ON NIELSEN REALIZATION AND MANIFOLD MODELS FOR CLASSIFYING SPACES

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Abstract. We consider the problem of whether, for a given virtually torsionfree discrete group Γ, there exists a cocompact proper topological Γ-manifold, which is equivariantly homotopy equivalent to the classifying space for proper actions. This problem is related to Nielsen Realization. We will make the assumption that the expected manifold model has a zero-dimensional singular set. Then we solve the problem in the case, for instance, that Γ contains a normal torsionfree subgroup π such that π is hyperbolic and π is the fundamental group of an aspherical closed manifold of dimension greater or equal to five and Γ/π is a finite cyclic group of odd order.

1. Introduction

If a group G acts effectively on a manifold X with fundamental group π, then there is a short exact sequence

$$1 \to \pi \xrightarrow{i} \Gamma \xrightarrow{p} G \to 1 \tag{1.1}$$

and a group action of Γ on the universal cover $\widetilde{X}$ so that the action of Γ/π on $\widetilde{X}/\pi$ recovers the G-action on X. (Here Γ is the subgroup of the homeomorphism group of $\widetilde{X}$ given by lifts of the elements of G.)

This paper makes progress on the two following interrelated questions. We will discuss these questions and then state our results.

Neilsen Realization Question. *If X is a closed aspherical manifold with fundamental group π, can any group monomorphism $\phi : G \to \operatorname{Out}(\pi)$ from a finite group to the outer automorphism group of π be realized by a G-action on X?*

Manifold Model Question. *Given a closed aspherical manifold X with fundamental group π and dimension d, and a short exact sequence*

$$1 \to \pi \to \Gamma \to G \to 1$$

with G finite, does there exist a d-dimensional manifold model for $\underline{E}\Gamma$, the classifying space for proper Γ-actions?

The Nielsen Realization Question was raised by Nielsen for 2-manifolds, and was answered affirmatively by Kerckhoff [35]. The answer to the Nielsen Realization Question is also yes for closed Riemannian manifolds with constant negative sectional curvature (see Subsection 10.6).

In considering the Nielsen Realization Question, the first step is to see if, given ϕ, there is an extension (1.1) realizing ϕ. There is a cohomological obstruction in $H^3(G; Z(\pi))$ to the existence of the extension [46, Theorem IV.8.7] and, if an extension realizing ϕ exists, $H^2(G; Z(\pi))$ classifies the extensions [46, Theorem IV.8.8]. Here $Z(\pi)$ is the center of the group π. Raymond and Scott [51] gave a negative

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answer to the Nielsen Realization Question, by giving examples of (X,ϕ) where the group extension does not exist. Block and Weinberger [5] gave negative answers to the Nielsen Realization Question where the center of π is trivial. However, Nielsen's original question concerned surfaces of genus > 1, so it is worth noting that there are no counterexamples known when X is negatively curved, or more generally when π is a hyperbolic group (in the sense of Gromov). An affirmative answer was given in [42, Remark 1.21] when $\dim X \geq 5, \pi = \pi_1 X$ is a hyperbolic group, and the extension Γ of π by G realizing ϕ is torsionfree. This generalized the analogous result of Farrell and Jones [32] in the case where X is a Riemannian manifold of negative curvature with $\dim X \geq 5$.

The answer to the Nielsen Realization Question is yes for closed Riemannian manifolds with constant negative sectional curvature (see Subsection 10.6).

Recall that for a discrete group Γ, *a model for* $\underline{E}\Gamma$ is a Γ-space M which is a Γ-CW-complex so that for every finite subgroup H, the fixed point set M^H is contractible and for every infinite subgroup H, the fixed point set M^H is empty. $\underline{E}\Gamma$ is the classifying space for proper actions in the sense that if Y is a proper Γ-CW-complex (i.e. Γ-CW-complex with finite isotropy), there is a Γ-map $Y \to \underline{E}\Gamma$, unique up to Γ-homotopy. For a survey on $\underline{E}\Gamma$ we refer to [41].

A *manifold model for* $\underline{E}\Gamma$ is simply a model M for $\underline{E}\Gamma$, so that M, ignoring the group action, is a topological manifold. (One could also include the hypothesis that M^H is a submanifold for non-trivial finite subgroups H, but we are interested in the case where the singular set is discrete, so this distinction is not relevant for us). A model M for $\underline{E}\Gamma$ is *cocompact* if M/Γ is compact. In the statement of the Manifold Model Question we could have replaced the words "d-dimensional" with "cocompact" and had an equivalent question (see 10.1).

Counterexamples to the Manifold Model Question have been given by Davis and Leary [29] and by Block and Weinberger [5, Theorem 1.5]. However, to the best of our knowledge, there are no counterexamples known in three cases of interest: (1) if the normalizer of each nontrivial finite subgroups of Γ is finite, or (2) if $\underline{E}\Gamma$ has a model which is a finite Γ-CW-complex, or (3) if π is a hyperbolic group.

The answer to the Manifold Model Question is yes for closed Riemannian manifolds with constant negative sectional curvature (see Subsection 10.6).

The Borel Conjecture for a closed aspherical manifold X states that any homotopy equivalence $N \to X$ where N is a closed manifold is homotopic to a homeomorphism. It has been proven for many manifolds X, but is open in general (see [55] and [43] for a discussion).

