $0 = s_0 < s_1 < \cdots < s_n = 1$ with $|s_{i+1} - s_i| < \delta$. Then $\widetilde{H} := \sum_{i=1}^n (H|_{D \times [s_{i-1}, s_i]})_*$ is a chain homotopy between $(f_0)_*$ and $(f_1)_*$. Now $\operatorname{supp} \widetilde{H}_\sigma^{\sigma'} \subseteq \bigcup_{i=1}^n \operatorname{supp}\big((H|_{D \times [s_{i-1}, s_i]})_*\big)$ and it is easy to check that $\widetilde{H}$ has the required properties. $\square$

10.D. **Triangulations.** Assume that the U-invariant subspace D of X is equipped with a U-invariant triangulation, i.e., $D = |K|$ for a simplicial complex K with a smooth U-action. Assume that the triangulation is locally ordered, i.e., there is a partial order on the set of vertices that restricts to a linear order on the vertex set of every simplex. This ensures that every simplex in the triangulation determines a unique singular simplex of D and therefore of X. In particular, we can view the set $\operatorname{simp}_n(D)$ of n-simplices of D as a subset of $S_n D$. We obtain the chain complex $\mathbf{C}_*(D) \in \operatorname{ch} \mathcal{S}^U(\operatorname{res}_G^U S(X))$, whose n-th chain module is $(\operatorname{simp}_n(D), c_n|_{\operatorname{simp}_n(D)})$. It comes with an inclusion $i\colon \mathbf{C}_*(D) \to \mathbf{S}_*(D)$ defined by

$$i_\sigma^{\sigma'} := \begin{cases} 1 & \text{if } \sigma' = \sigma; \\ 0 & \text{else.} \end{cases}$$

Let d_X be a metric on X. For a singular simplex $\sigma\colon \Delta^n \to X$ we write $\operatorname{im}\sigma$ for its image and $(\operatorname{im}\sigma)^\epsilon$ for the open ϵ-neighborhood of $\operatorname{im}\sigma$.

Lemma 10.5. *Assume that all simplices of the triangulation of D are of diameter $< \epsilon$. In $\operatorname{ch}\mathcal{S}^U(S(X))$ there exists a chain map $r\colon \mathbf{S}_*(D) \to \mathbf{C}_*(D)$ with $r \circ i = \operatorname{id}_{\mathbf{C}_*(D)}$ and a chain homotopy $H\colon \operatorname{id}_{\mathbf{S}_*(D)} \simeq i \circ r$ such that if $\binom{\sigma'}{\sigma} \in \operatorname{supp}_2 r \cup \operatorname{supp}_2 H$, then $\operatorname{im}\sigma' \subseteq (\operatorname{im}\sigma)^\epsilon$.*

Proof. This a minor variation of [9, Lem. 6.9]. Let $\mathcal{K}$ be the poset of subcomplexes of K ordered by inclusion. We view $\mathcal{K}$ as a category and work now in the abelian category of $\mathbb{Z}\mathcal{K}$-modules[55]. For $K_0 \in \mathcal{K}$ let $\underline{C}_*(K_0)$ be the simplicial chain complex of K_0 and $\underline{S}_*(K_0)$ be the singular chain complex of $|K_0|$. This defines chain complexes $\underline{C}_*$ and $\underline{S}_*$ of $\mathbb{Z}\mathcal{K}$-modules and it is not difficult to check that the underlying $\mathbb{Z}\mathcal{K}$-modules $\underline{C}_n$, $\underline{S}_n$ are free (and thus projective) for all n. Write $\underline{i}\colon \underline{C}_* \to \underline{S}_*$ for the inclusion. Each $\underline{i}_n\colon \underline{C}_n \to \underline{S}_n$ is the inclusion of a direct summand. Moreover, $\underline{i}$ induces an isomorphism in homology (taken in the category of $\mathbb{Z}\mathcal{K}$-modules). By general results in homological algebra (in abelian categories) it follows that there exists a chain map $\underline{r}\colon \underline{S}_* \to \underline{C}_*$ with $\underline{r} \circ \underline{i} = \operatorname{id}_{C_*}$ and a homotopy

[54] The support of a graded map is the union of the supports of the maps in all degrees.

[55] That is in the category of functors $\mathcal{K} \to \mathbb{Z}$-$\mathsf{Mod}$.

$\underline{H}\colon \mathrm{id}_{S_*} \simeq i \circ r$. We can set $r := \underline{r}(K)$, $H := \underline{H}(K)$. The additional properties of r and H follow from the functoriality in $\mathcal{K}$ of $\underline{r}$ and $\underline{H}$: Suppose $r_\sigma^{\sigma'} \neq 0$. Let K_σ be the smallest subcomplex of K such that $|K_\sigma|$ contains $\operatorname{im}\sigma$, so $\sigma \in \underline{S}_*(K_\sigma)$. As $\underline{r}(K_\sigma)\colon \underline{S}_*(K_\sigma) \to \underline{C}_*(K_\sigma)$ it follows that σ' is a simplex of K_σ. Now $|K_\sigma| \subseteq (\operatorname{im}\sigma)^\epsilon$ as the diameter of simplices in K are of diameter $< \epsilon$. Thus $\operatorname{im}\sigma' \subseteq (\operatorname{im}\sigma)^\epsilon$. The same argument applies to H. □

11. Diagonal tensor products

Throughout this section we fix a G-set Λ and a smooth G-space X. Later we will have $\Lambda = |\mathbf{M}|^\wedge$ for $\mathbf{M} \in \mathcal{R}^0$, while for X we will take the extended Bruhat-Tits building associated to a reductive p-adic group. We also fix a smooth G-set Ω. Later $\Omega = S(X)$ will be the set of singular simplices of X.

To simplify the discussion we assume throughout this section that $\mathcal{B}$ is the category $\mathcal{B}(G;R)$ from Example 3.3. This means in particular that if $\mathbf{B} = (S, \pi, B)$ is an object from $\mathcal{B}_G(\Lambda)$, then the $B(s)$ for $s \in S$ are compact open subgroups of G. Also, if $\varphi\colon \mathbf{B} = (S,\pi,B) \to (S',\pi',S')$ is a morphism in $\mathcal{B}_G(\Lambda)$, then the $\varphi_s^{s'}$ are elements of the Hecke algebra $\mathcal{H}(G;R)$ satisfying $\varphi_s^{s'}(a'ga) = \varphi_s^{s'}(g)$ for all $a \in B(s)$, $a' \in B(s')$.

Everything[56] we need also works for general Hecke categories with G-support and is treated in detail in [7, Sec. 7].

11.A. **Precursor.** Our first goal in this section is the construction of a bilinear functor

$$- \otimes - : \mathcal{S}^G(\Omega) \times \mathcal{B}_G(\Lambda) \to \operatorname{Idem}\mathcal{B}_G(\Omega \times \Lambda).$$

The construction is easier under some simplifying assumptions. So we assume for this subsection that G is discrete and consider the full subcategory $\mathcal{B}_G^1(\Lambda)$ of $\mathcal{B}_G(\Lambda)$ on the objects (S,π,B), for which $B(s)$ is the trivial subgroup for all s. For $\mathbf{V} = (\Sigma, c) \in \mathcal{S}^G(X)$ and $\mathbf{B} = (S,\pi,B) \in \mathcal{B}_G^1(\Lambda)$, we can then define

$$\mathbf{V} \otimes \mathbf{B} := (\Sigma \times S, c \times \pi, (\sigma, s) \mapsto B(s)). \tag{11.1}$$

For $\rho\colon (\Sigma, c) \to (\Sigma', c')$ and $\varphi\colon (S,\pi,B) \to (S',\pi',B')$ we can define

$$(\rho \otimes \varphi)_{\sigma,s}^{\sigma',s'}(g) := \rho_{g\sigma}^{\sigma'} \cdot \varphi_s^{s'}(g). \tag{11.2}$$

The general idea is that we use the G-action on Σ to twist morphisms in $\mathcal{B}_G^1(\Lambda)$[57]. In the general case these formulas do not define a functor; for example we can have $(\rho\otimes\varphi)_{\sigma,s}^{\sigma',s'} \neq (\rho\otimes\varphi)_{\sigma,s}^{\sigma',s'}(a'ga)$ for $a' \in B'(s')$, $a \in B(s)$. To account for this we will replace $(\sigma, s) \mapsto B(s)$ with $(\sigma, s) \mapsto G_\sigma \cap B(s)$. The only remaining drawback is that the functor obtained this way does not preserve units. We will correct this using the idempotent completion.

Remark 11.3 (Non-unital categories)**.** In this remark we will contemplate categories without units. Let $\mathcal{B}^{\mathrm{nu}} := \underline{\mathcal{H}(G;R)}$ be the $\mathbb{Z}$-linear category with exactly one object whose endomorphism ring is the Hecke algebra $\mathcal{H}(G;R)$. As $\mathcal{H}(G;R)$ does not have a unit, $\mathcal{B}^{\mathrm{nu}}$ is a non-unital category, but its idempotent completion $\operatorname{Idem}\mathcal{B}^{\mathrm{nu}}$ has units. Via $U \mapsto \frac{\chi_U}{\mu(U)}$ the category $\mathcal{B} = \mathcal{B}(G;R)$ can be identified with a full subcategory of $\operatorname{Idem}\mathcal{B}^{\mathrm{nu}}$. Definition 4.1 also makes sense for $\mathcal{B}^{\mathrm{nu}}$ in place of $\mathcal{B}$

[56] More precisely, there is a diagonal tensor product such that Lemmas 11.7, and 11.9, and the identities (11.12), (11.13) are still valid, see [7, Lem. 7.41, 7.43, 7.44]. Everything else in this section just depends on these results.

[57] The category $\mathcal{B}_G^1(\Lambda)$ is equivalent to the category of free $R[G]$-modules. Under this equivalence our functor is equivalent to $((\Sigma, c), M) \mapsto \mathbb{Z}[\Sigma] \otimes_{\mathbb{Z}} M$, where the tensor product over $\mathbb{Z}$ is equipped with the diagonal action of G.

and we obtain the non-unital category $\mathcal{B}_G^{\mathrm{nu}}(X)$. Now the formulas (11.1) and (11.2) define a functor (of non-unital categories)

$$\mathcal{S}^G(\Omega) \times \mathcal{B}_G^{\mathrm{nu}}(\Lambda) \to \mathcal{B}_G^{\mathrm{nu}}(\Omega \times \Lambda).$$

Of course this functor sends idempotents to idempotents and so induces a functor (of unital categories)

$$\mathcal{S}^G(\Omega) \times \operatorname{Idem} \mathcal{B}_G^{\mathrm{nu}}(\Lambda) \to \operatorname{Idem} \mathcal{B}_G^{\mathrm{nu}}(\Omega \times \Lambda).$$

One can now identify $\mathcal{B}_G^{\mathrm{nu}}(\Omega)$ with a full subcategory of $\operatorname{Idem} \mathcal{B}_G^{\mathrm{nu}}(\Omega)$ and use this to obtain

$$\mathcal{S}^G(\Omega) \times \mathcal{B}_G(\Lambda) \to \operatorname{Idem} \mathcal{B}_G(\Omega \times \Lambda).$$

This is essentially what we will do in Subsections 11.B and 11.C, although we will avoid non-unital categories and instead give the resulting formulas more directly.

11.B. **The diagonal tensor product $\otimes_0$.** We define

$$- \otimes_0 - : \mathcal{S}^G(\Omega) \times \mathcal{B}_G(\Lambda) \to \mathcal{B}_G(\Omega \times \Lambda)$$

as follows. For $\mathbf{V} = (\Sigma, c) \in \mathcal{S}^G(X)$ and $\mathbf{B} = (S, \pi, B) \in \mathcal{B}_G(\Lambda)$ we set

$$\mathbf{V} \otimes_0 \mathbf{B} := (\Sigma \times S, c \times \pi, (\sigma, s) \mapsto G_\sigma \cap B(s)).$$

For morphisms $\rho\colon \mathbf{V} = (\Sigma, c) \to \mathbf{V}' = (\Sigma', c')$ in $\mathcal{S}^G(\Omega)$ and $\varphi\colon \mathbf{B} = (S, \pi, B) \to \mathbf{B}' = (S', \pi', B')$ in $\mathcal{B}_G(\Lambda)$ we define

$$(\rho \otimes_0 \varphi)_{s,\sigma}^{s,'\sigma'}(g) := \rho_{g\sigma}^{\sigma'} \cdot \varphi_s^{s'}(g)$$

as in (11.2).

We will check in Lemmas 11.4 and 11.5 below that $\rho \otimes_0 \varphi$ is well defined and compatible with composition. While we do not claim that $\mathrm{id}_{\mathbf{V}} \otimes_0 \mathrm{id}_{\mathbf{B}}$ is $\mathrm{id}_{\mathbf{V} \otimes_0 \mathbf{B}}$, compatibility with composition implies that $\mathrm{id}_{\mathbf{V}} \otimes_0 \mathrm{id}_{\mathbf{B}}$ is an idempotent endomorphism of $\mathbf{V} \otimes_0 \mathbf{B}$.

Lemma 11.4. *Let $\rho\colon \mathbf{V} = (\Sigma, c) \to \mathbf{V}' = (\Sigma', c')$ in $\mathcal{S}^G(\Omega)$ and $\varphi\colon \mathbf{B} = (S, \pi, B) \to \mathbf{B}' = (S', \pi', B')$ in $\mathcal{B}_G(\Lambda)$. Then $\rho \otimes_0 \varphi\colon \mathbf{V} \otimes_0 \mathbf{B} \to \mathbf{V}' \otimes \mathbf{B}'$ is a morphism in $\mathcal{B}_G(\Omega \times \Lambda)$*

Proof. For $a \in G_\sigma \cap B(s)$, $a' \in G_{\sigma'} \cap B(s')$ we have

$$\begin{aligned}(\rho \otimes_0 \varphi)_{\sigma,s}^{\sigma',s'}(a'ga) = \rho_{a'ga\sigma}^{\sigma'} \cdot \varphi_s^{s'}(a'ga) &= \rho_{g\sigma}^{(a')^{-1}\sigma'} \cdot \varphi_s^{s'}(g) \\ &= \rho_{g\sigma}^{\sigma'} \cdot \varphi_s^{s'}(g) = (\rho \otimes_0 \varphi)_{\sigma,s}^{\sigma',s'}(g),\end{aligned}$$

so for fixed σ, σ', s, s', $(\rho \otimes_0 \varphi)_{\sigma,s}^{\sigma',s'}\colon G_\sigma \cap B(s) \to G_{\sigma'} \cap B(s')$ is a morphism in $\mathcal{B} = \mathcal{B}(G; R)$. We also need to check that $\rho \otimes_0 \varphi$ is column finite. Fix (σ, s). We need to check that there are only finitely many (σ', s') with $(\rho \otimes \varphi)_{\sigma,s}^{\sigma',s'} \neq 0$. As φ is column finite there is $S'_0 \subset S'$ finite such that $\varphi_s^{s'} \neq 0$ implies $s' \in S'_0$. The $\varphi_s^{s'}$ are compactly supported. Thus there is $M \subseteq G$ compact such that $\varphi_s^{s'}(g) \neq 0$ implies $g \in M$. As Σ is a smooth G-set, the set $M \cdot \sigma \subseteq \Sigma$ is finite. As ρ is column finite there is $\Sigma'_0 \subset \Sigma'$ finite such that $\rho_{g\sigma}^{\sigma'} \neq 0$ with $g \in M$ implies $\sigma' \in \Sigma'_0$. Now if $(\rho \otimes \varphi)_{s,\sigma}^{s',\sigma'} \neq 0$, then for some $g \in G$ we have $\rho_{g\sigma}^{\sigma'} \neq 0$ and $\varphi_s^{s'}(g) \neq 0$. Thus $(\sigma', s') \in \Sigma'_0 \times S'_0$. ☐

Lemma 11.5. *Let $(S, \pi) \xrightarrow{\varphi} (S', \pi') \xrightarrow{\varphi'} (S'', \pi'')$ be composable morphisms in $\mathcal{B}_G(\Lambda)$ and $(\Sigma, c) \xrightarrow{\rho} (\Sigma', c') \xrightarrow{\rho'} (\Sigma'', c'')$ be composable morphisms in $\mathcal{S}^G(\Omega)$. Then*

$$(\rho' \otimes_0 \varphi') \circ (\rho \otimes_0 \varphi) = (\rho' \circ \rho) \otimes_0 (\varphi' \circ \varphi).$$

Proof.

$$\begin{aligned}
\big((\rho' \otimes_0 \varphi') \circ (\rho \otimes_0 \varphi)\big)_{\sigma,s}^{\sigma'',s''}(g) &= \sum_{\sigma' \in \Sigma', s \in S'} \Big((\rho' \otimes_0 \varphi')_{\sigma',s'}^{\sigma'',s''} \circ (\rho \otimes_0 \varphi)_{\sigma,s}^{\sigma',s'}\Big)(g) \\
&= \sum_{\sigma' \in \Sigma', s \in S'} \int_{x \in G} (\rho' \otimes_0 \varphi')_{\sigma',s'}^{\sigma'',s''}(gx) \circ (\rho \otimes_0 \varphi)_{\sigma,s}^{\sigma',s'}(x^{-1}) \\
&= \sum_{\sigma' \in \Sigma', s \in S'} \int_{x \in G} (\rho')_{gx\sigma'}^{\sigma''} \cdot (\varphi')_{s'}^{s''}(gx) \cdot \rho_{x^{-1}\sigma}^{\sigma'} \cdot \varphi_s^{s'}(x^{-1}) \\
&= \sum_{\sigma' \in \Sigma', s \in S'} \int_{x \in G} (\rho')_{gx\sigma'}^{\sigma''} \cdot \rho_{x^{-1}\sigma}^{\sigma'} \cdot (\varphi')_{s'}^{s''}(gx) \cdot \varphi_s^{s'}(x^{-1}) \\
&= \sum_{\sigma' \in \Sigma', s \in S'} \int_{x \in G} (\rho')_{gx\sigma'}^{\sigma''} \cdot \rho_{g\sigma}^{gx\sigma'} \cdot (\varphi')_{s'}^{s''}(gx) \cdot \varphi_s^{s'}(x^{-1}) \\
&= \sum_{\sigma' \in \Sigma', s \in S'} \int_{x \in G} (\rho')_{\sigma'}^{\sigma''} \cdot \rho_{g\sigma}^{\sigma'} \cdot (\varphi')_{s'}^{s''}(gx) \cdot \varphi_s^{s'}(x^{-1}) \\
&= (\rho' \circ \rho)_{g\sigma}^{\sigma''} \cdot (\varphi' \circ \varphi)_s^{s''}(g) \\
&= \big((\rho' \circ \rho) \otimes_0 (\varphi' \circ \varphi)\big)_{\sigma,s}^{\sigma'',s''}(g).
\end{aligned}$$

□

11.C. **The diagonal tensor product $\otimes$.** For $\mathbf{V} \in \mathcal{S}^G(\Omega)$ and $\mathbf{B} \in \mathcal{B}_G(\Lambda)$ we set

$$\mathbf{V} \otimes \mathbf{B} := (\mathbf{V} \otimes_0 \mathbf{B}, \mathrm{id}_{\mathbf{V}} \otimes_0 \mathrm{id}_{\mathbf{B}}).$$

For morphisms $\rho\colon \mathbf{V} \to \mathbf{V}'$ in $\mathcal{S}^G(X)$ and $\varphi\colon \mathbf{B} \to \mathbf{B}'$ in $\mathcal{B}_G(\Lambda)$ we define

$$(\rho \otimes \varphi) := (\rho \otimes_0 \varphi)\colon \mathbf{V} \otimes \mathbf{B} \to \mathbf{V}' \otimes \mathbf{B}'.$$

Now $\mathrm{id}_{\mathbf{V}} \otimes \mathrm{id}_{\mathbf{B}} = \mathrm{id}_{\mathbf{V} \otimes \mathbf{B}}$. Altogether we have now defined a bilinear functor

$$- \otimes - : \mathcal{S}^G(\Omega) \times \mathcal{B}_G(\Lambda) \to \operatorname{Idem} \mathcal{B}_G(\Omega \times \Lambda). \tag{11.6}$$

The following observation will often allow us to get rid of idempotent completions.

Lemma 11.7. *Let $\mathbf{V} = (\Sigma, c) \in \mathcal{S}^G(\Omega)$ and $\mathbf{B} = (S, \pi, B) \in \mathcal{B}_G(\Lambda)$. If Σ is fixed pointwise by all $B(s)$, then $\mathbf{V} \otimes \mathbf{B} = \mathbf{V} \otimes_0 \mathbf{B}$.*

Proof. The content of the lemma is that $\mathrm{id}_{\mathbf{V}} \otimes_0 \mathrm{id}_{\mathbf{B}}$ is the identity of $\mathbf{V} \otimes_0 \mathbf{B}$, not just an idempotent. Indeed, since $G_\sigma \cap B(s) = B(s)$ for all σ and s we have

$$(\mathrm{id}_{\mathbf{V}} \otimes_0 \mathrm{id}_{\mathbf{B}})_{\sigma,s}^{\sigma',s'}(g) = (\mathrm{id}_{\mathbf{V}})_{g\sigma}^{\sigma'} (\mathrm{id}_{\mathbf{B}})_s^{s'}(g) = (\mathrm{id}_{\mathbf{V} \otimes_0 \mathbf{B}})_{\sigma,s}^{\sigma',s'}(g).$$

□

For $E \subseteq \Omega \times \Omega$ and $E' \subseteq \Lambda \times \Lambda$ we use the following convention

$$E \times E' := \left\{ \left(\begin{smallmatrix} x', \lambda' \\ x, \lambda \end{smallmatrix}\right) \;\middle|\; \left(\begin{smallmatrix} x' \\ x \end{smallmatrix}\right) \in E, \left(\begin{smallmatrix} \lambda' \\ \lambda \end{smallmatrix}\right) \in E' \right\} \subseteq (\Omega \times \Lambda)^{\times 2}. \tag{11.8}$$

Lemma 11.9. *Let $\mathbf{V} = (\Sigma, c) \in \mathcal{S}^G(\Omega)$ and $\mathbf{B} = (S, \pi, B) \in \mathcal{B}_G(\Lambda)$.*

(11.9a) *If $\mathbf{V}$ and $\mathbf{B}$ are finite, then $\mathbf{V} \otimes_0 \mathbf{B}$ is finite as well;*
(11.9b) $\operatorname{supp}_1(\mathbf{V} \otimes_0 \mathbf{B}) = \operatorname{supp}_1 \mathbf{V} \times \operatorname{supp}_1 \mathbf{B}$.

Let $\rho\colon \mathbf{V} = (\Sigma, c) \to \mathbf{V}' = (\Sigma', c')$ in $\mathcal{S}^G(\Omega)$, $\varphi\colon \mathbf{B} = (S, \pi, B) \to \mathbf{B}' = (S', \pi', B')$ in $\mathcal{B}_G(\Lambda)$. Then for $\rho \otimes \varphi$ in $\mathcal{B}_G(\Omega \times \Lambda)$ we have

(11.9c) $\operatorname{supp}_2(\rho \otimes \varphi) \subseteq \operatorname{supp}_2 \rho \times \operatorname{supp}_2 \varphi$;
(11.9d) $\operatorname{supp}_G(\rho \otimes \varphi) \subseteq \operatorname{supp}_G \varphi$.

Proof. These claims are straight forward from the definitions. We give some details for (11.9c). Let $\left(\begin{smallmatrix} x',\lambda' \\ x,\lambda \end{smallmatrix}\right) \in \operatorname{supp}_2(\rho \otimes \varphi)$. Then there are $\sigma \in \Sigma$, $\sigma' \in \Sigma'$, $s \in S$, $s' \in S'$, $g \in G$ with $(\rho \otimes \varphi)^{\sigma',s'}_{\sigma,s}(g) \neq 0$ and $x = gc(\sigma)$, $x' = c'(\sigma')$, $\lambda = g\pi(s)$, $\lambda' = \pi'(s')$. By definition $(\rho \otimes \varphi)^{\sigma',s'}_{\sigma,s}(g) = \rho^{\sigma'}_{g\sigma} \cdot \varphi^{s'}_{s}(g)$ and thus $\left(\begin{smallmatrix} x' \\ x \end{smallmatrix}\right) = \left(\begin{smallmatrix} c'(\sigma') \\ gc(\sigma) \end{smallmatrix}\right) = \left(\begin{smallmatrix} c'(\sigma') \\ c(g\sigma) \end{smallmatrix}\right) \in \operatorname{supp}_2 \rho$ and $\left(\begin{smallmatrix} \lambda' \\ \lambda \end{smallmatrix}\right) = \left(\begin{smallmatrix} \pi'(s') \\ g\pi(s) \end{smallmatrix}\right) \in \operatorname{supp}_2 \varphi$. □

11.D. **Tensor product with a singular chain complex.** Consider the singular chain complex $\mathbf{S}_*(X) \in \operatorname{ch} \mathcal{S}^G(S(X))$ from Subsection 10.B. Using the diagonal tensor product (11.6) we obtain a functor

$$\mathbf{S}_*(X) \otimes - : \mathcal{B}_G(\Lambda) \to \operatorname{ch} \operatorname{Idem}\big(\mathcal{B}_G(S(X) \times \Lambda)\big).$$

Note that for a morphism $\varphi\colon \mathbf{B} \to \mathbf{B}'$ in $\mathcal{B}_G(\Lambda)$ by (11.9c) we have

$$\operatorname{supp}_2\big(\operatorname{id}_{\mathbf{S}_*(X)} \otimes \varphi\big) \subseteq \Big\{ \left(\begin{smallmatrix} \sigma,\lambda' \\ \sigma,\lambda \end{smallmatrix}\right) \,\Big|\, \sigma \in S(X), \left(\begin{smallmatrix} \lambda' \\ \lambda \end{smallmatrix}\right) \in \operatorname{supp}_2 \varphi \Big\}. \tag{11.10}$$

Remark 11.11 (Shortcomings of $\mathbf{S}_*(X)$)**.** Let $J := |\mathbf{J}_{\mathcal{F}}(G)|^\wedge$, $\mathbf{M} \in \mathcal{R}^0$, and $\Lambda := |\mathbf{M}|^\wedge$. Let $f\colon S(X) \to |\mathbf{J}_{\mathcal{F}}(G)|$ be a G-equivariant map. Suppose also that $\widehat{f}\colon S(X) \to |\mathbf{J}_{\mathcal{F}}(G)|^\wedge$ is a lift of f. In light of (11.10) one might hope, that $(\widehat{f} \times \operatorname{id}_{|P|^\wedge})_*(\mathbf{S}_*(X) \otimes -)$ induces a functor[58]

$$\begin{array}{rcl} \mathrm{D}^0_G(P) & \to & \operatorname{ch} \operatorname{Idem} \mathrm{D}^0_G\big(\mathbf{J}_{\mathcal{F}}(G) \times P\big), \\ (\mathbf{B}_{\underline{t}})_{\underline{t} \in \mathbb{N}^{\times 2}} & \mapsto & \big((\widehat{f} \times \operatorname{id}_{|P|^\wedge})_*(\mathbf{S}_*(X) \otimes \mathbf{B}_{\underline{t}})\big)_{\underline{t} \in \mathbb{N}^{\times 2}}, \end{array}$$

but this is not the case. Let $(\mathbf{B}_{\underline{t}})_{\underline{t} \in \mathbb{N}^{\times 2}} = (S_{\underline{t}}, \pi_{\underline{t}}, B_{\underline{t}})_{\underline{t} \in \mathbb{N}^{\times 2}} \in \mathrm{D}^0_G(P)$. Typically $\big((\widehat{f} \times \operatorname{id}_{|P|^\wedge})_*(\mathbf{S}_*(X) \otimes \mathbf{B}_{\underline{t}})\big)_{\underline{t} \in \mathbb{N}^{\times 2}}$ will fail in two ways to define a chain complex in $\operatorname{Idem} \mathrm{D}^0_G(P \times \mathbf{J}_{\mathcal{F}})$.

