

# RECIPES TO COMPUTE THE ALGEBRAIC $K$ -THEORY OF HECKE ALGEBRAS OF REDUCTIVE $p$ -ADIC GROUPS

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ABSTRACT. We compute the algebraic  $K$ -theory of the Hecke algebra of a reductive  $p$ -adic group  $G$  using the fact that the Farrell-Jones Conjecture is known in this context. The main tool will be the properties of the associated Bruhat-Tits building and an equivariant Atiyah-Hirzebruch spectral sequence. In particular the projective class group can be written as the colimit of the projective class groups of the compact open subgroups of  $G$ .

## 1. INTRODUCTION

We begin with stating the main theorem of this paper, explanation will follow:

**Theorem 1.1** (Main Theorem). *Let  $G$  be a  $td$ -group which is modulo a normal compact subgroup a subgroup of a reductive  $p$ -adic group. Let  $R$  be a uniformly regular ring with  $\mathbb{Q} \subseteq R$ . Choose a model  $E_{\text{Cop}}(G)$  for the classifying space for proper smooth  $G$ -actions. Let  $\mathcal{I} \subseteq \text{Cop}$  be the set of isotropy groups of points in  $E_{\text{Cop}}(G)$ .*

*Then*

- (i) *The map induced by the projection  $E_{\text{Cop}}(G) \rightarrow G/G$  induces for every  $n \in \mathbb{Z}$  an isomorphism*

$$H_n^G(E_{\text{Cop}}(G); \mathbf{K}_R) \rightarrow H_n^G(G/G; \mathbf{K}_R) = K_n(\mathcal{H}(G; R));$$

- (ii) *There is a (strongly convergent) spectral sequence*

$$E_{p,q}^2 = SH_p^{G,\mathcal{I}}(E_{\text{Cop}}(G); \overline{K_q(\mathcal{H}(\cdot; R))}) \implies K_{p+q}(\mathcal{H}(G; R)),$$

*whose  $E^2$ -term is concentrated in the first quadrant;*

- (iii) *The canonical map induced by the various inclusions  $K \subseteq G$*

$$\text{colim}_{K \in \text{Sub}_{\mathcal{I}}(G)} K_0(\mathcal{H}(K; R)) \rightarrow K_0(\mathcal{H}(G; R))$$

*can be identified with the isomorphism appearing in assertion (i) in degree  $n = 0$  and hence is bijective;*

- (iv) *We have  $K_n(\mathcal{H}(G; R)) = 0$  for  $n \leq -1$ .*

Note that assertion (i) of Theorem 1.1 is proved in [3, Corollary 1.8]. So this papers deals with implications of it concerning computations of the algebraic  $K$ -groups  $K_n(\mathcal{H}(G))$  of the Hecke algebra of  $G$ .

A  $td$ -group  $G$  is a locally compact second countable totally disconnected topological Hausdorff group. It is *modulo a normal compact subgroup a subgroup of a reductive  $p$ -adic group* if it contains a (not necessarily open) normal compact subgroup  $K$  such that  $G/K$  is isomorphic to a subgroup of some reductive  $p$ -adic group.

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A ring is called *uniformly regular*, if it is Noetherian and there exists a natural number  $l$  such that any finitely generated  $R$ -module admits a resolution by projective  $R$ -modules of length at most  $l$ . We write  $\mathbb{Q} \subseteq R$ , if for any integer  $n$  the element  $n \cdot 1_R$  is a unit in  $R$ . Examples for uniformly regular rings  $R$  with  $\mathbb{Q} \subseteq R$  are fields of characteristic zero.

We denote by  $\mathcal{H}(G; R)$  the *Hecke algebra* consisting of locally constant functions  $s: G \rightarrow R$  with compact support, where the additive structure comes from the additive structure of  $R$  and the multiplicative structure from the convolution product. Note that  $\mathcal{H}(G; R)$  is a ring without unit.

We denote by  $E_{\text{Cop}}(G)$  a model for the *classifying space for proper smooth  $G$ -actions*, i.e., a  $G$ -CW-complex, whose isotropy groups are all compact open subgroups of  $G$  and for which  $E_{\text{Cop}}(G)^H$  is weakly contractible for any compact open subgroup  $H \subseteq G$ . Two such models are  $G$ -homotopy equivalent. Hence  $H_n^G(E_{\text{Cop}}(G); \mathbf{K}_R)$  is independent of the choice of a model. If  $G$  is a reductive  $p$ -adic group with compact center, then its Bruhat-Tits building is a model for  $E_{\text{Cop}}(G)$ . If the center is not compact, one has to pass to the extended Bruhat-Tits building.

We will construct a *smooth  $G$ -homology theory*  $H_*^G(-; \mathbf{K}_R)$  in Section 3. It assigns to a smooth  $G$ -CW-pair  $(X, A)$  a collection of abelian groups  $\mathcal{H}_n^G(X, A; \mathbf{K}_R)$  for  $n \in \mathbb{Z}$  that satisfies the expected axioms, i.e., long exact sequence of a pair,  $G$ -homotopy invariance, excision, and the disjoint union axiom. Moreover, for every open subgroup  $U \subseteq G$  and  $n \in \mathbb{Z}$  we have

$$(1.2) \quad H_n^G(G/U; \mathbf{K}_R) \cong K_n(\mathcal{H}(U; R)).$$

Let  $\mathcal{F}$  be a collection of open subgroups of  $G$  which is closed under conjugation. Examples are the set  $\text{Cop}$  of compact open subgroups of  $G$  and the set  $\mathcal{I}$  of isotropy groups of points of some model for  $E_{\text{Cop}}(G)$ . The subgroup category  $\text{Sub}_{\mathcal{F}}(G)$  appearing in Theorem 1.1 (iii) has  $\mathcal{F}$  as set of objects and will be described in detail in Subsection 2.A.

The abelian groups  $SH_p^{G, \mathcal{F}}(E_{\mathcal{F}}(G); \overline{K_q(\mathcal{H}(\cdot; R))})$  appearing in Theorem 1.1 (ii) will be defined for the covariant functor  $\overline{K_q(\mathcal{H}(\cdot; R))}: \text{Sub}_{\mathcal{F}}(G) \rightarrow \mathbb{Z}\text{-Mod}$ , whose value at  $U \in \mathcal{F}$  is  $K_n(\mathcal{H}(U; R))$ , in Subsection 2.B. They are closely related to the *Bredon homology groups*  $BH_p^{G, \mathcal{F}}(E_{\mathcal{F}}(G); K_q(\mathcal{H}(\cdot; R)))$ .

The proof of the Main Theorem 1.1 will be given in Section 4.

The relevance of the Hecke algebra  $\mathcal{H}(G; R)$  is that the category of non-degenerate modules over it is isomorphic to the category of smooth  $G$ -representations with coefficients in  $R$ , see for instance [5, 13]. Hence in particular its projective class group  $K_0(\mathcal{H}(G; R))$  is important. The various inclusions  $K \rightarrow G$  for  $K \in \text{Cop}$  induce a map

$$(1.3) \quad \bigoplus_{K \in \text{Cop}} K_0(\mathcal{H}(K; R)) \rightarrow K_0(\mathcal{H}(G; R)),$$

which factorizes over the canonical epimorphism from  $\bigoplus_{K \in \text{Cop}} K_0(\mathcal{H}(K; R))$  to  $\text{colim}_{K \in \text{Sub}_{\mathcal{I}}(G)} K_0(\mathcal{H}(K; R))$  to the isomorphism appearing in Theorem 1.1 (iii) and is hence surjective. Dat [10] has shown that the map (1.3) is rationally surjective for  $G$  a reductive  $p$ -adic group and  $R = \mathbb{C}$ . In particular, the cokernel of it is a torsion group. Dat [9, Conj. 1.11] conjectured that this cokernel is  $\tilde{w}_G$ -torsion. Here  $\tilde{w}_G$  is a certain multiple of the order of the Weyl group of  $G$ . Dat [9, Prop. 1.13] proved this conjecture for  $G = \text{GL}_n(F)$  for a  $p$ -adic field  $F$  of characteristic zero and asked about the integral version, see the comment following [9, Prop. 1.10], which is now proven by Theorem 1.1 (iii).

The computations simplify considerably in the case of a reductive  $p$ -adic group thanks to the associated (extended) Bruhat-Tits building, see Sections 5 and 7.

As an illustration we analyze the projective class groups of the Hecke algebras of  $\mathrm{SL}_n(F)$ ,  $\mathrm{PGL}_n(F)$  and  $\mathrm{GL}_n(F)$  in Section 6.

One of our main tools will be the *smooth equivariant Atiyah-Hirzebruch spectral sequence*, which we will establish and examine in Section 2.

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2. THE SMOOTH EQUIVARIANT HIRZEBRUCH SPECTRAL SEQUENCE

Throughout this section we fix a set  $\mathcal{F}$  of open subgroups of  $G$  which is closed under conjugation. Our main examples for  $\mathcal{F}$  are the family  $\mathcal{O}_p$  of all open subgroups and the family  $\mathcal{C}_p$  of all compact open subgroups. A  $\mathcal{F}$ - $G$ - $CW$ -complex  $X$  is a  $G$ - $CW$ -complex  $X$  such that for every  $x \in X$  its isotropy group  $G_x$  belongs to  $\mathcal{F}$ . A smooth  $G$ - $CW$ -complex is the same as a  $\mathcal{O}_p$ - $CW$ - $CW$ -complex and a proper

smooth  $G$ -CW-complex is the same as a Cop-CW-complex. Let  $\mathcal{H}_*^G$  be a smooth  $G$ -homology theory.

The main result of this section is

**Theorem 2.1.** *Consider a pair  $(X, A)$  of  $\mathcal{F}$ - $G$ -CW-complexes and a smooth  $G$ -homology theory  $\mathcal{H}_*^G$ . Then there is an equivariant Atiyah-Hirzebruch spectral sequence converging to  $\mathcal{H}_{p+q}^G(X, A)$ , whose  $E^2$ -term is given by*

$$E_{p,q}^2 = BH_p^{G,\mathcal{F}}(X, A; \mathcal{H}_q^G)$$

for the Bredon homology  $BH_p^{G,\mathcal{F}}(X, A; \mathcal{H}_q^G)$  of  $(X, A)$  with coefficients in the co-variant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $\mathcal{H}_q^G$  that sends  $G/H$  to  $\mathcal{H}_q^G(G/H)$ .

The remainder of this section is devoted to the definition of the Bredon homology, the construction of the equivariant Atiyah-Hirzebruch spectral sequence, and some general calculations concerning the  $E^2$ -term. Convergence means that there is an ascending filtration  $F_{l,m-l}\mathcal{H}_m^G(X, A)$  for  $l = 0, 1, 2, \dots$  of  $\mathcal{H}_m^G(X, A)$  such that  $F_{p,q}\mathcal{H}_{p+q}^G(X, A)/F_{p-1,q+1}\mathcal{H}_{p+q}^G(X, A) \cong E_{p,q}^\infty$  holds for  $E_{p,q}^\infty = \text{colim}_{r \rightarrow \infty} E_{p,q}^r$ .

**2.A. The smooth orbit category and the smooth subgroup category.** The  $\mathcal{F}$ -orbit category  $\text{Or}_{\mathcal{F}}(G)$  has as objects homogeneous  $G$ -spaces  $G/H$  with  $H \in \mathcal{F}$ . Morphisms from  $G/H$  to  $G/K$  are  $G$ -maps  $G/H \rightarrow G/K$ . We will put no topology on  $\text{Or}_{\mathcal{F}}(G)$ . For any  $G$ -map  $f: G/H \rightarrow G/K$  of smooth homogeneous spaces, there is an element  $g \in G$  such that  $gHg^{-1} \subseteq K$  holds and  $f$  is the  $G$ -map  $R_{g^{-1}}: G/H \rightarrow G/K$  sending  $g'H$  to  $g'g^{-1}K$ . Given two elements  $g_0, g_1 \in G$  such that  $g_iHg_i^{-1} \subseteq K$  holds for  $i = 0, 1$ , we have  $R_{g_0^{-1}} = R_{g_1^{-1}} \iff g_1g_0^{-1} \in K$ . We get a bijection

$$(2.2) \quad K \setminus \{g \in G \mid gHg^{-1} \subseteq K\} \xrightarrow{\cong} \text{map}_G(G/H, G/K), \quad g \mapsto R_{g^{-1}}.$$

The  $\mathcal{F}$ -subgroup category  $\text{Sub}_{\mathcal{F}}(G)$  has  $\mathcal{F}$  as the set of objects. For  $H, K \in \mathcal{F}$  denote by  $\text{conhom}_G(H, K)$  the set of group homomorphisms  $f: H \rightarrow K$ , for which there exists an element  $g \in G$  with  $gHg^{-1} \subseteq K$  such that  $f$  is given by conjugation with  $g$ , i.e.,  $f = c(g): H \rightarrow K$ ,  $h \mapsto ghg^{-1}$ . Note that  $c(g) = c(g')$  holds for two elements  $g, g' \in G$  with  $gHg^{-1} \subseteq K$  and  $g'Hg'^{-1} \subseteq K$ , if and only if  $g^{-1}g'$  lies in the centralizer  $C_G H = \{g \in G \mid gh = hg \text{ for all } h \in H\}$  of  $H$  in  $G$ . The group of inner automorphisms  $\text{Inn}(K)$  of  $K$  acts on  $\text{conhom}_G(H, K)$  from the left by composition. Define the set of morphisms

$$\text{mor}_{\text{Sub}_{\text{Cop}}(G)}(H, K) := \text{Inn}(K) \setminus \text{conhom}_G(H, K).$$

There is an obvious bijection

$$(2.3) \quad K \setminus \{g \in G \mid gHg^{-1} \subseteq K\} / C_G H \xrightarrow{\cong} \text{Inn}(K) \setminus \text{conhom}_G(H, K), \\ KgC_G H \mapsto [c(g)],$$

where  $[c(g)] \in \text{Inn}(K) \setminus \text{conhom}_G(H, K)$  is the class represented by the element  $c(g): H \rightarrow K$ ,  $h \mapsto ghg^{-1}$  in  $\text{conhom}_G(H, K)$  and  $K$  acts from the left and  $C_G H$  from the right on  $\{g \in G \mid gHg^{-1} \subseteq K\}$  by the multiplication in  $G$ .