An affirmative answer to the Manifold Model Question implies an affirmative answer to the Nielsen Realization Question in the following sense. Suppose X is a closed aspherical manifold with fundamental group π and $\phi : G \to \mathrm{Out}(\pi)$ is a group monomorphism with G finite. Suppose, in addition, that the Borel Conjecture holds for X and that ϕ is realized by a group extension

$$1 \to \pi \to \Gamma \to G \to 1.$$

If there is a cocompact manifold model M for $\underline{E}\Gamma$, then M/π is a closed manifold with a G-action realizing ϕ, and the Borel Conjecture gives a homeomorphism between X and M/π inducing the identity on the fundamental group.

Under very special circumstances an affirmative answer to the Nielsen Realization Question implies an affirmative answer to the Manifold Model Question, see Section 10.5.

For the rest of the introduction we focus on the Manifold Model Question. We note that the Manifold Model Question is an existence question. The corresponding uniqueness question is: are two d-dimensional manifold models for $\underline{E}\Gamma$ equivariantly homeomorphic?

The simplest case of the existence and uniqueness questions is when Γ is torsionfree, equivalently when Γ acts freely on M. The uniqueness question is the famous Borel Conjecture. The existence question was solved when $\dim X \geq 5$ and X is negatively curved in [32] and extended to the case where $\dim X \geq 5$ and π is hyperbolic in [42].

The next level in complexity (compared to free actions) is the pseudo-free case. A Γ-space M is *pseudo-free* if the singular set $M^{>1} = \{x \in M \mid \Gamma_x \neq 1\}$ is discrete, or, equivalently the Γ-space $M^{>1}$ is the disjoint union of its Γ-orbits. If M is pseudo-free model for $\underline{E}\Gamma$ and H is a non-trivial finite subgroup, then M^H is a point, fixed by its normalizer $N_\Gamma H$, hence the normalizer is finite. Conversely, Proposition 2.3 of [19] asserts that if Γ is a virtually torsionfree group where the normalizers of non-trivial finite subgroups are finite, and if Γ acts properly and cocompactly on a contractible manifold, then the action is pseudo-free. In summary, a cocompact manifold model for $\underline{E}\Gamma$ is pseudo-free if and only if the normalizer of each non-trivial finite subgroup is finite. Thus a geometric condition is equivalent to an algebraic condition.

This is our basic assumption in this paper. The uniqueness question in this case was studied extensively in [18] and [19]. We improve some of the techniques from these papers and extend their uniqueness results.

A question related to the manifold model question was posed by Brown [7, page 32]. It asks that, given extension (1.1) with G finite, if $\underline{E}\pi$ has a d-dimensional CW-model, then does $\underline{E}\Gamma$ have a d-dimensional CW-model? The paper [42] studied this question in the pseudo-free case and the results of that paper are a key input for our paper.

A recent book that discussed topics connected to the themes of this paper is [55].

1.1. **A special case.** As an illustration we state a special case of our main theorem. Recall that $E\Gamma$ is a free Γ-CW-complex, which is contractible after forgetting the Γ-action, or, equivalently, $E\Gamma \to B\Gamma := E\Gamma/\Gamma$ is the universal principal Γ-bundle. Recall that $\underline{E}\Gamma$ is a Γ-CW-complex such that $\underline{E}\Gamma^H$ is contractible for every finite subgroup $H \subseteq \Gamma$ and all its isotropy groups are finite, or, equivalently $\underline{E}\Gamma$ is the classifying space for proper Γ-actions. Two models for $E\Gamma$ or for $\underline{E}\Gamma$ are Γ-homotopy equivalent.

Notation 1.2. Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal finite subgroups of Γ. Put

$$\begin{aligned}
\partial E\Gamma &:= \coprod_{F \in \mathcal{M}} \Gamma \times_F EF; \\
\partial \underline{E}\Gamma &:= \coprod_{F \in \mathcal{M}} \Gamma/F; \\
\partial B\Gamma &:= \coprod_{F \in \mathcal{M}} BF; \\
\underline{B}\Gamma &:= \underline{E}\Gamma/\Gamma.
\end{aligned}$$

Recall that a virtually cyclic group is finite, surjects onto the infinite cyclic group with finite kernel (type I), or surjects onto the infinite dihedral group with finite kernel (type II).

We may impose some of the following conditions on a group Γ.

Definition 1.3 (Conditions on Γ)**.**

(M) Every non-trivial finite subgroup of Γ is contained in a unique maximal finite subgroup;

(NM) If F is a non-trivial maximal finite subgroup, then its normalizer satisfies $N_\Gamma F = F$;

(OH) The composite

$$H_d^\Gamma(E\Gamma, \partial E\Gamma) \xrightarrow{\partial} H_{d-1}^\Gamma(\partial E\Gamma) \xrightarrow{\cong} \bigoplus_{F \in \mathcal{M}} H_{d-1}^F(EF) \to H_{d-1}^F(EF)$$

of the boundary map, the inverse of the obvious isomorphism and the projection to the summand of $F \in \mathcal{M}$ is surjective for all $F \in \mathcal{M}$;

(F) If $H \subseteq \Gamma$ is finite and non-trivial, then $N_\Gamma H$ is finite;

(V) Every infinite virtually cyclic subgroup lies in a unique maximal infinite virtually cyclic subgroup;

(NV) Every maximal infinite virtually cyclic subgroup V satisfies $N_\Gamma V = V$;

(V_{II}) Every virtually cyclic subgroup of type II lies in a unique maximal virtually cyclic subgroup of type II;

($\mathrm{NV}_{\mathrm{II}}$) Every maximal virtually cyclic subgroup V of type II satisfies lies $N_\Gamma V = V$.