Firstly, the boundary maps $(\widehat{f} \times \operatorname{id}_{|P|^\wedge})_*(\partial_n \otimes \operatorname{id}_{\mathbf{B}_t})$ do not satisfy the required control conditions to define morphisms in $\mathrm{D}^0_G(\mathbf{J}_{\mathcal{F}}(G) \times P)$. For example the ϵ-control condition over $|J_{\mathcal{F}}|$ from Remark 7.9 will typically fail. The support of $(\widehat{f} \times \operatorname{id}_P)_*(\partial_n \otimes \operatorname{id}_{\mathbf{B}_{\underline{t}}})$ is the set of all

$$\left(\begin{smallmatrix} \widehat{f}(x'),\lambda \\ g\widehat{f}(x),g\lambda \end{smallmatrix}\right) \in (|\mathbf{J}_{\mathcal{F}}|^\wedge \times |P|^\wedge)^{\times 2}$$

with $g \in \operatorname{supp}_G B_{\underline{t}}(s)$ for some $s \in S$ with $\pi_{\underline{t}}(s) = x$ the barycenter of a singular n-simplex σ of X, and x' the barycenter of a face of σ. Since we can always arrange for the G-supports of the $B_{\underline{t}}(s)$ to be small, the appearance of g in the above formula is not the main problem. The real problem comes from the difference between $\widehat{f}(x)$ and $\widehat{f}(x')$. This difference shrinks if we use only small singular simplices X. Therefore, in the construction of a functor $\mathrm{D}^0_G(P) \to \mathrm{D}^0_G(\mathbf{J}_{\mathcal{F}} \times P)$ we will need to work with chain complexes of simplices that get smaller as $\underline{t} \to \infty$.

Secondly, $\big((\widehat{f} \times \operatorname{id}_{|P|^\wedge})_*(\mathbf{S}_n(X) \otimes B_{\underline{t}})\big)_{\underline{t} \in \mathbb{N}^{\times 2}}$ typically does not define an object of $\mathrm{D}^0_G(\mathbf{J}_{\mathcal{F}} \times P)$. The singular chain complex is simply to big: $S_n(X)$ is infinite and so (7.18d) will fail. In order to overcome this problem we will replace X with a suitable large ball D in X (which is compact). We will also replace singular simplices with simplices from a triangulation of D. The set of n-simplices $\operatorname{simp}_n D$ is no longer a smooth G-set, as D is not G-invariant in X, but $\operatorname{simp}_n D$ will be invariant for a compact open subgroup of G. Moreover, the G-action on X still induces a homotopy coherent action of G on D. Theorem D.1 provides maps $X \to |\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge$. For the restrictions of these maps to large balls we control the failure of equivariance relative to this homotopy coherent action. Again we will need the construction to vary in $\underline{t} \in \mathbb{N}^{\times 2}$.

[58] We use the sequence description of $\mathrm{D}^0_G(-)$ from Remark 7.17.

11.E. **Restriction to open subgroups.** As discussed in Remark 11.11, in applications of the tensor product later on we will not always be able to work with smooth G-sets, but will also need to consider U-invariant sets for an open subgroup of G. To formalize this we discuss restrictions and inductions.

Let U be an open subgroup of G. Write $\operatorname{res}_G^U\colon G\text{-}\mathsf{Sets} \to U\text{-}\mathsf{Sets}$ for the restriction functor. It induces a restriction functor

$$\begin{array}{rcl} \mathcal{S}^G(\Omega) & \to & \mathcal{S}^U(\operatorname{res}_G^U \Omega), \\ (\Sigma, c) & \mapsto & (\operatorname{res}_G^U \Sigma, \operatorname{res}_G^U c), \end{array}$$

that we will also denote by res_G^U. We write $\operatorname{ind}_U^G\colon \mathcal{B}_U(\operatorname{res}_G^U \Lambda) \to \mathcal{B}_G(\Lambda)$ for the canonical inclusion[59]. This inclusion identifies $\mathcal{B}_U(\operatorname{res}_G^U \Lambda)$ with the subcategory of $\mathcal{B}_G(\Lambda)$ consisting of all objects and morphisms with G-support in U. Let us briefly write

$$\begin{array}{lllll} \otimes^G\colon & \mathcal{S}^G(\Omega) & \times\ \mathcal{B}_G(\Lambda) & \to & \operatorname{Idem} \mathcal{B}_G(\Omega \times \Lambda); \\ \otimes^U\colon & \mathcal{S}^U(\operatorname{res}_G^U \Omega) & \times\ \mathcal{B}_U(\operatorname{res}_G^U \Lambda) & \to & \operatorname{Idem} \mathcal{B}_U(\operatorname{res}_G^U(\Omega \times \Lambda)), \end{array}$$

for the tensor products. Directly from the definition it follows that for $\mathbf{V} \in \mathcal{S}^G(\Omega)$ and $\mathbf{B} \in \mathcal{B}_U(\operatorname{res}_G^U \Lambda)$ we have

$$\operatorname{ind}_U^G(\operatorname{res}_G^U \mathbf{V} \otimes^U \mathbf{B}) = \mathbf{V} \otimes^G \operatorname{ind}_U^G \mathbf{B}. \tag{11.12}$$

Similarly, for morphisms $\rho\colon \mathbf{V} \to \mathbf{V}'$ in $\mathcal{S}^G(\Omega)$ and $\varphi\colon \mathbf{B} \to \mathbf{B}'$ in $\mathcal{B}_U(\operatorname{res}_G^U \Lambda)$ we have

$$\operatorname{ind}_U^G(\operatorname{res}_G^U \rho \otimes^U \varphi) = \rho \otimes^G \operatorname{ind}_U^G \varphi. \tag{11.13}$$

Because of these identities we will later often drop ind_U^G and res_G^U from the notation and simply write $\otimes = \otimes^G = \otimes^U$.

11.F. **The category $\mathcal{B}_{G,U}(\Lambda)$.** Let U be an open subgroup of G. We write $\mathcal{B}_{G,U}(\Lambda)$ for the full subcategory of $\mathcal{B}_G(\Lambda)$ on all objects with G-support in U. The induction $\operatorname{ind}_U^G\colon \mathcal{B}_U(\operatorname{res}_G^U \Lambda) \to \mathcal{B}_G(\Lambda)$ from Subsection 11.E factors through the inclusion $\mathcal{B}_{G,U}(\Lambda) \subseteq \mathcal{B}_G(\Lambda)$. Moreover, $\mathcal{B}_{G,U}(\Lambda)$ is the full subcategory on objects in the image of ind_U^G. Because of support cofinality (3.2d) for $\mathcal{B}$ the inclusion $\mathcal{B}_{G,U}(\Lambda) \subseteq \mathcal{B}_G(\Lambda)$ induces an equivalence on idempotent completions. Thus we can often work with $\mathcal{B}_{G,U}(\Lambda)$ in place of $\mathcal{B}_G(\Lambda)$.

11.G. **Tensor product with subcomplexes of $\mathbf{S}_*(X)$.** Assume we are given

(11.14a) a compact open subgroup U of G and a compact U-invariant subspace D of the U-fixed points X^U with a locally ordered triangulation.

Then we obtain the simplicial chain complex $\mathbf{C}_*(D) \in \operatorname{ch} \mathcal{S}^U(\operatorname{res}_G^U S(X))$, see Subsection 10.D. The tensor product with $\mathbf{C}_*(D)$ then yields a functor

$$\mathbf{C}_*(D) \otimes^U - \colon \mathcal{B}_U(\operatorname{res}_G^U \Lambda) \to \operatorname{ch} \mathcal{B}_U(\operatorname{res}_G^U(S(X) \times \Lambda)).$$

Here we do not need the idempotent completion because of Lemma 11.7. Write $i\colon \mathbf{C}_*(D) \to \operatorname{res}_G^U \mathbf{S}_*(X)$ for the inclusion in $\operatorname{ch} \mathcal{S}^U(\operatorname{res}_G^U S(X))$. Assume we are given in addition

(11.14b) a chain map $r\colon \operatorname{res}_G^U \mathbf{S}_*(X) \to \mathbf{C}_*(D)$ in $\operatorname{ch} \mathcal{S}^U(\operatorname{res}_G^U S(X))$ with $r \circ i = \operatorname{id}_{\mathbf{C}_*(D)}$ and a chain homotopy $H\colon i \circ r \simeq \operatorname{id}_{\operatorname{res}_G^U \mathbf{S}_*(X)}$.

[59]More precisely, writing $\operatorname{res}_G^U \mathcal{B}$ for the subcategory of $\mathcal{B}$ whose morphisms have support in U we have $\operatorname{ind}_U^G\colon \operatorname{res}_G^U \mathcal{B}_U(\operatorname{res}_G^U \Lambda) \to \mathcal{B}_G(\Lambda)$.

In $\operatorname{ch}\mathcal{B}_U(\operatorname{res}_G^U(S(X)\times\Lambda))$ we now obtain for each $\mathbf{B}\in\mathcal{B}_U(\operatorname{res}_G^U\Lambda)$

$$\mathbf{C}_*(D)\otimes^U\mathbf{B} \underset{r\otimes^U\mathrm{id}_{\mathbf{B}}}{\overset{i\otimes^U\mathrm{id}_{\mathbf{B}}}{\rightleftarrows}} \operatorname{res}_G^U\mathbf{S}_*(X)\otimes^U\mathbf{B}$$

where $(r\otimes^U\mathrm{id}_{\mathbf{B}})\circ(i\otimes^U\mathrm{id}_{\mathbf{B}})=\mathrm{id}_{\mathbf{C}_*(D)\otimes^U\mathbf{B}}$ and $H\otimes^U\mathrm{id}_{\mathbf{B}}\colon(i\otimes^U\mathrm{id}_{\mathbf{B}})\circ(r\otimes^U\mathrm{id}_{\mathbf{B}})\simeq\mathrm{id}_{\operatorname{res}_G^U\mathbf{S}_*(X)\otimes\mathbf{B}}$. Applying ind_U^G and using (11.12) we obtain

$$\operatorname{ind}_U^G\big(\mathbf{C}_*(D)\otimes^U\mathbf{B}\big) \underset{\operatorname{ind}_U^G(r\otimes^U\mathrm{id}_{\mathbf{B}})}{\overset{\operatorname{ind}_U^G(i\otimes^U\mathrm{id}_{\mathbf{B}})}{\rightleftarrows}} \mathbf{S}_*(X)\otimes^G\operatorname{ind}_U^G\mathbf{B}$$

in $\operatorname{ch}\operatorname{Idem}\mathcal{B}_G(S(X)\times\Lambda)$. The main advantage of $\operatorname{ind}_U^G(\mathbf{C}_*(D)\otimes^U\mathbf{B})$ over $\mathbf{S}_*(X)\otimes^G\operatorname{ind}_U^G\mathbf{B}$ is that it is smaller and has better chances of satisfying control conditions. Its disadvantage is that, as D is only U-invariant and not G-invariant, morphisms $\varphi\colon\operatorname{ind}_U^G\mathbf{B}\to\operatorname{ind}_U^G\mathbf{B}'$ in $\mathcal{B}_{G,U}(\Lambda)$ do not induce maps $\operatorname{ind}_U^G(\mathbf{C}_*(D)\otimes^U\mathbf{B})\to\operatorname{ind}_U^G(\mathbf{C}_*(D)\otimes^U\mathbf{B}')$. But $\mathrm{id}_{\mathbf{S}_*(X)}\otimes\varphi$ is defined and we can use the composition

(11.15)
$$\begin{tikzcd}
\operatorname{ind}_U^G\big(\mathbf{C}_*(D)\otimes^U\mathbf{B}\big) \arrow[r, "\operatorname{ind}_U^G(i\otimes^U\mathrm{id}_{\mathbf{B}})"] & \mathbf{S}_*(X)\otimes^G\operatorname{ind}_U^G\mathbf{B} \arrow[d, "\mathrm{id}_{\mathbf{S}_*(X)}\otimes^G\varphi"] \\
\operatorname{ind}_U^G\big(\mathbf{C}_*(D)\otimes^U\mathbf{B}'\big) & \mathbf{S}_*(X)\otimes^G\operatorname{ind}_U^G\mathbf{B}' \arrow[l, "\operatorname{ind}_U^G(r\otimes^U\mathrm{id}_{\mathbf{B}})"']
\end{tikzcd}$$

instead. While this is not strictly compatible with composition, the homotopies $\operatorname{ind}_U^G(H\otimes^U\mathrm{id}_{\mathbf{B}})$ guarantee that it is compatible with composition up to coherent homotopy. From now on we will simply the notation as alluded to in Subsection 11.E and drop ind_U^G and res_G^U from the notation and simply write $\otimes=\otimes^G=\otimes^U$. Thus (11.15) abbreviates to

$$\begin{tikzcd}
\mathbf{C}_*(D)\otimes\mathbf{B} \arrow[r, "i\otimes\mathrm{id}_{\mathbf{B}}"] & \mathbf{S}_*(X)\otimes\mathbf{B} \arrow[d, "\mathrm{id}_{\mathbf{S}_*(X)}\otimes\varphi"] \\
\mathbf{C}_*(D)\otimes\mathbf{B}' & \mathbf{S}_*(X)\otimes\mathbf{B}'. \arrow[l, "r\otimes\mathrm{id}'_{\mathbf{B}}"']
\end{tikzcd}$$

In summary, we can use the data chosen in (11.14a) and (11.14b) to define a homotopy coherent functor, see Definition C.1,

(11.16) $$F=(F^0,F^1,\dots):\mathcal{B}_{G,U}(\Lambda)\to\operatorname{ch}\mathcal{B}_G(S(X)\times\Lambda)$$

as follows. For $\mathbf{B}\in\mathcal{B}_{G,U}(\Lambda)$ we set

$$F^0(\mathbf{B}):=\mathbf{C}_*(D)\otimes\mathbf{B}.$$

For a chain

$$\mathbf{B}_0\xleftarrow{\varphi_1}\mathbf{B}_1\xleftarrow{\varphi_2}\dots\xleftarrow{\varphi_n}\mathbf{B}_n$$

of composable morphisms in $\mathcal{B}_{G,U}(\Lambda)$ we define

$$\begin{aligned}F^n(\varphi_1,\dots,\varphi_n):=(r\otimes\mathrm{id}_{\mathbf{B}_0})\circ(\mathrm{id}_{\mathbf{S}_*(X)}\otimes\varphi_1)\circ(H\otimes\mathrm{id}_{\mathbf{B}_1})\circ\dots\\ \dots\circ(H\otimes\mathrm{id}_{\mathbf{B}_{n-1}})\circ(\mathrm{id}_{\mathbf{S}_*(X)}\otimes\varphi_n)\circ(i\otimes\mathrm{id}_{\mathbf{B}_n}).\end{aligned}$$

It is not difficult to check that this defines a homotopy coherent functor, compare Example C.3.

In Section 12 we will discuss the effect of (11.16) on supp_2. For now we record the following easy facts.

Lemma 11.17.

(11.17a) If $\mathbf{B}$ is finite, then all chain modules $(F^0(\mathbf{B}))_n$ of $F^0(\mathbf{B})$ are finite as well;

(11.17b) *We have* $\operatorname{supp}_1(F^0(\mathbf{B}))_n \subseteq \operatorname{simp}_n(D) \times \operatorname{supp}_1 \mathbf{B}$ *and* $\operatorname{supp}_G(F^0(\mathbf{B}))_n \subseteq \operatorname{supp}_G \mathbf{B}$*;*

(11.17c) *The n-th boundary map* $\partial_n^{F^0(\mathbf{B})}$ *of* $F^0(\mathbf{B})$ *satisfies* $\operatorname{supp}_G \partial_n^{F^0(\mathbf{B})} \subseteq \operatorname{supp}_G \mathbf{B}$.

Proof. This follows from (11.9a), (11.9b) and (11.9d). □

11.H. **Projection back to Λ.** The projection $\mathrm{pr}\colon S(X)\times\Lambda \to \Lambda$ induces a functor

$$P\colon \operatorname{ch}\mathcal{B}_G(X\times\Lambda) \to \operatorname{ch}\mathcal{B}_G(\Lambda).$$

We are interested in the composition of P with F from (11.16)

$$P\circ F\colon \mathcal{B}_{G,U}(\Lambda) \to \operatorname{ch}\mathcal{B}_G(\Lambda).$$

Let

$$I\colon \mathcal{B}_{G,U}(\Lambda) \to \operatorname{ch}\mathcal{B}_G(\Lambda)$$

be the inclusion[60]. We construct a natural transformation $\tau\colon P\circ F \to I$. Write $\star$ for the one-point space. Let $p(\mathbf{C}_*(D))$ be the image of $\mathbf{C}_*(D)$ under the functor $\mathcal{S}^U(X) \to \mathcal{S}^U(\star)$ induced by the projection $X \to \star$. We can identify $(P\circ F)(\mathbf{A}) \cong p(\mathbf{C}_*(D)) \otimes \mathbf{B}$, for $\mathbf{B} \in \mathcal{B}_{G,U}(\Lambda)$. Let $\mathbb{1}_G := (\star, \mathrm{id}_\star) \in \mathcal{S}^G(\star)$. We have $\mathbb{1}_U := \operatorname{res}_G^U \mathbb{1}_G = (\star, \mathrm{id}_\star) \in \mathcal{S}^U(\star)$. We can identify $I(\mathbf{B}) \cong \mathbb{1}_U \otimes \mathbf{B}$. The projection $D \to \star$ induces an augmentation $\epsilon\colon p(\mathbf{C}_*(D)) \to \mathbb{1}_U$. Now for $\mathbf{B} \in \mathcal{B}_{G,U}(\Lambda)$ we define

$$\tau_{\mathbf{B}} := \epsilon \otimes \mathrm{id}_{\mathbf{B}}\colon p(C_*(D)) \otimes \mathbf{B} \to \mathbb{1}_U \otimes \mathbf{B}. \tag{11.18}$$

We will need the notion of a strict natural transformation between homotopy coherent functors, see Definition C.4.

Lemma 11.19. *Under the canonical identifications* $(P\circ F)(\mathbf{B}) \cong p(\mathbf{C}_*(D))\otimes\mathbf{B}$ *and* $I(\mathbf{B}) \cong \mathbb{1}_U\otimes\mathbf{B}$ *the maps* (11.18) *define a strict natural transformation* $\tau\colon P\circ F \to I$*, see Example C.5.*

Proof. This is a straight forward exercise in the definitions. □

Lemma 11.20. *Suppose that D is contractible. Let $\mathbf{B} \in \mathcal{B}_{G,U}(\Lambda)$. Then in* $\operatorname{ch}\operatorname{Idem}\mathcal{B}_{G,U}(\Lambda)$ *there are a chain map* $f\colon \mathbb{1}_U \otimes \mathbf{B} \to p(\mathbf{C}_*(D)) \otimes \mathbf{B}$*, and chain homotopies* $h\colon f\circ\tau_{\mathbf{B}} \simeq \mathrm{id}_{p(\mathbf{C}_*(D))\otimes\mathbf{B}}$*,* $k\colon \tau_{\mathbf{B}}\circ f \simeq \mathrm{id}_{\mathbb{1}U} \otimes \mathbf{B}$ *such that*

$$\operatorname{supp}_2 f, \operatorname{supp}_2 h, \operatorname{supp}_2 h' \subseteq \operatorname{supp}_2 \mathbf{B},$$

$$\operatorname{supp}_G f, \operatorname{supp}_G h, \operatorname{supp}_G h' \subseteq \operatorname{supp}_G \mathbf{B} \subseteq U.$$

In particular, $\tau_{\mathbf{B}}$ *is a homotopy equivalence.*

Proof. Under the assumptions on D, $p(\mathbf{C}_*(D))$ and $\mathbb{1}_U$ are homotopy equivalent in $\operatorname{ch}\mathcal{S}^U(\star)$ (because the simplicial chain complex of D is homotopy equivalent to the simplicial chain complex of a point). Thus in $\operatorname{ch}\mathcal{S}^U(\star)$ there are a chain map $f_0\colon \mathbb{1}_U \to p(\mathbf{C}_*(D))$, and chain homotopies $h_0\colon f_0\circ\epsilon \simeq \mathrm{id}_{p(\mathbf{C}_*(D))}$, $k_0\colon \epsilon\circ f_0 \simeq \mathrm{id}_{\mathbb{1}_U}$. Now set $f := f_0\otimes\mathrm{id}_{\mathbf{B}}$, $h := h_0\otimes\mathrm{id}_{\mathbf{B}}$, and $k := k_0\otimes\mathrm{id}_{\mathbf{B}}$. The claims about supp_2 and supp_G follow from (11.9c) and (11.9d). □

12. Support estimates for homotopy coherent functors

Let X be a G-space equipped with a G-invariant metric d_X. We assume that for $K \subseteq X$ compact the pointwise isotropy group G_K is open in G. Let J and Λ be further G-spaces. Let $\mathcal{B}$ be a Hecke category with G-support.

We will refine the construction of the homotopy coherent functor from Subsection 11.G and will be interested in its effect on supports of objects and morphisms. Its construction and analysis will depend on a list of data. This section is very formal and will be used later to check that a sequence of homotopy coherent functors does descend to $\mathrm{D}^0_G(-)$ as discussed in Subsection 9.A.

[60]Recall that $\mathcal{B}_{G,U}(\Lambda)$ is a subcategory of $\mathcal{B}_G(\Lambda)$.

12.A. **Set-up.** Throughout this section we fix

- numbers $L \in \mathbb{N}$, $\rho > 0$;
- a compact open subgroup $U \subseteq G$ and a compact subset $M \subseteq G$ with $U \subseteq M$;
- a subspace $D \subseteq X$ with a locally ordered triangulation all whose simplices are of diameter $< \rho$;
- a sequence of further subspaces $D = D^{(0)} \subseteq D^{(1)} \subseteq \cdots \subseteq D^{(L)}$ with $M \cdot D^{(l-1)} \subseteq D^{(l)}$ for $l = 1, \ldots, L$; moreover we require that $D^{(L)}$ is pointwise fixed by U;
- a retraction $r^0 \colon X \to D$ for the inclusion $i^0 \colon D \to X$, i.e., $r^0 \circ i^0 = \mathrm{id}_D$;
- a homotopy $H^0 \colon i^0 \circ r^0 \simeq \mathrm{id}_X$; we will assume that H^0 is non-expanding, i.e., for all $\tau \in [0,1]$, $x, x' \in X$ we require $d_X(H^0(x,\tau), H^0(x',\tau)) \leq d_X(x,x')$; we also assume that H^0 preserves the $D^{(l)}$, i.e., we require $H^0(D^{(l)} \times [0,1]) \subseteq D^{(l)}$;
- a map $f \colon X \to J$;
- a subset $E \subseteq J \times J$.

We do *not* assume that f is G-equivariant.

Remark 12.1 (Some explanations for the list of data)**.** Later

- X will be the extended Bruhat-Tits building for the reductive p-adic group G;
- the $D^{(i)}$ will be an increasing sequence of balls around a common center;
- r^0 will be the radial projection and H^0 will be the associated radial homotopy;
- U will be the pointwise isotropy group of $D^{(L)}$;
- Λ will be $|\mathbf{M}|^\wedge$ for some $\mathbf{M} \in \mathcal{R}^0$;
- J will be $|\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge$;
- $f \colon X \to J$ will come from Theorem D.1.

Below U, B, r^0, and H^0 will be used to construct a homotopy coherent functor

$$\mathcal{B}_{G,U}(\Lambda) \to \mathrm{ch}\,\mathcal{B}_G(S(X) \times \Lambda)$$

as in Subsection 11.G. Let bary: $S(X) \to X$ be the map that sends a singular simplex to its barycenter. We can then compose with $(f \circ \mathrm{bary} \times \mathrm{id}_\Lambda)_*$ and obtain a homotopy coherent functor

$$\mathcal{B}_{G,U}(\Lambda) \to \mathrm{ch}\,\mathcal{B}_G(J \times \Lambda)$$

and will be interested in estimates for the support of objects and morphisms under this latter functor, see Proposition 12.8 below. For these estimates we will bound the G-support of morphisms and objects by M and we will treat chains of at most L-composable morphisms. The upper bound for supp_2 will be in terms of E.

We will work under the following assumptions throughout this section.

Assumption 12.2.

(12.2a) $\left\{ \left(\begin{smallmatrix} f(gx) \\ gf(x) \end{smallmatrix} \right) \;\middle|\; x \in D^{(L)}, g \in M \right\} \subseteq E;$

(12.2b) $\left\{ \left(\begin{smallmatrix} f(H^0(x,\tau)) \\ f(x) \end{smallmatrix} \right) \;\middle|\; x \in D^{(L)}, \tau \in [0,1] \right\} \subseteq E;$

(12.2c) $\left\{ \left(\begin{smallmatrix} f(x') \\ f(x) \end{smallmatrix} \right) \;\middle|\; x, x' \in D^{(L)}, d_X(x,x') \leq (L+1)\rho \right\} \subseteq E.$

12.B. **From $S(X)$ to J.** Set

$$\begin{aligned} E_X \;&:=\; \left\{ \left(\begin{smallmatrix} x' \\ x \end{smallmatrix} \right) \;\middle|\; x, x' \in D^{(L)}, d_X(x,x') \leq (L+1)\rho \right\} \\ &\qquad \cup \left\{ \left(\begin{smallmatrix} H^0(x,\tau) \\ x \end{smallmatrix} \right) \;\middle|\; x \in D^{(L)}, \tau \in [0,1] \right\} \quad \subseteq X \times X; \\ E_S \;&:=\; (\mathrm{bary} \times \mathrm{bary})^{-1}(E_X) \qquad \subseteq S(X) \times S(X). \end{aligned}$$

We note that $E_X \subseteq E_X^{\circ 2}$ and $E_S \subseteq E_S^{\circ 2}$. We use again the convention for products from (11.8).

Lemma 12.3. *Let* $E' \subseteq \Lambda \times \Lambda$, $\Phi\colon \mathbf{B} \to \mathbf{B}' \in \mathcal{B}_G(S(X) \times \Lambda)$, *and* $\nu \in \mathbb{N}$. *Suppose* $\operatorname{supp}_1 \mathbf{B} \subseteq S(D^{(L)}) \times \Lambda$, $\operatorname{supp}_2 \Phi \subseteq E_S^{\circ\nu} \times E'$, *and* $\operatorname{supp}_G \Phi \subseteq M$. *Then* $\operatorname{supp}_2(f \circ \operatorname{bary} \times \operatorname{id}_\Lambda)_* \Phi \subseteq E^{\circ(\nu+1)} \times E'$.