Let

$$(2.4) \quad P: \text{Or}_{\mathcal{F}}(G) \rightarrow \text{Sub}_{\mathcal{F}}(G)$$

be the canonical projection which sends an object  $G/H$  to  $H$  and is given on morphisms by the obvious projection under the identifications (2.2) and (2.3).

**2.B. Cellular chain complexes and Bredon homology.** Given an  $\mathcal{F}$ - $G$ - $CW$ -complex  $X$ , we obtain a contravariant  $\text{Or}_{\mathcal{F}}(G)$ -space  $O_X: \text{Or}_{\mathcal{F}}(G) \rightarrow \text{Spaces}$  by sending  $G/H$  to  $\text{map}_G(G/H, X) = X^H$ . We get a contravariant  $\text{Sub}_{\mathcal{F}}(G)$ -space  $S_X: \text{Sub}_{\mathcal{F}}(G) \rightarrow \text{Spaces}$  by sending  $H$  to  $C_G H \setminus \text{map}_G(G/H, X) = C_G H \setminus X^H$ . A morphism  $H \rightarrow K$  given by an element  $g \in G$  satisfying  $gHg^{-1} \subseteq K$  is sent to the map  $C_G K \setminus X^K \rightarrow C_G H \setminus X^H$  induced by the map  $X^K \rightarrow X^H$ ,  $x \mapsto g^{-1}x$ .

Given a pair  $(Y, A)$  with a filtration  $A = Y_{-1} \subseteq Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y$  with  $Y = \text{colim}_{n \rightarrow \infty} Y_n$ , we associate to it a  $\mathbb{Z}$ -chain complex  $C_*^c(Y, A)$ , whose  $n$ -th chain module is the singular homology  $H_n^{\text{sing}}(Y_n, Y_{n-1})$  of the pair  $(Y_n, Y_{n-1})$  (with coefficients in  $\mathbb{Z}$ ) and whose  $n$ th differential is given by the composite

$$H_n^{\text{sing}}(Y_n, Y_{n-1}) \xrightarrow{\partial_n} H_{n-1}^{\text{sing}}(Y_{n-1}) \xrightarrow{H_{n-1}^{\text{sing}}(i_{n-1})} H_{n-1}^{\text{sing}}(Y_{n-1}, Y_{n-2})$$

for  $\partial_n$  the boundary operator of the pair  $(Y_n, Y_{n-1})$  and the inclusion  $i_{n-1}: Y_{n-1} = (Y_{n-1}, \emptyset) \rightarrow (Y_{n-1}, Y_{n-2})$ .

Given a pair of  $\mathcal{F}$ - $G$ - $CW$ -complexes  $(X, A)$ , the filtration by its skeletons induces filtrations on the spaces  $X^H$  and  $C_G H \setminus X^H$  for every  $H \in \mathcal{F}$ . We get a contravariant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -chain complex  $C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A): \text{Or}_{\mathcal{F}}(G) \rightarrow \mathbb{Z}\text{-Ch}$  and a contravariant  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -chain complex  $C_*^{\text{Sub}_{\mathcal{F}}(G)}(X, A): \text{Sub}_{\mathcal{F}}(G) \rightarrow \mathbb{Z}\text{-Ch}$  by putting

$$\begin{aligned} C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A)(G/H) &:= C_*^c(O_X(G/H), O_A(G/H)) = C_*^c(X^H, A^H); \\ C_*^{\text{Sub}_{\mathcal{F}}(G)}(X, A)(H) &:= C_*^c(S_X(X)(H), S_A(H)) = C_*^c(C_G H \setminus X^H, C_G H \setminus A^H). \end{aligned}$$

Choose a  $G$ -pushout

$$(2.5) \quad \begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_{n-1}. \end{array}$$

It induces for every closed subgroup  $H \subseteq G$  pushouts

$$\begin{array}{ccc} \coprod_{i \in I_n} (G/H_i)^H \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} (q_i^n)^H} & X_{n-1}^H \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} (G/H_i)^H \times D^n & \xrightarrow{\coprod_{i \in I_n} (Q_i^n)^H} & X_{n-1}^H \end{array}$$

and

$$\begin{array}{ccc} \coprod_{i \in I_n} C_G H \setminus (G/H_i)^H \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} C_G H \setminus (q_i^n)^H} & C_G H \setminus X_{n-1}^H \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} C_G H \setminus (G/H_i)^H \times D^n & \xrightarrow{\coprod_{i \in I_n} C_G H \setminus (Q_i^n)^H} & C_G H \setminus X_{n-1}^H. \end{array}$$

Note that  $(G/H_i)^H$  agrees with  $\text{mor}_{\text{Or}_{\mathcal{F}}(G)}(G/H, G/H_i) = \text{map}_G(G/H, G/H_i)$ . In the sequel we denote by  $\mathbb{Z}S$  for a set  $S$  the free  $\mathbb{Z}$ -module with the set  $S$  as basis. Since singular homology satisfies the disjoint union axiom, homotopy invariance and excision, we obtain an isomorphism of contravariant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -modules

$$(2.6) \quad \bigoplus_{i \in I_n} \mathbb{Z} \text{mor}_{\text{Or}_{\mathcal{F}}(G)}(?, G/H_i) \xrightarrow{\cong} C_n^{\text{Or}_{\mathcal{F}}(G)}(X, A),$$

where  $\mathbb{Z}\text{mor}_{\text{Or}_{\mathcal{F}}(G)}(? , G/H_i)$  is the free  $\mathbb{Z}\text{Or}(G)$ -module based at the object  $G/H_i$ , see [14, Example 9.8 on page 164], and analogously an isomorphism of contravariant  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -modules

$$(2.7) \quad \bigoplus_{i \in I_n} \mathbb{Z}\text{mor}_{\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)}(? , H_i) \xrightarrow{\cong} C_n^{\text{Sub}_{\mathcal{F}}(G)}(X, A).$$

If  $P_*C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A)$  is the  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -chain complex obtained by induction with  $P: \text{Or}_{\mathcal{F}}(G) \rightarrow \text{Sub}_{\mathcal{F}}(G)$  from  $C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A)$ , see [14, Example 9.15 on page 166], we conclude from (2.6) and (2.7) that the canonical map of  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -chain complexes

$$(2.8) \quad P_*C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A) \xrightarrow{\cong} C_*^{\text{Sub}_{\mathcal{F}}(G)}(X, A)$$

is an isomorphism.

For a covariant  $\mathbb{Z}\text{Or}(G)$ -module  $M$ , we get from the tensor product over  $\text{Or}_{\mathcal{F}}(G)$ , see [14, 9.13 on page 166], a  $\mathbb{Z}$ -chain complex  $C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\text{Or}_{\mathcal{F}}(G)} M$ .

**Definition 2.9** (Bredon homology). We define the  $n$ -th *Bredon homology* to be the  $\mathbb{Z}$ -module

$$BH_n^{G, \mathcal{F}}(X, A; M) = H_n(C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\text{Or}_{\mathcal{F}}(G)} M).$$

Given a covariant  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -module  $N$ , define analogously

$$SH_n^{G, \mathcal{F}}(X, A; N) = H_n(C_*^{\text{Sub}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)} N).$$

Given a covariant  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -module  $N$ , define the covariant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $P^*N$  to be  $N \circ P$ . We get from the adjunction of [14, 9.22 on page 169] and (2.8) a natural isomorphism of  $\mathbb{Z}$ -chain complexes

$$(2.10) \quad C_*^{\text{Sub}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)} N \xrightarrow{\cong} C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\text{Or}_{\mathcal{F}}(G)} P^*N$$

and hence natural isomorphism of  $\mathbb{Z}$ -modules

$$(2.11) \quad BH_n^{G, \mathcal{F}}(X, A; P^*N) \xrightarrow{\cong} SH_n^{G, \mathcal{F}}(X, A; N).$$

Let  $(X, A)$  be a pair of  $\mathcal{F}$ -CW-complexes. Denote by  $\mathcal{I}$  the set of isotropy groups of points in  $X$ . Let  $M$  be a covariant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module and  $N$  be a covariant  $\text{Sub}_{\mathcal{F}}(G)$ -module. Denote by  $M|_{\mathcal{I}}$  and  $N|_{\mathcal{I}}$  their restrictions to  $\text{Or}_{\mathcal{I}}(G)$  and  $\text{Sub}_{\mathcal{I}}(G)$ . Then one easily checks using [11, Lemma 1.9] that there are canonical isomorphisms

$$(2.12) \quad BH_n^{G, \mathcal{I}}(X, A; M|_{\mathcal{I}}) \cong BH_n^{G, \mathcal{F}}(X, A; M);$$

$$(2.13) \quad SH_n^{G, \mathcal{I}}(X, A; N|_{\mathcal{I}}) \cong SH_n^{G, \mathcal{F}}(X, A; N).$$

## 2.C. The construction of the equivariant Atiyah-Hirzebruch spectral sequence.

*Proof of Theorem 2.1.* Since  $(X, A)$  comes with the skeletal filtration, there is by a general construction a spectral sequence

$$E_{p,q}^r, \quad d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r$$

converging to  $\mathcal{H}_{p+q}^G(X, A)$ , whose  $E_1$ -term is given by

$$E_{p,q}^1 = \mathcal{H}_{p+q}^G(X_p, X_{p-1})$$

and the first differential is the composite

$$d_{p,q}^1 : E_{p,q}^1 = \mathcal{H}_{p+q}^G(X_p, X_{p-1}) \rightarrow \mathcal{H}_{p+q-1}^G(X_{p-1}) \rightarrow \mathcal{H}_{p+q-1}^G(X_{p-1}, X_{p-2}) = E_{p-1, q}^1,$$

where the first map is the boundary operator of the pair  $(X_p, X_{p-1})$  and the second is induced by the inclusion. The elementary construction is explained for trivial  $G$

for instance in [18, 15.6 on page 339]. The construction carries directly over to the equivariant setting.