Theorem 1.4 (Oriented manifold models)**.** *Suppose there is a short exact sequence of groups*

$$1 \to \pi \xrightarrow{i} \Gamma \xrightarrow{p} G \to 1$$

with G finite.

Suppose that the following conditions are satisfied:

- *There exists a closed d-dimensional oriented manifold, which is homotopy equivalent to $B\pi$;*
- *The natural number d satisfies $d \geq 5$;*
- *The group π is hyperbolic;*
- *Every non-trivial finite subgroup of Γ is odd order cyclic;*
- *The group Γ satisfies conditions (M), (NM), and (OH), see Definition 1.3.*

Then:

1. *There exists a proper cocompact oriented d-dimensional topological manifold M, which is a model for $\underline{E}\Gamma$;*
2. *Any Γ-manifold appearing in assertion* (1) *is pseudo-free;*
3. *Any two Γ-manifolds appearing in assertion* (1) *are Γ-homeomorphic.*

If we require that our manifold model for $\underline{E}\Gamma$ is pseudo-free, then the conditions (M), (NM), and (OH) are automatically satisfied as explained in [42, Lemma 1.9]. Hence these conditions have to appear in Theorem 1.4.

1.2. Slice manifold systems and slice manifold models.

Definition 1.5. Let F be a nontrivial finite group. A *Swan complex of type* $(F, d-1)$ is a $(d-1)$-dimensional free F-CW-complex S_F such that S_F, after forgetting the F-action, is homotopy equivalent to the sphere S^{d-1}. The Swan complex is *oriented*, if we have chosen a generator $[S_F]$ for the infinite cyclic group $H_{d-1}(S_F)$.

Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal finite subgroups of Γ. The following definition is taken from [42, Definition 3.1].

Definition 1.6. A *d-dimensional free slice system* $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$, or just *slice system*, consists of a Swan complex S_F of type $(F, d-1)$ for every $F \in \mathcal{M}$. We call $\mathcal{S}$ *oriented*, if each Swan complex is oriented.

We need the following manifold version of it.

Definition 1.7. A *d-dimensional free slice manifold system* or just *slice manifold system* $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$ is a d-dimensional free slice system $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$ so that each S_F, after forgetting the F-action, is homeomorphic to S^{d-1}.

Remark 1.8. Swan complexes were introduced in [53]. For a Swan complex of type $(F, d-1)$, the Lefschetz Fixed Point Theorem implies that if $d-1$ is even, that F is cyclic of order 2 and acts reversing orientation, and that if $d-1$ is odd, then F acts preserving orientation. Any two Swan complexes of type $(C_2, 2k)$ are equivariantly homotopy equivalent.

Now assume $d-1$ is odd. There is a Swan complex of type $(F, d-1)$ if and only if F has periodic cohomology with period d (see [53, Proposition 4.1]), which is equivalent to $H_{d-1}(BF)$ being cyclic of order $|F|$ (see [14, Proposition 11.1]). There exists a Swan complex $(F, d-1)$ for some d if and only if the sylow 2-subgroup is cyclic or generalized quaternionic and for p odd the sylow p-subgroups are cyclic (see [14, Theorem 11.6]).

Let S_F be an oriented Swan complex of type $(F, d-1)$. Let $[S_F/F] \in H_{d-1}(S_F/F)$ be chosen so that the covering map $S_F \to S_F/F$ sends $[S_F]$ to $|F| \cdot [S_F/F]$. Let $c_F \colon S_F \to EF$ and $\bar{c}_F \colon S_F/F \to BF$ be classifying maps. Define the *k-invariant*

$$\kappa(S_F/F) = H_{d-1}(\bar{c}_F)[S_F/F] \in H_{d-1}(BF).$$

It is a generator of this cyclic group.

Two oriented Swan complexes S_F and S'_F of type $(F, d-1)$ are *oriented homotopy equivalent* if there is an orientation preserving equivariant homotopy equivalence. This occurs if and only if their k-invariants are equal (see [27, Proposition 2.21]). Furthermore, any additive generator of $H_{d-1}(BF)$ is realized as the k-invariant of an oriented Swan complex (see [27, Lemma 2.22]).

Remark 1.9. Thus for d odd, a d-dimensional slice system exists if and only if all $F \in \mathcal{M}$ have order 2, and all slice systems are homotopy equivalent. For d even, the existence of a d-dimensional slice system is equivalent to every $F \in \mathcal{M}$ having periodic cohomology of period $d+1$, and the homotopy type of a slice system is determined by the k-invariants. Thus if $G := \Gamma/\pi$ has periodic cohomology of period $d+1$, then a d-dimensional slice system exists. The existence of d-dimensional manifold slice system is equivalent to S^{d-1} admitting a free F-action for every element $F \in \mathcal{M}$. This occurs if G acts freely on S^{d-1}, for example if G is cyclic and d is even.