Proof. By (4.7), and since bary and $\operatorname{id}_\Lambda$ are G-equivariant, $\operatorname{supp}_2(f \circ \operatorname{bary} \times \operatorname{id}_\Lambda)_* \Phi$ is contained in

$$(f \circ \operatorname{bary} \operatorname{id}_\Lambda)^{\times 2}(E_S^{\circ\nu} \times E') \circ \Big\{ \Big(\begin{smallmatrix} f(g \operatorname{bary}(\sigma)), g\lambda \\ gf(\operatorname{bary}(\sigma)), g\lambda \end{smallmatrix} \Big) \,\Big|\, (\sigma, \lambda) \in S(D^{(L)}) \times \Lambda, g \in M \Big\}.$$

We have

$$\begin{aligned} (f \circ \operatorname{bary} \operatorname{id}_\Lambda)^{\times 2}(E_S^{\circ\nu} \times E') &\subseteq ((f \circ \operatorname{bary})^{\times 2}(E_S^{\circ\nu})) \times E' \\ &\subseteq ((f \circ \operatorname{bary})^{\times 2}(E_S))^{\circ\nu} \times E' \subseteq E^{\circ\nu} \times E' \end{aligned}$$

and, by (12.2a)

$$\Big\{ \Big(\begin{smallmatrix} f(g \operatorname{bary}(\sigma)), g\lambda \\ gf(\operatorname{bary}(\sigma)), g\lambda \end{smallmatrix} \Big) \,\Big|\, (\sigma, \lambda) \in S(D^{(L)}) \times \Lambda, g \in M \Big\} \subseteq E \times \Big\{ \big(\begin{smallmatrix} \lambda \\ \lambda \end{smallmatrix} \big) \mid \lambda \in \Lambda \Big\}.$$

Thus

$$\operatorname{supp}_2(f \circ \operatorname{bary} \times \operatorname{id}_\Lambda)_* \Phi \subseteq \Big(E^{\circ\nu} \times E' \Big) \circ \Big(E \times \Big\{ \big(\begin{smallmatrix} \lambda \\ \lambda \end{smallmatrix} \big) \mid \lambda \in \Lambda \Big\} \Big) \subseteq E^{\circ(\nu+1)} \times E'.$$

□

12.C. **Construction of r and H.** We write $\mathbf{C}_*(D) \in \operatorname{ch} \mathcal{S}^U(S(X))$ for the simplicial chain complex of D and $\mathbf{S}_*(X) \in \operatorname{ch} \mathcal{S}^G(S(X))$ for the singular chain complex of X. Let $i\colon \mathbf{C}_*(D) \to \mathbf{S}_*(X)$ be the inclusion. For $k, l \leq L$ we define $S^{k,l} \subseteq S(X)$ as the collection of all singular simplices in X that are contained in $D^{(l)}$ and are of diameter $< k\rho$.

Lemma 12.4. *There is a chain map* $r\colon \mathbf{S}_*(X) \to \mathbf{C}_*(D)$ *and a chain homotopy* $H\colon \mathbf{S}_*(X) \to \mathbf{S}_*(X)$ *in* $\operatorname{ch} \mathcal{S}^U(S(X))$ *with* $r \circ i = \operatorname{id}_{\mathbf{C}_*(B)}$, $H\colon i \circ r \simeq \operatorname{id}_{\mathbf{S}_*(X)}$ *satisfying the following: if* $\big(\begin{smallmatrix} \sigma' \\ \sigma \end{smallmatrix} \big) \in \operatorname{supp}_2 r \cup \operatorname{supp}_2 H$ *with* $\sigma \in S^{k,l}$ *and* $k \leq L-1, l \leq L$, *then*

(12.4a) $\sigma' \in S^{k+1,l}$;
(12.4b) $\big(\begin{smallmatrix} \sigma' \\ \sigma \end{smallmatrix} \big) \in E_S^{\circ 2}$.

Proof. We will use $\operatorname{im} \sigma$ and $(\operatorname{im} \sigma)^\rho$ as introduced in Subsection 10.D. We write $i^1\colon \mathbf{C}_*(D) \to \mathbf{S}_*(D)$ for the inclusion. By Lemma 10.5 there is $r^1\colon \mathbf{S}_*(D) \to \mathbf{C}_*(D)$ with $r^1 \circ i^1 = \operatorname{id}_{\mathbf{C}_*(D)}$ and a chain homotopy $H^1\colon \mathbf{S}_*(D) \to \mathbf{S}_{*+1}(D)$ for $i^1 \circ r^1 \simeq \operatorname{id}_{\mathbf{S}_*(D)}$. Moreover, Lemma 10.5 also yields

(12.4c) if $\big(\begin{smallmatrix} \sigma' \\ \sigma \end{smallmatrix} \big) \in \operatorname{supp}_2(r^1) \cup \in \operatorname{supp}_2(H^1)$, then $\operatorname{im} \sigma' \subseteq (\operatorname{im} \sigma)^\rho$.

Next we use that H^0 is not expanding and apply Lemma 10.4 to the homotopy H^0 and obtain a chain homotopy $\widetilde{H}^0\colon (i^0)_* \circ (r^0)_* \simeq \operatorname{id}_{\mathbf{S}_*(X)}$ such that

(12.4d) if $\big(\begin{smallmatrix} \sigma' \\ \sigma \end{smallmatrix} \big) \in \operatorname{supp}_2(\widetilde{H}^0)$, then $\operatorname{diam} \operatorname{im}(\sigma') < \operatorname{diam} \operatorname{im}(\sigma) + \rho$ and $\operatorname{im} \sigma' \subseteq \operatorname{im}(H^0 \circ (\sigma \times \operatorname{id}_{[0,1]}))$.

We have $i = (i^0)_* \circ i^1\colon \mathbf{C}_*(D) \to \mathbf{S}_*(X)$ and set

$$\begin{aligned} r &:= r^1 \circ (r^0)_*\colon \mathbf{S}_*(X) \to \mathbf{C}_*(D); \\ H &:= (i^0)_* \circ H^1 \circ (r^0)_* + (\widetilde{H}^0)_*\colon i \circ r \simeq \operatorname{id}_{\mathbf{S}_*(X)}. \end{aligned}$$

Suppose $\big(\begin{smallmatrix} \sigma' \\ \sigma \end{smallmatrix} \big) \in \operatorname{supp}_2 r$ with $\sigma \in S^{k,l}$. As r is a map to $\mathbf{C}_*(D)$, σ' is simplex in the triangulation of D, in particular $\sigma' \in S^{k+1,l}$. Now $r_\sigma^{\sigma'} \neq 0$ implies that there is τ with $(r^1)_\tau^{\sigma'} \neq 0$ and $((r^0)_*)_\sigma^\tau \neq 0$. The latter means $\tau = r^0 \circ \sigma = H^0(\sigma(-), 1)$. In particular, $\big(\begin{smallmatrix} \tau \\ \sigma \end{smallmatrix} \big) \in E_S$. As H^0 is non-expanding, $\operatorname{diam} \operatorname{im} \tau \leq \operatorname{diam} \operatorname{im} \sigma < k\rho$. By (12.4c), $\operatorname{im} \sigma' \subseteq (\operatorname{im} \tau)^\rho$. Thus $d_X(\operatorname{bary}(\sigma'), \operatorname{bary}(\tau)) \leq \rho + \operatorname{diam} \operatorname{im} \tau < (k+1)\rho \leq L\rho$ and $\big(\begin{smallmatrix} \sigma' \\ \tau \end{smallmatrix} \big) \in E_S$. Therefore $\big(\begin{smallmatrix} \sigma' \\ \sigma \end{smallmatrix} \big) \in E_S^{\circ 2}$.

Suppose $\binom{\sigma'}{\sigma} \in \operatorname{supp}_2 H$ with $\sigma \in S^{k,l}$. Then $\binom{\sigma'}{\sigma} \in \operatorname{supp}_2 \widetilde{H}$ or $\binom{\sigma'}{\sigma} \in \operatorname{supp}_2((i^0)_* \circ H^1 \circ (r^0))$. In the first case, (12.4d) implies $\operatorname{diam} \sigma' < \operatorname{diam} \sigma + \rho < (k+1)\rho$. Moreover, $\operatorname{im} \sigma' \subseteq H_0(\operatorname{im} \sigma \times [0,1])$. As H^0 preserves $D^{(l)}$, this implies $\sigma' \in S^{k+1,l}$. Also $\operatorname{bary}(\sigma') = H_0(x,t)$ for some $t \in [0,1]$ and $x \in \operatorname{im} \sigma$. Let $\tau := H_0(\sigma(-),t)$. Then $\binom{\sigma}{\tau} \in E_S$. Also $\operatorname{bary}(\sigma') \in \operatorname{im} \tau$. As $H_0(-,t)$ is non-expanding, $d_X(\operatorname{bary}(\sigma'), \operatorname{bary}(\tau)) < \operatorname{diam} \operatorname{im} \tau < \operatorname{diam} \operatorname{im} \sigma < k\rho$. Thus $\binom{\tau}{\sigma} \in E_S$. Altogether, $\binom{\sigma'}{\sigma} \in E_S^{\circ 2}$.

In the second case there are τ, τ' with $\binom{\sigma'}{\tau'} \in \operatorname{supp}_2 (i^0)_*$, $\binom{\tau'}{\tau} \in \operatorname{supp}_2 H^1$, and $\binom{\tau}{\sigma} \in \operatorname{supp}_2 (r^0)_*$. Now $\binom{\sigma'}{\tau'} \in \operatorname{supp}_2 (i^0)_*$ means $\tau' = \sigma'$, while $\binom{\tau}{\sigma} \in \operatorname{supp}_2 (r^0)_*$ means $\tau = r^0 \circ \sigma = H^0(\sigma(-),1)$. In particular, $\binom{\tau}{\sigma} \in E_S$. Also, as H^0 is non-expanding, $\operatorname{diam} \operatorname{im} \tau \leq \operatorname{diam} \operatorname{im} \sigma < k\sigma$. Using (12.4c), $\binom{\tau'}{\tau} \in \operatorname{supp}_2 H^1$ implies $\operatorname{im} \tau' \subseteq (\operatorname{im} \tau)^\rho$. In particular, $d_X(\operatorname{bary}(\tau'), x) < \rho$ for some $x \in \operatorname{im} \tau$. Moreover, $\operatorname{diam} \operatorname{im} \sigma' = \operatorname{diam} \operatorname{im} \tau' < \operatorname{diam} \operatorname{im} \tau + \rho < (k+1)\rho$. As H^1 is a map to $\mathbf{S}_*(D)$, we have $\operatorname{im} \sigma' = \operatorname{im} \tau' \subseteq D$. In particular, $\sigma' \in S^{k+1,l}$. Now

$$\begin{aligned} d_X(\operatorname{bary}(\sigma'), \operatorname{bary}(\tau)) &= d_X(\operatorname{bary}(\tau'), \operatorname{bary}(\tau)) \\ &\leq d_X(\operatorname{bary}(\tau'), x) + d_X(x, \operatorname{bary}(\tau)) < \rho + \operatorname{diam} \operatorname{im} \tau < \rho + k\rho = (k+1)\rho \end{aligned}$$

implies $\binom{\sigma'}{\tau} \in E_S$. Thus $\binom{\sigma'}{\tau} \in E_S^{\circ 2}$. $\square$

12.D. **Estimates over $S(X) \times \Lambda$.** We now fix r and H as in Lemma 12.4. We obtain a homotopy coherent functor as in (11.16)

$$F = (F^0, F^1, \ldots,) \colon \mathcal{B}_{G,U}(\Lambda) \to \operatorname{ch} \mathcal{B}_G(S(X) \times \Lambda).$$

Recall that $F_0(\mathbf{B}) = \mathbf{C}(D)_* \otimes \mathbf{B}$. Its n-th chain module is $F_0(\mathbf{B})_n = \mathbf{C}(D)_n \otimes \mathbf{B}$. Its n-th boundary map is $\partial_n^{F_0(\mathbf{B})} = \partial_n^{\mathbf{C}_*(D)} \otimes \operatorname{id}_{\mathbf{B}}$.

Lemma 12.5. *Let $\mathbf{B} \in \mathcal{B}_{G,U}(\Lambda)$. Then*

(12.5a) $\operatorname{supp}_G F_0(\mathbf{B})_n \subseteq U$, $\operatorname{supp}_2 F_0(\mathbf{B})_n \subseteq E_S \times \operatorname{supp}_2 \mathbf{B}$;

(12.5b) $\operatorname{supp}_G \partial_n^{F_0(\mathbf{B})} \subseteq U$, $\operatorname{supp}_2 \partial_n^{F_0(\mathbf{B})} \subseteq E_S \times \operatorname{supp}_2 \mathbf{B}$.

Proof. We have $\operatorname{supp}_2 \operatorname{id}_{\mathbf{C}_n(D)} \subseteq \{\binom{\sigma}{\sigma} \mid \sigma \in S(D)\} \subset E_S$. Diameters of simplices in the triangulation of D are $< \rho$. If $(\partial_n^{\mathbf{C}_*(D)})_\sigma^{\sigma'} \neq 0$, then $\operatorname{im} \sigma' \subseteq \operatorname{im} \sigma$ and so $d_X(\operatorname{bary} \sigma', \operatorname{bary} \sigma) < \rho$. Therefore $\operatorname{supp}_2 \partial_n^{\mathbf{C}_*(D)} \subseteq E_S$. We have $\operatorname{supp}_G \mathbf{B} = \operatorname{supp}_G \operatorname{id}_{\mathbf{B}} \subseteq U$, as $\mathbf{B} \in \mathcal{B}_{G,U}(\Lambda)$. Both (12.5a), and (12.5b) follow now from Lemma 11.9. $\square$

We will now use summands and corners as in Remarks 4.14 and 4.15 relative to $S^{k,l} \subseteq S(X)$. For $\mathbf{B} \in \mathcal{B}_{G,U}(\Lambda)$ we abbreviate $(\mathbf{S}_n(X) \otimes \mathbf{B})_{k,l} := (\mathbf{S}_n(X) \otimes \mathbf{B})|_{S^{s,l} \times \Lambda}$ and write

$$(\mathbf{S}_n(X) \otimes \mathbf{B})_{k,l} \xrightarrow{i_{k,l}} \mathbf{S}_n(X) \otimes \mathbf{B} \xrightarrow{r^{k,l}} (\mathbf{S}_n(X) \otimes \mathbf{B})_{k,l}$$

for the corresponding inclusion and retraction. For $\Phi \colon \mathbf{S}_n(X) \otimes \mathbf{B} \to \mathbf{S}_{n'}(X) \otimes \mathbf{B}'$ we set $\Phi_{k,l}^{k',l'} := r^{k',l'} \circ \Phi \circ i_{k,l}$. We also set $(i \otimes \operatorname{id}_{\mathbf{B}})^{k',l'} := r^{k',l'} \circ (i \otimes \operatorname{id}_{\mathbf{B}})$, and $(r \otimes \operatorname{id}_{\mathbf{B}})_{k,l} := (r \otimes \operatorname{id}_{\mathbf{B}}) \circ i^{k,l}$. This means

$$\begin{aligned} (\Phi_{k,l}^{k',l'})_{\sigma,s}^{\sigma',s'} &= \begin{cases} \Phi_{\sigma,s}^{\sigma',s'} & \sigma \in S^{k,l}, \sigma' \in S^{k',l'}; \\ 0 & \text{else}; \end{cases} \\ ((i \otimes \operatorname{id}_{\mathbf{B}})^{k',l'})_{\sigma,s}^{\sigma',s'} &= \begin{cases} (i \otimes \operatorname{id}_{\mathbf{B}})_{\sigma,s}^{\sigma',s'} & \sigma' \in S^{k',l'}; \\ 0 & \text{else}; \end{cases} \\ ((r \otimes \operatorname{id}_{\mathbf{B}})_{k,l})_{\sigma,s}^{\sigma',s'} &= \begin{cases} (r \otimes \operatorname{id}_{\mathbf{B}})_{\sigma,s}^{\sigma',s'} & \sigma \in S^{k,l}; \\ 0 & \text{else}. \end{cases} \end{aligned}$$

Lemma 12.6. *Let $k, l \leq L$, $\mathbf{B}, \mathbf{B}' \in \mathcal{B}_{G,U}(\Lambda)$, $\varphi\colon \mathbf{B} \to \mathbf{B}' \in \mathcal{B}_{G,U}(\Lambda)$ with $\operatorname{supp}_G \varphi \subseteq M$. Set $E' := \operatorname{supp}_2 \varphi \cup \operatorname{supp}_2 \mathbf{B}$. Then*

(12.6a) $i_{k,l+1} \circ (\mathrm{id}_{\mathbf{S}_*(X)} \otimes \varphi)^{k,l+1}_{k,l} = (\mathrm{id}_{\mathbf{S}_*(X)} \otimes \varphi) \circ i_{k,l}$ *(provided $l+1 \leq L$);*

(12.6b) $\operatorname{supp}_2 \big((\mathrm{id}_{\mathbf{S}_*(X)} \otimes \varphi)^{k,l+1}_{k,l} \big) \subseteq E_S \times E'$ *(provided $l+1 \leq L$);*

(12.6c) $i_{k+1,l} \circ (H \otimes \mathrm{id}_{\mathbf{B}})^{k+1,l}_{k,l} = (H \otimes \mathrm{id}_{\mathbf{B}}) \circ i_{k,l}$ *(provided $k+1 \leq L$);*

(12.6d) $\operatorname{supp}_2 \big((H \otimes \mathrm{id}_{\mathbf{B}})^{k+1,l}_{k,l} \big) \subseteq E_S^{\circ 2} \times E'$ *(provided $k+1 \leq L$);*

(12.6e) $i_{k,l} \circ (i \otimes \mathrm{id}_{\mathbf{B}})^{k,l} = i \otimes \mathrm{id}_{\mathbf{B}}$*;*

(12.6f) $\operatorname{supp}_2 \big((i \otimes \mathrm{id}_{\mathbf{B}})^{k,l} \big) \subseteq E_S^{\circ 2} \times E'$*;*

(12.6g) $(r \otimes \mathrm{id}_{\mathbf{B}})_{k,l} = (r \otimes \mathrm{id}_{\mathbf{B}}) \circ i_{k,l}$*;*

(12.6h) $\operatorname{supp}_2 \big((r \otimes \mathrm{id}_{\mathbf{B}})_{k,l} \big) \subseteq E_S^{\circ 2} \times E'$.

Proof. From (11.9d) we obtain $\operatorname{supp}_G \mathrm{id}_{\mathbf{S}_*(X)} \otimes \varphi \subseteq \operatorname{supp}_G \varphi \subseteq M$. Using $M \cdot D^{(l)} \subseteq D^{(l+1)}$ we obtain $(\operatorname{supp}_G \mathrm{id}_{\mathbf{S}_*(X)} \otimes \varphi) \cdot S^{k,l} \subseteq M \cdot S^{k,l} \subseteq S^{k,l+1}$. If $\left(\begin{smallmatrix} \sigma', \lambda' \\ \sigma, \lambda \end{smallmatrix} \right) \in \operatorname{supp}_2 \mathrm{id}_{\mathbf{S}_*(X)} \otimes \varphi$, then, by (11.9c), $\left(\begin{smallmatrix} \sigma' \\ \sigma \end{smallmatrix} \right) \in \operatorname{supp}_2 \mathrm{id}_{\mathbf{S}_*(X)}$ and so $\sigma = \sigma'$. Now Lemma 4.17 gives (12.6a). Using (11.9c) and (4.16) we obtain

$$\operatorname{supp}_2 (\mathrm{id}_{\mathbf{S}_*(X)} \otimes \varphi)^{k,l+1}_{k,l} \subseteq \left\{ \left(\begin{smallmatrix} \sigma, \lambda' \\ \sigma, \lambda \end{smallmatrix} \right) \middle| \sigma \in S^{k,l+1}, \left(\begin{smallmatrix} \lambda' \\ \lambda \end{smallmatrix} \right) \in \operatorname{supp}_2 \varphi \right\}$$

and this implies (12.6b).

By (11.9d) $\operatorname{supp}_G H \otimes \mathrm{id}_{\mathbf{B}}$, $\operatorname{supp}_G r \otimes \mathrm{id}_{\mathbf{B}}$, and $\operatorname{supp}_G i \otimes \mathrm{id}_{\mathbf{B}}$ are all contained in $\operatorname{supp}_G \mathbf{B} \subseteq U$. As U fixes $D^{(l)}$, we have $U \cdot S^{k,l} = S^{k,l}$. If $\left(\begin{smallmatrix} \sigma', \lambda' \\ \sigma, \lambda \end{smallmatrix} \right) \in \operatorname{supp}_2 H \otimes \mathrm{id}_{\mathbf{B}}$, then, by (11.9c), $\left(\begin{smallmatrix} \sigma' \\ \sigma \end{smallmatrix} \right) \in \operatorname{supp}_2 H$. If, in addition, $\sigma \in S^{k,l}$, then, by (12.4a), $\sigma' \in S^{k+1,l}$ and by (12.4b) $\left(\begin{smallmatrix} \sigma' \\ \sigma \end{smallmatrix} \right) \in E_S^{\circ 2}$. Now (12.6c) follows from Lemma 4.17 and (12.6d) follows from (11.9c) and (4.16). As (12.4a) and (12.4b) also apply to r, and hold directly by definition for i, (12.6e), (12.6f), and (12.6h) follow from the same argument. Finally, (12.6g) is the definition of $(r \otimes \mathrm{id}_{\mathbf{B}})_{k,l}$. □

Proposition 12.7. *Let $\mathbf{B}_0 = (S_0, \pi_0, B_0) \xleftarrow{\varphi_1} \dots \xleftarrow{\varphi_l} \mathbf{B}_l = (S_l, \pi_l, B_l)$ be a chain of composable morphisms in $\mathcal{B}_{G,U}(\Lambda)$. Assume $l \leq L$ and $\operatorname{supp}_G(\varphi_j) \subseteq M$ for all j. Let $E' := \bigcup \operatorname{supp}_2 \varphi_j \cup \bigcup_j \operatorname{supp}_2 \mathbf{B}_j \subseteq \Lambda \times \Lambda$. Then there are $\Phi_i\colon \mathbf{B}'_i \to \mathbf{B}'_{i+1} \in \mathcal{B}_{G,U}(S(X) \times \Lambda)$, $i = 0, \dots, 2l$ such that*

$$F^l(\varphi_1, \dots, \varphi_l) = \Phi_{2l} \circ \dots \circ \Phi_0$$

and $\operatorname{supp}_G \Phi_i \subseteq M$, $\operatorname{supp}_2 \Phi \subseteq E_S^{\circ 2} \times E'$, $\operatorname{supp}_1 \mathbf{B}'_i \subseteq S(D) \times \Lambda$ for all i.

Proof. Recall that by construction

$$\begin{aligned} F^l(\varphi_1, \dots, \varphi_l) = \\ & \big(r \otimes \mathrm{id}_{\mathbf{B}_0}\big) \circ \big(\mathrm{id}_{\mathbf{S}_*(X)} \otimes \varphi_1\big) \circ \big(H \otimes \mathrm{id}_{\mathbf{B}_1}\big) \circ \dots \\ & \quad \dots \circ \big(H \otimes \mathrm{id}_{\mathbf{B}_{l-1}}\big) \circ \big(\mathrm{id}_{\mathbf{S}_*(X)} \otimes \varphi_l\big) \circ \big(i \otimes \mathrm{id}_{\mathbf{B}_l}\big). \end{aligned}$$

Using (12.6a),(12.6c),(12.6e), and (12.6g) we can rewrite this as

$$\begin{aligned} F^l(\varphi_1, \dots, \varphi_l) = \\ & \big(r \otimes \mathrm{id}_{\mathbf{B}_0}\big)_{l,l} \circ \big(\mathrm{id}_{\mathbf{S}_*(X)} \otimes \varphi_1\big)^{l,l}_{l,l-1} \circ \big(H \otimes \mathrm{id}_{\mathbf{B}_1}\big)^{l,l-1}_{l-1,l-1} \circ \dots \\ & \quad \dots \circ \big(H \otimes \mathrm{id}_{\mathbf{B}_{l-1}}\big)^{2,1}_{1,1} \circ \big(\mathrm{id}_{\mathbf{S}_*(X)} \otimes \varphi_l\big)^{1,1}_{1,0} \circ \big(i \otimes \mathrm{id}_{\mathbf{B}_l}\big)^{1,0}. \end{aligned}$$

By (11.9d) (and since $U \subseteq M$) for each of these factors supp_G is contained in M. By (12.6b),(12.6d),(12.6f), and (12.6h), supp_2 of each factor, is contained in

$E_S^{\circ 2} \times E'$[61]. Finally, the domains of the factors are of the form $(\mathbf{S}_n(X) \otimes \mathbf{B}_i)_{k,l}$ (or $\mathbf{C}(D) \otimes \mathbf{B}_0$) and have therefore supp_1 contained in $S(D^{(L)})$. $\square$

12.E. **Estimates over $J \times \Lambda$.** We now consider the homotopy coherent functor

$$\widetilde{F} := (f \circ \operatorname{bary} \times \operatorname{id}_\Lambda)_* \circ F \colon \mathcal{B}_{G,U}(\Lambda) \to \operatorname{ch} \mathcal{B}_G(J \times \Lambda).$$

Proposition 12.8.

(12.8a) *Let $\mathbf{B} \in \mathcal{B}_{G,U}(\Lambda)$ and $E' := \operatorname{supp}_2 \mathbf{B}$. Then*

$$\begin{aligned} \operatorname{supp}_G \widetilde{F}_0(\mathbf{B})_n &\subseteq U, & \operatorname{supp}_2 \widetilde{F}_0(\mathbf{B})_n &\subseteq E^{\circ 2} \times E', \\ \operatorname{supp}_G \partial_n^{\widetilde{F}_0(\mathbf{B})} &\subseteq U, & \operatorname{supp}_2 \partial_n^{\widetilde{F}_0(\mathbf{B})} &\subseteq E^{\circ 2} \times E'; \end{aligned}$$

(12.8b) *Let $\mathbf{B}_0 = (S_0, \pi_0, B_0) \xleftarrow{\varphi_1} \dots \xleftarrow{\varphi_l} \mathbf{B}_l = (S_l, \pi_l, B_l)$ be a chain of composable morphisms in $\mathcal{B}_{G,U}(\Lambda)$. Assume $l \leq L$ and $\operatorname{supp}_G(\varphi_j) \subseteq M$ for all j. Let $E' := \bigcup \operatorname{supp}_2 \varphi_j \cup \bigcup_j \operatorname{supp}_2 \mathbf{B}_j$. Then*

$$\begin{aligned} \operatorname{supp}_2 \widetilde{F}^l(\varphi_1, \dots, \varphi_l) &\subseteq (M^{2l}(E^{\circ 3} \times E'))^{\circ(2l+1)}, \\ \operatorname{supp}_G \widetilde{F}^l(\varphi_1, \dots, \varphi_l) &\subseteq M^{2l+1}. \end{aligned}$$

The point here is that the upper bounds are in terms of E, E', M and l and independent of $\mathbf{B}$ and the φ_i.