The straightforward proof of the identification of  $E_{p,q}^2$  with  $BH_p^{G,\mathcal{F}}(X, A; \mathcal{H}_q)$  is left to the reader.  $\square$

## 2.D. Passing to the subgroup category.

**Condition 2.14** ( $\text{Sub}|_{\mathcal{F}}$ ). *Let  $\mathcal{H}_*^G(-)$  be a smooth  $G$ -homology theory. Then  $\mathcal{H}_*^G(-)$  satisfies the Condition ( $\text{Sub}|_{\mathcal{F}}$ ) if for any  $H \in \mathcal{F}$  and  $g \in C_G H$  the  $G$ -map  $R_{g^{-1}}: G/H \rightarrow G/H$  sending  $g'H$  to  $g'g^{-1}H$  induces the identity on  $\mathcal{H}_q^G(G/H)$ , i.e.,  $\mathcal{H}_q^G(R_{g^{-1}}) = \text{id}_{\mathcal{H}_q^G(G/H)}$ .*

**Remark 2.15.** Suppose that the  $G$ -homology theory  $\mathcal{H}_*^G$  satisfies the Condition ( $\text{Sub}|_{\mathcal{F}}$ ). Then the covariant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $\mathcal{H}_q^G$  sending  $G/H$  with  $H \in \mathcal{F}$  to  $\mathcal{H}_q^G(G/H)$  defines a covariant  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -module  $\overline{\mathcal{H}}_q^G: \text{Sub}_{\mathcal{F}}(G) \rightarrow \mathbb{Z}\text{-Mod}$  uniquely determined by  $\mathcal{H}_q^G = \overline{\mathcal{H}}_q^G \circ P$  for the projection  $P: \text{Or}_{\mathcal{F}}(G) \rightarrow \text{Sub}_{\mathcal{F}}(G)$ . Moreover, we obtain from (2.11) for every pair  $(X, A)$  of  $\mathcal{F}$ - $G$ -CW-complexes natural isomorphisms

$$BH_n^{G,\mathcal{F}}(X, A; \mathcal{H}_q^G(-)) \xrightarrow{\cong} SH_n^{G,\mathcal{F}}(X, A; \overline{\mathcal{H}}_q^G(-)).$$

Note that the right hand side is often easier to compute than the left hand side. One big advantage of  $\text{Sub}(G)$  in comparison with  $\text{Or}(G)$  is that for a finite subgroup  $H \subseteq G$  the set of automorphisms of  $H$  is the group  $N_G H/H \cdot C_G H$ , which is finite, whereas the set of automorphisms of  $G/H$  in  $\text{Or}(G)$  for a finite group  $H$  is the group  $N_G H/H$ , which is not necessarily finite. This is a key ingredient in the construction of an equivariant Chern character for discrete groups  $G$  and proper  $G$ -CW-complexes in [15, 16].

If  $G$  is abelian,  $\text{Sub}_{\mathcal{F}}(G)$  reduces to the poset of open subgroups of  $G$  ordered by inclusion.

## 2.E. The connective case.

**Theorem 2.16.** (i) *Suppose that  $\mathcal{H}_q^G(G/H) = 0$  for every  $H \in \mathcal{F}$  and  $q \in \mathbb{Z}$  with  $q < 0$ . Then we get for every pair  $(X, A)$  of  $\mathcal{F}$ - $G$ -CW-complexes and every  $q \in \mathbb{Z}$  with  $q < 0$*

$$\mathcal{H}_q^G(X, A) = 0;$$

(ii) *Choose a model  $E_{\text{COP}}(G)$  for the classifying space of smooth proper  $G$ -actions. Let  $\mathcal{I}$  be the set of isotropy groups of points in  $E_{\text{COP}}(G)$ . Suppose that  $\mathcal{H}_q^G(G/H) = 0$  for every open  $H \in \mathcal{I}$  and  $q \in \mathbb{Z}$  with  $q < 0$ .*

(a) *Then for every  $q < 0$  we have  $\mathcal{H}_q^G(E_{\text{COP}}(G)) = 0$ , the edge homomorphism induces an isomorphism*

$$BH_0^G(E_{\text{COP}}(G); \mathcal{H}_q^G(-)) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\mathcal{F}}(G))$$

and the canonical map

$$\text{colim}_{G/H \in \text{Or}_{\mathcal{F}}(G)} \mathcal{H}_0^G(G/H) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\mathcal{F}}(G))$$

is bijective;

(b) *Suppose additionally that  $\mathcal{H}_*^G$  satisfies Condition ( $\text{Sub}_{\mathcal{I}}$ ), see Condition 2.14. Then the edge homomorphism induces an isomorphism*

$$SH_0^G(E_{\text{COP}}(G); \overline{\mathcal{H}}_q^G(-)) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\mathcal{F}}(G))$$

and the canonical map

$$\text{colim}_{H \in \text{Sub}_{\mathcal{I}}(G)} \overline{\mathcal{H}}_0^G(H) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\text{COP}}(G))$$

is bijective.

*Proof.* (i) This follows directly from the smooth equivariant Atiyah-Hirzebruch spectral sequence of Theorem 2.1

(ii)a We get  $\mathcal{H}_q^G(E_{\text{COP}}(G)) = 0$  for  $q < 0$  from assertion (i).

We get from the the smooth equivariant Atiyah-Hirzebruch spectral sequence of Theorem 2.1 an isomorphism

$$BH_0^{G, \mathcal{I}}(E_{\text{COP}}(G); \mathcal{H}_0^G) = H_0(C_*^{\text{Or}_{\mathcal{I}}(G)}(E_{\text{COP}}(G)) \otimes_{\mathbb{Z}\text{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G) \xrightarrow{\cong} \mathcal{H}_0^G(E_{\text{COP}}(G)).$$

since  $E_{p,q}^2 = BH_0^{G, \mathcal{I}}(E_{\text{COP}}(G); \mathcal{H}_q^G) = 0$  is valid for  $p, q \in \mathbb{Z}$  if  $p < 0$  or  $q < 0$  holds. Since the  $\mathbb{Z}\text{Or}_{\mathcal{I}}(G)$ -module  $C_n^{\text{Or}_{\mathcal{I}}(G)}(E_{\text{COP}}(G))$  is free in the sense of [14, 9.16 on page 167] for  $n \geq 0$  by (2.6) and  $E_{\text{COP}}(G)^H$  is weakly contractible for  $H \in \mathcal{I}$ , the  $\mathbb{Z}\text{Or}_{\mathcal{I}}(G)$ -chain complex  $C_*^{\text{Or}_{\mathcal{I}}(G)}(E_{\text{COP}}(G))$  is a projective  $\mathbb{Z}\text{Or}_{\mathcal{I}}(G)$ -resolution of the constant contravariant  $\mathbb{Z}\text{Or}_{\mathcal{I}}(G)$ -module  $\mathbb{Z}$ , whose value is  $\mathbb{Z}$  at each object and assigns to any morphism  $\text{id}_{\mathbb{Z}}$ . Since  $- \otimes_{\mathbb{Z}\text{Or}_{\mathcal{I}}(G)} \mathcal{H}_q^G$  is right exact by [14, 9.23 on page 169], we get a isomorphism

$$H_0(C_*^{\text{Or}_{\mathcal{I}}(G)}(E_{\text{COP}}(G)) \otimes_{\mathbb{Z}\text{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G) \cong \mathbb{Z} \otimes_{\mathbb{Z}\text{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G.$$

We conclude from the adjunction appearing in [14, 9.21 on page 169] and the universal property of the colimit that there is a canonical isomorphism

$$\text{colim}_{G/H \in \text{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G(G/H) \cong \mathbb{Z} \otimes_{\mathbb{Z}\text{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G.$$

This finishes the proof of assertion (ii)a.

(ii)b This follows from assertion (ii)a, since we get from Condition (Sub $_{\mathcal{I}}$ ) a canonical isomorphism

$$\text{colim}_{G/H \in \text{Or}_{\mathcal{I}}(G)} \mathcal{H}_0^G(G/H) \xrightarrow{\cong} \text{colim}_{H \in \text{Sub}_{\mathcal{I}}(G)} \overline{\mathcal{H}}_q^G(H).$$

for the covariant  $\mathbb{Z}\text{Sub}_{\mathcal{I}}(G)$ -module  $\overline{\mathcal{H}}_q^G$  determined by the covariant  $\mathbb{Z}\text{Or}_{\mathcal{I}}(G)$ -module  $\mathcal{H}_q^G$ , see Remark 2.15.  $\square$

**2.F. The first differential.** Let  $X$  be an  $\mathcal{F}$ - $G$ -CW-complex. Suppose that  $X_0 = \coprod_{j \in J} G/V_j$  and that  $X_1$  is given by the  $G$ -pushout

$$(2.17) \quad \begin{array}{ccc} \coprod_{i \in I} G/U_i \times S^0 & \xrightarrow{\coprod_{i \in I} q_i} & X_0 \\ \downarrow & & \downarrow \\ \coprod_{i \in I} G/U_i \times D^1 & \xrightarrow{\coprod_{i \in I} Q_i} & X_1. \end{array}$$

We want to figure out the map of  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -modules  $\gamma$  making the following diagram commute

$$\begin{array}{ccc} \bigoplus_{i \in I} \mathbb{Z} \text{mor}_{\text{Or}_{\mathcal{F}}(G)}(?, G/U_i) & \xrightarrow{\gamma} & \bigoplus_{j \in J} \mathbb{Z} \text{mor}_{\text{Or}_{\mathcal{F}}(G)}(?, G/V_j) \\ \cong \downarrow & & \downarrow \cong \\ C_1^{\text{Or}_{\mathcal{F}}(G)}(X) & \xrightarrow{c_n} & C_0^{\text{Or}_{\mathcal{F}}(G)}(X) \end{array}$$

where the vertical isomorphisms come from the isomorphisms (2.6). In order to describe  $\gamma$ , we have to define for each  $i \in I$  and  $j \in J$  a map of  $\mathbb{Z}\text{Or}(G)$ -modules

$$\gamma_{i,j}: \mathbb{Z} \text{mor}_{\text{Or}_{\mathcal{F}}(G)}(?, G/H_i) \rightarrow \mathbb{Z} \text{mor}_{\text{Or}_{\mathcal{F}}(G)}(?, G/K_j)$$



such that  $\{j \in I_{n-1} \mid \gamma_{i,j} \neq 0\}$  is finite for every  $i \in I_n$ . Note that  $\gamma_{i,j}$  is determined by the image of  $\text{id}_{G/H_i}$ . Hence we need to specify for  $i \in I$  and  $j \in J$  an element

$$(2.18) \quad \overline{\gamma_{i,j}} \in \mathbb{Z} \text{mor}_{\text{Or}_{\mathcal{F}}(G)}(G/U_i, G/V_j) = \mathbb{Z} \text{map}_G(G/U_i, G/V_j).$$

For each  $i \in I$  there are two elements  $j_-(i)$  and  $j_+(i)$  in  $J$  such that the image of  $G/H_i \times \{\pm 1\}$  under the map  $q_i$  appearing in (2.17) is the summand  $G/K_{j_{\pm}}(i)$  belonging to  $j_{\pm}(i)$  of  $\coprod_{j \in I_0} G/K_j$ , if we write  $S^0 = \{-1, 1\}$ . Denote by  $(q_i^{\pm})_{\pm 1}: G/H_i \rightarrow G/K_{j_{\pm}}(i)$  the restriction of  $q_i^{\pm}$  to  $G/H_i \times \{\pm 1\}$ . We leave the elementary proof of the next lemma to the reader.

**Lemma 2.19.** *We get in  $\mathbb{Z} \text{map}_G(G/H_i, G/K_j)$*

$$\overline{\gamma_{i,j}} = \begin{cases} \pm[(q_i^{\pm})_{\pm 1}] & \text{if } j = j_{\pm}(i) \text{ and } j_-(i) \neq j_+(i); \\ [(q_i^{\pm})_{+1}] - [(q_i^{\pm})_{-1}] & \text{if } j = j_-(i) = j_+(i); \\ 0 & \text{if } j \notin \{j_-(i), j_+(i)\}. \end{cases}$$

**Remark 2.20.** This implies for the  $\mathbb{Z}$ -chain complex  $C_*^{\text{Or}_{\mathcal{F}}(G)}(X, A) \otimes_{\text{Or}_{\mathcal{F}}(G)} M$  for a covariant  $\mathbb{Z}\text{Or}_{\mathcal{F}}(G)$ -module  $M$  that its first differential agrees with the  $\mathbb{Z}$ -homomorphism

$$\alpha = (\alpha_{i,j})_{i \in I, j \in J}: \bigoplus_{i \in I} M(G/U_i) \rightarrow \bigoplus_{j \in J} M(G/V_j),$$

where the  $\mathbb{Z}$ -homomorphisms  $\alpha_{i,j}: M(G/U_i) \rightarrow M(G/V_j)$  are given as follows. We get in the notation of Lemma 2.19

$$\alpha_{i,j} = \begin{cases} \pm M((q_i^{\pm})_{\pm}) & \text{if } j = j_{\pm}(i) \text{ and } j_-(i) \neq j_+(i); \\ M((q_i^{\pm})_{+1}) - M((q_i^{\pm})_{-1}) & \text{if } j = j_-(i) = j_+(i); \\ 0 & \text{if } j \notin \{j_-(i), j_+(i)\}. \end{cases}$$

Note that the cokernel of  $\alpha$  is  $BH_0^{G, \mathcal{F}}(X; M)$ .