Let $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$ be a slice manifold system. We denote by D_F the cone over S_F. So we get a compact d-dimensional topological manifold D_F with boundary $\partial D_F = S_F$ together with a topological F-action such that the F-action is free outside one point 0_F in the interior of D_F, whose isotropy group is F, the pair (D_F, S_F) is a finite F-CW-pair, and (D_F, S_F) is homeomorphic to (D^d, S^{d-1}).

In dimension $d \geq 6$ the desired F-CW-complex structure on S_F comes for free in Definition 1.7. Namely, the closed topological manifold S_F/F has a handlebody structure and hence a CW-structure, if $\dim(S_F/F) = d-1 \geq 5$, see [34, Section 9.2] and [37, III.2], and therefore S_F is a free F-CW-complex. Note that it is an open question, whether every closed 4-manifold carries a CW-structure. There are examples of closed 4-manifolds, which admit no triangulation, see [47].

Notation 1.10. Given a space Z, with path components $\pi_0(Z)$, let $C(Z)$ be its *path componentwise cone*, i.e, $C(Z) := \coprod_{C \in \pi_0(Z)} \operatorname{cone}(C)$.

One may describe $C(Z)$ also by the pushout

$$\begin{tikzcd} Z \arrow[r, "p"] \arrow[d, "i_0"'] & \pi_0(Z) \arrow[d] \\ Z \times [0,1] \arrow[r] & C(Z) \end{tikzcd}$$

where $i_0\colon Z \to Z \times [0,1]$ sends z to $(z,0)$, $p\colon Z \to \pi_0(Z)$ is the projection, and $\pi_0(Z)$ is equipped with the discrete topology. If Z is a Γ-CW-complex, then $C(Z)$ inherits a Γ-CW-structure. If $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$ is a free d-dimensional slice manifold system, we get an identification of Γ-manifolds

$$C(\coprod_{F\in\mathcal{M}} \Gamma \times_F S_F) = \coprod_{F\in\mathcal{M}} \Gamma \times_F D_F.$$

Definition 1.11 (Slice manifold model)**.** We call a proper Γ-manifold M without boundary a *slice manifold model for* $\underline{E}\Gamma$, or just *slice manifold model*, with respect to the slice manifold system $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$, if there exists a proper cocompact free d-dimensional Γ-manifold N with boundary $\partial N = \coprod_{F\in\mathcal{M}} \Gamma \times_F S_F$ and a Γ-pushout

$$\begin{array}{ccc} \partial N = \coprod_{F\in\mathcal{M}} \Gamma \times_F S_F & \longrightarrow & N \\ \big\downarrow & & \big\downarrow \\ C(\partial N) = \coprod_{F\in\mathcal{M}} \Gamma \times_F D_F & \longrightarrow & M, \end{array}$$

where the left vertical arrow is the obvious inclusion, such that M is Γ-homotopy equivalent to $\underline{E}\Gamma$.

We call the pair $(N, \partial N)$ a *slice manifold complement*.

Note that for a slice manifold model M we have specified an open Γ-neighborhood of the singular set $M^{>1}$. Such an open neighborhood exists automatically in the smooth category. We will discuss this assumption in the topological category in Section 10.

We will frequently use that a slice manifold model comes with a Γ-pushout

(1.12)
$$\begin{array}{ccc} \partial N & \longrightarrow & N \\ \big\downarrow & & \big\downarrow \\ \partial\underline{E}\Gamma & \longrightarrow & \underline{E}\Gamma, \end{array}$$

where the left vertical arrow is the disjoint union over $F \in \mathcal{M}$ of the canonical projections $\Gamma \times_F S_F \to \Gamma/F$.

1.3. **Main theorems.** We introduce some notation and then formulate our main theorems. Let

$$1 \to \pi \xrightarrow{i} \Gamma \to G \to 1$$

be a group extension with G finite as in (1.1). Assume that $B\pi$ is a Poincaré complex of dimension $d > 0$ (e.g. $B\pi$ is a closed d-manifold.). Then by Poincaré duality, $H^d_\pi(E\pi;\mathbb{Z}\pi)$ is infinite cyclic as an abelian group, hence is isomorphic to $\mathbb{Z}^v$ as a $\mathbb{Z}\pi$-module for a unique homomorphism

$$v : \pi \to \{\pm 1\}.$$

Here $\mathbb{Z}^v$ is the $\mathbb{Z}\pi$-module which is infinite cyclic as an abelian group, but where $\gamma x = v(\gamma)x$ for $\gamma \in \pi$ and $x \in \mathbb{Z}^v$. It follows that $H_d^\pi(E\pi;\mathbb{Z}^v)$ is infinite cyclic as an abelian group. Choose a generator $[B\pi]$ (a "fundamental class").

Shapiro Lemma [8, Proposition III.6.2] gives an isomorphism of $\mathbb{Z}\pi$-modules $H^d_\Gamma(E\Gamma;\mathbb{Z}\Gamma) \cong H^d_\pi(E\pi;\mathbb{Z}\pi)$. Hence the $\mathbb{Z}\Gamma$-module $H^d_\Gamma(E\Gamma;\mathbb{Z}\Gamma)$ is infinite cyclic as an abelian group and thus determines a homomorphism

(1.13)
$$w : \Gamma \to \{\pm 1\}$$

which restricts to v. It has already been defined in [42, Notation 6.7].