Proof. Lemma 12.5 directly implies $\operatorname{supp}_G \widetilde{F}_0(\mathbf{B})_n, \operatorname{supp}_G \partial_n^{\widetilde{F}_0(\mathbf{B})} \subseteq U$. Lemma 12.5 together with Lemma 12.3 implies $\operatorname{supp}_2 \widetilde{F}_0(\mathbf{B})_n \subseteq E^{\circ 2} \times E'$, $\operatorname{supp}_2 \partial_n^{\widetilde{F}_0(\mathbf{B})} \subseteq E^{\circ 2} \times E'$[62]. Thus (12.8a) holds.

Applying $(f \circ \operatorname{bary} \times \operatorname{id}_\Lambda)_*$ to the factorization from Proposition 12.7 gives a factorization

$$\widetilde{F}^l(\varphi_1, \dots, \varphi_l) = \widetilde{\Phi}_{2l} \circ \dots \circ \widetilde{\Phi}_0.$$

The estimates for the Φ_i from Proposition 12.7 translate to the $\tilde{\Phi}_i$. We get $\operatorname{supp}_G \widetilde{\Phi}_i \subseteq M$ directly from Proposition 12.7. Using in addition Lemma 12.3 we get $\operatorname{supp}_2 \widetilde{\Phi}_i \subseteq E^{\circ 3} \times E'$. As supp_G is submultiplicative we get $\operatorname{supp}_G \widetilde{F}^l(\varphi_1, \dots, \varphi_l) \subseteq M^{2l+1}$. Using the composition formula 4.3 for supp_2 it is not difficult to obtain an explicit bound for $\operatorname{supp}_2 \widetilde{F}^l(\varphi_1, \dots, \varphi_l)$, for example $(M^{2l}(E^{\circ 3} \times E'))^{\circ(2l+1)}$ works. Thus (12.8b) holds. $\square$

13. Construction of the transfer

Let G be a reductive p-adic group and X be the associated extended Bruhat-Tits building. For each $\underline{t} \in \mathbb{N}^{\times 2}$ we will construct the data considered in Section 12, i.e.,

- numbers $L_{\underline{t}} \in \mathbb{N}$, $\rho_{\underline{t}} > 0$;
- compact open subgroups $U_{\underline{t}} \subseteq G$ and a compact subset $M_{\underline{t}} \subseteq G$ with $U_{\underline{t}} \subseteq M$;
- a subspace $D_{\underline{t}} \subseteq X$ with a locally ordered triangulation all whose simplices are of diameter $< \rho_{\underline{t}}$;
- a sequence of further subspaces $D_{\underline{t}} = D_{\underline{t}}^{(0)} \subseteq D_{\underline{t}}^{(1)} \subseteq \dots \subseteq D_{\underline{t}}^{(L_{\underline{t}})}$ with $M_{\underline{t}} \cdot D_{\underline{t}}^{l-1} \subseteq D_{\underline{t}}^l$ for $l = 1, \dots, L_{\underline{t}}$, and such that $D_{\underline{t}}^L$ is pointwise fixed by $U_{\underline{t}}$;
- a retraction $r_{\underline{t}}^0 \colon X \to D_{\underline{t}}$ for the inclusion $i_{\underline{t}}^0 \colon D_{\underline{t}} \to X$, i.e., $r_{\underline{t}}^0 \circ i_{\underline{t}}^0 = \operatorname{id}_{D_{\underline{t}}}$;
- a homotopy $H_{\underline{t}}^0 \colon i_{\underline{t}}^0 \circ r_{\underline{t}}^0 \simeq \operatorname{id}_X$, that is not expanding, i.e., for all $\tau \in [0,1]$, $x, x' \in X$ we will have $d_X(H_{\underline{t}}^0(x,\tau), H_{\underline{t}}^0(x',\tau)) \leq d_X(x,x')$;
- a map $f_{\underline{t}} \colon X \to |\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge$;
- a subset $E_{\underline{t}} \subseteq |\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge \times |\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge$.

[61] Here we use that $E_S \subseteq E_S^{\circ 2}$ for our specific E_S.

[62] As U fixes D pointwise we could here use E instead of $E^{\circ 2}$, but the precise form of the estimates is not important.

Let $\mathbf{M} \in \mathcal{R}^0$. As in Section 12 (with $\Lambda = |\mathbf{M}|^\wedge$) we obtain then for every $\underline{t}$ a homotopy coherent functor

$$F_{\underline{t}} = (F^0_{\underline{t}}, F^1_{\underline{t}}, \dots)\colon \mathcal{B}_{G,U_{\underline{t}}}(|\mathbf{M}|^\wedge) \to \operatorname{ch}\mathcal{B}_G(S(X) \times |\mathbf{M}|^\wedge)$$

and its composition

$$\widetilde{F}_{\underline{t}} := ((f_{\underline{t}} \circ \operatorname{bary}) \times \operatorname{id}_{|\mathbf{M}|^\wedge})_* \circ F_{\underline{t}}\colon \mathcal{B}_{G,U_{\underline{t}}}(|\mathbf{M}|^\wedge) \to \operatorname{ch}\mathcal{B}_G(|\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge \times |\mathbf{M}|^\wedge).$$

We will verify Assumption 12.2 for each $\underline{t}$. Thus the support estimates from Proposition 12.8 will be available. These estimates will allow us to combine the $(f_{\underline{t}} \times \operatorname{id}_{|\mathbf{M}|^\wedge})_* \circ F_{\underline{t}}$ to obtain a homotopy coherent functor

$$\widetilde{F}_{\mathbf{M}}\colon \mathrm{D}^0_G(\mathbf{M}) \to \operatorname{ch}\mathrm{D}^0_G(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G))$$

and Lemmas 11.19 and 11.20 will imply that the $\tilde{F}_{\mathbf{M}}$ induce the desired transfer for

$$\mathbf{K}\mathrm{D}^0_G\big(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times -\big) \to \mathbf{K}\mathrm{D}^0_G(-)$$

in $\mathcal{R}^0$-$\mathsf{Spectra}$.

13.A. **Retractions to balls in X.** We write d_X for the CAT(0)-metric of X. We fix a base point $x_0 \in X$ and write D_R for the closed ball in X centered at x_0. We write $\pi_R\colon X \to D_R$ for the radial projection. We define the radial homotopy $H_R\colon i_R \circ r_R \simeq \operatorname{id}_X$ with $H_R(x,\tau) = \pi_{\tau d_X(x,x_0)+(1-\tau)R}(x)$. The CAT(0)-condition implies that H_R is contracting, i.e.,

$$d_X\big(H^X_R(x,\tau), H^X_R(x',\tau)\big) \le d_X(x,x')$$

for all $R \ge 0$, $\tau \in [0,1]$ and $x,x' \in X$. Let $U := G_{x_0}$ be the stabilizer of x_0 in G. As the action of G on X is smooth and proper, this is a compact open subgroup of G.

13.B. **Choosing the data.** We choose for $j \in \mathbb{N}$ numbers $\epsilon_j > 0$, $\eta_j > 0$, $L_j \in \mathbb{N}$ and compact subsets $M_j \subseteq G$ such that

$$\epsilon_j \to 0, \eta_j \to 0, L_j \to \infty \text{ as } j \to \infty$$

and that for any $K \subseteq G$ compact we have $K \subseteq M_j$ for all but finitely many j. We also assume that each M_j contains an open subgroup[63]. We set $L^+_j := L_j \cdot \max\{d_X(x_0, gx_0) \mid g \in M_j\}$.

Let N be the first number appearing in Theorem D.1. Given $t_0 \in \mathbb{N}$, we obtain from Theorem D.1 applied to $M = M_{t_0}$ and $\epsilon = \epsilon_{t_0}$ a number β_{t_0} and $\mathcal{V}_{t_0} \subseteq \mathcal{C}\mathrm{vcy}$ finite. Given further $t_1 \in \mathbb{N}$, we obtain again from Theorem D.1 applied to $\eta = \eta_{t_1}$ and $L = L^+_{t_1}$ numbers $R_{t_0,t_1} > 0$, $\rho'_{t_0,t_1} > 0$ and a map

$$f_{t_0,t_1}\colon X \to |\mathbf{J}^N_{\mathcal{V}_{t_0}}(G)|^\wedge$$

such that

(13.1a) for $x \in D_{R_{t_0,t_1}+L^+_{t_1}}$, $g \in M_{t_0}$ we have

$$d_{\mathbf{J}\text{-fol}}\big(f_{t_0,t_1}(gx), gf_{t_0,t_1}(x)\big) < (\beta_{t_0}, \eta_{t_1}, \epsilon_{t_0});$$

(13.1b) for $x \in D_{R_{t_0,t_1}+L^+_{t_1}}$, $R' \ge R_{t_0,t_1}$ we have

$$d_{\mathbf{J}\text{-fol}}\big(f_{t_0,t_1}(x), f_{t_0,t_1}(\pi_{R'}(x))\big) < (\beta_{t_0}, \eta_{t_1}, \epsilon_{t_0});$$

(13.1c) for all $x, x' \in X$ with $d_X(x,x') < \rho'_{t_1}$ we have

$$d_{\mathbf{J}\text{-fol}}\big(f_{t_0,t_1}(x), f_{t_0,t_1}(x')\big) < (\beta_{t_0}, \eta_{t_1}, \epsilon_{t_0}).$$

[63]To construct M_j we can choose a metric on G and take M_j to be the closed ball of radius j around the unit in G. Since G is a td-group, M_j contains a compact open subgroup of G.

Here and later we abbreviate $d_{\mathbf{J}\text{-fol}} = d_{\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\text{-fol}}$. For $\underline{t} = (t_0, t_1)$ we now define respectively choose

- $L_{\underline{t}} := L_{t_1}$, $\rho_{\underline{t}} := \rho'_{\underline{t}}/(L_t + 1)$, $M_{\underline{t}} := M_{t_0}$, $D_{\underline{t}} := D_{R_t}$; as balls in a building the $D_{\underline{t}}$ can be triangulated;
- $D^{(l)}_{\underline{t}} := D_{R_{t_0,t_1} + \frac{l}{L_{\underline{t}}} L^{+}_{t_1}}$ for $l = 0, \ldots, L_t$;
- $U_{\underline{t}}$ a compact open subgroup that fixes $D^{(L_{\underline{t}})}_{\underline{t}}$ pointwise and is contained in $M_{\underline{t}}$;
- $i^0_{\underline{t}} \colon D_{\underline{t}} \to X$ the inclusion;
- $r^0_{\underline{t}} := \pi_{R_{\underline{t}}} \colon X \to D_{\underline{t}}$, the radial projection;
- $f_{\underline{t}} := f_{t_0,t_1}$;
- $H^0_{\underline{t}} := H_{R_{\underline{t}}}$ the radial homotopy $i^0_{\underline{t}} \circ r^0_{\underline{t}} \simeq \mathrm{id}_X$;
- $E_{\underline{t}} := \left\{ \left(\begin{smallmatrix} z \\ z' \end{smallmatrix} \right) \;\middle|\; d_{\mathbf{J}\text{-fol}}(z, z') < (\beta_{t_0}, \eta_{t_1}, \epsilon_{t_0}) \right\}$.

We note that (13.1a), (13.1b) and (13.1c) imply that Assumption 12.2 holds. We can now apply the construction from Section 12 and obtain for $\mathbf{M} \in \mathcal{R}^0$ the homotopy coherent functor

$$((f_{\underline{t}} \circ \mathrm{bary}) \times \mathrm{id}_{|\mathbf{M}|^\wedge})_* \circ F_{\underline{t}} \colon \mathcal{B}_{G,U_{\underline{t}}}(|\mathbf{M}|^\wedge) \to \mathrm{ch}\,\mathcal{B}_G(|\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge \times |\mathbf{M}|^\wedge).$$

13.c. **The $\tilde{F}_t$ combine to a homotopy coherent functor on $\mathrm{D}^0_{G,\mathbf{U}}(\mathbf{M})$.** We will now use the G-control structure $\mathfrak{D}^0(\mathbf{\Sigma})$ from Definition 7.7 for $\mathbf{\Sigma} = \mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)$.

Lemma 13.2. *Let* $E := \left\{ \left(\begin{smallmatrix} z', \underline{t} \\ z, \underline{t} \end{smallmatrix} \right) \;\middle|\; \underline{t} \in \mathbb{N}^{\times 2}, \left(\begin{smallmatrix} z' \\ z \end{smallmatrix} \right) \in E_{\underline{t}} \right\}$. *Then* $E \in \mathfrak{D}^0_2(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G))$.

Proof. We need to verify the foliated control condition from Definition 7.7. Let $\epsilon > 0$ be given. Choose k_0 such that $\epsilon_{t_0} < \epsilon$ for all $t_0 \geq k_0$. Let $t_0 \geq k_0$ be given. Set $\beta := \beta_{t_0}$. Let $\eta > 0$ be given. Choose k_1 such that $\eta_{t_1} < \eta$ for all $t_1 \geq k_1$. Let $t_1 \geq k_1$ be given. Set $\underline{t} := (t_0, t_1)$. Let $z, z' \in |\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge$ be given with $\left(\begin{smallmatrix} z', \underline{t} \\ z, \underline{t} \end{smallmatrix} \right) \in E$. By definition of E we then have $\left(\begin{smallmatrix} z' \\ z \end{smallmatrix} \right) \in E_{\underline{t}}$. By definition of $E_{\underline{t}}$ we then have $d_{\mathbf{J}\text{-fol}}(z, z') < (\beta_{t_0}, \eta_{t_1}, \epsilon_{t_0})$. Since $\beta = \beta_{t_0}$, $\eta_{t_1} < \eta$ and $\epsilon_{t_0} < \epsilon$ this implies

$$d_{\mathbf{J}\text{-fol}}(z, z') < (\beta, \eta, \epsilon),$$

as required. $\square$

Following Remark 7.17 we will view $\mathrm{D}^0_G(\mathbf{M})$ and $\mathrm{D}^0_G(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times \mathbf{M})$ respectively as subcategories of $\prod'_{\mathbb{N}^{\times 2}} \mathcal{B}_G(|\mathbf{M}|^\wedge)$ and $\prod'_{\mathbb{N}^{\times 2}} \mathcal{B}_G(|\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times \mathbf{M}|^\wedge)$ respectively. Thus objects are sequences $\mathbf{B} = (\mathbf{B}_{\underline{t}})_{\underline{t} \in \mathbb{N}^\times}$ of objects in $\mathcal{B}_G(|\mathbf{M}|^\wedge)$ and $\mathcal{B}_G(|\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times \mathbf{M}|^\wedge)$ respectively satisfying the conditions spelled out in Remarks 7.17. Similarly morphisms are equivalence classes of sequences $(\varphi_{\underline{t}})_{\underline{t} \in \mathbb{N}^{\times 2}}$. We recall that sequences $(\varphi_{\underline{t}})_{\underline{t} \in \mathbb{N}^{\times 2}}$ and $(\varphi'_{\underline{t}})_{\underline{t} \in \mathbb{N}^{\times 2}}$ are equivalent, if there is k_0 such that for all $t_0 \geq k_0$ there is k_1 such that for all $t_1 \geq k_1$ we have $\varphi_{t_0,t_1} = \varphi'_{t_0,t_1}$. In particular, we can ignore all φ_{t_0,t_1} with t_0 or t_1 small. We define $\mathrm{D}^0_{G,\mathbf{U}}(\mathbf{M})$ as the full subcategory of $\mathrm{D}^0_G(\mathbf{M})$ on all objects $\mathbf{B} = (\mathbf{B}_{\underline{t}})_{\underline{t} \in \mathbb{N}^{\times 2}}$ with $\mathrm{supp}_G \mathbf{B}_{\underline{t}} \subseteq U_{\underline{t}}$ for all $\underline{t}$. We can now define the homotopy coherent functor

$$F_{\mathbf{M}} = (F^0_{\mathbf{M}}, F^1_{\mathbf{M}}, \ldots) \colon \mathrm{D}^0_{G,\mathbf{U}}(\mathbf{M}) \to \mathrm{ch}\,\mathrm{D}^0_G\big(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times \mathbf{M}\big).$$

Let $\mathbf{B} = (\mathbf{B}_{\underline{t}})_{\underline{t} \in \mathbb{N}^{\times 2}}$ be an object of $\mathrm{D}^0_G(\mathbf{M})$. We define

$$F^0_{\mathbf{M}}(\mathbf{B}) := \Big(((f_{\underline{t}} \circ \mathrm{bary}) \times \mathrm{id}_{|\mathbf{M}|^\wedge})_* (F^0_{\underline{t}}(\mathbf{B}_{\underline{t}})) \Big)_{\underline{t} \in \mathbb{N}^{\times 2}}.$$

Let $\mathbf{B}_0 \xleftarrow{\varphi_1} \ldots \xleftarrow{\varphi_l} \mathbf{B}_l$ be a chain of composable morphisms in $\mathrm{D}^0_G(\mathbf{M})$. Write $\varphi_i = \big((\varphi_i)_{\underline{t}}\big)_{\underline{t} \in \mathbb{N}^{\times 2}}$. We define

$$F^l_{\mathbf{M}}(\varphi_1, \ldots, \varphi_l) := \Big(((f_{\underline{t}} \circ \mathrm{bary}) \times \mathrm{id}_{|\mathbf{M}|^\wedge})_* \big(F^l_{\underline{t}}\big((\varphi_1)_{\underline{t}}, \ldots, (\varphi_l)_{\underline{t}}\big)\big) \Big)_{\underline{t} \in \mathbb{N}^{\times 2}}.$$

We write $\big(F^0_{\mathbf{M}}(\mathbf{B})\big)_n$ for the n-chain module of $F^0_{\mathbf{M}}(\mathbf{B})$ and $\partial_n^{F^0_{\mathbf{M}}(\mathbf{B})}$ for the n-th boundary map.

Lemma 13.3. *(a) $\big(F^0_{\mathbf{M}}(\mathbf{B})\big)_n$ as above is an object in $\mathsf{D}_G(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\times\mathbf{M})$;*

(b) $\partial_n^{F^0_{\mathbf{M}}(\mathbf{B})}$ as above is a morphism in $\mathsf{D}_G(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\times\mathbf{M})$;

(c) $F^l_{\mathbf{M}}(\varphi_1,\dots,\varphi_l)$ as above is a morphism in $\operatorname{ch}\mathsf{D}_G(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\times\mathbf{M})$.

Proof. (a) We need to verify the four conditions (7.18a) to (7.18d).

For (7.18a) we need to check $\operatorname{supp}_1\big(F^0_{\mathbf{M}}(\mathbf{B})\big)_n\in\mathsf{D}^0_1(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\times\mathbf{M})$. Let

$$F:=\bigcup_{\underline{t}\in\mathbb{N}^{\times 2}} f_{\underline{t}}(\mathrm{bary}_n(D_{\underline{t}}))\times\{\underline{t}\}.$$

We claim that $F\in\mathfrak{D}^0_1(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G))$. This amounts to checking the three conditions in (7.7a) for F: Finiteness over $\mathbb{N}^{\times 2}$ is clear as $D_{\underline{t}}$ is compact and has only finitely many simplices. For the other two conditions we use that $f_{\underline{t}}=f_{t_0,t_1}$ is a map to $|\mathbf{J}^N_{\mathcal{V}_{t_0}}(G)|^\wedge\subseteq|\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge$. Thus F has finite dimensional support. As $\mathcal{V}_{t_0}$ is finite $|J^N_{\mathcal{V}_{t_0}}(G)|$ is a finite subcomplex of $|J_{\mathcal{C}\mathrm{vcy}}(G)|$. It follows that F has compact support in $|J_{\mathcal{C}\mathrm{vcy}}(G)|$. So $F\in\mathfrak{D}^0_1(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G))$. Set $F':=\operatorname{supp}_1\mathbf{B}\in\mathfrak{D}^0_1(\mathbf{M})$. By (11.9b)

$$\operatorname{supp}_1((f_{\underline{t}}\circ\mathrm{bary})\times\mathrm{id}_{|\mathbf{M}|^\wedge})_*(F^0_{\underline{t}}(\mathbf{B}_{\underline{t}}))=f_{\underline{t}}(\mathrm{bary}_n(D_{\underline{t}}))\times\operatorname{supp}_1\mathbf{B}_t.$$

Thus

$$\begin{aligned}\operatorname{supp}_1\big(F^0_{\mathbf{M}}(\mathbf{B})\big)_n &= \bigcup_{\underline{t}\in\mathbb{N}^{\times 2}}\operatorname{supp}_1((f_{\underline{t}}\circ\mathrm{bary})\times\mathrm{id}_{|\mathbf{M}|^\wedge})_*(F^0_{\underline{t}}(\mathbf{B}_{\underline{t}})\times\{\underline{t}\})\\ &= \big\{(z,\lambda,\underline{t})\mid(z,\underline{t})\in F,(\lambda,\underline{t})\in F'\big\}.\end{aligned}$$

Thus $\operatorname{supp}_1\big(F^0_{\mathbf{M}}(\mathbf{B})\big)_n\in\mathfrak{D}^0_1(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\times\mathbf{M})$ by (7.19a).

For (7.18b) we need to check $\operatorname{supp}_2\big(F^0_{\mathbf{M}}(\mathbf{B})\big)_n\in\mathfrak{D}^0_2(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\times\mathbf{M})$. By (12.8a)

$$\operatorname{supp}_2(f_{\underline{t}}\times\mathrm{bary})\subseteq E^{\circ 2}_{\underline{t}}\times\operatorname{supp}_2\mathbf{B}_{\underline{t}}.$$

We now use $E\in\mathfrak{D}^0_2(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G))$ from Lemma 13.2. Then

$$\operatorname{supp}_2(F^0_{\mathbf{M}}(\mathbf{B}))_n\subseteq\left\{\left(\begin{smallmatrix}z',\lambda',\underline{t}\\ z,\lambda,\underline{t}\end{smallmatrix}\right)\,\middle|\,\left(\begin{smallmatrix}z',\underline{t}\\ z,\underline{t}\end{smallmatrix}\right)\in E^{\circ 2},\left(\begin{smallmatrix}\lambda',\underline{t}\\ \lambda,\underline{t}\end{smallmatrix}\right)\in\operatorname{supp}_2\mathbf{B}\right\}$$

which belongs to $\mathfrak{D}^0_2(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\times\mathbf{M})$ by (7.19b).

For (7.18c) we need to check $\operatorname{supp}_G\big(F^0_{\mathbf{M}}(\mathbf{B})\big)_n\in\mathfrak{D}^0_G(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\times\mathbf{M})$. By (11.17b) we have $\operatorname{supp}_G\big(F^0_{\mathbf{M}}(\mathbf{B})\big)_n\subseteq\operatorname{supp}_G\mathbf{B}$. The latter is relatively compact as $\mathbf{B}\in\mathsf{D}^0_G(\mathbf{M})$. Thus $\operatorname{supp}_G\big(F^0_{\mathbf{M}}(\mathbf{B})\big)_n$ is relatively compact as well.

Finally, the finiteness condition (7.18d) for $\big(F^0_{\mathbf{M}}(\mathbf{B})\big)_n$ follows from (11.17a) as the $\mathbf{B}_{\underline{t}}$ are finite.

(b) We need to verify (7.18e) and (7.18f) for $\partial_n^{F^0_{\mathbf{M}}(\mathbf{B})}$. This can be done exactly in the same way as for the corresponding properties (7.18b) and (7.18c) for $\big(F^0_{\mathbf{M}}(\mathbf{B})\big)_n$ under (a). For (7.18f) this uses (11.17c) in place of (11.17b).

(c) We need to verify (7.18e) and (7.18f) for $F^l_{\mathbf{M}}(\varphi_1,\dots,\varphi_l)$.

Choose k_0 such that $\operatorname{supp}_G\varphi_i\subseteq M_{k_0}$ for $i=1,\dots,l$. Since we are working in $\mathsf{D}^0_G(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\times\mathbf{M})$ we can ignore all $\underline{t}=(t_0,t_1)$ with $t_0\le k_0$. Thus for the rest of the argument we can and will always assume $t_0\ge k_0$. From (12.8b) we obtain that

$$\operatorname{supp}_G\big(((f_{\underline{t}}\,\mathrm{bary})\times\mathrm{id}_{|\mathbf{M}|^\wedge})_*(F^l_{\underline{t}})\big((\varphi_1)_{\underline{t}},\dots,(\varphi_l)_{\underline{t}}\big)\big)\subseteq M^{2l+1}_{k_0}$$

for all $\underline{t}$. In particular, the union of these G-supports is relatively compact in G. Thus $\operatorname{supp}_G F^l_{\mathbf{M}}(\varphi_1,\dots,\varphi_l)\in\mathfrak{D}^0_G(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)\times\mathbf{M})$ as required by (7.18f).