We get a computation of the first differential of  $C_*^{\text{Sub}_{\mathcal{F}}(G)}(X, A) \otimes_{\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)} N$  for a covariant  $\mathbb{Z}\text{Sub}_{\mathcal{F}}(G)$ -module  $N$  from the isomorphism (2.10). Explicitly the first differential is given by

$$\beta = (\beta_{i,j})_{i \in I, j \in J}: \bigoplus_{i \in I_n} N(U_i) \rightarrow \bigoplus_{j \in I_{n-1}} N(V_j),$$

where the  $\mathbb{Z}$ -homomorphisms  $\beta_{i,j}: N(G/U_i) \rightarrow N(G/V_j)$  are given as follows. Choose for the map  $(q_i)_{\pm}: G/U_i \rightarrow G/V_j$  an element  $(g_i)_{\pm}$  with  $(q_i)_{\pm}(eU_i) = (g_i)_{\pm}^{-1}V_j$ . Let  $[c(g_i)_{\pm}]: U_i \rightarrow V_j$  be the morphism in  $\text{Sub}_{\mathcal{F}}(G)$  represented by  $c(g_i)_{\pm}: U_i \rightarrow V_j$  sending  $u$  to  $gu_g^{-1}$ . Then

$$\beta_{i,j} = \begin{cases} \pm N([c(g_i)_{\pm}]) & \text{if } j = j_{\pm}(i) \text{ and } j_-(i) \neq j_+(i); \\ N([c(g_i)_{+}]) - N([c(g_i)_{-}]) & \text{if } j = j_-(i) = j_+(i); \\ 0 & \text{if } j \notin \{j_-(i), j_+(i)\}. \end{cases}$$

Note that the cokernel of  $\beta$  is  $SH_0^{G, \mathcal{F}}(X; N)$ .

### 3. A BRIEF REVIEW OF THE FARRELL JONES CONJECTURE FOR THE ALGEBRAIC $K$ -THEORY OF HECKE ALGEBRAS

In this section we give a review of the Farrell Jones Conjecture for the algebraic  $K$ -theory of Hecke algebras. Further information can be found in [2, 3].

Let  $R$  be a (not necessarily commutative) associative unital ring with  $\mathbb{Q} \subseteq R$ . Let  $G$  be a td-group. Let  $\mathcal{H}(G; R)$  be the associated Hecke algebra.

One can construct a covariant functor

$$\mathbf{K}_R: \text{Or}_{\mathcal{O}_p}(G) \rightarrow \text{Spectra};$$

such that  $\pi_n(\mathbf{K}_R(Q'/U')) \cong K_n(\mathcal{H}(U; R))$  holds for any  $n \in \mathbb{Z}$  and open subgroup  $U \subseteq Q$ . Associated to it is a smooth  $G$ -homology theory  $H_*^G(-; \mathbf{K}_R)$  such that

$$(3.1) \quad H_n^G(G/U; \mathbf{K}_R) \cong K_n(\mathcal{H}(U; R))$$

holds for every  $n \in \mathbb{Z}$  and every open subgroup  $U \subseteq Q$ .

The next result follows from [3, Corollary 1.8].

**Theorem 3.2.** *Let  $G$  be a td-group which is modulo a normal compact subgroup a subgroup of a reductive  $p$ -adic group. Let  $R$  be a uniformly regular ring with  $\mathbb{Q} \subseteq R$ .*

*Then the map induced by the projection  $E_{\text{cop}}(G) \rightarrow G/G$  induces for every  $n \in \mathbb{Z}$  an isomorphism*

$$H_n^G(E_{\text{cop}}(G); \mathbf{K}_R) \xrightarrow{\cong} H_n^G(G/G; \mathbf{K}_R) = K_n(\mathcal{H}(G; R)).$$

#### 4. PROOF OF THE MAIN THEOREM 1.1

*Proof of Theorem 1.1.* (i) This is exactly Theorem 3.2.

(ii) Since an open group homomorphism  $U \rightarrow V$  between two td-groups induces a ring homomorphism  $\mathcal{H}(U; R) \rightarrow \mathcal{H}(V; R)$  between the Hecke algebras and hence a homomorphism  $K_n(\mathcal{H}(U; R)) \rightarrow K_n(\mathcal{H}(V; R))$  and inner automorphisms of a td-group  $U$  induce the identity on  $K_n(\mathcal{H}(U; R))$ , we get a covariant  $\mathbb{Z}\text{Sub}_{\text{com}}(G)$ -module  $K_n(\mathcal{H}(?; R))$  whose value at  $U$  is  $K_n(\mathcal{H}(U; R))$ . Since the isomorphism (3.1) is natural, we get an isomorphisms of covariant  $\mathbb{Z}\text{Or}_{\mathcal{O}_p}(G)$ -modules

$$P^* K_n(\mathcal{H}(?; R)) \xrightarrow{\cong} \pi_n(\mathbf{K}_R)$$

for the projection  $P: \text{Or}_{\mathcal{O}_p}(G) \rightarrow \text{Sub}_{\mathcal{O}_p}(G)$  of (2.4). So the smooth equivariant Atiyah-Hirzebruch spectral sequence applied to the smooth homology theory  $H_*^G(-; \mathbf{K}_R)$  takes for a  $\mathcal{F}$ - $G$ -CW-complexes  $X$  the form

$$(4.1) \quad E_{p,q}^2 = SH_q^{G, \mathcal{F}}(X; K_q(\mathcal{H}(?; R))) \implies H_{p+q}^G(X; \mathbf{K}_R).$$

Now assertion (ii) follows from the special case  $X = E_{\text{cop}}(G)$  and assertion (i).

(iii) and (iv) As  $K_q(\mathcal{H}(K; R))$  vanishes for every compact td-group  $K$  and every  $q \leq -1$ , see [2, Lemma 8.1], assertions (iii), and (iv) follow from Theorem 2.16 applied in the case  $X = E_{\text{cop}}(G)$  and from assertion (i). This finishes the proof of the Main Theorem 1.1.  $\square$

#### 5. THE MAIN RECIPE FOR THE COMPUTATION OF THE PROJECTIVE CLASS GROUP

Throughout this section  $G$  will be a td-group and  $R$  be a uniformly regular ring with  $\mathbb{Q} \subseteq R$ , e.g., a field of characteristic zero. We will assume that the assembly map  $H_n^G(E_{\text{cop}}(G); \mathbf{K}_R) \rightarrow H_n^G(G/G; \mathbf{K}_R) = K_n(\mathcal{H}(G; R))$  is bijective for all  $n \in \mathbb{Z}$ . This is known to be true for subgroups of reductive  $p$ -adic groups by Theorem 3.2.

**5.A. The general case.** Let  $X$  be an abstract simplicial complex with a simplicial  $G$ -action such that all isotropy groups are compact open, the  $G$ -action is cellular, and  $|X|^K$  is non-empty and connected for every compact open subgroup  $K$  of  $G$ .

We can choose a subset  $V$  of the set of vertices of  $X$  such that the  $G$ -orbit through any vertex in  $X$  meets  $V$  in precisely one element. Fix a total ordering on  $V$ . Let  $E$  be the subset of  $V \times V$  consisting of those pairs  $(v, w)$  such that  $v \leq w$  holds and there exists  $g \in G$  for which  $v$  and  $gw$  satisfy  $v \neq gw$  and span an edge  $[v, gw]$  in  $X$ . For  $(v, w) \in E$  define  $\overline{F(v, w)}$  to be the subset of  $G_v \backslash G/G_w$  consisting of elements  $x$  for which  $v$  and  $gw$  satisfy  $v \neq gw$  and span an edge  $[v, gw]$  in  $X$  for

some (and hence all) representative  $g$  of  $x$ . Choose a subset  $F(v, w)$  of  $G$  such that the projection  $G \rightarrow G_v \backslash G/G_w$  induces a bijection  $F(v, w) \rightarrow \overline{F(v, w)}$ .

Then for every edge of  $X$  the  $G$ -orbit through it meets the set  $\{[v, gw] \mid (v, w) \in E, g \in F(v, w)\}$  in precisely one element. Moreover, the 0-skeleton of  $|X|$  is given by  $|X|_0 = \coprod_{u \in V} G/G_u$  and  $|X|_1$  is given by the  $G$ -pushout

$$\begin{array}{ccc} \coprod_{(v,w) \in E} \coprod_{g \in F(v,w)} G/(G_v \cap G_{gw}) \times S^0 & \xrightarrow{\coprod_{(v,w) \in E} \coprod_{g \in F(v,w)} q_{(v,w),g}} & |X|_0 \\ \downarrow & & \downarrow \\ \coprod_{(v,w) \in E} \coprod_{g \in F(v,w)} G/(G_v \cap G_{gw}) \times D^1 & \longrightarrow & |X|_1 \end{array}$$

where  $q_{(v,w),g}: G/(G_v \cap G_{gw}) \times S^0 \rightarrow |X|_0 = \coprod_{u \in V} G/G_u$  is defined as follows. Write  $S^0 = \{-1, 1\}$ . The restriction of  $q_{(v,w),g}$  to  $G/(G_v \cap G_{gw}) \times \{-1\}$  lands in the summand  $G/G_v$  and is given by canonical projection. The restriction of  $q_{(v,w),g}$  to  $G/(G_v \cap G_{gw}) \times \{1\}$  lands in the summand  $G/G_w$  and is given by the  $G$ -map  $R_{g^{-1}}: G/(G_v \cap G_{gw}) \rightarrow G/G_w$  sending  $z(G_v \cap G_{gw})$  to  $zgG_w$ .

Next we define a map

$$\beta = (\beta_{(v,w),g,u}): \bigoplus_{(v,w) \in E} \bigoplus_{g \in F(v,w)} K_0(\mathcal{H}(G_v \cap G_{gw}; R)) \rightarrow \bigoplus_{u \in V} K_0(\mathcal{H}(G_u; R)).$$

If  $u = v$ , then  $\beta_{(v,w),g,v}: K_0(\mathcal{H}(G_v \cap G_{gw}; R)) \rightarrow K_0(\mathcal{H}(G_v; R))$  is the map induced by the inclusion  $G_v \cap G_{gw} \rightarrow G_v$  multiplied with  $(-1)$ . If  $u = w$ , then  $\beta_{(v,w),g,w}: K_0(\mathcal{H}(G_v \cap G_{gw}; R)) \rightarrow K_0(\mathcal{H}(G_w; R))$  is the map induced by the group homomorphism  $G_v \cap G_{gw} \rightarrow G_w$  sending  $z$  to  $g^{-1}zg$ . If  $u \notin \{v, w\}$ , then  $\beta_{(v,w),g,u} = 0$ .

**Lemma 5.1.** *The cokernel of  $\beta$  is isomorphic to  $K_0(\mathcal{H}(G; R))$ .*

*Proof.* We conclude from Remark 2.20 that the cokernel of  $\beta$  is  $SH_0^{G, \text{COP}}(X; \overline{K_0^G(-)})$ . The up to  $G$ -homotopy unique  $G$ -map  $f: X \rightarrow E_{\text{COP}}(G)$  induces for every compact open subgroup  $K \subset G$  a 1-connected map  $f^K: |X|^K \rightarrow E_{\text{COP}}(G)^K$ . This implies that the map  $SH_0^{G, \text{COP}}(X; \overline{K_0^G(-)}) \rightarrow SH_0^{G, \text{COP}}(E_{\text{COP}}(G); \overline{K_0^G(-)})$  induced by  $f$  is an isomorphism, see [14, Proposition 23 (iii) on page 35]. Theorem 2.16 (ii)b implies  $SH_0^G(E_{\text{COP}}(G); \overline{K_0^G(-)}) \cong H_0^G(E_{\text{COP}}(G); \mathbf{K}_R)$ . Since by assumption we have  $H_0^G(E_{\text{COP}}(G); \mathbf{K}_R) \cong K_0(\mathcal{H}(G; R))$ , Lemma 5.1 follows.  $\square$

**Remark 5.2.** Suppose additionally that  $X$  possesses a strict fundamental domain  $\Delta$ , i.e., a simplicial subcomplex  $\Delta$  that contains exactly one simplex from each orbit for the  $G$ -action on the set of simplices of  $X$ . Then one can take  $V$  to be the set of vertices of  $\Delta$  and for  $(v, w) \in E$  the set  $F(v, w)$  to be  $\{e\}$ . Moreover,  $\beta$  reduces to the map

$$\beta = (\beta_{(v,w),u}): \bigoplus_{(v,w) \in E} K_0(\mathcal{H}(G_v \cap G_w; R)) \rightarrow \bigoplus_{u \in V} K_0(\mathcal{H}(G_u; R)).$$

where  $\beta_{(v,w),u}$  is the map induced by the inclusion  $G_v \cap G_w \rightarrow G_v$  multiplied with  $(-1)$  for  $u = v$ , the map induced by the inclusion  $G_v \cap G_w \rightarrow G_w$  for  $u = w$ , and zero for  $u \notin \{v, w\}$ . Note that  $E$  is the subset of  $V \times V$  consisting of elements  $(v, w)$  for which  $v < w$  holds and  $v$  and  $w$  span an edge  $[v, w]$  in  $\Delta$ .