Assume Γ satisfies conditions (M) and (NM), in other words, assume that every non-trivial finite subgroup is contained in a unique maximal finite subgroup F and that $N_\Gamma F = F$. Assume also that every such F has cohomology of period $d+1$ (equivalently, there is a d-dimensional slice system) Then each of the two maps below are rational isomorphisms

$$H_d^\pi(E\pi;\mathbb{Z}^v) \to H_d^\Gamma(E\Gamma;\mathbb{Z}^w) \to H_d^\Gamma(E\Gamma,\partial E\Gamma;\mathbb{Z}^w)$$

and each of the three groups are infinite cyclic (see [42, Lemma 6.21 and diagram (6.4)]). The specified generator $[B\pi]$ for the first group specifies generators $[B\Gamma]$ and $[B\Gamma,\partial B\Gamma]$ for the second and third groups by requiring that they are a positive multiple of the image of $[B\pi]$.

Definition 1.14 (Condition (H))**.** The composite

$$H_d^\Gamma(E\Gamma,\partial E\Gamma;\mathbb{Z}^w) \xrightarrow{\partial} H_{d-1}^\Gamma(\partial E\Gamma;\mathbb{Z}^w) \xrightarrow{\cong} \bigoplus_{F\in\mathcal{M}} H_{d-1}^F(EF;\mathbb{Z}^{w|F}) \to H_{d-1}^F(EF;\mathbb{Z}^{w|F})$$

of the boundary map, the inverse of the obvious isomorphism and the projection to the summand of $F\in\mathcal{M}$ is surjective for all $F\in\mathcal{M}$.

We now review the condition (S) on an oriented slice system $\mathcal{S}$. This condition was introduced in Section 7 of [42], where further details and explanations are given.

Definition 1.15 (Condition (S))**.** Let $\mathcal{S} = \{S_F,[S_F] \mid F\in\mathcal{M}\}$ be an oriented d-dimensional free slice system with d even and suppose $w:\Gamma\to\{\pm1\}$ satisfies condition (H). Let $\kappa_F\in H_{d-1}(BF)$ be the image of $[B\Gamma,\partial B\Gamma]$ under the composite

$$H_d^\Gamma(E\Gamma,\partial E\Gamma;\mathbb{Z}^w) \xrightarrow{\partial} H_{d-1}^\Gamma(\partial E\Gamma;\mathbb{Z}^w) \xrightarrow{\cong} \bigoplus_{F\in\mathcal{M}} H_{d-1}(BF) \to H_{d-1}(BF).$$

Condition (S) on the oriented slice system $\mathcal{S}$ says that

$$\kappa[S_F/F] = \kappa_F \in H_{d-1}(BF).$$

Note that if assumption (H) holds and d is even, there is always an oriented slice system satisfying condition (S), unique up to oriented homotopy equivalence (see Remark 1.9).

So regardless what slice manifold model M we get out of Theorem 1.16, its underlying slice manifold system $\mathcal{S}'$ has the property that the F-homotopy type of S'_F is uniquely determined by the group Γ itself and is independent of M.

Let $\mathrm{UNil}_d(\mathbb{Z};\mathbb{Z}^{\pm1},\mathbb{Z}^{\pm1})$ the UNil-groups defined by Cappell, see Remark 5.23.

Theorem 1.16 (Existence)**.** *Suppose there is a short exact sequence of groups*

$$1\to\pi\xrightarrow{i}\Gamma\xrightarrow{p}G\to1$$

with G finite. Suppose that the following conditions are satisfied:

1. *There is a closed manifold of dimension d, which is homotopy equivalent to $B\pi$. Fix a generator $[B\pi]$ of the infinite cyclic group $H_d^\pi(E\pi;\mathbb{Z}^v)$;*
2. *The natural number d satisfies $d\geq5$;*
3. *For every $F\in\mathcal{M}$ the restriction of the homomorphism $w:\Gamma\to\{\pm1\}$ to F is trivial, if d is even, and is non-trivial, if d is odd;*
4. *The group Γ satisfies conditions (M), (NM), and (H), see Definitions 1.3 and 1.14;*
5. *One of the following assertions holds:*
   - (a) *There exists a finite Γ-CW-model for $\underline{E}\Gamma$, the group π satisfies the Full Farrell-Jones Conjecture, see Subsection 3.4, and Γ satisfies condition (V_{II}), see Definition 1.3;*
   - (b) *The group π is hyperbolic;*
   - (c) *The group Γ acts cocompactly, properly, and isometrically on a proper* CAT(0)*-space;*

(6) There exists an oriented free d-dimensional slice system $\mathcal{S}$ in the sense of Definition 1.6, which satisfies condition (S). Fix such a choice;

(7) One of the following conditions is satisfied:
   (a) The groups $\mathrm{UNil}_d(\mathbb{Z};\mathbb{Z}^{(-1)^d},\mathbb{Z}^{(-1)^d})$ and $\mathrm{UNil}_{d+1}(\mathbb{Z};\mathbb{Z}^{(-1)^d},\mathbb{Z}^{(-1)^d})$ vanish;
   (b) We have $d \equiv 0 \mod (4)$;
   (c) The group Γ contains no subgroup isomorphic to D_∞;
   (d) Every element $F \in \mathcal{M}$ has odd order;
   (e) The order of G is odd.

Then there exists a d-dimensional free slice manifold system $\mathcal{S}' = \{S'_F \mid F \in \mathcal{M}\}$ in the sense of Definition 1.7 and a slice manifold model for $\underline{E}\Gamma$ with respect to the slice system $\mathcal{S}'$ in the sense of Definition 1.11. Moreover, for any such pair $(M, \mathcal{S}')$, the F-CW-complexes S_F and S'_F are F-homotopy equivalent.