Set $E'_{\underline{t}} := \bigcup_j \operatorname{supp}_2((\varphi_j)_{\underline{t}}) \cup \bigcup_j \operatorname{supp}_2((\mathbf{B}_j)_{\underline{t}})$. From (12.8b) we obtain

$$\begin{aligned} \text{(13.4)} \quad & \operatorname{supp}_2(((f_{\underline{t}} \circ \mathrm{bary}) \times \mathrm{id}_{|\mathbf{M}|^\wedge})_*(F^l_{\underline{t}})\big((\varphi_1)_{\underline{t}}, \ldots, (\varphi_l)_{\underline{t}}\big) \\ & \subseteq \Big\{ \left(\begin{smallmatrix} z',\lambda' \\ z,\lambda \end{smallmatrix} \right) \Big| \left(\begin{smallmatrix} z' \\ z \end{smallmatrix} \right) \in \big(M^{2l}_{k_0} \cdot E^{\circ 3}_{\underline{t}}\big)^{\circ(2l+1)}, \left(\begin{smallmatrix} \lambda' \\ \lambda \end{smallmatrix} \right) \in \big(M^{2l}_{k_0} \cdot E'_{\underline{t}}\big)^{\circ(2l+1)} \Big\}. \end{aligned}$$

We use again $E \in \mathfrak{D}^0_2(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G))$ from Lemma 13.2 and set

$$\tilde{E} := \big(M^{2l}_{k_0} \cdot E^{\circ 3}\big)^{\circ(2l+1)} \in \mathfrak{D}^0_2(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)).$$

We also have, as the φ_j and the $\mathbf{B}_j$ are from $\mathsf{D}^0_{G,\mathbf{U}}(\mathbf{M})$,

$$E' := \Big\{ \left(\begin{smallmatrix} \lambda',\underline{t} \\ \lambda,\underline{t} \end{smallmatrix} \right) \;\Big|\; \left(\begin{smallmatrix} \lambda' \\ \lambda \end{smallmatrix} \right) \in E'_{\underline{t}} \Big\} \in \mathfrak{D}^0_2(\mathbf{M})$$

and also $\tilde{E}' := (M^{2l}_{k_0} \cdot E')^{\circ(2l+1)} \in \mathfrak{D}^0_2(\mathbf{M})$. Now (13.4) implies

$$\operatorname{supp}_2 F^l_{\mathbf{M}}(\varphi_1, \ldots, \varphi_l) \subseteq \Big\{ \left(\begin{smallmatrix} z',\lambda',\underline{t} \\ z,\lambda,\underline{t} \end{smallmatrix} \right) \;\Big|\; \left(\begin{smallmatrix} z',\underline{t} \\ z,\underline{t} \end{smallmatrix} \right) \in \tilde{E}, \left(\begin{smallmatrix} \lambda',\underline{t} \\ \lambda,\underline{t} \end{smallmatrix} \right) \in \tilde{E}' \Big\}.$$

The latter belongs to $\mathfrak{D}^0_2(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times \mathbf{M})$ by (7.19b). Thus $\operatorname{supp}_2 F^l_{\mathbf{M}}(\varphi_1, \ldots, \varphi_l) \in \mathfrak{D}^0_2(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times \mathbf{M})$ as required by (7.18e). □

Conclusion of the proof of Theorem 6.7. We have the following diagram.

$$\begin{tikzcd} \mathsf{D}^0_{G,\mathbf{U}}(\mathbf{M}) \arrow[r, "F_{\mathbf{M}}"] \arrow[rr, bend left, "I"] & \mathrm{ch}_{\mathrm{fin}}\, \mathsf{D}^0_G(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times \mathbf{M}) \arrow[r, "(\mathrm{pr}_{\mathbf{M}})_*"] & \mathrm{ch}_{\mathrm{fin}}\, \mathsf{D}^0_G(\mathbf{M}) \end{tikzcd}$$

The functor $(\mathrm{pr}_{\mathbf{M}})_*$ is induced by the projection $\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times \mathbf{M} \to \mathbf{M}$, $F_{\mathbf{M}}$ is the homotopy coherent functor defined above (well-defined by Lemma 13.3), and I is the inclusion. We can then apply K-theory and obtain

$$\begin{tikzcd} \mathbf{K}\big(\mathsf{D}^0_{G,\mathbf{U}}(\mathbf{M})\big) \arrow[r, "\mathbf{K}(F_{\mathbf{M}})"] \arrow[rr, bend left, "\mathbf{K}(I)"] & \mathbf{K}\big(\mathsf{D}^0_G(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times \mathbf{M})\big) \arrow[r, "\mathbf{p}_{\mathbf{M}} = \mathbf{K}((\mathrm{pr}_{\mathbf{M}})_*)"] & \mathbf{K}\big(\mathsf{D}^0_G(\mathbf{M})\big). \end{tikzcd}$$

Strictly speaking the homotopy coherent functor $F_{\mathbf{M}}$ induces a zig-zag in K-theory, see Remark C.24. Since $\mathsf{D}^0_{G,\mathbf{U}}(\mathbf{M})\big) \to \mathsf{D}^0_G(\mathbf{M})$ induces an equivalence of idempotent completions, compare Subsection 11.F, the map $\mathbf{K}(I)$ is a weak equivalence. It is not difficult to see from the construction that the diagram is natural in $\mathbf{M}$, basically because the tensor product functor is natural for induced functors in both entries. There is a strict natural transformation $\tau_{\mathbf{M}} \colon (\mathrm{pr}_{\mathbf{M}})_* \circ F_{\mathbf{M}} \to I$ by weak equivalences. It is given by applying the construction from Subsection 11.H for each $t \in \mathbb{N}^{\times 2}$, see Lemma 11.19. On each object $\tau_{\mathbf{M}}$ evaluates to a chain homotopy equivalence, see Lemma 11.20. It follows that $(\mathrm{pr}_{\mathbf{M}})_* \circ F_{\mathbf{M}}$ and I induce homotopic maps in K-theory, see Remark C.26. Altogether,

$$\mathbf{K}\big(\mathsf{D}^0_G(\mathbf{M})\big) \xleftarrow[\sim]{\mathbf{K}(I)} \mathbf{K}\big(\mathsf{D}^0_{G,\mathbf{U}}(\mathbf{M})\big) \xrightarrow{\mathbf{K}(F_{\mathbf{M}})} \mathbf{K}\big(\mathsf{D}^0_G(\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G) \times \mathbf{M})\big)$$

is the required section $\mathbf{tr}_{\mathbf{M}}$ for $\mathbf{p}_{\mathbf{M}}$. □

This concludes the proof of the $\mathcal{C}$vcy-Farrell–Jones Conjecture for reductive p-adic groups, i.e., of Theorem 5.15.

14. Reduction from $\mathcal{C}$vcy to $\mathcal{C}$om

Theorem 14.1 (Reduction). *Let G be a td-group and let $\mathcal{B}$ be Hecke category with G-support satisfying (Reg) from Definition 3.11. Assume that the $\mathcal{C}$vcy-assembly map* (5.11) *for $\mathcal{B}$*

$$\operatorname*{hocolim}_{P \in \mathsf{P}\mathcal{C}\mathrm{vcy}(G)} \mathbf{K}\big(\mathsf{C}_G(P)\big) \;\to\; \mathbf{K}\big(\mathsf{C}_G(*)\big)$$

is an equivalence. Then the $\mathcal{C}$op-assembly map (3.9) *for $\mathcal{B}$*

$$\operatorname*{hocolim}_{G/U \in \mathsf{Or}_{\mathcal{C}\mathrm{op}}(G)} \mathbf{K}\big(\mathcal{B}[G/U]\big) \to \mathbf{K}\big(\mathcal{B}[G/G]\big) \simeq \mathbf{K}\mathcal{B}$$

is also an equivalence.

Modulo further results the proof of Theorem 14.1 will be given in Subsection 14.B.

14.A. **Functoriality in the orbit category.** Recall that the definition of $\mathsf{C}_G(P)$ used the (free) G-space $|P|^\wedge$. The construction of this space is not functorial for $P \in \mathsf{P}_+\mathsf{Or}(G)$, only for $P \in \mathsf{P}_+\mathcal{A}\mathrm{ll}(G)$. For this reason, $\mathsf{C}_G(P)$ is not functorial in $\mathsf{P}_+\mathsf{Or}(G)$. But this is not a serious issue and we will now tweak the definitions to remedy this.

Let $\mathcal{B}$ be a category with G-support and $\mathfrak{C} = (\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_G)$ be a G-control structure on X. Let $\mathcal{Y}$ be a collection of subsets of X as in Definition 4.19. We say that two maps $\pi, \rho\colon S \to X$ are $\mathfrak{C}_2$-*equivalent* if

$$\Big\{ \Big(\begin{smallmatrix} \rho(s) \\ \pi(s) \end{smallmatrix} \Big) \;\Big|\; s \in S \Big\} \in \mathfrak{C}_2.$$

In the definition of $\mathcal{B}_G(\mathfrak{C}, \mathcal{Y})$ we can replace the maps $\pi\colon S \to X$ with their $\mathfrak{C}_2$-equivalence classes and obtain the category $\overline{\mathcal{B}}_G(\mathfrak{C}, \mathcal{Y})$. In more detail, objects of $\overline{\mathcal{B}}_G(\mathfrak{C}, \mathcal{Y})$ are triples $\mathbf{B} = (S, [\pi], B)$ such that (S, π, B) is an object of $\mathcal{B}_G(\mathfrak{C}, \mathcal{Y})$. The point here is that, if π and ρ are $\mathfrak{C}_2$-equivalent, then (S, π, B) is an object of $\mathcal{B}_G(\mathfrak{C}, \mathcal{Y})$, iff (S, ρ, B) is an object of $\mathcal{B}_G(\mathfrak{C}, \mathcal{Y})$. A morphism $\mathbf{B} = (S, [\pi], B) \to \mathbf{B}' = (S', [\pi'], B')$ in $\overline{\mathcal{B}}_G(\mathfrak{C}, \mathcal{Y})$ is a matrix $\varphi = (\varphi_s^{s'}\colon B(s) \to B'(s'))_{s \in S, s' \in S'}$ of morphisms in $\mathcal{B}$ that also defines a morphism $(S, \pi, B) \to (S', \pi', B')$ in $\mathcal{B}_G(\mathfrak{C}, \mathcal{Y})$. Again, the point here is that, if π and ρ are $\mathfrak{C}_2$-equivalent, π' and ρ' are $\mathfrak{C}_2$-equivalent, then φ defines a morphism in $(S, \pi, B) \to (S', \pi', B')$ in $\mathcal{B}_G(\mathfrak{C}, \mathcal{Y})$, iff φ defines a morphism in $(S, \rho, B) \to (S', \rho', B')$ in $\mathcal{B}_G(\mathfrak{C}, \mathcal{Y})$. The functor

$$\mathcal{B}_G(\mathfrak{C}, \mathcal{Y}) \xrightarrow{\sim} \overline{\mathcal{B}}_G(\mathfrak{C}, \mathcal{Y}), \quad (S, \pi, B) \mapsto (S, [\pi], B) \tag{14.2}$$

is an equivalence. Essentially, $\overline{\mathcal{B}}_G(\mathfrak{C}, \mathcal{Y})$ is obtained from $\mathcal{B}_G(\mathfrak{C}, \mathcal{Y})$ by identifying (S, π, B) and (S, ρ, B) along a canonical isomorphism, whenever π and ρ are $\mathfrak{C}_2$-equivalent.

Recall $\mathsf{C}_G(P) = \mathcal{B}_G(\mathfrak{C}(P), \mathcal{Y}(P))$ from Definition 5.9. We now define $\overline{\mathsf{C}}_G(P) := \overline{\mathcal{B}}_G(\mathfrak{C}(P), \mathcal{Y}(P))$. From (14.2) we obtain a natural equivalence

$$\mathsf{C}_G(P) \xrightarrow{\sim} \overline{\mathsf{C}}_G(P). \tag{14.3}$$

Let $f\colon P \to P'$ be a map in $\mathsf{P}_+\mathsf{Or}G$. It induces a map $|f|\colon |P| \to |P'|$. There is always a lift of $|f|$ to $|P|^\wedge \to |P'|^\wedge$, but it is typically not unique.

Lemma 14.4. *Let $\widehat{f}_0, \widehat{f}_1\colon |P|^\wedge \to |P|^\wedge$ be lifts of $|f|$. Then $\widehat{f}_0 \times \mathrm{id}_{\mathbb{N}}, \widehat{f}_1 \times \mathrm{id}_{\mathbb{N}}\colon |P|^\wedge \times \mathbb{N} \to |P|^\wedge \times \mathbb{N}$ are $\mathfrak{C}_2(P)$-equivalent.*

Proof. To simplify notation, we just treat the case $P = G/H$, $P' = G/H'$. Then $f = |f|\colon G/H = |G/H| \to G/H' = |G/H'|$. Any G-map $f\colon G/H \to G/H'$ is of the form $xH \mapsto xgH'$ where $g^{-1}Hg \subseteq H'$. Now for the two lifts $\widehat{f}_0, \widehat{f}_1\colon G = |G/H|^\wedge \to G = |G/H'|^\wedge$ there are $h_0', h_1' \in H'$ with $\widehat{f}_0(x) = xgh_0'$, $\widehat{f}_1(x) = xgh_1'$ for all $x \in G$. With $\beta := d(h_0', h_1')$ then for all $x \in G$

$$d_{H'\text{-fol}}(\widehat{f}_0(x), \widehat{f}_1(x)) \le (\beta, 0).$$

This implies that $\widehat{f_0} \times \mathrm{id}_{\mathbb{N}}, \widehat{f_1} \times \mathrm{id}_{\mathbb{N}}$ are $\mathfrak{C}_2(P)$-equivalent. □

Lemma 14.4 implies that the assignment

$$P \mapsto \overline{\mathsf{C}}_G(P)$$

is functorial in $\mathsf{P_+Or}(G)$. This allows us to consider the bottom row in the following diagram

$$\begin{tikzcd}
\operatorname*{hocolim}_{P \in \mathsf{P}\mathcal{C}\mathrm{op}(G)} \mathbf{K}\big(\mathsf{C}_G(P)\big) \arrow[r] \arrow[d, "\sim"] & \operatorname*{hocolim}_{P \in \mathsf{P}\mathcal{C}\mathrm{vcy}(G)} \mathbf{K}\big(\mathsf{C}_G(P)\big) \arrow[r] \arrow[d, "\sim"] & \mathbf{K}\big(\mathsf{C}_G(*)\big) \arrow[d, "\sim"] \\
\operatorname*{hocolim}_{P \in \mathsf{P}\mathcal{C}\mathrm{op}(G)} \mathbf{K}\big(\overline{\mathsf{C}}_G(P)\big) \arrow[r] \arrow[d, "\sim"] & \operatorname*{hocolim}_{P \in \mathsf{P}\mathcal{C}\mathrm{vcy}(G)} \mathbf{K}\big(\overline{\mathsf{C}}_G(P)\big) \arrow[r] \arrow[d, "\sim"] & \mathbf{K}\big(\overline{\mathsf{C}}_G(*)\big) \arrow[d, "="] \\
\operatorname*{hocolim}_{P \in \mathsf{POr}_{\mathcal{C}\mathrm{op}}(G)} \mathbf{K}\big(\overline{\mathsf{C}}_G(P)\big) \arrow[r] & \operatorname*{hocolim}_{P \in \mathsf{POr}_{\mathcal{C}\mathrm{vcy}}(G)} \mathbf{K}\big(\overline{\mathsf{C}}_G(P)\big) \arrow[r] & \mathbf{K}\big(\overline{\mathsf{C}}_G(*)\big).
\end{tikzcd} \tag{14.5}$$

We claim that the vertical maps in this diagram are all equivalences. For the first three vertical maps this follows from (14.3). For the first two vertical maps in the bottom row, this is an application of Lemma A.6.

14.B. **The category $\mathsf{C}^+_G(P)$.** We will need slightly bigger categories than $\mathsf{C}_G(P)$ and $\overline{\mathsf{C}}_G(P)$, where we relax the condition on the supports in G. This will be important in Subsection 14.H. For smooth P this will not change the K-theory, but we do not know this for general P.

Define the G-control structure $\mathfrak{C}^+(P) = (\mathfrak{C}^+_1(P), \mathfrak{C}^+_2(P), \mathfrak{C}^+_G(P))$ by $\mathfrak{C}^+_1(P) := \mathfrak{C}_1(P)$, $\mathfrak{C}^+_2(P) := \mathfrak{C}_2(P)$ and

$$\mathfrak{C}^+_G(P) := \text{All subsets of } G.$$

We define

$$\mathsf{C}^+_G(P) := \mathcal{B}_G(\mathfrak{C}^+(P), \mathcal{Y}(P)) \quad \text{and} \quad \overline{\mathsf{C}}^+_G(P) := \overline{\mathcal{B}}_G(\mathfrak{C}^+(P), \mathcal{Y}(P)).$$

As $\mathfrak{C}_G(P) \subseteq \mathfrak{C}^+_G(P)$, there are inclusions $\mathsf{C}_G(P) \to \mathsf{C}^+_G(P)$ and $\overline{\mathsf{C}}_G(P) \to \overline{\mathsf{C}}^+_G(P)$.

Lemma 14.6. *Suppose that P is smooth, i.e., that $P \in \mathsf{P_+}\mathcal{O}\mathrm{p}(G)$. Then the inclusions $\mathsf{C}_G(P) \to \mathsf{C}^+_G(P)$ and $\overline{\mathsf{C}}_G(P) \to \overline{\mathsf{C}}^+_G(P)$ induce equivalences in K-theory.*

Proof. Proposition 5.16 holds, with exactly the same proof, for $\mathsf{C}^+_G(P)$ in place of $\mathsf{C}_G(P)$ as well[64]. This implies the result for $\mathsf{C}_G(P) \to \mathsf{C}^+_G(P)$. For $\overline{\mathsf{C}}_G(P) \to \overline{\mathsf{C}}^+_G(P)$ it follows now from (14.2). □

Theorem 14.7. *Suppose that $\mathcal{B}$ satisfies (Reg) from Definition 3.11. Then for all $P \in \mathsf{POr}_{\mathcal{C}\mathrm{vcy}}(G)$ the canonical map*

$$\operatorname*{hocolim}_{(Q,f) \in \mathsf{POr}_{\mathcal{C}\mathrm{om}}(G) \downarrow P} \mathbf{K}\big(\overline{\mathsf{C}}^+_G(Q)\big) \xrightarrow{\sim} \mathbf{K}\big(\overline{\mathsf{C}}^+_G(P)\big) \tag{14.8}$$

is an equivalence.

The proof of Theorem 14.7 will be given later. We do not know whether or not Theorem 14.7 holds also for $\overline{\mathsf{C}}_G(-)$ instead of $\overline{\mathsf{C}}^+_G(-)$. We note that for Theorem 14.7 it is important that we used $\mathsf{POr}_{\mathcal{C}\mathrm{om}}(G)$; in contrast the category $\mathsf{P}\mathcal{C}\mathrm{om}(G) \downarrow P$ is typically empty.

[64] This works in the generality of categories with G-support in place of $\mathcal{H}_G(\mathcal{A})$.

Proof of Theorem 14.1 modulo Theorem 14.7. Consider the following diagram.

$$\begin{tikzcd}
\operatorname*{hocolim}_{P \in \mathsf{POr}_{\mathcal{C}\mathrm{op}}(G)} \mathbf{K}\big(\overline{\mathsf{C}}_G(P)\big) \arrow[r, "\sim"] \arrow[d, "\alpha_1"', "\sim"] & \operatorname*{hocolim}_{P \in \mathsf{POr}_{\mathcal{C}\mathrm{op}}(G)} \mathbf{K}\big(\overline{\mathsf{C}}^{+}_G(P)\big) \arrow[d, "\widehat{\alpha}_1"', "\sim"] \\
\operatorname*{hocolim}_{P \in \mathsf{POr}_{\mathcal{C}\mathrm{om}}(G)} \mathbf{K}\big(\overline{\mathsf{C}}_G(P)\big) \arrow[r] \arrow[d, "\alpha_2"'] & \operatorname*{hocolim}_{P \in \mathsf{POr}_{\mathcal{C}\mathrm{om}}(G)} \mathbf{K}\big(\overline{\mathsf{C}}^{+}_G(P)\big) \arrow[d, "\widehat{\alpha}_2"', "\sim"] \\
\operatorname*{hocolim}_{P \in \mathsf{POr}_{\mathcal{C}\mathrm{vcy}}(G)} \mathbf{K}\big(\overline{\mathsf{C}}_G(P)\big) \arrow[r] \arrow[d, "FJ"', "\sim"] & \operatorname*{hocolim}_{P \in \mathsf{POr}_{\mathcal{C}\mathrm{vcy}}(G)} \mathbf{K}\big(\overline{\mathsf{C}}^{+}_G(P)\big) \arrow[d] \\
\mathbf{K}\big(\overline{\mathsf{C}}_G(*)\big) \arrow[r, "\sim"] & \mathbf{K}\big(\overline{\mathsf{C}}^{+}_G(*)\big)
\end{tikzcd}$$

We will argue below that all maps labeled with an $\sim$ are equivalences. A diagram chase shows then that all other maps in the diagram are equivalences as well. In particular the left vertical composition is an equivalence. Proposition 3.13 in combination with the equivalences from (14.5) implies that this vertical composition is equivalent to (3.9).

The analog of the map labeled FJ for $\mathsf{C}_G(P)$ instead of $\overline{\mathsf{C}}_G(P)$ and $\mathsf{P}\mathcal{C}\mathrm{vcy}(G)$ instead of $\mathsf{POr}_{\mathcal{C}\mathrm{vcy}}(G)$ is an equivalence by assumption of Theorem 14.1. The equivalences from (14.5) imply that the map labeled FJ is an equivalence as well.

Lemma 14.6 implies that the top and bottom horizontal maps are equivalences. Any compact subgroup of a td-group is contained in a compact open subgroup, see Lemma 2.3. This implies that for all $P \in \mathsf{P}\mathcal{C}\mathrm{om}(G)$ the category $P \downarrow \mathsf{P}\mathcal{C}\mathrm{op}(G)$ is non-empty. Thus Lemma A.6 implies that α_1 and $\widehat{\alpha}_1$ are equivalences.

The map $\widehat{\alpha}_2$ is an equivalence by Theorem 14.7 and the transitivity Lemma A.2 for homotopy colimits. $\square$

14.C. **The category $\overline{\mathsf{C}}^{+,0}_G(P)$.** To prove Theorem 14.7 we introduce a further category as an intermediate step. In many ways this is similar to the passage from $\mathsf{D}_G(\mathbf{\Sigma};\mathcal{B})$ to $\mathsf{D}^0_G(\mathbf{\Sigma};\mathcal{B})$, see Definition 7.15.

For $P \in \mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G)$ we define the G-control structure

$$\mathfrak{C}^{+,0}(P) = (\mathfrak{C}^{+,0}_1(P), \mathfrak{C}^{+,0}_2(P), \mathfrak{C}^{+,0}_G(P))$$

as follows. Set $\mathfrak{C}^{+,0}_1(P) := \mathfrak{C}^{+}_1(P) = \mathfrak{C}_1(P)$, $\mathfrak{C}^{+,0}_G(P) := \widehat{\mathfrak{C}}_G(P)$. We define $\mathfrak{C}^{+,0}_2(P)$ to consist of all $E \in \mathfrak{C}^{+}_2(P) = \mathfrak{C}_2(P)$ satisfying

$$\left(\begin{smallmatrix} \lambda', t' \\ \lambda, t \end{smallmatrix} \right) \in E \implies t' = t.$$

Set

$$\mathsf{C}^{+,0}_G(P) := \mathcal{B}_G(\mathfrak{C}^{+,0}(P), \mathcal{Y}(P)) \quad \text{and} \quad \overline{\mathsf{C}}^{+,0}_G(P) := \overline{\mathcal{B}}_G(\mathfrak{C}^{+,0}(P), \mathcal{Y}(P)).$$

Proposition 14.9. *There are homotopy pushout diagrams of functors* $\mathsf{P}_{+}\mathcal{A}\mathrm{ll}(G) \to \mathsf{Spectra}$

$$\begin{tikzcd}
\mathbf{K}\big(\mathsf{C}^{+,0}_G(-)\big) \arrow[r] \arrow[d] & \mathbf{K}\big(\mathsf{C}^{+,0}_G(-)\big) \arrow[d] \\
\mathbf{K}\big(\mathsf{C}^{+,0}_G(-)\big) \arrow[r] & \mathbf{K}\big(\mathsf{C}^{+}_G(-)\big)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{K}\big(\overline{\mathsf{C}}^{+,0}_G(-)\big) \arrow[r] \arrow[d] & \mathbf{K}\big(\overline{\mathsf{C}}^{+,0}_G(-)\big) \arrow[d] \\
\mathbf{K}\big(\overline{\mathsf{C}}^{+,0}_G(-)\big) \arrow[r] & \mathbf{K}\big(\overline{\mathsf{C}}^{+}_G(-)\big)
\end{tikzcd}$$

Proof. For the diagram on the left this is analog to the proof of Proposition 8.9, more precisely to the construction of either of the pushout squares from Lemma 8.11. For the diagram on the right it follows now from (14.2). ☐

Proposition 14.9 implies that it suffices to prove Theorem 14.7 for $\overline{\mathsf{C}}_G^{+,0}(-)$ in place of $\overline{\mathsf{C}}_G^{+}(-)$.

14.D. **Structure of the remainder of the proof.** For the proof of Theorem 14.7 for $\overline{\mathsf{C}}_G^{+,0}(-)$ we will introduce further variations of $\mathsf{C}_G(-)$ and argue along the following diagram

$$\begin{tikzcd}
\operatorname*{hocolim}_{(Q,f)\in\mathsf{POr}_{\mathcal{C}\mathrm{om}}(G)\downarrow P}\mathbf{K}\big(\overline{\mathsf{C}}_G^{+,0}(Q)\big) \arrow[r] & \mathbf{K}\big(\overline{\mathsf{C}}_G^{+,0}(P)\big) \\
\operatorname*{hocolim}_{\underline{\Gamma}}\mathbf{K}\big(\overline{\mathsf{C}}_G^{+,0}(M)\big) \arrow[r] \arrow[u, "\sim"', "(1)"] \arrow[d, "\sim"', "(3)"] & \mathbf{K}\big(\overline{\mathsf{C}}_G^{+,0}(M)[\Gamma]\big) \arrow[u, "\sim"', "(2)"] \arrow[d, "\sim"', "(4)"] \\
\operatorname*{hocolim}_{\underline{\Gamma}}\mathbf{K}\big(\overline{\mathsf{C}}_G^{+,0,\sharp}(M)\big) \arrow[r] & \mathbf{K}\big(\overline{\mathsf{C}}_G^{+,0,\sharp}(M)[\Gamma]\big) \\
\operatorname*{hocolim}_{\underline{\Gamma}}\mathbf{K}\big(\mathsf{C}_G^{+,0,\sharp}(M)\big) \arrow[r, "\sim", "(7)"'] \arrow[u, "\sim"', "(4)"] & \mathbf{K}\big(\mathsf{C}_G^{+,0,\sharp}(M)[\Gamma]\big). \arrow[u, "\sim"', "(6)"]
\end{tikzcd} \tag{14.10}$$

Details will be worked out in the remainder of this section.

14.E. **The group Γ.** Here we discuss the equivalences (1) and (2) from (14.10).

Fix $P = (G/V_1, \dots, G/V_n)$ with $V_i \in \mathcal{C}\mathrm{vcy}$. Let $K_i \subseteq V_i$ be the maximal compact open subgroup of V_i. Set $M := (G/K_1, \dots, G/K_n)$. The quotients $\Gamma_i := V_i/K_i$ are either infinite cyclic or trivial. Let $\Gamma := \Gamma_1 \times \cdots \times \Gamma_n$. Then Γ is a finitely generated free abelian group of rank at most n. There are canonical maps $h_i \colon \Gamma_i \to \mathrm{end}_{\mathsf{Or}(G)}(G/K_i)$, sending $cK_i \in C_i$ to $G/K_i \to G/K_i, gK_i \mapsto gcK_i$. These combine to an action of Γ on M by morphisms in $\mathsf{POr}(G)$. This induces an action of Γ on $\overline{\mathsf{C}}_G^{+,0}(M)$ and we can form $\overline{\mathsf{C}}_G^{+,0}(M)[\Gamma]$. The projection $\pi \colon M \to P$ is Γ-equivariant for the trivial action of Γ on P. Thus $\pi_* \colon \overline{\mathsf{C}}_G^{+,0}(M) \to \overline{\mathsf{C}}_G^{+,0}(P)$ is Γ-equivariant for the trivial action on $\overline{\mathsf{C}}_G^{+,0}(P)$ and induces a functor

$$R \colon \overline{\mathsf{C}}_G^{+,0}(M)[\Gamma] \ \to\ \overline{\mathsf{C}}_G^{+,0}(P).$$

The functor R induces (2) in (14.10). We write $\underline{\Gamma}$ for the category with exactly one object $*_\Gamma$ whose endomorphisms are given by Γ. The action of Γ on M determines a functor $h \colon \underline{\Gamma} \to \mathsf{POr}_{\mathcal{C}\mathrm{om}}(G) \downarrow P$ that sends $*_\Gamma$ to $\pi \colon M \to P$. In turn h induces a map

$$\operatorname*{hocolim}_{\underline{\Gamma}}\mathbf{K}\big(\overline{\mathsf{C}}_G^{+,0}(M)\big) \ \to \operatorname*{hocolim}_{(Q,f)\in\mathsf{POr}_{\mathcal{C}\mathrm{om}}(G)\downarrow P}\mathbf{K}\big(\overline{\mathsf{C}}_G^{+,0}(Q)\big);$$

this is (1) in (14.10). The inclusion $\overline{\mathsf{C}}_G^{+,0}(M) \to \overline{\mathsf{C}}_G^{+,0}(M)[\Gamma]$ induces the assembly map from the second row of (14.10)

$$\operatorname*{hocolim}_{\underline{\Gamma}}\mathbf{K}\big(\overline{\mathsf{C}}_G^{+,0}(M)\big) \to \mathbf{K}\big(\overline{\mathsf{C}}_G^{+,0}(M)[\Gamma]\big). \tag{14.11}$$

Lemma 14.12.