**5.B. A variation.** Consider a central extension  $1 \rightarrow \tilde{C} \rightarrow \tilde{G} \xrightarrow{\text{pr}} G \rightarrow 1$  of td-groups together with a group homomorphism  $\mu: \tilde{G} \rightarrow \mathbb{Z}$  such that  $\tilde{C} \cap \tilde{M}$  is compact for  $\tilde{M} := \ker(\mu)$ . We still consider the abstract simplicial complex  $X$  of Subsection 5.A coming with a simplicial  $G$ -action such that all isotropy

groups are compact open, and  $|X|^K$  is non-empty and connected for every compact open subgroup  $K$  of  $G$ . Furthermore, we will assume that the assembly map  $H_n^{\tilde{G}}(E_{\text{COP}}(\tilde{G}); \mathbf{K}_R) \rightarrow H_n^{\tilde{G}}(\tilde{G}/\tilde{G}; \mathbf{K}_R) = K_0(\mathcal{H}(\tilde{G}; R))$  is bijective for all  $n \in \mathbb{Z}$ .

If  $\tilde{C}$  is compact, then we can consider  $X$  as a  $\tilde{G}$ -CW-complex by restricting the  $G$ -action with  $\text{pr}$  and Subsection 5.A applies. Hence we will assume that  $\tilde{C}$  is not compact, or, equivalently, that  $\tilde{C}$  is not contained in the kernel  $\tilde{M} := \ker(\mu)$ . Then the index  $m := [\mathbb{Z} : \mu(C)]$  is a natural number  $m \geq 1$ . We fix an element  $\tilde{c} \in \tilde{C}$  with  $\mu(\tilde{c}) = m$ . In sequel we choose for every  $g \in G$  an element  $\tilde{g}$  in  $\tilde{G}$  satisfying  $\text{pr}(\tilde{g}) = g$  and denote for an open subgroup  $U \subseteq G$  by  $\tilde{U} \subseteq \tilde{G}$  its preimage under  $\text{pr}: \tilde{G} \rightarrow G$ . Let

$$\gamma: \bigoplus_{(v,w) \in E} \bigoplus_{g \in F(v,w)} K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{G}_{gw} \cap \tilde{M}; R)) \rightarrow \bigoplus_{u \in V} K_0(\mathcal{H}(\tilde{G}_u \cap \tilde{M}; R))$$

be the map whose component for  $(v, w) \in E$ ,  $g \in F(v, w)$ , and  $u \in V$  is the map

$$(5.3) \quad \gamma_{(v,w),g,u}: K_0(\mathcal{H}(\tilde{G}_v \cap \tilde{G}_{gw} \cap \tilde{M}; R)) \rightarrow K_0(\mathcal{H}(\tilde{G}_u \cap \tilde{M}; R))$$

defined next. If  $u = v$ , it is the map coming from the inclusion  $\tilde{G}_v \cap \tilde{G}_{gw} \cap \tilde{M} \rightarrow \tilde{G}_v \cap \tilde{M}$  multiplied with  $(-1)$ . If  $u = w$ , it is the map coming from the group homomorphism  $\tilde{G}_v \cap \tilde{G}_{gw} \cap \tilde{M} \rightarrow \tilde{G}_w \cap \tilde{M}$  sending  $x$  to  $\tilde{g}x\tilde{g}^{-1}$ . If  $u \notin \{v, w\}$ , it is trivial. Note that this definition is independent of the choice of  $\tilde{g} \in \tilde{G}$  satisfying  $\text{pr}(\tilde{g}) = g$  for  $g \in F(v, w)$ .

**Lemma 5.4.** *The cokernel of  $\gamma$  is  $K_0(\mathcal{H}(\tilde{G}; R))$ .*

*Proof.* Note that  $|X| \times \mathbb{R}$  carries the  $G \times \mathbb{Z}$ -CW-complex structure coming from the product of the  $G$ -CW-complex structure on  $|X|$  and the standard free  $\mathbb{Z}$ -CW-structure on  $\mathbb{R}$ . Since the  $\mathbb{Z}$ -CW-complex  $\mathbb{R}$  has precisely one equivariant 1-cell and one equivariant 0-cell, the set of equivariant 0-cells of the  $G \times \mathbb{Z}$ -CW-complex  $|X| \times \mathbb{R}$  can be identified with the set  $V$  and the set of equivariant 1-cells can be identified with the disjoint union of  $V$  and the set  $\coprod_{(v,w) \in E} F(v, w)$ . Now the 0-skeleton of  $|X| \times \mathbb{R}$  is given by the disjoint union  $\coprod_{u \in V} \tilde{G}/\tilde{G}_u \times \mathbb{Z}$  and the 1-skeleton of  $|X| \times \mathbb{R}$  is given by the  $G \times \mathbb{Z}$ -pushout

$$(5.5) \quad \begin{array}{ccc} \coprod_{v \in V} \tilde{G}/\tilde{G}_v \times \mathbb{Z} \times S^0 & & \\ \coprod_{(v,w) \in E} \coprod_{g \in F(v,w)} \tilde{G}/(\tilde{G}_v \cap \tilde{G}_{gw}) \times \mathbb{Z} \times S^0 & \xrightarrow{\tilde{q}} & \coprod_{u \in V} \tilde{G}/\tilde{G}_u \times \mathbb{Z} \\ \downarrow & & \downarrow \\ \coprod_{v \in V} \tilde{G}/\tilde{G}_v \times \mathbb{Z} \times D^1 & & \\ \coprod_{(v,w) \in E} \coprod_{g \in F(v,w)} \tilde{G}/(\tilde{G}_v \cap \tilde{G}_{gw}) \times \mathbb{Z} \times S^0 & \longrightarrow & (|X| \times \mathbb{R})_1 \end{array}$$

where  $\tilde{q}$  is given as follows. Write  $S^0 = \{-1, 1\}$ . Fix  $u \in V$ . The restriction of  $\tilde{q}$  to the summand  $\tilde{G}/\tilde{G}_v \times \mathbb{Z} \times \{\epsilon\}$  lands in the summand  $\tilde{G}/\tilde{G}_v \times \mathbb{Z}$  and is given by  $\text{id}$  for  $\epsilon = -1$  and by  $\text{id} \times \text{sh}_1$  for  $\epsilon = 1$ , where  $\text{sh}_a: \mathbb{Z} \rightarrow \mathbb{Z}$  sends  $b$  to  $a + b$  for  $a, b \in \mathbb{Z}$ . Fix  $(v, w) \in E$  and  $g \in F(v, w)$ . The restriction of  $\tilde{q}$  to the summand  $\tilde{G}/(\tilde{G}_v \cap \tilde{G}_{gw}) \times \mathbb{Z} \times \{-1\}$  belonging to  $(v, w)$  and  $g$  lands in the summand for  $u = v$  and is the canonical projection  $\tilde{G}/(\tilde{G}_v \cap \tilde{G}_{gw}) \times \mathbb{Z} \rightarrow \tilde{G}/\tilde{G}_v \times \mathbb{Z}$ . The restriction of  $\tilde{q}$  to the summand  $\tilde{G}/(\tilde{G}_v \cap \tilde{G}_{gw}) \times \mathbb{Z} \times \{1\}$  belonging to  $(v, w)$  and  $g$  lands in the

summand for  $u = w$  and is the map  $R_{\tilde{g}^{-1}} \times \text{id}_{\mathbb{Z}}: \tilde{G}/(\tilde{G}_v \cap \tilde{G}_{gw}) \times \mathbb{Z} \rightarrow \tilde{G}/\tilde{G}_w \times \mathbb{Z}$ , where  $R_{\tilde{g}^{-1}}$  sends  $\tilde{z}(\tilde{G}_v \cap \tilde{G}_{gw})$  to  $\tilde{z}\tilde{g}^{-1}\tilde{G}_w$ .

We have the group homomorphism

$$\iota := \text{pr} \times \mu: \tilde{G} \rightarrow G \times \mathbb{Z}.$$

Its kernel is  $\tilde{C} \cap \tilde{M}$ . Its image has finite index in  $G \times \mathbb{Z}$ , which agrees with the index  $m$  of the image of  $\mu$  in  $\mathbb{Z}$ .

We are interested in the  $\tilde{G}$ -CW-complex  $\iota^*(|X| \times \mathbb{R})$  obtained by restriction with  $\iota$  from the  $G \times \mathbb{Z}$ -CW-complex  $|X| \times \mathbb{R}$ . So we have to analyze how the  $G \times \mathbb{Z}$ -cells in  $\iota^*(|X| \times \mathbb{R})$  viewed as  $\tilde{G}$ -spaces decompose as disjoint union of  $\tilde{G}$ -cells. Consider any open subgroup  $U \subseteq G$ . Then we obtain a  $\tilde{G}$ -homeomorphism

$$\alpha(U): \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{U} \cap \tilde{M}) \xrightarrow{\cong} \iota^*(G/U \times \mathbb{Z})$$

by sending the element  $\tilde{z}(\tilde{U} \cap \tilde{M})$  in the  $p$ -th summand to  $(\text{pr}(\tilde{z})U, \mu(\tilde{z}) + p)$ . Next we have to analyze the naturality properties of  $\alpha(U)$ . The following diagram commutes for  $a \in \mathbb{Z}$

$$\begin{array}{ccc} \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{U} \cap \tilde{M}) & \xrightarrow{\alpha(U)} & \iota^*(G/U \times \mathbb{Z}) \\ \hat{\pi}^a \downarrow & & \downarrow \text{id} \times \text{sh}_a \\ \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{U} \cap \tilde{M}) & \xrightarrow{\alpha(U)} & \iota^*(G/U \times \mathbb{Z}) \end{array}$$

where  $\hat{\pi}$  sends the summand for  $p = 0, \dots, m-2$  by the identity to the summand for  $p+1$  and sends the summand for  $p = m-1$  to the summand for  $p = 0$  by the map  $R_{\tilde{c}}: \tilde{G}/(\tilde{U} \cap \tilde{M}) \rightarrow \tilde{G}/(\tilde{U} \cap \tilde{M})$  for  $\tilde{c} \in \tilde{C}$  satisfying  $\mu(\tilde{c}) = m$ . Note for the sequel that the endomorphism  $\pi_n(\mathbf{K}_R(R_{\tilde{c}}))$  of  $\pi_n(\mathbf{K}_R(\tilde{G}/(\tilde{U} \cap \tilde{M}))) = K_0(\mathcal{H}(\tilde{U} \cap \tilde{M}))$  is the identity, since conjugation with  $\tilde{c}$  induces the identity on  $\tilde{U} \cap \tilde{M}$ .

Consider two open subgroups  $U$  and  $V$  of  $G$  and an element  $g \in G$  with  $gUg^{-1} \subseteq V$ . Then we get well-defined  $\tilde{G}$ -maps  $R_{\tilde{g}^{-1}}: \tilde{G}/(\tilde{U} \cap \tilde{M}) \rightarrow \tilde{G}/(\tilde{V} \cap \tilde{M})$  sending  $\tilde{z}(\tilde{U} \cap \tilde{M})$  to  $\tilde{z}\tilde{g}^{-1}(\tilde{V} \cap \tilde{M})$  and  $R_{\tilde{g}^{-1}} \times \text{id}: \iota^*(G/U \times \mathbb{Z}) \rightarrow \iota^*(G/V \times \mathbb{Z})$  sending  $(zU, n)$  to  $(zg^{-1}V, n)$  and the following diagram commutes

$$\begin{array}{ccc} \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{U} \cap \tilde{M}) & \xrightarrow[\cong]{\alpha(U)} & \iota^*(G/U \times \mathbb{Z}) \\ \coprod_{p=0}^{m-1} R_{\tilde{g}^{-1}} \downarrow & & \downarrow R_{\tilde{g}^{-1}} \times \text{sh}_{\mu(\tilde{g}^{-1})} \\ \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{V} \cap \tilde{M}) & \xrightarrow[\cong]{\alpha(V)} & \iota^*(G/V \times \mathbb{Z}). \end{array}$$

In particular the following diagram commutes

$$\begin{array}{ccc} \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{U} \cap \tilde{M}) & \xrightarrow[\cong]{\alpha(U)} & \iota^*(G/U \times \mathbb{Z}) \\ \pi^{\mu(\tilde{g})} \circ (\coprod_{p=0}^{m-1} R_{\tilde{g}^{-1}}) \downarrow & & \downarrow R_{\tilde{g}^{-1}} \times \text{id} \\ \coprod_{p=0}^{m-1} \tilde{G}/(\tilde{V} \cap \tilde{M}) & \xrightarrow[\cong]{\alpha(V)} & \iota^*(G/V \times \mathbb{Z}). \end{array}$$

Now we obtain from the  $G \times \mathbb{Z}$ -pushout (5.5) by applying restriction with  $\iota$  and the maps  $\alpha_U$  above a  $\tilde{G}$ -pushout describing how the 1-skeleton of the  $\tilde{G}$ -CW-complex  $\iota^*(|X| \times \mathbb{R})$  is obtained from its 0-skeleton and explicit descriptions of the attaching maps.