Theorem 1.16 is a direct consequence of Remark 1.19, Lemma 2.1, Remark 5.23, Theorem 7.2 3, and Theorem 8.1.

Remark 1.17 (The role of w)**.** Consider the situation of Theorem 1.16. If M is any slice manifold model for $\underline{E}\Gamma$ and N is any slice manifold complement in the sense of Definition 1.11, then N is simply connected and the homomorphism w of (1.13) is automatically the first Stiefel-Whitney class of N/Γ under the obvious identification $\Gamma = \pi_1(N/\Gamma)$. Moreover, the restriction of w to π is the first Stiefel-Whitney class of any closed manifold model for $B\pi$.

Remark 1.18 (Orientation preserving)**.** One can improve Theorem 1.16 by taking the orientations of the slice systems into account. For simplicity we consider only the case, where d is even. Namely, the choice of a generator $[B\pi] \in H_d^\pi(E\pi;\mathbb{Z}^w)$ yields a choice of a generator $[N,\partial N]$ of the infinite cyclic group $H_d^\pi(N,\partial N;\mathbb{Z}^w)$, see [42, Notation 6.22]. Its image under the obvious composite

$$H_d^\pi(N,\partial N;\mathbb{Z}^w) \to H_{d-1}^\pi(\partial N;\mathbb{Z}^w) \xrightarrow{\cong} \bigoplus_{F\in\mathcal{M}} H_{d-1}(S'_F/F)$$

yields orientations on S'_F for every $F \in \mathcal{M}$. Then one can show that S_F and S'_F are oriented F-homotopy equivalent for every $F \in \mathcal{M}$.

Remark 1.19 (Some redundance)**.** Consider condition (5) in Theorem 1.16. Conditions (5b) or (5c) imply condition (5a), provided that conditions (M) and (NM) are satisfied and there is a finite CW-model for $B\pi$, see [3, Theorem B], [42, Theorem 1.12] and Lemma 2.1. Hence it suffices to treat conditions (5a) when dealing with condition (5).

Similarly, consider condition (7) appearing in Theorem 1.16. Obviously the implications (7e) $\implies$ (7d) $\implies$ (7c) hold. The implication (7b) $\implies$ (7a) has been proved in [21, 1, 20, 22]. Hence it suffices to treat condition (7a) and (7c), when dealing with condition (7).

Remark 1.20 (Some conditions are necessary)**.** If we want to find a slice manifold model for $\underline{E}\Gamma$ in the sense of Definition 1.11, then conditions (3), (4), (1), and (6) appearing in Theorem 1.16 are necessary and there must be a finite Γ-CW-model for $\underline{E}\Gamma$, see [42, Lemma 1.9, Lemma 3.3, and Lemma 7.10].

Condition (2) stems from the well-known problem that the Whitney trick and hence surgery theory works without further assumptions only in high dimensions.

The Farrell-Jones Conjecture, condition ($\mathrm{V_{II}}$), and condition (7a) will enter in the proof that certain periodic structure sets are trivial.

Theorem 1.21 (Uniqueness)**.** *Suppose there is a short exact sequence of groups*

$$1 \to \pi \xrightarrow{i} \Gamma \xrightarrow{p} G \to 1$$

with G finite. Let d be a natural number. Consider two d-dimensional free slice manifold systems $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$ and $\mathcal{S}' = \{S'_F \mid F \in \mathcal{M}\}$. Let M and M' be two slice manifold models for $\underline{E}\Gamma$ with respect to $\mathcal{S}$ and $\mathcal{S}'$. Suppose that the following conditions are satisfied:

- *The natural number d satisfies $d \geq 5$,*
- *One of the following assertions holds:*
  - *The group π is a Farrell-Jones group, see Subsection 3.4, and Γ satisfies condition (V_{II}), see Definition 1.3;*
  - *The group π is hyperbolic;*
  - *The group Γ acts cocompactly, properly, and isometrically on a proper* CAT(0)*-space;*
- *One of the following conditions is satisfied:*
  - *The group $\mathrm{UNil}_{d+1}(\mathbb{Z};\mathbb{Z}^{(-1)^d},\mathbb{Z}^{(-1)^d})$ vanishes;*
  - *We have $d \equiv 0 \mod (4)$ or $d \equiv 1 \mod (4)$;*
  - *The group Γ contains no subgroup isomorphic to D_∞;*
  - *Every element $F \in \mathcal{M}$ has odd order;*
  - *The order of G is odd.*

Then:

1. *There exists a Γ-homeomorphism $M \xrightarrow{\cong} M'$;*
2. *For every F there exists an F-h-cobordism between S_F and S'_F;*
3. *Suppose additionally that for every $F \in \mathcal{M}$ the 2-Sylow subgroup of F is cyclic. Let $(N, \partial N)$ and $(N', \partial N')$ be slice manifold complements of M and M'. Suppose that S_F and S'_F are simple F-homotopy equivalent for every $F \in \mathcal{M}$.*

   Then there exists a Γ-homeomorphism $(N, \partial N) \xrightarrow{\cong} (N', \partial N')$. In particular S_F and S'_F are F-homeomorphic for every $F \in \mathcal{M}$.