(a) The functor R induces an equivalence in K-theory, i.e., (2) in (14.10) *is an equivalence;*

(b) the functor h induces an equivalence, i.e., (1) in (14.10) *is an equivalence.*

Proof. (a) We will first show that for $\mathbf{B} \in \overline{\mathsf{C}}_G^{+,0}(P)$, there is $(\widetilde{\mathbf{B}}, p) \in \operatorname{Idem} \overline{\mathsf{C}}_G^{+,0}(M)$ such that $\mathbf{B} \cong R(\widetilde{\mathbf{B}}, p)$ in $\operatorname{Idem} \overline{\mathsf{C}}_G^{+,0}(P)$. Write $\mathbf{B} = (S, [\pi], B)$. Let $U_0 \subseteq U_1 \subset \dots$ be a sequence of compact open subgroups of G with $\operatorname{diam} U_i \to 0$ as $i \to \infty$. Write $\pi(s) := (g_1(s), \dots, g_n(s), t(s)) \in G^n \times \mathbb{N} = |P|^\wedge \times \mathbb{N}$. Set

$$K(s) := \operatorname{supp} B(s) \cap g_1(s) U_{t(s)} (g_1(s))^{-1} \cap \dots \cap g_n(s) U_{t(s)} (g_n(s))^{-1}.$$

Using (3.2d) we set $\widetilde{\mathbf{B}} := (s, \Pi, B|_{K(s)})$. By design $\widetilde{\mathbf{B}} \in \overline{\mathsf{C}}_G^{+,0}(P)$. Define $i \colon \mathbf{B} \to \widetilde{\mathbf{B}}$ and $r \colon \widetilde{\mathbf{B}} \to \mathbf{B}$ with

$$i_s^{s'} = \begin{cases} i_{B(s),K(s)} & s = s' \\ 0 & \text{else} \end{cases} \quad \text{and} \quad r_s^{s'} = \begin{cases} r_{B(s),K(s)} & s = s' \\ 0 & \text{else} \end{cases}.$$

Again by (3.2d) we have $r \circ i = \mathrm{id}_{\mathbf{B}}$. Thus $p := i \circ r$ is an idempotent on $\widetilde{\mathbf{B}}$ and $\mathbf{B} \cong R(\widetilde{\mathbf{B}}, p)$ as promised. It is an exercise in the definitions to check that R is bijective on morphism sets. Altogether, R induces an equivalence on idempotent completions and thus in K-theory.

(b) This follows from cofinality Lemma A.1 for homotopy colimits; we need to verify that h is right cofinal. Fix $f \colon Q \to P$ from $\mathsf{POr}_{\mathcal{C}\mathrm{om}}(G) \downarrow P$. We need to show that $(Q, f) \downarrow h$ is contractible. Objects in $(Q, f) \downarrow h$ are morphisms $Q \xrightarrow{g} M$ in $\mathsf{POr}_{\mathcal{C}\mathrm{om}}(G)$ such that $f = \pi \circ g$. Morphisms in $(Q, f) \downarrow h$ are commutative diagrams

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, "="] & M \arrow[r, "\pi"] \arrow[d, "\gamma"] & P \arrow[d, "="] \\ Q \arrow[r, "g'"] & M \arrow[r, "\pi"] & P \end{tikzcd}$$

We check that $(Q, f) \downarrow h$ is equivalent to a point. Recall $P = (G/V_1, \dots, G/V_n)$ and $M = (G/K_1, \dots, G/K_n)$, where K_i is the maximal compact open subgroup of V_i. Write $Q = (G/L_1, \dots, G/L_m)$ and $f = (u, \varphi)$ with $u \colon \{1, \dots, n\} \to \{1, \dots, m\}$ and $\varphi(i) \colon G/L_{u(i)} \to G/V_i$. Then there are $x_1, \dots, x_n \in G$ with $\varphi(i) y L_{u(i)} = y x_i V_i$ for all $y \in G$, and $x_i^{-1} L_{u(i)} x_i \subseteq V_i$. As $x_i^{-1} L_{u(i)} x_i$ is compact and K_i is the unique maximal compact subgroup, we have $x_i^{-1} L_{u(i)} x_i \subseteq K_i$. Let $g := (u, \psi) \colon Q \to M$ with $\psi(i) \colon G/L_{u(i)} \to G/K_i$ given by $\psi(i) y L_{u(i)} = y x_i K_i$ for all $y \in G$. Then $f = \pi \circ g$. Thus $(Q, f) \downarrow h$ is non-empty. If $g' = (u', \psi') \colon Q \to M$ is another map with $f = \pi \circ g'$, then $u = u'$ and there are $v_i \in V_i$ with $\psi'(i) y L_{u(i)} = y x_i v_i K_i$. Now the v_i define an isomorphism $\gamma \colon M \to M$ with $\gamma \circ g = g'$ and this γ is the only morphism with this property. Thus any two objects in $(Q, f) \downarrow h$ are uniquely isomorphic. $\square$

14.F. **The category $\overline{\mathsf{C}}^{+,0,\sharp}(M)$.** Here we discuss the equivalences (3), (4), (5) and (6) from (14.10).

Let $M = (G/K_1 \times \dots \times G/K_n)$ be as before. We give a slight variation of the categories $\mathsf{C}_G^{+,0}(M)$ and $\overline{\mathsf{C}}_G^{+,0}(M)$. The fact that $|M|^\wedge \to |M|$ has compact fibers (since $M \in P\mathsf{Or}_{\mathcal{C}\mathrm{om}}$) will allow us to base the definition on $|M|$ instead of $|M|^\wedge$. See also Remark 5.4.

Let μ_i be a Haar measure on K_i. We can integrate the left-invariant metric d_G on G to a left-invariant metric $d_{G,i}$ on G that is in addition right K-invariant, $d_{G,i}(g, g') := \int_{K_i} d_G(gk, g'k) d\mu_i(k)$ and obtain a left-invariant metric

$$d_{G/K_i}(gK_i, g'K_i) := \min_{k \in K} d_{G,i}(gk, g')$$

on G/K_i. We obtain a left-invariant metric $d_{|M|}$ on $|M|$ with

$$d_{|M|}\big((g_1K_1,\dots,g_nK_n),(g_1'K_1,\dots,g_n'K_n)\big) := \max_i d_{G/K_i}(g_iK_i,g_i'K_i).$$

We define the G-control structure $\mathfrak{C}^{+,0,\sharp}(M) = \big(\mathfrak{C}_1^{+,0,\sharp}(M),\mathfrak{C}_2^{+,0,\sharp}(M),\mathfrak{C}_G^{+,0,\sharp}(M)\big)$ on $|M| \times \mathbb{N}$ as follows:

- $\mathfrak{C}_1^{+,0,\sharp}(M)$ consists of all subsets F of $|M| \times \mathbb{N}$ for which $F \cap |M| \times \{t\}$ is finite for all $t \in \mathbb{N}$;
- $\mathfrak{C}_2^{+,0,\sharp}(M)$ consists of all subsets E of $\big(|M| \times \mathbb{N}\big)^{\times 2}$ satisfying the following two conditions
  - 0-*control over* $\mathbb{N}$: if $\left(\begin{smallmatrix}\lambda',t'\\ \lambda,t\end{smallmatrix}\right) \in E$ then $t = t'$;
  - *metric control over* $|M|$: for any $\eta > 0$ there is t_0 such that for all $t \geq t_0$ and all λ,λ' we have
  $$\left(\begin{smallmatrix}\lambda',t\\ \lambda,t\end{smallmatrix}\right) \in E \quad\implies\quad d_{|M|}(\lambda,\lambda') < \eta;$$
- $\mathfrak{C}_G^{+,0,\sharp}(M)$ consists of all subsets of G.

Let $\mathcal{Y}^\sharp(M)$ be the collection of all subsets Y of $|M| \times \mathbb{N}$, for which there is d with $Y \subseteq |M| \times \mathbb{N}_{\leq d}$. We define

$$\begin{aligned} \mathsf{C}^{+,0,\sharp}(M) &:= \mathcal{B}_G(\mathfrak{C}^{+,0,\sharp}(M),\mathcal{Y}^\sharp(M)); \\ \overline{\mathsf{C}}^{+,0,\sharp}(M) &:= \overline{\mathcal{B}}_G(\mathfrak{C}^{+,0,\sharp}(M),\mathcal{Y}^\sharp(M)). \end{aligned}$$

Let $p\colon |M|^\wedge \to |M|$ be the canonical projection. As the K_i are compact and therefore of finite diameter, it is not difficult to check that $\mathfrak{C}_2^{+,0,\sharp}(M)$ is exactly the image of $\mathfrak{C}_2^{+,0}(M)$ under $(p \times \mathrm{id}_\mathbb{N})^{\times 2}\colon (|M|^\wedge \times \mathbb{N})^{\times 2} \to (|M| \times \mathbb{N})^{\times 2}$. This in turn implies that $p \times \mathrm{id}$ induces an equivalence $\mathsf{C}_G^{+,0}(M) \xrightarrow{\sim} \mathsf{C}_G^{+,0,\sharp}(M)$. Applying (14.2) we obtain the equivalence $\mathsf{C}_G^{+,0}(M) \xrightarrow{\sim} \mathsf{C}_G^{+,0,\sharp}(M)$ and this yields the equivalences (3) and (4) from (14.10). Also from (14.2) we obtain an equivalence $\mathsf{C}_G^{+,0,\sharp}(M) \xrightarrow{\sim} \overline{\mathsf{C}}_G^{+,0,\sharp}(M)$ and this induces the equivalences (5) and (6) from (14.10).

14.G. **The limit category.** We briefly digress to recall a result from [4]. Consider a nested sequence of categories

$$\mathcal{C} = \mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \cdots .$$

Associated to this is the *sequence category* $\mathcal{S}(\mathcal{C}_*)$. This is the subcategory of $\prod_{m\geq 0}\mathcal{C}$, whose objects are sequences $(C_m)_{m\geq 0}$ such that for any l there is m_0 with $C_m \in \mathcal{C}_l$ for all $m \geq m_0$. Morphisms are sequences $(\varphi_m)_{m\geq 0}$ such that for any l there is m_0 with $\varphi_m \in \mathcal{C}_l$ for all $m \geq m_0$. The sum $\bigoplus_{m\geq 0}\mathcal{C}$ is a subcategory of $\mathcal{S}(\mathcal{C}_*)$ and we call the quotient category

$$\mathcal{L}(\mathcal{C}_*) := \mathcal{S}(\mathcal{C}_*)\Big/\bigoplus_{m\geq 0}\mathcal{C}_0$$

the *limit category*. We reviewed l-uniform regular coherence and exactness in Subsection 3.B. Assume now that

(14.13a) for any $d \in \mathbb{N}$ there is m_0 such that the inclusion $\mathcal{C}_{m+1}[\mathbb{Z}^d] \to \mathcal{C}_m[\mathbb{Z}^d]$ is exact for all $m \geq m_0$;

(14.13b) for any $d \in \mathbb{N}$ there are $m_0, l \in \mathbb{N}$ such that $\mathcal{C}_m[\mathbb{Z}^d]$ is l-regular coherent for all $m \geq m_0$.

Here we use the trivial action to form $\mathcal{C}_m[\mathbb{Z}^d]$. Then [4, Thm. 14.1] asserts the following. Consider an action of a finitely generated free abelian group Γ on $\mathcal{C}$ with the following property: for $\gamma \in \Gamma$, $j \in \mathbb{N}$ there is $i \in \mathbb{N}$ such that $\gamma(\mathcal{C}_i) \subseteq \mathcal{C}_j$. Then the assembly map

$$\operatorname*{hocolim}_{\underline{\Gamma}} \mathbf{K}\big(\mathcal{L}(\mathcal{C}_*)\big) \xrightarrow{\sim} \mathbf{K}\big(\mathcal{L}(\mathcal{C}_*)[\Gamma]\big) \tag{14.14}$$

is an equivalence. (In [4] the result is formulated using an equivalent homology theory instead of a homotopy colimit, but the two formulations are equivalent, see [22].)

14.H. **Analysis of $\mathsf{C}_G^{+,0,\sharp}(M)$.** Finally, we discuss the equivalence (7) from (14.10).

Fix again $M \in \mathsf{POr}_{\mathcal{C}\mathrm{om}}(G)$ and write $M = (G/K_1, \ldots, G/K_n)$ with $K_r \in \mathcal{C}\mathrm{om}$. Using Lemma 2.3 we fix a choice of compact open subgroups $(U_{r,i})_{r=1,\ldots,n, i\in\mathbb{N}_{\geq 1}}$ of G such that for each $r = 1, \ldots, n$ we have

$$U_{r,1} \supseteq U_{r,2} \supseteq \ldots \ \supseteq \bigcap_i U_{r,i} = K_r.$$

Set $U_{r,0} := G$[65] and $Q_i := (G/U_{1,i}, \ldots, G/U_{n,i})$. We write $p^i \colon |M| \to |Q_i|$ and, for $j \geq i$, $p^i_j \colon |Q_j| \to |Q_i|$ for the canonical projections. We define the G-control structure $\mathfrak{E}^i = (\mathfrak{E}^i_1, \mathfrak{E}^i_2, \mathfrak{E}^i_G)$ on $|M|$, where $\mathfrak{E}^i_1$ is the collection of all finite subsets of $|M|$, $\mathfrak{E}^i_2$ consists of all $E \subseteq |M| \times |M|$ satisfying

$$\left(\begin{smallmatrix} \lambda' \\ \lambda \end{smallmatrix}\right) \in E \implies p^i(\lambda) = p^i(\lambda'), \tag{14.15}$$

and $\mathfrak{E}^i_G$ is the collection of all subsets of G. We abbreviate $\mathcal{C}_i := \mathcal{B}_G(\mathfrak{E}^i)$ and obtain a nested sequence of categories $\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \ldots$. The metric control condition for $E \in \widetilde{\mathfrak{E}}^0_2(M)$ is equivalent to

- for any i there is t_0 such that for all $t \geq t_0$ and all λ, λ' we have
$$\left(\begin{smallmatrix} \lambda', t \\ \lambda, t \end{smallmatrix}\right) \in E \implies p^i(\lambda) = p^i(\lambda').$$

From this observation it is easy to deduce that there is an equivalence $\mathsf{C}_G^{+,0,\sharp}(M) \xrightarrow{\sim} \mathcal{L}(\mathcal{C}_*)$ that sends an object $\mathbf{B} = (S, \pi, B)$ to the sequence $(S_i, \pi_i, B_i)_{i\in\mathbb{N}}$, where $S_i = \pi^{-1}(|M| \times \{i\})$, $B_i = B|_{S_i}$ and π_i is the composition

$$S_i \xrightarrow{\pi|_{S_i}} |M| \times \{i\} \xrightarrow{\equiv} |M|.$$

We point out that for the equivalence $\mathsf{C}_G^{+,0,\sharp}(M) \xrightarrow{\sim} \mathcal{L}(\mathcal{C}_*)$ it is important that we work with $\mathfrak{C}^+_G(P) = \mathfrak{C}^{+,0,\sharp}_G(M) =$ All subsets of G; if one were to use only compact subsets of G instead, then the limit category would be strictly bigger.

Let now $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be as in Subsection 14.E. The group Γ_r does not necessarily normalize $U_{r,i}$, but for each $\gamma \in \Gamma_r$ we still have $\bigcap_i (U_{r,i})^\gamma = (K_r)^\gamma = K_r$. Using the compactness of the $U_{r,i}$ for $i > 0$, it is easy to check that for fixed γ and j there is i with $(U_{r,i})^\gamma \subseteq U_{r,j}$. This implies that the action of Γ on M induces an action acts on the nested sequence C_* as in Subsection 14.G. Moreover $\mathsf{C}_G^{+,0,\sharp}(M) \to \mathcal{L}(\mathcal{C}_*)$ is Γ-equivariant and an equivalence. It induces therefore an equivalence $\mathsf{C}_G^{+,0,\sharp}(M)[\Gamma] \to \mathcal{L}(\mathcal{C}_*)[\Gamma]$. It follows that (7) from (14.10) is an equivalence, if and only if (14.14) is an equivalence for our present choice of $\mathcal{C}_*$.

Lemma 14.16. *The nested sequence $\mathcal{C}_*$ defined above satisfies (14.13a). If $\mathcal{B}$ satisfies (Reg) from Definition 3.11, then $\mathcal{C}_*$ also satisfies (14.13b).*

Proof. We start by examining $\mathcal{C}_i = \mathcal{B}_G(\mathfrak{E}^i)$ for $i > 0$. For $\lambda \in |Q_i|$ let G_λ be the isotropy group of λ. This is a finite intersection of conjugates of the $U_{i,r}$ and thus compact open in G. Let $\mathbf{B} = (S, \pi, B) \in \mathcal{C}_i$ and $s \in S$. The control condition (14.15)

[65] This ensure that $\mathcal{C}_0$ defined later carries a Γ-action.

implies $\operatorname{supp} B(s) \subseteq G_{p_i(\pi(s))}$, in other words $B(s) \in \mathcal{B}|_{G_{p_i(\pi(s))}}$. For $\lambda \in |Q_i|$ we obtain, still using (14.15), a functor

$$\begin{array}{rcll} F_{i,\lambda}\colon & \mathcal{C}_i & \to & (\mathcal{B}|_{G_\lambda})_\oplus \\ & \mathbf{B} & \mapsto & \displaystyle\bigoplus_{p_i(\pi(s))=\lambda} B(s). \end{array}$$

By definition of $\mathfrak{E}_1^i$ the set S is finite and so the above sum is finite and $F_{i,\lambda}(\mathbf{B}) = 0$ for all but finitely many λ. The $F_{i,\lambda}$ combine to an equivalence

$$\begin{array}{rcl} \mathcal{C}_i & \xrightarrow{\sim} & \bigoplus_{\lambda \in |Q_i|} (\mathcal{B}|_{G_\lambda})_\oplus \\ \mathbf{B} & \mapsto & (F_{i,\lambda}(\mathbf{B}))_{\lambda \in |Q_i|}. \end{array} \tag{14.17}$$

Since the property l-uniformly regular passes to direct sums over arbitrary index sets and is invariant under equivalence of additive categories and the passage going from an additive category $\mathcal{A}$ to $\mathcal{A}[\mathbb{Z}^d]$ is compatible with infinite direct sums over arbitrary index sets, (Reg) from Definition 3.11 implies (14.13b).

For $j > i$ and $\lambda \in |Q_j|$ we have $G_\lambda \subseteq G_{p_j^i(\lambda)}$ as p_j^i is G-equivariant. The inclusion $\mathcal{C}_j \subseteq \mathcal{C}_i$ corresponds under (14.17) to the functor

$$\bigoplus_{\lambda \in |Q_j|} (\mathcal{B}|_{G_\lambda})_\oplus \to \bigoplus_{\kappa \in |Q_i|} (\mathcal{B}|_{G_\kappa})_\oplus$$

that sends the λ-summand into the κ-summand by the functor induced from the inclusion $G_\lambda \subseteq G_\kappa$ where $\kappa = p_j^i(\lambda)$. By [7, Lem. 7.51] each of these is exact. This implies (14.13a). ☐

Lemma 14.16 allows us to apply [4, Thm. 14.1] as reviewed in Subsection 14.G to the nested sequence constructed above. So (14.14) and therefore also (7) appearing in (14.10) is an equivalence.

Formal conclusion of the proof of Theorem 14.7 and Theorem 14.1. Altogether we have shown that the top horizontal map in (14.10) is an equivalence. As noted before, because of the pushout from Proposition 14.9, this also implies that (14.8) is an equivalence as claimed in Theorem 14.7. We have already explained in Subsection 14.B that Theorem 14.1 follows from Theorem 14.7. ☐

Remark 14.18. At least in the case $M \in \mathsf{POr}_{\mathcal{C}\mathrm{om}}(G)$ we can give an explanation of the value of $\mathbf{K}(\mathsf{C}_G(M))$. Write $M = (G/K_1, \dots, G/K_n)$ with $K_r \in \mathcal{C}\mathrm{om}$ and fix a choice of compact open subgroups $(U_{r,i})_{r=1,\dots,n, i \in \mathbb{N}_{\geq 1}}$ of G such that for each $r = 1, \dots, n$ we have $U_{r,1} \supseteq U_{r,2} \supseteq \dots \quad \supseteq \bigcap_i U_{r,i} = K_r$. Put $Q_i = (G/K_{1,i}, \dots, G/K_{n,i})$ for $i \in \mathbb{N}_{\geq 1}$.

Then there is zigzag of weak homotopy equivalences from $\mathbf{K}(\mathsf{C}_G(M))$ to the homotopy inverse limit $\operatorname{holim}_{i\to\infty} \mathbf{K}(\mathsf{C}_G(M_i))$. In particular there is for every $n \in \mathbb{Z}$ a short exact sequence

$$\begin{aligned} 0 \to \operatorname{invlim}^1_{i\to\infty} \pi_{n+1}\big(\mathbf{K}(\mathsf{C}_G(Q_i))\big) &\to \pi_n\big(\mathbf{K}(\mathsf{C}_G(M))\big) \\ &\to \operatorname{invlim}_{i\to\infty} \pi_n\big(\mathbf{K}(\mathsf{C}_G(Q_i))\big) \to 0. \end{aligned}$$

Since we do not need this result in this paper, we omit its proof. It is very unlikely that the corresponding statement holds, if we drop the assumption that each K_i is compact.

Appendix A. Homotopy colimits

A.i. **Cofinality and transitivity.** We recall two basic facts about homotopy colimits [25, §9].

Let $F\colon \mathcal{A} \to \mathcal{B}$ be functor. Let $B \in \mathcal{B}$. We write $B \downarrow F$ for the following category. Objects are pairs (A, β) with $A \in \mathcal{A}$ and $\beta\colon B \to F(A)$ in $\mathcal{B}$. Morphisms $(A,\beta) \to (A',\beta')$ are morphisms $\alpha\colon A \to A'$ in $\mathcal{A}$ with $F(\alpha) \circ \beta = \beta'$. We write $F \downarrow B$ for the following category. Objects are pairs (A,β) with $A \in \mathcal{A}$ and $\beta\colon F(A) \to B$ in $\mathcal{B}$. Morphisms $(A,\beta) \to (A',\beta')$ are morphisms $\alpha\colon A \to A'$ in $\mathcal{A}$ with $\beta \circ F(\alpha) = \beta'$. If F is the inclusion of the subcategory $\mathcal{A}$, then we write $B \downarrow \mathcal{A}$ and $\mathcal{A} \downarrow B$ instead of $B \downarrow F$ and $F \downarrow B$.

A functor $F\colon \mathcal{A} \to \mathcal{B}$ between small categories is said to be *right cofinal*, if the nerve of $B \downarrow F$ is contractible (and in particular non-empty) for every $B \in \mathcal{B}$.

Lemma A.1 (Cofinality)**.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right cofinal functor between small categories. Then for any $\mathbf{D}\colon \mathcal{B} \to \mathsf{Spectra}$ the canonical map*

$$\operatorname*{hocolim}_{\mathcal{A}} F^*\mathbf{D} \xrightarrow{\sim} \operatorname*{hocolim}_{\mathcal{B}} \mathbf{D}$$

is an equivalence.

Proof. This is [25, 9.4]. □

Lemma A.2 (Transitivity)**.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between small categories. Consider functors $\mathbf{D}\colon \mathcal{A} \to \mathsf{Spectra}$, $\mathbf{E}\colon \mathcal{B} \to \mathsf{Spectra}$ and a natural transformation $\tau\colon \mathbf{D} \to F^*\mathbf{E}$. For $B \in \mathcal{B}$ write $v_B\colon F \downarrow B \to \mathcal{A}$ for the forgetful functor. Suppose that for any $B \in \mathcal{B}$ the canonical map*

$$\operatorname*{hocolim}_{F\downarrow B} v_B^*\mathbf{D} \xrightarrow{\sim} \mathbf{E}(B)$$

is an equivalence. Then the canonical map

$$\operatorname*{hocolim}_{\mathcal{A}} \mathbf{D} \xrightarrow{\sim} \operatorname*{hocolim}_{\mathcal{B}} \mathbf{E}$$

is an equivalence as well.

Proof. The assumption implies that $\mathbf{E}$ is the homotopy push down $F_*\mathbf{D}$ of $\mathbf{D}$ along F. The assertion follows from [25, 9.4]. □

A.ii. **Homotopy colimits and categories with product.** Let $\mathcal{P}$ be a small category with all non-empty finite products. Let $\mathcal{P}_+$ be the category obtained from $\mathcal{P}$ by adding a terminal object $*$. For example for $\mathcal{P} = \mathsf{P}\mathcal{A}$, we have $\mathcal{P}_+ = \mathsf{P}_+\mathcal{A}$. Let $\mathbf{D}\colon \mathcal{P}_+ \to \mathsf{Spectra}$ be a functor. The unique maps $P \to *$ induce the canonical map

$$\operatorname*{hocolim}_{P\in\mathcal{P}} \mathbf{D}(P) \to \mathbf{D}(*).$$

Lemma A.3. *Fix $Q \in \mathcal{P}$. Then the canonical map*

$$\operatorname*{hocolim}_{P\in\mathcal{P}} \mathbf{D}(P \times Q) \to \mathbf{D}(Q)$$

is an equivalence.

Proof. Consider

$$\mathcal{P} \xrightarrow{f_Q} \big(\mathcal{P} \downarrow Q\big) \xrightarrow{v_Q} \mathcal{P} \xrightarrow{\mathbf{D}} \mathsf{Spectra}$$

where f_Q sends P to the canonical projection $P \times Q \to Q$ and v_Q is the forgetful functor. It is easy to verify that f_Q is right cofinal. Obviously (Q, id_Q) is terminal in $\mathcal{P} \downarrow Q$. Thus, using Lemma A.1,

$$\operatorname*{hocolim}_{P\in\mathcal{P}} \mathbf{D}(P\times Q) = \operatorname*{hocolim}_{\mathcal{P}} f_Q^* v_Q^*\mathbf{D} \xrightarrow{\sim} \operatorname*{hocolim}_{\mathcal{P}\downarrow Q} v_Q^*\mathbf{D} \xrightarrow{\sim} v_Q^*\mathbf{D}(Q, \mathrm{id}_Q) = \mathbf{D}(Q).$$

□

Proposition A.4. *Let $\mathcal{A}$ be a small category. Let $\mathcal{E}$ be the smallest collection of functors $\mathbf{E}\colon \mathcal{P}_+\mathcal{A} \to \mathsf{Spectra}$ that is closed under hocolimits, retracts and contains all functors of the form $P \mapsto \mathbf{D}(P \times Q)$ with $Q \in \mathcal{P}$ and $\mathbf{D}\colon \mathcal{P} \to \mathsf{Spectra}$. Then for any $\mathbf{E} \in \mathcal{E}$ the canonical map*

$$\operatorname*{hocolim}_{P \in \mathcal{P}} \mathbf{E}(P) \xrightarrow{\sim} \mathbf{E}(*).$$

is an equivalence.