In the sequel  $A^m$  stands for the  $m$ -fold direct sum of copies of  $A$  for an abelian group  $A$  and  $\pi: A^m \rightarrow A^m$  denotes the permutation map sending  $(a_1, a_2, \dots, a_m)$  to  $(a_m, a_1, \dots, a_{m-1})$  and  $\text{aug}: A^m \rightarrow A$  denotes the augmentation map sending  $(a_1, \dots, a_m)$  to  $a_1 + \dots + a_m$ .

Let  $\delta$  be the map given by the direct sum

$$\delta = \bigoplus_{v \in V} \delta_v: \bigoplus_{v \in V} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m \rightarrow \bigoplus_{v \in V} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m$$

where  $\delta_v: K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m \rightarrow K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m$  is  $\pi - \text{id}$ . Let

$$\epsilon: \bigoplus_{(v,w) \in E} \bigoplus_{g \in F(v,w)} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M}; R))^m \rightarrow \bigoplus_{u \in V} K_0(\mathcal{H}(\widetilde{G}_u \cap \widetilde{M}; R))^m$$

be the map given by the components  $\epsilon_{(v,w),g,u}$  defined as follows. For  $u = v$  the map  $\epsilon_{(v,w),g,v}$  is the  $m$ -fold direct sum  $\gamma_{(v,w),g,v}^m$  of the maps  $\gamma_{(v,w),g,v}$  defined in (5.3). For  $u = w$  we put

$$\begin{aligned} \epsilon_{(v,w),g,w}: K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M}; R))^m &\xrightarrow{\gamma_{(v,w),g,w}^m} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m \\ &\xrightarrow{\pi^{\mu(\tilde{g})}} K_0(\mathcal{H}(\widetilde{G}_w \cap \widetilde{M}; R))^m. \end{aligned}$$

Since  $\pi^m = \text{id}$ , the map  $\pi^{\mu(\tilde{g})}$  depends only on  $\bar{\mu}(g)$ , where  $\bar{\mu}: G \rightarrow \mathbb{Z}/m$  sends  $g$  to the image of  $\tilde{g}$  under the projection  $\mathbb{Z} \rightarrow \mathbb{Z}/m$  for any choice of an element  $\tilde{g} \in \widetilde{G}$  with  $\text{pr}(\tilde{g}) = g$ .

The cokernel of the map

$$\begin{aligned} \delta \oplus \epsilon: \left( \bigoplus_{v \in V} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m \right) \oplus \left( \bigoplus_{(v,w) \in E} \bigoplus_{g \in F(v,w)} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M}; R))^m \right) \\ \rightarrow \bigoplus_{u \in V} K_0(\mathcal{H}(\widetilde{G}_u \cap \widetilde{M}; R))^m \end{aligned}$$

is  $K_0(\mathcal{H}(\widetilde{G}; R))$  because of Theorem 2.16 (ii)b and Remark 2.20 by the same argument as it appears in the proof of Lemma 5.1 since  $(\iota^*(|X| \times \mathbb{R}))^K$  is connected for every compact open subgroup  $K$  of  $\widetilde{G}$ . It does not matter that  $\iota^*(|X| \times \mathbb{R})$  is a  $\widetilde{G}$ -CW-complex but not a simplicial complex, since in the description of  $\beta_{i,j}$  appearing in Remark 2.20 the case  $j_i(+) = j_-(i)$  never occurs.

We can identify  $\bigoplus_{v \in V} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))$  and the cokernel of  $\delta$ , since we have the exact sequence  $A^m \xrightarrow{\pi - \text{id}} A^m \xrightarrow{\alpha} A \rightarrow 0$  for every abelian group  $A$ . The cokernel of  $\delta \oplus \epsilon$  is isomorphic to the cokernel of the composite of  $\epsilon$  with the map

$$\bigoplus_{v \in V} \text{aug}: \bigoplus_{v \in V} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R))^m \rightarrow \bigoplus_{v \in V} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{M}; R)) = \text{cok}(\delta).$$

For every  $(v, w) \in E$ ,  $g \in F(v, w)$ , and  $u \in V$  the diagram

$$\begin{array}{ccc} K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M}; R))^m & \xrightarrow{\epsilon_{(v,w),u}} & K_0(\mathcal{H}(\widetilde{G}_u \cap \widetilde{M}; R))^m \\ \text{aug} \downarrow & & \downarrow \text{aug} \\ K_0(\mathcal{H}(\widetilde{G}_v \cap \widetilde{G}_{gw} \cap \widetilde{M}; R)) & \xrightarrow{\gamma_{(v,w),u}} & K_0(\mathcal{H}(\widetilde{G}_u \cap \widetilde{M}; R)) \end{array}$$

commutes, since  $\alpha \circ \pi = \alpha$  holds. This finishes the proof of Lemma 5.4.  $\square$

6. THE PROJECTIVE CLASS GROUP OF THE HECKE ALGEBRAS OF  $\mathrm{SL}_n(F)$ ,  
 $\mathrm{PGL}_n(F)$  AND  $\mathrm{GL}_n(F)$ 

Next we apply the recipes of Sections 5 to some prominent reductive  $p$ -adic groups  $G$  as an illustration. For the remainder of this section  $R$  is a uniformly regular ring with  $\mathbb{Q} \subseteq R$ .

Note that for a reductive  $p$ -adic groups  $G$  the assembly map  $H_n^G(E_{\mathrm{cop}}(G); \mathbf{K}_R) \rightarrow H_n^G(G/G; \mathbf{K}_R) = K_n(\mathcal{H}(G; R))$  is bijective for all  $n \in \mathbb{Z}$  by Theorem 3.2. Moreover, the Bruhat-Tits building  $X$  of  $G$  or of  $G/\mathrm{cent}(G)$  can serve as the desired simplicial complex  $X$  appearing in Section 5. The original construction of the Bruhat-Tits building can be found in [8]. For more information about buildings we refer to [1, 6, 7, 17]. The space  $X$  carries a CAT(0)-metric, which is invariant under the action of  $G$  or  $G/\mathrm{cent}(G)$ , see [6, Theorem 10A.4 on page 344], Hence  $|X|^H$  is contractible for any compact open subgroup  $H$  of  $G$  or  $G/\mathrm{cent}(G)$ , since  $X^H$  is a convex non-empty subset of  $X$  and hence contractible by [6, Corollary II.2.8 on page 179]. Therefore the geometric realization of the Bruhat-Tits building  $X$  is (after possibly subdividing to achieve a cellular action) a model for  $E_{\mathrm{cop}}(G)$  or of  $E_{\mathrm{cop}}(G/\mathrm{cent}(G))$ .

6.A.  $\mathrm{SL}_n(F)$ . We begin with computing  $K_0(\mathcal{H}(\mathrm{SL}_n(F); R))$ , where  $F$  is a non-Archimedean local field with valuation  $v: F \rightarrow \mathbb{Z} \cup \{\infty\}$ . The following claims about the Bruhat-Tits building  $X$  for  $\mathrm{SL}_n(F)$  (and later about  $X'$ ) can all be verified from the description of  $X$  in [1, Sec. 6.9].

For  $l = 0, \dots, n-1$  let  $U_l^S$  be the compact open subgroup of  $\mathrm{SL}_n(F)$  consisting of all matrices  $(a_{ij})$  in  $\mathrm{SL}_n(F)$  satisfying  $v(a_{i,j}) \geq -1$  for  $1 \leq i \leq n-l < j \leq n$ ,  $v(a_{i,j}) \geq 1$  for  $1 \leq j \leq n-l < i \leq n$  and  $v(a_{i,j}) \geq 0$  for all other  $i, j$ . In particular  $U_0^S = \mathrm{SL}_n(\mathcal{O})$ , where  $\mathcal{O} = \{z \in F \mid v \geq 0\}$ . The intersection of the  $U_l^S$ -s is the Iwahori subgroup  $I^S$  of  $\mathrm{SL}_n(F)$ . It is given by those matrices  $A$  in  $\mathrm{SL}_n(F)$  for which  $v(a_{i,j}) \geq 1$  for  $i > j$  and  $v(a_{i,j}) \geq 0$  for  $i \leq j$  hold.

The  $(n-1)$ -simplex  $\Delta$  can be chosen with an ordering on its vertices such that the isotropy group of its  $l$ -th vertex  $v_l$  is  $U_l^S$ . The isotropy group of a face  $\sigma$  of  $\Delta$  is the intersection of the isotropy groups of the vertices of  $\sigma$ . In particular, the isotropy group of  $\Delta$  is the Iwahori subgroup  $I^S$  of  $\mathrm{SL}_n(F)$ . Consider the map

$$d^{\mathrm{SL}_n(F)}: \bigoplus_{0 \leq i < j \leq n-1} K_0(\mathcal{H}(U_i^S \cap U_j^S; R)) \rightarrow \bigoplus_{0 \leq l \leq n-1} K_0(\mathcal{H}(U_l^S; R)),$$

for which the component  $d_{i < j, l}^{\mathrm{SL}_n(F)}: K_0(\mathcal{H}(U_i^S \cap U_j^S; R)) \rightarrow K_0(\mathcal{H}(U_l^S; R))$  is given by  $-K_0(\mathcal{H}(f_{i < j}^i; R))$ , if  $l = i$ , by  $K_0(\mathcal{H}(f_{i < j}^j; R))$ , if  $l = j$ , and is zero, if  $l \notin \{i, j\}$ , where  $f_{i < j}^k: U_i^S \cap U_j^S \rightarrow U_k^S$  is the inclusion for  $k = i, j$ .

Then the cokernel of  $d^{\mathrm{SL}_n(F)}$  is  $K_0(\mathcal{H}(\mathrm{SL}_n(F); R))$  by Lemma 5.1 and Remark 5.2.

6.B.  $\mathrm{PGL}_n(F)$ . Next we compute  $K_0(\mathcal{H}(\mathrm{PGL}_n(F); R))$ . The action of  $\mathrm{SL}_n(F)$  on  $X$  extends to an action of  $\mathrm{GL}_n(F)$ . This action factors through the canonical projection  $\mathrm{pr}: \mathrm{GL}_n(F) \rightarrow \mathrm{PGL}_n(F)$  to an action of  $\mathrm{PGL}_n(F)$ . These actions are still simplicial, but no longer cellular. Let

$$\widehat{h} := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ \zeta & & & \end{pmatrix} \in \mathrm{GL}_n(F)$$

where we chose a uniformizer  $\zeta \in F$ , i.e., an element in  $F$  satisfying  $v(\zeta) = 1$ . Obviously  $\widehat{h}^n$  is the diagonal matrix  $\zeta \cdot I_n$ , all whose diagonal entries are  $\zeta$ , and

hence is central in  $\mathrm{GL}_n(F)$ . Define  $h \in \mathrm{PGL}_n(F)$  by  $h = \mathrm{pr}(\widehat{h})$ . Then  $hv_l = v_{l+1}$  for  $l = 0, \dots, n-2$  and  $hv_{n-1} = v_0$  and  $h^n$  is the unit in  $\mathrm{PGL}_n(F)$ . In particular, the action of  $\mathrm{PGL}_n(F)$  is transitive on the vertices of  $X$ . To obtain a cellular action,  $X$  can be subdivided to  $X'$  as follows. The  $(n-2)$ -skeleton of  $X$  is unchanged, while the  $(n-1)$ -simplices of  $X$  are in  $X'$  replaced with cones on their boundary. More formally, the vertices of  $X'$  are the vertices of  $X$  and the barycenters  $b_\sigma$  of  $(n-1)$ -simplices  $\sigma$  of  $X$ . A set  $S$  of vertices of  $X'$  is a simplex of  $X'$ , if and only if  $S$  is a  $k$ -simplex of  $X$  and  $k < n-1$  or if  $S$  contains exactly one barycenter  $b_\sigma$  and for all  $v \in S \setminus \{b_\sigma\}$  are vertices of  $\sigma$  (in the simplicial structure of  $X$ ). The action of  $\mathrm{PGL}_n(F)$  on  $X'$  is then cellular and is transitive on  $(n-1)$ -simplices of  $X'$ . There are two orbits of vertices, represented by  $v_0$  and  $b_\Delta$ . Let  $k := \lfloor n/2 \rfloor$ . There are  $k+1$  orbits of 1-simplices, represented by  $\{v_0, v_1\}, \dots, \{v_0, v_k\}$  and  $\{v_0, b_\Delta\}$ . Next we describe some isotropy groups.