Theorem 1.21 is now a direct consequence of Remark 1.19, Remark 5.23, and Theorem 9.1. This theorem is similar to the main results of [18], [36], and [19]; some discussion of the variant statements is given in 10.3.

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The paper is organized as follows:

Contents

2. Relating some conditions on Γ

Lemma 2.1. *Let Γ be a group.*

(1) *Suppose that Γ satisfies (M). Then Γ satisfies (F), if and only if it satisfies (NM);*
(2) *Suppose that Γ satisfies condition (F). Then every virtually cyclic subgroup of type I is infinite cyclic;*
(3) *Suppose that Γ satisfies condition (F). Let V be a virtually cyclic subgroup of type II. Then V and $N_\Gamma V$ are both isomorphic to the infinite dihedral group D_∞;*
(4) *Suppose Γ acts properly, cocompactly, and effectively on a contractible manifold M. The following are equivalent:*
   (a) *The Γ action on M is pseudo-free;*
   (b) *Γ satisfies (F);*
   (c) *Γ satisfies (M) and (NM);*
(5) *Suppose that Γ is hyperbolic or, more generally, that any infinite subgroup, which is not virtually cyclic, contains a copy of $\mathbb{Z} * \mathbb{Z}$ as subgroup. Then Γ satisfies conditions (V) and (NV);*
(6) *Suppose that Γ satisfies condition (F) and (V_{II}). Then Γ satisfies condition (NV_{II});*
(7) *Suppose that Γ acts cocompactly, properly, and isometrically on a proper* CAT(0)*-space X and that Γ satisfies (F).*
   Then Γ-satisfies conditions (V_{II}) and (NV_{II}).

Proof. (1) Suppose that (M) and (NM) hold. Choose a maximal finite subgroup M of Γ satisfying $H \subseteq M$. Consider $\gamma \in N_\Gamma H$. We get $\{1\} \neq H = H \cap \gamma H \gamma^{-1} \subseteq M \cap \gamma M \gamma^{-1}$. We conclude $M = \gamma M \gamma^{-1}$ from condition (M). This implies $\gamma \in N_\Gamma M$.

Therefore $N_\Gamma H$ is contained in the finite subgroup M by condition (NM). Hence (F) holds.

Suppose that (M) and (F) hold. Consider a non-trivial maximal finite subgroup $F \subseteq \Gamma$. Since $N_\Gamma F$ is finite by assumption and F is maximal, we get $N_\Gamma F = F$. Hence (NM) holds.

(2) We can find a normal finite subgroup K of H such that V/K is infinite cyclic. Since $V \subseteq N_\Gamma K$ holds and V is infinite, K must be trivial by condition (F). Hence V is infinite cyclic.

(3) Choose an epimorphism $p\colon V \to D_\infty$ with finite kernel K. Since $V \subseteq N_\Gamma K$ holds and V is infinite, K must be trivial by condition (F). Hence V is isomorphic to D_∞.

Let V be a virtually cyclic subgroup of type II. We have already proved that V contains two cyclic subgroups C_1 and C_2 of order two such that $C_1 * C_2 = V$. We have the short exact sequence $1 \to C_\Gamma V \to N_\Gamma V \to \operatorname{aut}(V)$ where the first term is the centralizer of V. Obviously $C_\Gamma V = C_\Gamma C_1 \cap C_\Gamma C_2 \subseteq N_\Gamma C_1 \cap N_\Gamma C_2$. The condition (F) implies that $N_\Gamma C_1 \cap N_\Gamma C_2$ is finite. Since $\operatorname{aut}(V)$ is the semi-direct product $D_\infty \rtimes \mathbb{Z}/2$, the group $N_\Gamma V$ is virtually cyclic. Since it does not have a central element of infinite order, it is a virtually cyclic subgroup of type II. We have already shown that any such group is isomorphic to D_∞.

(4) Proposition 2.3 of [19] shows that (a) and (b) are equivalent, and, furthermore if either of these are satisfied, then the fixed-point set of a finite non-trivial subgroup is a point. It follows that if (a) and (b) are satisfied, then for non-trivial finite subgroups $F \subseteq H$, $M^H = M^F$ and both are contained in the isotropy group of this point, thus conditions (M) and (NM) are satisfied. But conditions (M) and (NM) imply condition (F) by assertion (1).

(5) This follows from [45, Theorem 3.1, Lemma 3.4, Example 3.6].

(6) Let V be a maximal virtually cyclic subgroup of type II. Since V is maximal and $N_\Gamma V$ is a virtually cyclic subgroup of type II by assertion (3) we must have $N_\Gamma V = V$.

(7) By (6), we only need prove that Γ satisfies condition ($\mathrm{V_{II}}$).