Proof. Homotopy colimits and retracts of equivalences are again equivalences. The result follows thus from Lemma A.3. □

Lemma A.5. *The nerv of $\mathcal{P}$ is contractible (provided that $\mathcal{P}$ is non-empty).*

Proof. Let Q be a fixed object of $\mathcal{P}$. Let $c_Q, x_Q\colon \mathcal{P} \to \mathcal{P}$ be the functors $c_Q(P) = Q$, $c_Q(\varphi) = \mathrm{id}_Q$, $x_Q(P) = P \times Q$, $x_Q(\varphi) = (\varphi \times \mathrm{id}_Q)$. There are evident natural transformations $\mathrm{id}_{\mathcal{P}} \leftarrow x_Q \to c_Q$. On the nerv $\mathrm{id}_{\mathcal{P}}$ induces the identity, c_Q induces a constant map, and the two maps are homotopic. □

Lemma A.6. *Let $\mathcal{Q}$ be a further category with all non-empty finite products. Let $F\colon \mathcal{Q} \to \mathcal{P}$ be a product preserving functor. Assume that $P \downarrow F$ is non-empty for all $P \in \mathcal{P}$. Then F is right cofinal. In particular, for any $\mathbf{D}\colon \mathcal{P} \to \mathsf{Spectra}$ the canonical map*

$$\operatorname*{hocolim}_{\mathcal{Q}} F^*\mathbf{D} \xrightarrow{\sim} \operatorname*{hocolim}_{\mathcal{P}} \mathbf{D}$$

is an equivalence.

Proof. Using that $\mathcal{Q}$ has all non-empty finite products and that F preserves finite products, it is not difficult to check that $P \downarrow F$ also has all non-empty finite products. Lemma A.5 implies that F is right cofinal. The statement about homotopy colimits follows from Lemma A.1. □

Appendix B. K-theory of dg-categories

In order to obtain induced maps in K-theory for homotopy coherent functors we use K-theory for dg-categories. We briefly review its construction. We write $\mathsf{Cat}_{\mathrm{dg}}$ for the category of small dg-categories [36]. For a dg-category $\mathcal{C}$ and objects $C, C' \in \mathcal{C}$, we write $\mathcal{C}(C, C')$ for the chain complex of morphisms from C to C'. For a dg-category $\mathcal{C}$ the category $H_0(\mathcal{C})$ has the same objects as $\mathcal{C}$ and for $C, C' \in \mathcal{C}$ the set of morphisms $C \to C'$ in $H_0(\mathcal{C})$ is given by $H_0(\mathcal{C}(C, C'))$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ of dg-categories is a *quasi-equivalence* if

$$F_*\colon \mathcal{C}(C, C') \to \mathcal{D}(F(C), F(C'))$$

is a quasi-isomorphism for all C, C', and it induces an equivalence $H_0(\mathcal{C}) \to H_0(\mathcal{C})$.

Theorem B.1 ([19, 36, 48])**.** *There exists a functor*

$$\mathbf{K}\colon \mathsf{Cat}_{\mathrm{dg}} \to \mathsf{Spectra}$$

with the following properties:

- *(a) Restricted to additive categories $\mathbf{K}$ is weakly equivalent to the usual (non-connective) K-theory functor;*
- *(b) For an additive category $\mathcal{A}$ the inclusion of $\mathcal{A}$ into the category $\mathrm{ch}_{\mathrm{fin}}\,\mathcal{A}$ of finite chain complexes induces a weak equivalence $\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathrm{ch}_{\mathrm{fin}}(\mathcal{A}))$;*
- *(c) If $F\colon \mathcal{C} \to \mathcal{C}'$ in $\mathsf{Cat}_{\mathrm{dg}}$ is a quasi-equivalence, then $\mathbf{K}(F)$ is a weak equivalence.*

Sketch of proof. K-theory of a dg-category $\mathcal{C}$ can be defined as the Waldhausen K-theory of the category of perfect $\mathcal{C}$-modules [36, Sec. 5.2] and [19].

We review the construction of Schlichting [48, Sec. 6.4][66]. Let $\mathcal{C}$ be a dg-category. A $\mathcal{C}$-module is a dg-functor $M\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{ch}(\mathbb{Z}\text{-}\mathsf{Mod})$. A map of $\mathcal{C}$-modules $M \to M'$ is a natural transformation by chain maps. A $\mathcal{C}$-module is free on $v \in M(C)_k$ if $f \mapsto f^*(v)$ defines an isomorphism $\mathcal{C}(-,C)[k] \to M(-)$ of $\mathcal{C}$-modules. A finite cell $\mathcal{C}$-module is a $\mathcal{C}$-module M together with a finite filtration $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ such that the quotients M_i/M_{i-1} are free on a finite number of generators. We write $\mathcal{C}\text{-}\mathsf{cell}$ for the category of finite cell $\mathcal{C}$-modules. If $\mathcal{A}$ is an additive category, then the category of finite cell $\mathcal{A}$-modules can be identified with the dg-category $\mathrm{ch}_{\mathrm{fin}}(\mathcal{A})$ of finite chain complexes over $\mathcal{A}$. For $\mathcal{C} = \mathrm{ch}_{\mathrm{fin}}(\mathcal{A})$ the Yoneda embedding of $\mathrm{ch}_{\mathrm{fin}}(\mathcal{A})$ into the category of $\mathrm{ch}_{\mathrm{fin}}(\mathcal{A})$-modules identifies $\mathrm{ch}_{\mathrm{fin}}(\mathcal{A})$ with $\mathrm{ch}_{\mathrm{fin}}(\mathcal{A})$-$\mathsf{cell}$. A sequence

$$M \xrightarrow{i} M' \xrightarrow{p} M''$$

in $\mathcal{C}\text{-}\mathsf{cell}$ is said to be *exact*, if $M(C) \to M'(C) \to M''(C)$ is exact in $\mathrm{ch}(\mathbb{Z})$ for each $C \in \mathcal{C}$. With this notion of exact sequences $\mathcal{C}\text{-}\mathsf{cell}$ is an exact category.

The exact category $\mathcal{C}\text{-}\mathsf{cell}$ has enough injectives and projectives.[67] The classes of projectives and injectives in this exact category coincide and are given by the contractible cell modules. A map $\mathcal{C}\text{-}\mathsf{cell}$ factors over an injective (projective), if and only if it chain nullhomotopic. (All this can be proven using cones of $\mathcal{C}$-modules exactly as in [48, Sec.6.4].)

A *Frobenius category* is an exact category $\mathcal{E}$ with enough projectives and injectives such that the classes of injectives and projectives coincide. The associated stable category is the quotient of $\mathcal{E}$ by the subcategory of projectives (injectives), i.e., morphisms are identified if they factor over a projective (injective). A map of Frobenius categories is an exact functor that preserves projectives (injectives). The category of cell modules $\mathcal{C}\text{-}\mathsf{cell}$ is a Frobenius category and the associated stable category is the homotopy category of $\mathcal{C}\text{-}\mathsf{cell}$.

Schlichting's [48, Sec. 12.1] non-connective K-theory functor

$$\mathbf{K}^{\mathrm{Frob}}\colon \text{Frobenius categories} \to \mathsf{Spectra}$$

has the following property: if a map $\mathcal{E} \to \mathcal{E}'$ of Frobenius categories induces an equivalence of stable categories, then $\mathbf{K}^{\mathrm{Frob}}(\mathcal{E}) \to \mathbf{K}^{\mathrm{Frob}}(\mathcal{E})$ is an equivalence [48, Cor. 1].

For a dg-category $\mathcal{C}$ one then defines

$$\mathbf{K}(\mathcal{C}) := \mathbf{K}^{\mathrm{Frob}}(\mathcal{C}\text{-}\mathsf{cell}).$$

For an additive category $\mathcal{A}$ we have $\mathcal{A}\text{-}\mathsf{cell} \simeq \mathrm{ch}_{\mathrm{fin}}(\mathcal{A})$, see above. Now (a) follows from [48, Thm. 5]. For (b) we can use that for an additive category $\mathcal{A}$ the inclusion $\mathcal{A} \to \mathrm{ch}_{fin}(\mathcal{A})$ induces an equivalence on cell modules and therefore in K-theory. For (c) we use that for a quasi-equivalence $\mathcal{C} \to \mathcal{C}'$ the induced functor on the stable categories associated to the categories of cell modules (i.e., on the homotopy categories of cell modules) is an equivalence. $\square$

We record the following consequence of the quasi-isomorphism invariance of K-theory.

[66] Strictly speaking Schlichting only considers dgas, not dg-categories, but his construction generalizes in a straight forward manner.

[67] I is injective (projective) in an exact category $\mathcal{E}$ if for any exact sequence $A \xrightarrow{i} B \xrightarrow{p} C$ the map $\mathcal{E}(B,I) \xrightarrow{i^*} \mathcal{E}(A,I)$ (the map $\mathcal{E}(P,B) \xrightarrow{p_*} \mathcal{E}(P,C)$) is onto [35, Sec. 5].

Corollary B.2. *Let $F, F' \colon \mathcal{C} \to \mathcal{D}$ be* dg*-functors and let $\tau \colon F \to F'$ be a natural transformation of* dg*-functors. Assume that $H_0(\tau) \colon H_0(F) \to H_0(F')$ is an isomorphism between the functors $H_0(F), H_0(F') \colon H_0(\mathcal{C}) \to H_0(\mathcal{D})$. Then $\mathbf{K}(F)$ and $\mathbf{K}(F')$ are homotopic.*

Proof. This follows from Theorem B.1 (c) using the path dg-category $P(\mathcal{D})$ associated to $\mathcal{D}$ (see [24, 2.9] and [50, Def. 4.1]) by a standard argument.

Objects of $P(\mathcal{D})$ are diagrams $D \xrightarrow{f} D'$ in $\mathcal{D}$ with f closed of degree 0 whose homology class is an isomorphism in $H_0(\mathcal{D})$. Given objects $D_0 \xrightarrow{f_0} D_0'$, $D_1 \xrightarrow{f_1} D_1'$, the complex $P(\mathcal{D})(D_0 \xrightarrow{f_0} D_0', D_1 \xrightarrow{f_1} D_1')$ is the homotopy fiber of

$$\begin{array}{rcl} \mathcal{D}(D_0, D_0') \oplus \mathcal{D}(D_1, D_1') & \to & \mathcal{D}(D_0, D_1') \\ (\varphi, \varphi') & \mapsto & f_1 \circ \varphi - \varphi' \circ f_0. \end{array}$$

Composition is defined by the formula

$$(\varphi_1, \varphi_1', s_1) \circ (\varphi_0, \varphi_0', s_0) := (\varphi_1 \circ \varphi_0, \varphi_1' \circ \varphi_0', \varphi_1' \circ s_0 + s_1 \circ \varphi_0).$$

There are dg-functors $p, p' \colon P(\mathcal{D}) \to \mathcal{D}$ with $p(D \xrightarrow{f} D') = D$, $p'(D \xrightarrow{f} D') = D'$. We also have $i \colon \mathcal{D} \to P(\mathcal{D})$ with $i(D) = (D \xrightarrow{\mathrm{id}_D} D)$ and $\Psi \colon \mathcal{C} \to P(\mathcal{D})$ with $\Psi(C) = (F(C) \xrightarrow{\tau_C} F'(C))$. Then $p \circ i = p' \circ i = \mathrm{id}_{\mathcal{D}}$. Moreover, i is a quasi-isomorphism [50, Lem. 4.3]. Using Theorem B.1 (c) this implies $\mathbf{K}(p) \simeq \mathbf{K}(p')$. We also have $F = p \circ \Psi$ and $F' = p' \circ \Psi$. Thus $\mathbf{K}(F) \simeq \mathbf{K}(F')$. □

Appendix C. Homotopy coherent functors

Our construction of the transfer depends on induced maps in K-theory for homotopy coherent functors. Here we give a self contained construction of such induced maps that is tailored to our specific needs. It is very pedestrian, by no means a complete theory, and makes no direct contact with the elegant language of stable ∞-categories. We include this for completeness.

Definition C.1. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. A *homotopy coherent functor* $F = (F^0, F^1, \dots) \colon \mathcal{A} \to \operatorname{ch} \mathcal{B}$ consists of the following data.

(C.1a) For any object $A \in \mathcal{A}$ an object $F^0(A) \in \operatorname{ch} \mathcal{B}$;

(C.1b) For any chain

$$A_0 \xleftarrow{\varphi_1} A_1 \xleftarrow{\varphi_2} \dots \xleftarrow{\varphi_n} A_n$$

of morphisms in $\mathcal{A}$ a map $F^n(\varphi_1, \dots, \varphi_n) \colon F^0(A_n) \to F^0(A_0)$ of degree $n-1$.

F is required to satisfy

$$dF^n(\varphi_1, \dots, \varphi_n) = \sum_{j=1}^{n-1} (-1)^j \Big(F^{n-1}(\varphi_1, \dots, \varphi_j \circ \varphi_{j+1}, \dots \varphi_n) \\ - F^j(\varphi_1, \dots, \varphi_j) \circ F^{n-j}(\varphi_{j+1}, \dots, \varphi_n) \Big).$$

Here d is the differential in $\hom_{\operatorname{ch} \mathcal{B}}(F(A_n), F(A_0))$[68]. The F^n are also be required to be multi-linear in the φ_i.

Example C.2. Every functor $f \colon \mathcal{A} \to \operatorname{ch} \mathcal{B}$ can be viewed as a homotopy coherent functor F with $F^0(A) = f(A)$, $F^1(\varphi) = f(\varphi)$, $F^n \equiv 0$ for $n \geq 2$.

[68]We use $d(\psi) = d_{B'} \circ \psi - (-1)^{|\psi|} \psi \circ d_B$ for $\psi \colon B \to B'$ in $\operatorname{ch} \mathcal{B}$ of degree $|\psi|$.

Example C.3 (Deformation of a functor to a homotopy coherent functor)**.** Let $f\colon \mathcal{A} \to \operatorname{ch}\mathcal{B}$ be a $\mathbb{Z}$-linear functor. Suppose we are given for each $A \in \mathcal{A}$ a diagram in $\operatorname{ch}\mathcal{B}$

$$B_A \xrightarrow{i_A} f(A) \xrightarrow{r_A} B_A$$

with $r_A \circ i_A = \mathrm{id}_{B_A}$ and a chain homotopy $H_A\colon i_A \circ r_A \simeq \mathrm{id}_{f(A)}$. Then we obtain a homotopy coherent functor $F = (F^0, F^1, \dots)$ with $F(A) = B_A$ for $A \in \mathcal{A}$ and

$$F^n(\varphi_1, \dots, \varphi_n) = \\ r_{A_0} \circ f(\varphi_1) \circ H_{A_1} \circ f(\varphi_2) \circ \cdots \circ f(\varphi_{n-1}) \circ H_{A_{n-1}} \circ f(\varphi_n) \circ i_{A_n}$$

for

$$A_0 \xleftarrow{\varphi_1} A_1 \xleftarrow{\varphi_2} \dots \xleftarrow{\varphi_n} A_n$$

in $\mathcal{A}$.

Definition C.4. Let $F, G\colon \mathcal{A} \to \operatorname{ch}\mathcal{B}$ be homotopy coherent functors. A strict natural transformation $\tau\colon F \to G$ consists of chain maps $\tau_A\colon F^0(A) \to G^0(A)$ for all $A \in \mathcal{A}$ such that for any chain

$$A_0 \xleftarrow{\varphi_1} A_1 \xleftarrow{\varphi_2} \dots \xleftarrow{\varphi_n} A_n$$

of morphisms in $\mathcal{A}$ the diagram

$$\begin{tikzcd} F(A_n) \arrow[r, "\tau_{A_n}"] \arrow[d, "F^n(\varphi_1,\dots,\varphi_n)"'] & G(A_n) \arrow[d, "G^n(\varphi_1,\dots,\varphi_n)"] \\ F(A_0) \arrow[r, "\tau_{A_0}"] & G(A_0) \end{tikzcd}$$

commutes.

Example C.5. Let $f\colon \mathcal{A} \to \operatorname{ch}\mathcal{B}$,

$$B_A \xrightarrow{i_A} f(A) \xrightarrow{r_A} B_A$$

and H_A be as in Example C.3. Let $F\colon \mathcal{A} \to \operatorname{ch}\mathcal{B}$ be the homotopy coherent functor associated to this data.

Let $g\colon \mathcal{A} \to \operatorname{ch}\mathcal{B}$ a further functor, which we also view as a homotopy coherent functor, see Example C.2.

Let $\tau\colon f \to g$ be a natural transformation. Suppose we are given chain maps $p_A\colon B_A \to g(A)$ such that $\tau_A = p_A \circ r_A$, $p_A = \tau_A \circ i_A$, $0 = \tau_A \circ H_A$. Then p_A determines a strict natural transformation $F \to g$.

Definition C.6. We write Int for the following category. The objects of Int are linearly ordered sets of the form $[0, n] := \{0 < 1 < \cdots < n\}$ with $n \in \mathbb{N}_{>0}$. Maps $[0, n] \to [0, m]$ are strictly order preserving maps $\sigma\colon \{0 < \cdots < n\} \to \{0 < \cdots < m\}$ with $\sigma(0) = 0$, $\sigma(n) = \sigma(m)$. In particular σ is injective and $n \le m$.

Remark C.7. There is an identification $\Delta_{\mathrm{inj}}^{\mathrm{op}} \cong \mathsf{Int}$. Here Δ is the usual simplicial category of finite ordered sets of the form $\{0 < \cdots < n\}$ and Δ_{inj} is the subcategory obtained by restricting to injective maps.

Remark C.8. Often we will identify $[0, n] \in \mathsf{Int}$ with the category associated to $[0, n]$ as a poset. Note that a composable chain of morphisms

$$A_0 \xleftarrow{\varphi_1} A_1 \xleftarrow{\varphi_2} \dots \xleftarrow{\varphi_n} A_n$$

in $\mathcal{A}$ is the same thing as a functor $[0, n]^{\mathrm{op}} \to \mathcal{A}$.

Moreover, any finite linearly order set I with at least two elements is canonically isomorphic to some $[0, n]$. In particular, we can evaluate any homotopy coherent functor $F\colon \mathcal{A} \to \operatorname{ch}\mathcal{B}$ on any functor $I^{\mathrm{op}} \to \mathcal{A}$.

Definition C.9. For $[0,n] \in \operatorname{ob}\mathsf{Int}$ we define the chain complex $C_*([0,n])$ as follows. As an abelian group $C_*([0,n])$ is generated by pairs (I,J) with $\{0<n\} \subseteq J \subseteq I \subseteq [0,n]$. The degree of (I,J) is the cardinality $|I \setminus J|$ of $I \setminus J$. The boundary d is determined as follows. Write $I \setminus J = \{i_1 < \cdots < i_l\}$. Then

$$(I,J) \mapsto \sum_{j=0}^{l} (-1)^j \Big((I \setminus \{i_j\}, J) - (I, J \cup \{i_j\}) \Big).$$

For $\sigma\colon [0,n] \to [0,m]$ we define $\sigma_*\colon C_*([0,n]) \to C_*([0,m])$ by

$$\sigma_*(I,J) := \begin{cases} (\sigma(I), \sigma(J)) & |I \setminus J| = |\sigma(I) \setminus \sigma(J)| \\ 0 & \text{else} \end{cases}.$$

Altogether we obtain the functor[69]

$$C_*\colon \mathsf{Int} \to \operatorname{ch}(\mathbb{Z}\text{-}\mathsf{Mod}).$$

Remark C.10. The chain complex $C_*([0,n])$ can also be described as the cellular chain complex of the $(n\text{-})1$-dimensional cube.

Remark C.11. In the category of functors $\mathsf{Int} \to \mathbb{Z}\text{-}\mathsf{Mod}$ the complex $C_*(-)$ is degreewise free; we have

$$C_k(-) \cong \bigoplus_{i=1}^{\infty} \bigoplus_{\substack{\{0<i\} \subseteq J \subseteq [0,i], \\ |[0,i] \setminus J| = k}} \mathbb{Z}\mathsf{Int}(J,-).$$

Remark C.12. There is a concatenation map

$$C_*([0,n]) \otimes C_*([0,m]) \to C_*([0,n+m]) \tag{C.13}$$

defined by $(I,J) \otimes (I',J') \mapsto (I \cup n+I', J \cup n+J')$.

Lemma C.14. *The augmentation map $\epsilon\colon C_*([0,n]) \to \mathbb{Z}$ with $\epsilon(I,I) = 1$ is a homology isomorphism.*

Proof. We proceed by induction on n. For $n=1$ the augmentation is an isomorphism. For $n>1$ we define $\tilde{C}_*([0,n])$ as the sub-complex of $C_*([0,n])$ spanned by all (I,J) with $1 \notin I$. Clearly $\tilde{C}_*([0,n]) \cong C_*([0,n-1])$. The map defined by

$$(I,J) \mapsto \begin{cases} (I, J \cup \{1\}) & 1 \in I \setminus J \\ 0 & \text{else} \end{cases}$$

induces a contraction on the quotient $C_*([0,n])/\tilde{C}_*([0,n])$. Thus the inclusion $\tilde{C}_*([0,n]) \to C_*([0,n])$ induces an isomorphism in homology. $\square$

In the following we abbreviate $\mathcal{A}(A,A') := \operatorname{mor}_{\mathcal{A}}(A,A')$.

Definition C.15. Let $\mathcal{A}$ be a small $\mathbb{Z}$-linear category. For $[0,n] \in \mathsf{Int}$ and $A, A' \in \operatorname{ob}\mathcal{A}$ we define

$$\mathcal{A}_{[0,n]}(A,A') := \bigoplus_{A_1,\ldots,A_{n-1} \in \mathcal{A}} \mathcal{A}(A_1,A') \otimes \mathcal{A}(A_2,A_1) \otimes \cdots \otimes \mathcal{A}(A,A_{n-1}).$$

There is an evident concatenation map

$$\mathcal{A}_{[0,n]}(A',A'') \otimes \mathcal{A}_{[0,m]}(A,A') \to \mathcal{A}_{[0,n+m]}(A,A''). \tag{C.16}$$

Using composition in $\mathcal{A}$, this construction is functorial in Int and we obtain for fixed $A, A' \in \mathcal{A}$ a contravariant functor

$$\mathcal{A}_-(A,A')\colon \mathsf{Int} \to \mathbb{Z}\text{-}\mathsf{Mod}; \quad [0,n] \mapsto \mathcal{A}_{[0,n]}(A,A').$$

---

[69] In the language of [22, end of Section 3] this is the covariant $\mathbb{Z}\mathsf{Int}$-chain complex $C_*(E^{\mathrm{bar}}\mathsf{Int})$ for $E^{\mathrm{bar}}\mathsf{Int}$ the bar-model for the covariant classifying Int-CW-complex of the category Int.

Definition C.17. Let $\mathcal{A}$ be a small $\mathbb{Z}$-linear category. The dg-category $\mathfrak{C}_\mathcal{A}$ is defined as follows. The objects of $\mathfrak{C}_\mathcal{A}$ are the objects of $\mathcal{A}$. For objects $A, A' \in \mathcal{A}$ the chain complex of morphisms is defined as

$$\mathfrak{C}_\mathcal{A}(A,A') := \mathcal{A}_-(A,A') \otimes_{\mathtt{Int}} C_*(-).$$

Here $\mathcal{A}_{[0,n]}(A,A')$ is viewed as a chain complex concentrated in degree 0. Composition is induced from the concatenation maps (C.13) and (C.16).

Construction C.18. We construct a functor $\mathrm{comp}\colon \mathfrak{C}_\mathcal{A} \to \mathcal{A}$ as follows. On objects the functor is the identity. Let $A, A' \in \mathcal{A}$. Then

$$\mathrm{comp}\colon \mathfrak{C}_\mathcal{A}(A,A') \to \mathcal{A}(A,A')$$

is defined as follows. For $f \in \big(\mathfrak{C}_\mathcal{A}(A,A')\big)_{>0}$ we set $\mathrm{comp}(\varphi) = 0$. Let $f \in \big(\mathfrak{C}_\mathcal{A}(A,A')\big)_0$. We can write $f = (\varphi_1, \ldots, \varphi_n) \otimes (I, I)$ with $n \in \mathbb{N}$ and $\{0,1\} \subseteq I \subseteq [0,n]$. Then $\mathrm{comp}(f) = \varphi_1 \circ \cdots \circ \varphi_n$.

Lemma C.19. *The functor* $\mathrm{comp}\colon \mathfrak{C}_\mathcal{A} \to \mathcal{A}$ *is a quasi-isomorphism.*

Proof. The objects of $\mathfrak{C}$ and $\mathcal{A}$ coincide. It remains to show that for all $A, A' \in \mathcal{A}$

$$\mathrm{comp}\colon \mathfrak{C}_\mathcal{A}(A,A') \to \mathcal{A}(A,A')$$

is a chain homotopy equivalence. Here $\mathcal{A}(A,A')$ is viewed as chain complex concentrated in degree 0.

By Lemma C.14 the augmentation $\epsilon\colon C_*([0,n]) \to \mathbb{Z}$ is a homology isomorphism. As $\mathtt{Int}([0,1],[0,n])$ consists of a single morphism we obtain a homology isomorphism

$$C_*(-) \to \mathbb{Z}\big[\mathtt{Int}([0,1],-)\big],$$

of chain complexes in the category of functors $\mathtt{Int} \to \mathbb{Z}\text{-}\mathsf{Mod}$. As both are free, see Remark C.11, it is a chain homotopy equivalence. Thus

$$\begin{aligned}
\mathfrak{C}_\mathcal{A}(A,A') &= \mathcal{A}_{[0,n]}(A,A') \otimes_{[0,n]\in\mathtt{Int}} C_*([0,n]) \\
&\simeq \mathcal{A}_{[0,n]}(A,A') \otimes_{[0,n]\in\mathtt{Int}} \mathbb{Z}\big[\mathtt{Int}([0,1],[0,n])\big] \\
&= \mathcal{A}_{[0,1]}(A,A') = \mathcal{A}(A,A').
\end{aligned}$$

□

Construction C.20. Let $F\colon \mathcal{A} \to \mathrm{ch}\,\mathcal{B}$ be a homotopy coherent functor. We define a functor $F_\mathfrak{C}\colon \mathfrak{C}_\mathcal{A} \to \mathrm{ch}\,\mathcal{B}$ as follows. On objects the functor is given by $A \mapsto F(A)$. For $A, A' \in \mathcal{A}$

$$\mathfrak{C}_\mathcal{A}(A,A') \to \mathrm{ch}\,\mathcal{B}(F(A),F(A'))$$

is induced from the maps

$$\mathcal{A}_{[0,n]}(A,A') \otimes C_*([0,n]) \to \mathrm{ch}\,\mathcal{B}\big(F(A),F(A')\big) \tag{C.21}$$

that we define next. Let $\varphi_1 \otimes \cdots \otimes \varphi_n \otimes (I,J) \in \mathcal{A}_{[0,n]}(A,A') \otimes C_*([0,n])$. We obtain a functor $\phi\colon [0,n]^{\mathrm{op}} \to \mathcal{A}$ that sends the map $l \to l+1$ in $[0,n]$ to f_{l+1}. Write $J = \{j_0 = 0 < j_1 < \cdots < j_k = n\}$ and set $J_{r-1,r} := \{j_{r-1} < j_{r-1}+1 < \cdots < j_r\} \cap I$ for $r = 1, \ldots, k$. Recall that we can evaluate F on any (contravariant) functor from a finite linear ordered set to $\mathcal{A}$. Now (C.21) sends $\varphi_1 \otimes \cdots \otimes \varphi_n \otimes (I,J)$ to

$$F(\phi|_{J_{k-1,k}}) \circ F(\phi|_{J_{k-2,k-1}}) \circ \cdots \circ F(\phi|_{J_{1,2}}) \circ F(\phi|_{J_{0,1}}).$$

Verifying that this yields a well-defined functor $F_\mathfrak{C}$ is a lengthy but not difficult computation, that we omit.