For an open subgroup  $W \subseteq \mathrm{PGL}_n(F)$  we denote by  $\widetilde{W}$  its preimage under the projection  $\mathrm{pr}: \mathrm{GL}_n(F) \rightarrow \mathrm{PGL}_n(F)$ . For  $l = 0, \dots, n-1$  let  $U_l^G$  be the compact open subgroup of  $\mathrm{GL}_n(F)$  given by  $\widehat{h}^l \mathrm{GL}_n(\mathcal{O}) \widehat{h}^{-l} = \mathrm{PGL}_n(F)_{v_l} = \mathrm{PGL}_n(F)_{h_l v_0}$ . In particular  $U_0^G = \mathrm{GL}_n(\mathcal{O})$ . Note that

$$U_l^G \cap \mathrm{SL}_n(F) = (\widehat{h}^l \mathrm{GL}_n(\mathcal{O}) \widehat{h}^{-l}) \cap \mathrm{SL}_n(F) = \widehat{h}^l \mathrm{SL}_n(\mathcal{O}) \widehat{h}^{-l} = U_l^S$$

holds. The intersection of the  $U_l^G$ -s is the Iwahori subgroup  $I^G$  of  $\mathrm{GL}_n(F)$ . Let  $U_l^P$  be the image of  $U_l^G$  in  $\mathrm{PGL}_n(F)$ . This is the isotropy groups of the vertex  $v_l$  for the action of  $\mathrm{PGL}_n(F)$ . The Iwahori subgroup  $I^P$  of  $\mathrm{PGL}_n(F)$  is the image of  $I^G$  under  $\mathrm{pr}$ . It is the pointwise isotropy subgroup for  $\Delta$ . Let  $H$  be the subgroup generated by the image of  $h$  in  $\mathrm{PGL}_n(F)$ . It is a cyclic subgroup of order  $n$  that cyclically permutes the vertices of  $\Delta$ . This subgroup normalizes  $I^P$  and the isotropy group of  $b_\Delta$  is the product  $HI^P$ . Recall that  $v_l = h^l v_0$  and hence  $U_l^P = h^l U_0^P h^{-l}$ .

Write  $i_H: I^P \rightarrow HI^P$ ,  $i_0: I^P \rightarrow U_0^P$ ,  $c_0: U_0^P \cap U_i^P \rightarrow U_0^P$  for the inclusions and define  $c_l: U_0^P \cap U_l^P \rightarrow U_0^P$  by  $z \mapsto h^{-l} z h^l$ . Let

$$\begin{aligned} d^{\mathrm{PGL}_n(F)}: K_0(\mathcal{H}(I^P; R)) \oplus \bigoplus_{l=1}^k K_0(\mathcal{H}(U_0^P \cap U_l^P; R)) \\ \rightarrow K_0(\mathcal{H}(HI^P; R)) \oplus K_0(\mathcal{H}(U_0^P; R)) \end{aligned}$$

be the map that is  $K_0(i_H) \times -K_0(i_0)$  on  $K_0(\mathcal{H}(I^P; R))$  and  $0 \times (K_0(c_l) - K_0(c_0))$  on  $K_0(\mathcal{H}(U_0^P \cap U_l^P; R))$ . The cokernel of the homomorphism  $d^{\mathrm{PGL}_n(F)}$  agrees with  $SH_0^{\mathrm{PGL}_n(F)}(X'; K_0(\mathcal{H}(?; R)))$  by Lemma 5.1, if, using the notation of Section 5.A, we put  $E = \{v_0, b_\Delta\}$  with  $v_0 < b_\Delta$ ,  $F(v_0, v_0) = \{h, h^2, \dots, h^k\}$ , and  $F(v_0, b_\Delta) = \{e\}$ .

6.c.  $\mathrm{GL}_n(F)$ . Next we compute  $K_0(\mathcal{H}(\mathrm{GL}_n(F); R))$ . Note that  $\mathrm{GL}_n(F)$  has a non-compact center. Hence Subsection 5.A does not apply and we have to pass to the setting of Subsection 5.B using the short exact sequence  $1 \rightarrow C = \mathrm{cent}(\mathrm{GL}_n(F)) \rightarrow \mathrm{GL}_n(F) \xrightarrow{\mathrm{pr}} \mathrm{PGL}_n(F) \rightarrow 1$ , the discussion in Subsection 6.B and Lemma 5.4.

Let  $\widetilde{M}$  be the kernel of the composite  $\mu: \mathrm{GL}_n(F) \xrightarrow{\det} F^\times \xrightarrow{\nu} \mathbb{Z}$ . Let  $\widehat{H} \subseteq \mathrm{GL}_n(F)$  be the infinite cyclic subgroup generated by the element  $\widehat{h}$ . Note that  $\widetilde{M} \cap C$  consists of those diagonal matrices whose entries on the diagonal are all the same and are sent to 0 under  $\nu$ . We conclude  $(\mathrm{GL}_n(\mathcal{O}) \cdot C) \cap \widetilde{M} = \mathrm{GL}_n(\mathcal{O})$  from  $C \cap \widetilde{M} \subseteq \mathrm{GL}_n(\mathcal{O}) \subseteq \widetilde{M}$ . Recall that for  $W \subseteq \mathrm{PGL}_n(F)$  we denote by  $\widetilde{W}$  its preimage under  $\mathrm{pr}: \mathrm{GL}_n(F) \rightarrow \mathrm{PGL}_n(F)$ . Since  $\mathrm{pr}(U_l^G) = U_l^P$ , we get for



$l = 0, \dots, n-1$

$$\begin{aligned} \widetilde{U}_i^{\mathbb{P}} \cap \widetilde{M} &= (U_i^{\mathbb{G}} \cdot C) \cap \widetilde{M} = (\widehat{h}^l \mathrm{GL}_n(\mathcal{O}) \widehat{h}^{-l} \cdot C) \cap \widetilde{M} \\ &= \widehat{h}^l ((\mathrm{GL}_n(\mathcal{O}) \cdot C) \cap \widetilde{M}) \widehat{h}^{-l} = \widehat{h}^l \mathrm{GL}_n(\mathcal{O}) \widehat{h}^{-l} = U_i^{\mathbb{G}}. \end{aligned}$$

Now one easily checks  $\widetilde{I}^{\mathbb{P}} \cap \widetilde{M} = I^{\mathbb{G}}$ . Finally we show  $\widetilde{HI}^{\mathbb{P}} \cap \widetilde{M} = I^{\mathbb{G}}$ . We get  $I^{\mathbb{G}} \subseteq \widetilde{HI}^{\mathbb{P}} \cap \widetilde{M}$  from  $\widetilde{I}^{\mathbb{P}} \cap \widetilde{M} = I^{\mathbb{G}}$ . Consider an element  $A \in \widetilde{HI}^{\mathbb{P}} \cap \widetilde{M}$ . We can find an integer  $b$ , an element  $B \in I^{\mathbb{G}}$ , and an element  $D \in C$  such that  $A = \widehat{h}^b B D$  and  $\nu(A) = 0$  holds. From  $I^{\mathbb{G}} \subseteq \widetilde{M}$  we conclude  $\widehat{h}^b D \in \widetilde{M}$ . Since  $\mu(D)$  is divisible by  $n$  and  $\mu(\widehat{h}) = 1$  holds,  $b$  is divisible by  $n$ . This implies  $\widehat{h}^b \in C$  and hence  $\widehat{h}^b D \in C \cap \widetilde{M}$ . As  $(C \cap \widetilde{M}) I^{\mathbb{G}} = I^{\mathbb{G}}$  holds, we conclude  $A \in I^{\mathbb{G}}$ . Hence  $\widetilde{HI}^{\mathbb{P}} \cap \widetilde{M} = I^{\mathbb{G}}$  holds.

Let  $\widetilde{i}_0: I^{\mathbb{G}} \rightarrow U_0^{\mathbb{G}}$  and  $\widetilde{c}_0: U_0^{\mathbb{G}} \cap U_i^{\mathbb{G}} \rightarrow U_0^{\mathbb{G}}$  be the inclusions and let  $\widetilde{c}_l: U_0^{\mathbb{G}} \cap U_i^{\mathbb{G}} \rightarrow U_0^{\mathbb{G}}$  be the map sending  $\widetilde{z}$  to  $\widehat{h}^{-l} \widetilde{z} \widehat{h}^l$ . Let

$$\begin{aligned} \overline{d}^{\mathrm{GL}_n(F)}: K_0(\mathcal{H}(I^{\mathbb{G}}; R)) \oplus \bigoplus_{l=1}^k K_0(\mathcal{H}(U_0^{\mathbb{G}} \cap U_l^{\mathbb{G}}; R)) \\ \rightarrow K_0(\mathcal{H}(I^{\mathbb{G}}; R)) \oplus K_0(\mathcal{H}(U_0^{\mathbb{G}}; R)) \end{aligned}$$

be the map that is  $\mathrm{id}_{K_0(I^{\mathbb{G}})} \times -K_0(\widetilde{i}_0)$  on  $K_0(\mathcal{H}(I^{\mathbb{G}}; R))$  and  $0 \times (K_0(\widetilde{c}_l) - K_0(\widetilde{c}_0))$  on  $K_0(\mathcal{H}(U_0^{\mathbb{G}} \cap U_l^{\mathbb{G}}; R))$ . The cokernel of the map  $\overline{d}^{\mathrm{GL}_n(F)}$  is  $K_0(\mathcal{H}(\mathrm{GL}_n(F); R))$  by Lemma 5.4 Let

$$\widetilde{d}^{\mathrm{GL}_n(F)}: \bigoplus_{l=1}^k K_0(\mathcal{H}(U_0^{\mathbb{G}} \cap U_l^{\mathbb{G}}; R)) \rightarrow K_0(\mathcal{H}(U_0^{\mathbb{G}}; R))$$

be the map which is given by  $K_0(\widetilde{c}_l) - K_0(\widetilde{c}_0)$  on  $K_0(\mathcal{H}(U_0^{\mathbb{G}} \cap U_l^{\mathbb{G}}; R))$ . Since  $\overline{d}^{\mathrm{GL}_n(F)}$  has the same cokernel as  $\widetilde{d}^{\mathrm{GL}_n(F)}$ , the cokernel of  $\overline{d}^{\mathrm{GL}_n(F)}$  is  $K_0(\mathcal{H}(\mathrm{GL}_n(F); R))$ .

## 7. HOMOTOPY COLIMITS

### 7.A. The Farrell-Jones assembly map as a map of homotopy colimits.

Next we want to extend the considerations of Section 6 to the higher  $K$ -groups. For this purpose and the proofs appearing in [3] it is worthwhile to write down the assembly map in terms of homotopy colimits. The projections  $G/U \rightarrow G/G$  for  $U$  compact open in  $G$  induce a map

$$(7.1) \quad \mathrm{hocolim}_{G/U \in \mathrm{Or}_{\mathrm{cop}}(G)} \mathbf{K}_R(G/U) \rightarrow \mathbf{K}_R(G/G) \simeq \mathbf{K}(\mathcal{H}(G; R)).$$

This map can be identified after applying  $\pi_n$  with the assembly map appearing in Theorem 1.1 (i) and Theorem 3.2. This follows from [11, Section 5].

**7.B. Simplifying the source of the Farrell Jones assembly map.** Let  $X$  be an abstract simplicial complex with simplicial  $G$ -action such that the isotropy group of each vertex is compact open and the  $G$ -action is cellular. Furthermore we assume that  $|X|^K$  is weakly contractible for any compact open subgroup of  $G$ . Then  $|X|$  is a model for  $E_{\mathrm{cop}}(G)$ .

Let  $C$  be a collection of simplices of  $X$  that contains at least one simplex from each orbit of the action of  $G$  on the set of simplices of  $X$ . Define a category  $\mathcal{C}(C)$  as follows. Its objects are the simplices from  $C$ . A morphism  $gG_\sigma: \sigma \rightarrow \tau$  is an element  $gG_\sigma \in G/G_\sigma$  satisfying  $g\sigma \subseteq \tau$ . The composite of  $gG_\sigma: \sigma \rightarrow \tau$  with  $hG_\tau: \tau \rightarrow \rho$  is  $hgG_\sigma: \sigma \rightarrow \rho$ . Define a functor

$$(7.2) \quad \iota_C: \mathcal{C}(C)^{\mathrm{op}} \rightarrow \mathrm{Or}_{\mathrm{cop}}(G)$$

by sending an object  $\sigma$  to  $G/G_\sigma$  and a morphism  $gG_\sigma: \sigma \rightarrow \tau$  to  $R_g: G/G_\tau \rightarrow G/G_\sigma$ ,  $g'G_\tau \mapsto g'gG_\sigma$ .

**Lemma 7.3.** *Under the assumptions above the map induced by the functor  $\iota_C$*

$$\operatorname{hocolim}_{\sigma \in \mathcal{C}(C)^{\text{op}}} \mathbf{K}_R(G/G_\sigma) \xrightarrow{\sim} \operatorname{hocolim}_{G/U \in \text{Or}_{\text{cop}}(G)} \mathbf{K}_R(G/U)$$

is a weak homotopy equivalence.

*Proof.* We want to apply the criterion [12, 9.4]. So we have to show that the geometric realization of the nerve of the category  $G/K \downarrow \iota_C$  is a contractible space for every object  $G/K$  in  $\text{Or}_{\text{cop}}(G)$ . An object in  $G/K \downarrow \iota_C$  is a pair  $(\sigma, u)$  consisting of an element  $\sigma \in C$  and a  $G$ -map  $u: G/K \rightarrow G/G_\sigma$ . A morphism  $(\sigma, u) \rightarrow (\tau, v)$  in  $G/K \downarrow \iota_C$  is given by a morphism  $gG_\tau: \tau \rightarrow \sigma$  in  $\mathcal{C}(C)$  such that the  $G$ -map  $R_g: G/G_\sigma \rightarrow G/G_\tau$  sending  $zG_\sigma$  to  $zgG_\tau$  satisfies  $v \circ R_g = u$ .

Let  $\mathcal{P}(X^K)$  be the poset given by the simplices of  $X^K$  ordered by inclusion. Then we get an equivalence of categories

$$F: \mathcal{P}(X^K)^{\text{op}} \xrightarrow{\cong} G/K \downarrow \iota_C$$

as follows. It sends a simplex  $\sigma$  to the object  $(\sigma, \text{pr}_\sigma: G/K \rightarrow G/G_\sigma)$  for the canonical projection  $\text{pr}_\sigma$ . A morphism  $\sigma \rightarrow \tau$  in  $\mathcal{P}(X^K)^{\text{op}}$  is sent to the morphism  $(\sigma, \text{pr}_\sigma) \rightarrow (\tau, \text{pr}_\tau)$  in  $G/K \downarrow \iota_C$  which is given by the morphism  $eG_\tau: \tau \rightarrow \sigma$  in  $\mathcal{C}(C)$ .

Consider an object  $(\sigma, u)$  in  $G/K \downarrow \iota_C$ . We want to show that it is isomorphic to an object in the image of  $F$ . Choose  $g \in G$  such that  $g^{-1}Kg \subseteq G_\sigma$  holds and  $u$  is the  $G$ -map  $R_g: G/K \rightarrow G/G_\sigma$  sending  $zK$  to  $zgG_\sigma$ . Then  $K \subseteq G_{g\sigma}$  and we can consider the object  $F(g\sigma) = (g\sigma, \text{pr}_{g\sigma})$  for the projection  $\text{pr}_{g\sigma}: G/K \rightarrow G_{g\sigma}$ . Now the isomorphism  $gG_\sigma: \sigma \rightarrow g\sigma$  in  $\mathcal{C}(C)$  induces an isomorphism  $F(g\sigma) \xrightarrow{\cong} (\sigma, u)$  in  $G/K \downarrow \iota_C$ .

Obviously  $F$  is faithful. It remains to show that  $F$  is full. Fix two objects  $\sigma$  and  $\tau$  in  $\mathcal{P}(X^K)$ . Consider a morphism  $f: F(\sigma) = (\sigma, \text{pr}_\sigma) \rightarrow F(\tau) = (\tau, \text{pr}_\tau)$  in  $G/K \downarrow \iota_C$ . It is given by a morphism  $gG_\tau: \tau \rightarrow \sigma$  in  $\mathcal{C}(C)$  such that the composite of  $R_g: G/G_\sigma \rightarrow G/G_\tau$  with  $\text{pr}_\sigma$  is  $\text{pr}_\tau$ . This implies  $gG_\tau = G_\tau$  and hence  $g \in G_\tau$ . Since  $g\tau \subseteq \sigma$  holds by the definition of a morphism in  $\mathcal{C}(C)$ , we get  $\tau \subseteq \sigma$ . Hence  $f$  is the image of the morphism  $\sigma \rightarrow \tau$  under  $F$ . This shows that  $F$  is full.

Hence it remains to show that geometric realization of the nerve of  $\mathcal{P}(X^K)^{\text{op}}$  is contractible. Since this is the barycentric subdivision of  $|X|^K$ , this follows from the assumptions.  $\square$

Suppose additionally that  $X$  admits a strict fundamental domain  $\Delta$ , i.e., a simplicial subcomplex  $\Delta$  that contains exactly one simplex from each orbit for the  $G$ -action on the set of simplices of  $X$ . Then we can take for  $C$  the simplices from  $\Delta$ . In this case  $\mathcal{C}(C)$  can be identified with the poset  $\mathcal{P}(\Delta)$  of simplices of  $\Delta$ . Recall that for any open subgroup  $U$  of  $G$ , there is an explicit weak homotopy equivalence  $\mathbf{K}(\mathcal{H}(U; R)) \xrightarrow{\cong} \mathbf{K}_R(G/U)$ , where the source is the  $K$ -theory spectrum  $\mathbf{K}(\mathcal{H}(U; R))$  of the Hecke algebra  $\mathcal{H}(U; R)$ , see [4, 5.6 and Remark 6.7]. Lemma 7.3 implies

**Theorem 7.4.** *Let  $X$  be an abstract simplicial complex with a simplicial  $G$ -action such that the isotropy group of each vertex is compact open, the  $G$ -action is cellular, and  $|X|^K$  is weakly contractible for every compact open subgroup  $K$  of  $G$ . Let  $\Delta$  be a strict fundamental domain.*

*Then the assembly map*

$$(7.5) \quad \operatorname{hocolim}_{\sigma \in \mathcal{P}(\Delta)^{\text{op}}} \mathbf{K}(\mathcal{H}(G_\sigma; R)) \rightarrow \operatorname{hocolim}_{G/U \in \text{Or}_{\text{cop}}(G)} \mathbf{K}_R(G/U)$$

that is induced by the functor  $\mathcal{P}(\Delta)^{\text{op}} \rightarrow \text{Or}_{\text{COP}}(G)$  sending a simplex  $\sigma$  to  $G_\sigma$ , is a weak homotopy equivalence,

**Example 7.6** ( $\text{SL}_n(F)$ ). Let  $X$  be the Bruhat-Tits building for  $\text{SL}_n(F)$ . Then the canonical  $\text{SL}_n(F)$  action on  $X$  is cellular. We will use again the notation introduced in Section 6. The  $(n-1)$ -simplex  $\Delta$ , viewed as a subcomplex of  $X$ , is a strict fundamental domain. Applying this in the case  $n=2$  yields the homotopy pushout diagram

$$\begin{array}{ccc} \mathbf{K}(\mathcal{H}(I^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(U_1^S; R)) \\ \downarrow & & \downarrow \\ \mathbf{K}(\mathcal{H}(U_0^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(\text{SL}_2(F); R)). \end{array}$$

For the  $K$ -groups this yields a Mayer-Vietoris sequence, infinite to the left,

$$(7.7) \quad \begin{aligned} \cdots \rightarrow K_n(\mathcal{H}(I^S; R)) &\rightarrow K_n(\mathcal{H}(U_1^S; R)) \oplus K_n(\mathcal{H}(U_0^S; R)) \rightarrow K_n(\mathcal{H}(\text{SL}_2(F); R)) \\ &\rightarrow K_{n-1}(\mathcal{H}(I^S; R)) \rightarrow K_{n-1}(\mathcal{H}(U_1^S; R)) \oplus K_{n-1}(\mathcal{H}(U_0^S; R)) \rightarrow \cdots \\ \cdots \rightarrow K_0(\mathcal{H}(I^S; R)) &\rightarrow K_0(\mathcal{H}(U_1^S; R)) \oplus K_0(\mathcal{H}(U_0^S; R)) \rightarrow K_0(\mathcal{H}(\text{SL}_2(F); R)) \rightarrow 0 \end{aligned}$$

and  $K_n(\mathcal{H}(\text{SL}_2(F); R)) = 0$  for  $n \leq -1$ .

For  $n=3$  we obtain the homotopy push-out diagram

$$\begin{array}{ccccc} & & \mathbf{K}(\mathcal{H}(U_{12}^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(U_2^S; R)) \\ & \nearrow & \downarrow & & \downarrow \\ \mathbf{K}(\mathcal{H}(I^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(U_{02}^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(U_2^S; R)) \\ \downarrow & & \downarrow & & \downarrow \\ & & \mathbf{K}(\mathcal{H}(U_1^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(\text{SL}_3(F); R)) \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ \mathbf{K}(\mathcal{H}(U_{01}^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(U_0^S; R)) & \longrightarrow & \mathbf{K}(\mathcal{H}(U_0^S; R)) \end{array}$$

where we abbreviated  $U_{ij}^S := U_i^S \cap U_j^S$ . In general, for  $\text{SL}_n(F)$  we obtain a homotopy push-out diagram whose shape is an  $n$ -cube.

To such an  $n$ -cube there is assigned a spectral sequence concentrated in the region for  $p \geq 0$  and  $0 \leq q \leq n-1$ , which corresponds to the spectral sequence appearing in Theorem 1.1 (ii)

## 8. ALLOWING CENTRAL CHARACTERS AND ACTIONS ON THE COEFFICIENTS

So far we have only considered the standard Hecke algebra  $\mathcal{H}(G; R)$ . There are more general Hecke algebras  $\mathcal{H}(G; R, \rho, \omega)$ , see [2], and all the discussions of this paper carry over to them in the obvious way.

## REFERENCES

- [1] P. Abramenko and K. S. Brown. *Buildings*, volume 248 of *Graduate Texts in Mathematics*. Springer, New York, 2008. Theory and applications.
- [2] A. Bartels and W. Lück. On the algebraic  $K$ -theory of Hecke algebras. preprint, arXiv:2204.07982 [math.KT], to appear in the Festschrift *Mathematics Going Forward*, Lecture Notes in Mathematics 2313, Springer, 2022.

- [3] A. Bartels and W. Lück. Algebraic  $K$ -theory of reductive  $p$ -adic groups. Preprint, arXiv:2306.03452 [math.KT], 2023.
- [4] A. Bartels and W. Lück. Inheritance properties of the  $K$ -theoretic Farrell-Jones Conjecture for totally disconnected groups. Preprint arXiv:2306.01518 [math.KT], 2023.
- [5] J. Bernstein. Draft of: Representations of  $p$ -adic groups. [http://www.math.tau.ac.il/~bernstei/Unpublished\\_texts/Unpublished\\_list.html](http://www.math.tau.ac.il/~bernstei/Unpublished_texts/Unpublished_list.html), 1992.
- [6] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*. Springer-Verlag, Berlin, 1999. Die Grundlehren der mathematischen Wissenschaften, Band 319.
- [7] K. S. Brown. *Buildings*. Springer-Verlag, New York, 1998. Reprint of the 1989 original.
- [8] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. *Inst. Hautes Études Sci. Publ. Math.*, 41:5–251, 1972.
- [9] J.-F. Dat. Quelques propriétés des idempotents centraux des groupes  $p$ -adiques. *J. Reine Angew. Math.*, 554:69–103, 2003.
- [10] J.-F. Dat. Théorie de Lubin-Tate non-abélienne et représentations elliptiques. *Invent. Math.*, 169(1):75–152, 2007.
- [11] J. F. Davis and W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in  $K$ - and  $L$ -theory. *K-Theory*, 15(3):201–252, 1998.
- [12] W. G. Dwyer and D. M. Kan. A classification theorem for diagrams of simplicial sets. *Topology*, 23(2):139–155, 1984.
- [13] P. Garrett. Smooth representations of totally disconnected groups. unpublished notes, [https://www-users.cse.umn.edu/~garrett/m/v/smooth\\_of\\_td.pdf](https://www-users.cse.umn.edu/~garrett/m/v/smooth_of_td.pdf), 2012.
- [14] W. Lück. *Transformation groups and algebraic K-theory*, volume 1408 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.
- [15] W. Lück. Chern characters for proper equivariant homology theories and applications to  $K$ - and  $L$ -theory. *J. Reine Angew. Math.*, 543:193–234, 2002.
- [16] W. Lück. The relation between the Baum-Connes conjecture and the trace conjecture. *Invent. Math.*, 149(1):123–152, 2002.
- [17] M. Ronan. *Lectures on buildings*. Academic Press Inc., Boston, MA, 1989.
- [18] R. M. Switzer. *Algebraic topology—homotopy and homology*. Springer-Verlag, New York, 1975. Die Grundlehren der mathematischen Wissenschaften, Band 212.

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