Remark C.22. The construction of $\mathrm{comp}\,\mathfrak{C}_\mathcal{A} \to \mathcal{A}$ is the special case of Construction C.20 applied to $\mathrm{id}_\mathcal{A}\colon \mathcal{A} \to \mathcal{A}$.

Remark C.23. Let $F, G\colon \mathcal{A} \to \operatorname{ch} \mathcal{B}$ be homotopy coherent functors and $\tau\colon F \to G$ be a strict natural transformation. Then τ also defines a natural transformation $F_{\mathfrak{C}} \to G_{\mathfrak{C}}$.

Remark C.24. Let $F\colon \mathcal{A} \to \operatorname{ch}_{\mathrm{fin}} \mathcal{B}$ be a homotopy coherent functor. We can apply K-theory to

$$\mathcal{A} \xleftarrow{\mathrm{comp}} \mathfrak{C}_{\mathcal{A}} \xrightarrow{F_{\mathfrak{C}}} \operatorname{ch}_{\mathrm{fin}} \mathcal{B} \xleftarrow{i} \mathcal{B}$$

where i is the inclusion. Using Theorem B.1 we obtain an induced zig-zag in K-theory

$$\mathbf{K}\mathcal{A} \xleftarrow{\simeq} \mathbf{K}\mathfrak{C}_{\mathcal{A}} \xrightarrow{\mathbf{K}(F_{\mathfrak{C}})} \mathbf{K} \operatorname{ch}_{\mathrm{fin}} \mathcal{B} \xleftarrow{\simeq} \mathbf{K}\mathcal{B}. \tag{C.25}$$

Remark C.26. Let $F, G\colon \mathcal{A} \to \operatorname{ch}_{\mathrm{fin}} \mathcal{B}$ be homotopy coherent functors and $\tau\colon F \to G$ be a strict natural transformation. Then τ also defines a natural transformation $F_{\mathfrak{C}} \to G_{\mathfrak{C}}$. Suppose that $\tau_A\colon F^0(A) \to G^0(A)$ is a weak equivalence for all $A \in \mathcal{A}$. Corollary B.2 implies that then $\mathbf{K}(F_{\mathfrak{C}}) \simeq \mathbf{K}(G_{\mathfrak{C}})$. In other words the zig-zags (C.25) induced in K-theory by F and G agree up to homotopy.

Appendix D. Mapping X to $|\mathbf{J}_{\mathcal{C}\mathrm{vcy}}(G)|^\wedge$

Let G be a reductive p-adic group and let X be the associated extended Bruhat-Tits building. The extended Bruhat-Tits building X can be given a simplicial structure for which the action of G is simplicial, proper and smooth. We will also use the existence of a CAT(0)-metric d_X on X that generates the topology of X and is G-invariant. We fix a base point $x_0 \in X$. We write B_R for the closed ball of radius R centered at x_0 and $\pi_R\colon X \to B_R$ for the radial projection.

For a collection of subgroups $\mathcal{V} \subseteq \mathcal{C}\mathrm{vcy}$ we will consider the $\mathsf{P}_+\mathcal{A}\mathrm{ll}(G)$-simplicial complex $\mathbf{J}_{\mathcal{V}}(G)$ from Example 6.4. For $N \in \mathbb{N}$ we will also use $\mathbf{J}^N_{\mathcal{V}}(G)$ from Example 6.4. Recall that the simplicial complex $J^N_{\mathcal{V}}(G)$ underlying $\mathbf{J}^N_{\mathcal{V}}(G)$ is a finite complex if $\mathcal{V}$ is finite. We will also need the foliated distance $d_{\mathbf{J}^N_{\mathcal{V}}(G)\text{-fol}}$ on $|\mathbf{J}^N_{\mathcal{V}}(G)|^\wedge$ (defined in Subsection 7.c). Recall that it is compatible with restrictions to subcomplexes, i.e., $d_{\mathbf{J}^N_{\mathcal{V}}(G)\text{-fol}} = d_{\mathbf{J}^{N'}_{\mathcal{V}'}(G)\text{-fol}}|_{|\mathbf{J}^N_{\mathcal{V}}(G)|^\wedge}$ for $N \leq N'$, $\mathcal{V} \subseteq \mathcal{V}'$. To ease notation we abbreviate $d_{\mathbf{J}\text{-fol}} := d_{\mathbf{J}^N_{\mathcal{V}}(G)\text{-fol}}$.

The following result is [6, Thm. 1.2] for td-groups with an action on CAT(0)-space satisfying a technical assumption. This assumption is satisfied for the action of a reductive p-adic group on its extended Bruhat-Tits building [6, Prop. A.7].

Theorem D.1 (X to J)**.** *There is $N \in \mathbb{N}$ such that for all $M \subseteq G$ compact and $\epsilon > 0$ there are $\beta > 0$ and $\mathcal{V} \subseteq \mathcal{C}\mathrm{vcy}$ finite with the following property. For all $\eta > 0$ and all $L > 0$ we find $R > 0$ and a (not necessarily continuous) map $f\colon X \to |\mathbf{J}^N_{\mathcal{V}}(G)|^\wedge$ satisfying:*

(a) for $x \in B_{R+L}$, $g \in M$ we have $d_{\mathbf{J}\text{-fol}}(f(gx), gf(x)) < (\beta, \eta, \epsilon)$;
(b) for $x \in B_{R+L}$, $R' \geq R$ we have $d_{\mathbf{J}\text{-fol}}(f(x), f(\pi_{R'}(x))) < (\beta, \eta, \epsilon)$;
(c) there is $\rho > 0$ such that for all $x, x' \in X$ with $d_X(x, x') < \rho$ we have $d_{\mathbf{J}\text{-fol}}(f(x), f(x')) < (\beta, \eta, \epsilon)$.

The three assertions appearing in Theorem D.1 correspond to (12.2a), (12.2b) and (12.2c) from Assumption 12.2.

The goal in this section is to outline the proof Theorem D.1. A detailed construction is given in [6, Thm. 1.2].

D.i. **The flow space.** We briefly recall the flow space FS associated to X from [3, Sec. 1]. It consists of all *generalized geodesic*, i.e., of continuous maps $c\colon \mathbb{R} \to X$

whose restriction to some close interval[70] is an isometric embedding and is locally constant on the complement of this interval. The metric on FS is given by

$$d_{FS}(c,c') := \int_{\mathbb{R}} \frac{d_X(c(t),c'(t))}{2e^{|t|}}\,dt.$$

In this metric the distance between c and c' is small, iff the restrictions $c|_{[-a,a]}$ and $c'|_{[-a,a]}$ are pointwise close for large a. We recall from [3, Prop. 1.7] that this metric generates the topology of uniform convergence on compact subsets. The flow Φ on FS is defined by

$$(\Phi_\tau c)(t) := c(t+\tau).$$

For $c,c' \in FS$, $\alpha,\delta > 0$ we write

$$d_{FS\text{-fol}}(c,c') < (\alpha,\delta)$$

if there is $t \in [-\alpha,\alpha]$ with $d_{FS}(\Phi_t(c),c') < \delta$.

The construction is natural for the action of the isometry group of X. In particular, G acts on FS, the flow Φ_τ is G-equivariant, and d_{FS} and $d_{FS\text{-fol}}$ are both G-invariant.

D.II. **V-foliated distance and the flow space.** There is a close relation between the foliated distance $d_{FS\text{-fol}}$ on the flow space and the V-foliated distance from Subsection 5.C. We discuss the case $G = \mathrm{SL}_2(F)$. We will not explicitly use elsewhere what follows, but this was our motivation for the definition of V-foliated distance.

The building X for $\mathrm{SL}_2(F)$ is a simplicial tree, the Bass-Serre tree. We normalize the metric on X such that each edge has length 1. Let us say that a *bi-infinite combinatorial geodesic* is an isometric embedding $c\colon \mathbb{R} \to X$ that sends $\mathbb{Z} \subseteq \mathbb{R}$ into the 0-skeleton X^0 of the Bass-Serre tree[71]. The bi-infinite combinatorial geodesics form a closed subspace $FS^\sharp$ of FS. The flow on FS restricts to an $\mathbb{Z}$ action on $FS^\sharp$. Fix $c \in FS^\sharp$ and let K_c be the (pointwise) stabilizer of c and V_c be the stabilizer of the image of c in the quotient of $FS^\sharp$ by the $\mathbb{Z}$-action, compare D.VI. There is a choice of c such that K_c is the subgroup of diagonal matrices in $\mathrm{SL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of F and V_c is the subgroup of diagonal matrices in $\mathrm{SL}_2(F)$, but this will not be important in the following. Define $\pi\colon G \to FS^\sharp$ as $g \mapsto gc$. As the action of $\mathrm{SL}_2(F)$ on X is strongly transitive, the induced action on $FS^\sharp$ is transitive. Thus we can identify $FS^\sharp$ with G/K_c via π. The group V_c/K_c acts on $\mathbb{Z} \cong c(\mathbb{Z}) \subseteq X^0$ by translation. This induces a group homomorphism $t\colon V_c \to \mathbb{Z}$, that in turn induces an isomorphism $\bar{t}\colon V_c/K_c \cong \mathbb{Z}$. The quotient V_c/K_c acts from the right on G/K_c and under $G/K_c \equiv FS^\sharp$ and $V_c/K_c \cong \mathbb{Z}$ this action is the $\mathbb{Z}$-action on $FS^\sharp$ induced from the flow on FS. Recall that $d_{V_c\text{-fol}}$ on G depended on the choice of a left invariant proper metric d_G on G. The metric d_{FS} restricts to a metric on $FS^\sharp$. As π is G-equivariant it is in particular uniformly continuous. As $d_G|_{V_c}$ is proper and V_c-invariant and as K_c is compact, $t\colon V_c \to \mathbb{Z}$ is a coarse equivalence. This means that for all $A > 0$ there is $B > 0$ such that for all $v,v' \in V_c$

$$\begin{aligned} d_G(v,v') \le A &\implies |t(v)-t(v')| \le B;\\ |t(v)-t(v')| \le A &\implies d_G(v,v') \le B. \end{aligned}$$

Combining all this we have the following: for all $\alpha > 0$ there is $\beta > 0$, such that for $\epsilon > 0$ there is $\delta > 0$ such that for all $g,g' \in G$,

$$\begin{aligned} d_{V_c\text{-fol}}(g,g') < (\alpha,\delta) &\implies d_{FS\text{-fol}}(\pi(g),\pi(g')) < (\beta,\epsilon);\\ d_{FS\text{-fol}}(\pi(g),\pi(g')) < (\alpha,\delta) &\implies d_{V_c\text{-fol}}(g,g') < (\beta,\epsilon). \end{aligned}$$

[70] By this we mean a subset of the form $(-\infty,b]$, $[a,\infty)$, $(-\infty,\infty)$, or $[a,b]$.

[71] Bi-infinite combinatorial geodesics are uniquely determined by their restriction to $\mathbb{Z}$ and any isometric embedding $\mathbb{Z} \to X^0$ extends uniquely to a bi-infinite combinatorial geodesic.

Thus we can translate between $d_{FS\text{-fol}}$ and $d_{V_c\text{-fol}}$.

D.III. **Factorization over the flow space.** The maps in Theorem D.1 are constructed in two steps as compositions

$$X \xrightarrow{f_0} FS \xrightarrow{f_1} |\mathbf{J}^N_{\mathcal{V}}(G)|^\wedge;$$

Theorem D.2 (X to FS)**.** *For all $M \subseteq G$ compact there is $\alpha > 0$ with the following property. For all $\delta > 0$, $L > 0$ there exists $R > 0$ and a uniformly continuous map $f_0 \colon X \to FS$ such that*

(a) for $x \in B_{R+L}$, $g \in M$ we have $d_{FS\text{-fol}}\big(f_0(gx), gf_0(x)\big) < (\alpha, \delta)$;
(b) for $x \in B_{R+L}$, $R' \geq R$ we have $d_{FS\text{-fol}}\big(f_0(x), f_0(\pi_{R'}(x))\big) < (\alpha, \delta)$, where $\pi_{R'}$ denotes the radial projection onto $\overline{B}_{R+L}$.

Theorem D.3 (FS to J)**.** *There is $N \in \mathbb{N}$ such that for any $\alpha > 0$ and any $\epsilon > 0$ there are $\beta > 0$ and $\mathcal{V} \subseteq \mathcal{C}\mathrm{vcy}$ finite such that for any $\eta > 0$ there are $\delta > 0$, $f_1 \colon FS \to |\mathbf{J}^N_{\mathcal{V}}|^\wedge$, satisfying the following properties.*

(a) For $c, c' \in FS$ with $d_{FS\text{-fol}}(c, c') < (\alpha, \delta)$ we have $d_{\mathbf{J}\text{-fol}}\big(f_1(c), f_1(c')\big) < (\beta, \eta, \epsilon)$;
(b) For $c \in FS$, $g \in G$ we have $d_{\mathbf{J}\text{-fol}}\big(f_1(gc), gf_1(c)\big) < (\beta, \eta, \epsilon)$.

Theorem D.2 is [6, Thm. 4.1]. Its proof uses the CAT(0)-geometry of X and associated dynamic properties of the flow space and is sketched in Subsection D.IV. Theorem D.3 is [6, Thm. 4.3]. Its proof uses so called long and thin covers for the flow space and is sketched in Subsection D.V.

Theorem D.1 is a formal consequence of Theorems D.2 and D.3. Formally this uses that for any $\alpha > 0$, $\delta > 0$ there is $\rho > 0$ such that $d_X(x, x') < \rho$ implies $d_{FS\text{-fol}}(f_1(x), f_1(x')) < (\alpha, \delta)$. This statement follows from uniform continuity of f_1 (even uniformly in α).

D.IV. **Dynamic of the flow.** Theorem D.2 is closely related to the results from [3, Sec. 3] and follows from similar estimates[72]. We sketch its proof here.

For $x, x' \in X$ we write $c_{x,x'} \in FS$ for the generalized geodesic from x to x', i.e., for the generalized geodesic characterized by

$$\begin{aligned} c_{x,x'}(t) &= x \quad t \in (-\infty, 0]; \\ c_{x,x'}(t) &= x \quad t \in [d(x,y), +\infty). \end{aligned}$$

For $T \geq 0$ consider the map $f_1^T \colon X \to FS$, $x \mapsto \Phi_T(c_{x_0,x})$. Recall that $x_0 \in X$ is our fixed base point. Both $x \mapsto c_{x_0,x}$ and Φ_T (for fixed T) are uniformly continuous, in particular, f_1^T is uniformly continuous.

Lemma D.4. *For all $\delta > 0$ there is $\Delta > 0$ such that for all R', T with $R' \geq T + \Delta$, $x \in X$ we have*

$$d_{FS}\big(f_1^T(x), f_1^T(\pi_{R'}(x))\big) < \delta.$$

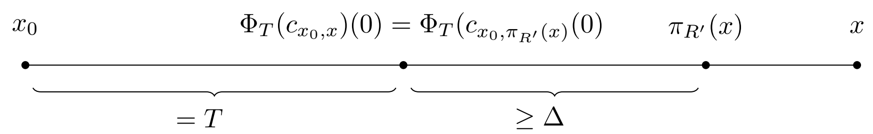

FIGURE 4. Schematic picture for Lemma D.4

[72]The present set-up is simpler and avoids the homotopy action in our estimates.

Sketch of proof for Lemma D.4. For large Δ the generalized geodesics $f_1^T(x) = \Phi_T(c_{x_0,x})$ and $f_1^T(\pi_{R'}(x)) = \Phi_T(c_{x_0,\pi_{R'}(x)})$ agree on a large interval $[-a,a]$, see Figure 4, and are therefore close to each other in FS. For more details see [6, Lem. 5.2]. $\square$

Lemma D.5. *For all $\alpha > 0$, $\Delta > 0$, $L > 0$, and $\delta > 0$ there are $R > 0$, $0 \le T \le R - \Delta$ such that for all $x \in \overline{B}_{R+L}(b)$ and $g \in G$ with $d_X(b, gb) \le \alpha$, we have*

$$d_{FS\text{-fol}}\big(gf_1^T(x), f_1^T(gx)\big) < (\alpha, \delta).$$

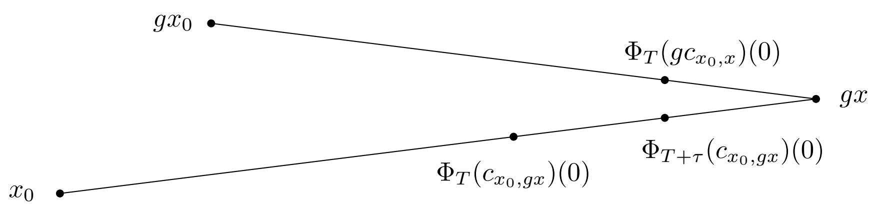

FIGURE 5. Schematic picture for Lemma D.5

Sketch of proof for Lemma D.5. We have $gf_1^T(x) = \Phi_T(c_{gx_0,gx})$ and $f_1^T(gx) = \Phi_T(c_{gx_0,x})$. Let $\tau := d_X(x_0, gx) - d_X(gx_0, gx)$. By assumption $\tau \in [-\alpha, \alpha]$. Now for $r >> 0$, $R >> 0$ and $T := R - r$, $\Phi_T(c_{gx_0,gx})$ and $\Phi_{T+\tau}(c_{gx_0,x})$ will be pointwise close on a large interval $[-a,a]$ by the CAT(0) inequality, see Figure 5. For more details see [6, Lem. 5.4]. $\square$

Sketch of proof for Theorem D.2. We set $\alpha := \max\{d_X(gx_0, x_0) \mid g \in M\}$. Given $\delta > 0$, $L > 0$ we first choose Δ as in Lemma D.4 and use then Lemma D.5 to choose R and T. Then $f_1^T \colon X \to FS$ has the desired properties: for (a) we use the estimate from Lemma D.5 and for (b) we use the estimate from Lemma D.4. $\square$

D.v. **Long thin covers.**

Definition D.6 (α-long cover)**.** A cover $\mathcal{U}$ of the flow space FS by open subsets is said to be *α-long* if for any $c_0 \in FS$ there exists $U \in \mathcal{U}$ such that $\Phi_{[-\alpha,\alpha]}(c_0) \subseteq U$.

Typically such covers are thin in directions transversal to the flow and are often referred to as long thin covers. The proof of Theorem D.3 uses the following three results for the flow space.

Proposition D.7 (Partition of unity)**.** *For all $\alpha > 0$, $\varepsilon > 0$, $N \in \mathbb{N}$ there is $\alpha' > 0$ such that the following holds. Let $\mathcal{U}$ be a α'-long cover of dimension $\le N$ by G-invariant open subsets of FS. Then there exists a partition of unity $\{t_U \colon FS \to [0,1] \mid U \in \mathcal{U}\}$ subordinate to $\mathcal{U}$ and $\delta > 0$ such that*

(a) for $U \in \mathcal{U}$, $c, c' \in FS$ with $d_{FS\text{-fol}}(c, c') < (\alpha, \delta)$ we have

$$|t_U(c) - t_U(c')| < \varepsilon;$$

(b) the t_U are G-invariant.

Proposition D.7 is [6, Prop. 6.4]. Its proof is not complicated. The fact that the cover is α'-long allows us to find a partition of unity whose functions only vary very slowly along flow lines.

Proposition D.8 (Dimension of long thin covers)**.** *There is $N \in \mathbb{N}$ such that for any $\alpha' > 0$ there is α'' such that the following is true. Let $\mathcal{W}$ be an α''-long cover of FS by G-invariant open subsets. Then there exist collections $\mathcal{U}_0, \dots, \mathcal{U}_N$ of open G-invariant subsets of FS such that*

(a) $\mathcal{U} := \mathcal{U}_0 \sqcup \ldots \sqcup \mathcal{U}_N$ *is an* α'*-long cover of* FS*, in particular* $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ *for* $i \neq j$*;*
(b) for each i *the open sets in* $\mathcal{U}_i$ *are pairwise disjoint;*
(c) for each $U \in \mathcal{U} = \mathcal{U}_0 \sqcup \ldots \sqcup \mathcal{U}_N$ *there is* $W \in \mathcal{W}$ *with* $U \subseteq W$*.*

Proposition D.8 is [6, Prop. 6.6]. It is closely related to other existence results for long and thin covers in the literature, although it is not stated in exactly this form elsewhere. To prove it one can pass to the quotient $G\backslash FS$ and adapt the strategy from [34]. The main difference is that here our constrain on $\mathcal{U}$ is formulated in terms of the given cover $\mathcal{W}$, while in [34] the constrain on the members of $\mathcal{U}$ are formulated in terms of the group action.

Proposition D.9 (Local structure)**.** *For all* $\alpha'' > 0$ *there are* $\beta > 0$ *and* $\mathcal{V} \subseteq \mathcal{C}\mathrm{vcy}$ *finite with the following property. For all* $\eta > 0$ *and all* $c_0 \in FS$ *there exist* $U \subseteq FS$ *open,* $h\colon U \to G$*,* $V \in \mathcal{V}$ *and* $\delta'' > 0$ *such that*
(a) for some neighborhood U_0 *of the orbit* Gc_0 *we have* $\Phi_{[-\alpha'',\alpha'']}(U_0) \subseteq U$*;*
(b) U *is* G*-invariant;*
(c) for $c, c' \in U$ *we have*

$$d_{FS\text{-fol}}(c,c') < (\alpha'',\delta'') \implies d_{V\text{-fol}}(h(c),h(c')) < (\beta,\eta);$$

(d) for $c \in U$*,* $g \in G$ *we have;*

$$d_{V\text{-fol}}(h(gc),gh(c)) < (\beta,\eta).$$

Proposition D.9 is [6, Prop. 6.8]. We will briefly discuss its proof in Subsection D.vi.

The map f_1 from Theorem D.3 can locally be constructed using Proposition D.9. Using the partition of unity from Proposition D.7 these can be patched together to a map $FS \to |\mathbf{J}^N_{\mathcal{V}}(G)|^\wedge$. As noted already in Subsection 1.m this patching procedure forces us to pass from orbits to products of orbits. Some care is needed to control the dimension of the image of theses maps; this is where Proposition D.8 is needed.

D.vi. **Local structure of the flow space.** For $c \in FS(X)$ we set

$$\begin{aligned} K_c &:= G_c = \{g \in G \mid gc = c\} = \{g \in G \mid gc(t) = c(t) \text{ for all } t \in \mathbb{R}\};\\ V_c &:= \{g \in G \mid \exists t \in \mathbb{R} \text{ such that } gc = \Phi_t(c)\};\\ \tau_c &:= \inf\{t > 0 \mid \exists v \in V_c \setminus K_c, \text{with } \Phi_t(c) = vc\}. \end{aligned}$$

We use $\inf \emptyset = \infty$. If $\tau_c < \infty$ then we say that c is *periodic*. We have $K_c \subseteq V_c$ as the flow is G-equivariant. For $\alpha > 0, \delta > 0$, $c \in FS$ we set

$$U^{\mathrm{fol}}_{\alpha,\delta}(c) := \{c' \in FS \mid d_{FS\text{-fol}}(c,c') < (\alpha,\delta)\}. \tag{D.10}$$

One may think of $U^{\mathrm{fol}}_{\alpha,\delta}(c)$ as an open ball around c with respect to $d_{FS\text{-fol}}$.

Proposition D.11. *Let* $FS_0 \subseteq FS$ *be compact. For all* $\alpha > 0$ *there is* $\beta > 0$ *such that the following is true: For all* $\eta > 0$*,* $c_0 \in FS_0$*, there are* $\delta > 0$ *and a (not necessarily continuous) map* $h\colon G \cdot U^{\mathrm{fol}}_{\alpha,\delta}(c_0) \to G$ *satisfying*
(a) for $c, c' \in G \cdot U^{\mathrm{fol}}_{\alpha,\delta}(c_0)$ *we have*

$$d_{FS\text{-fol}}(c,c') < (\alpha,\delta) \quad \implies \quad d_{V_{c_0}\text{-fol}}(h(c),h(c')) < (\beta,\eta);$$

(b) for $g \in G$*,* $c \in G \cdot U^{\mathrm{fol}}_{\alpha,\delta}(c_0)$ *we have*

$$d_{V_{c_0}\text{-fol}}(h(gc),gh(c)) < (\beta,\eta).$$

Sketch of proof. Let $\alpha > 0$ be given. Using compactness of FS_0, it is not difficult to show the following: there is $\beta > 0$ such that for $g \in G$, $c \in FS_0$ we have

$$d_{FS}(gc,c) < 3\alpha \quad \implies \quad d_G(g,e) < \beta.$$

Next let $\eta > 0$ and $c_0 \in FS_0$ be given. For $n \in \mathbb{N}$ choose $h_n \colon G \cdot U^{\mathrm{fol}}_{\alpha,1/n}(c_0) \to G$ such that $c \in h_n(c) \cdot U^{\mathrm{fol}}_{\alpha,1/n}(c_0)$ for all $c \in G \cdot U^{\mathrm{fol}}_{\alpha,\delta_n}(c_0)$. It is not difficult to check that for all sufficiently large n the map h_n satisfies (a) and (b). For more details see [6, Prop. 9.2]. □

Remark D.12. In the assertion of Proposition D.11 the estimates can be strengthened to use $d_{K_{c_0}\text{-fol}}$ in place of $d_{V_{c_0}\text{-fol}}$ provided $\tau_{c_0} > \ell$ where ℓ is a constant only depending on α.

On its own Proposition D.11 is not quite strong enough to imply Proposition D.9, because it is not quite clear yet that we only need a finite set $\mathcal{V}$ of subgroups. To resolve this one needs to understand how V_{c_0} varies in c_0. Ideally we would like for V_{c_0} not to increase in small neighborhoods of c_0, at least up to conjugation. While we do not know this, we have the following result.

Proposition D.13. *There exists $FS_0 \subseteq FS$ compact such that*

(a) $G \cdot FS_0 = FS$;

(b) for $\ell > 0$ and $c_0 \in FS_0$ there exists an open neighborhood U of c_0 in FS_0 such that for all $c \in U$ with $\tau_c \leq \ell$ we have $V_c \subseteq V_{c_0}$.

Proposition D.13 is [6, Prop. A.7]. We will not discuss its proof in detail. But we want to point out that this proof uses the combinatorial structure of the building X. Here we fix an apartment A for X and use that the G-action translates any geodesic to a geodesic in A. To study the groups V_{c_0} for $c_0 \in FS(A)$, one can then use the combinatorial structure of A as a Coxeter complex. Here $FS(A)$ is the flow space for A. This is the only point where the proof of Theorem D.1 uses the combinatorial structure of X.

To prove Proposition D.9 one combines Proposition D.11, Remark D.12 and Proposition D.13. This concludes the discussion of Proposition D.9.

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