Note that any virtually cyclic subgroup of type II is isomorphic to D_∞ by assertion (3). Let $V \subseteq \Gamma$ be a subgroup isomorphic to $D_\infty = \langle t, s \mid t^2 = 1, tst = s^{-1}\rangle$. First we show that there is a geodesic $c\colon \mathbb{R} \to X$ such that $\operatorname{im}(c)$ is V-invariant. Consider the isometry $l_s\colon X \to X$ given by multiplication with s. It is semi-simple, see [6, Proposition 6.10 (2) in II.6 on page 233]. Obviously it has no fixed point. Therefore it has to be a hyperbolic isometry in the sense of [6, Definition 6.3 in II.6 on page 229]. Hence there is an axis for l_s, i.e., a geodesic $c\colon \mathbb{R} \to X$ such that $s \cdot c(\tau) = c(\tau + |l_s|)$ holds for all $\tau \in \mathbb{R}$, where $|l_s| > 0$ is the translation length of l_s, see [6, Theorem 6.8 (1) in II.6 on page 231]. Let Y be the subspace of X appearing in [6, Theorem 6.8 (4) in II.6 on page 231]. Since Y is closed and convex in X, it is itself a CAT(0)-space. Since we have for $\tau \in \mathbb{R}$

$$s \cdot (t \cdot c)(\tau) = t \cdot s^{-1} \cdot c(\tau) = t \cdot c(\tau - |l_s|),$$

we get by $\tau \mapsto (t \cdot c)(-\tau)$ another axis for l_s. This implies that $l_t\colon X \to X$ leaves the subspace $\operatorname{Min}(l_s)$ invariant, see [6, Theorem 6.8 (3) in II.6 on page 231]. Hence there is a point $(y_0, \tau_0) \in Y \times \mathbb{R}$ such that $t \cdot (y_0, \tau_0) = (y_0, \tau_0)$ holds, see [6, Corollary II.2.8 on page 179]. Now the geodesic $c\colon \mathbb{R} \to X$ given by y_0 is an axis for l_s and satisfies $tc(\tau_0) = c(\tau_0)$. We get for all $m \in \mathbb{Z}$

$$t \cdot c(\tau_0 + m \cdot |l_s|) = t \cdot s^m \cdot c(\tau_0) = s^{-m} \cdot t \cdot c(\tau_0) = s^{-m} \cdot c(\tau_0) = c(\tau_0 - m \cdot |l_s|) \in \operatorname{im}(c).$$

This implies that $l_t(\operatorname{im}(c)) = \operatorname{im}(c)$. Since $l_s(\operatorname{im}(c)) = \operatorname{im}(c)$ holds, we conclude that c is V-invariant.

Next we show for two subgroups $V \subseteq V' \subseteq \Gamma$ and a geodesic $c\colon \mathbb{R} \to X$ such that V and V' are isomorphic to D_∞ and $\operatorname{im}(c)$ is V-invariant, that $\operatorname{im}(c)$ is V'-invariant. We can find a presentation of $V' = \langle t, s \mid t^2 = 1, tst = s^{-1}\rangle$ and a natural number $m \geq 1$ such that V is generated by t and s^m. Hence it suffices to show that an axis for s^m is automatically an axis for s. This follows by inspecting the proof of [6, Theorem 6.8 (2) in II.6 on page 231].

Now consider any virtually cyclic subgroup V of Γ of type II. We know already that V has to be isomorphic to D_∞. Choose a geodesic $c\colon \mathbb{R} \to X$ such that $\operatorname{im}(c)$ is V-invariant. Let Γ_c be the subgroup of Γ consisting of elements, for which $l_\gamma(\operatorname{im}(c)) = \operatorname{im}(c)$ holds. Note that Γ_c is non-abelian, since it contains V. Since the non-abelian group Γ_c acts properly and cocompactly on $\mathbb{R}$, it must be isomorphic to D_∞, hence is virtually cyclic of type II. Clearly Γ_c is a maximal virtually cyclic subgroup of type II leaving $\operatorname{im}(c)$ invariant, and, in fact, it is the unique such maximal subgroup. The preceding paragraph implies that Γ_c is the unique maximal virtually cyclic subgroup of type II containing V.

Hence Γ satisfies condition ($\mathrm{V_{II}}$). $\square$

Lemma 2.1 (7) was proved in [19, Remark 1.2] in the case where Γ acts cocompactly, properly, and isometrically on a contractible Riemannian manifold X of non-positive sectional curvature and in the CAT(0)-case in the announcement [36].

3. Equivariant homology theories, spectra over groupoids and the Full Farrell-Jones Conjecture

3.1. **Equivariant homology theories.** For the definition of a *G-homology theory* $\mathcal{H}_*^G$ for G-CW-pairs we refer for instance to [40, Chapter 1]. This is extended to the notion of an *equivariant homology theory* $\mathcal{H}_*^?$ in [40, Chapter 1]. Roughly speaking, an equivariant homology theory assigns to every group G a G-homology theory $\mathcal{H}_*^G$ and comes with a so-called induction structure, i.e., for any group homomorphism $\alpha\colon G \to G'$ and G-CW-pair (X, A), there is a natural homomorphism $\operatorname{ind}_\alpha\colon \mathcal{H}_*^G(X, A) \to \mathcal{H}_*^{G'}(\alpha_*(X, A))$ of $\mathbb{Z}$-graded abelian groups, where $\alpha_*(X, A)$ is the induced G'-CW-pair. It satisfies certain naturality conditions and is compatible with the long exact sequence of pairs and disjoint unions. If the kernel of α acts freely on (X, A), the map $\operatorname{ind}_\alpha\colon \mathcal{H}_*^G(X, A) \to \mathcal{H}^{G'}(\alpha_*(X, A))$ is an isomorphism. In particular we get for every group G and subgroup H a natural isomorphism $\mathcal{H}_*^G(G/H) \cong \mathcal{H}_*^H(\{\bullet\})$ and for every free G-CW-pair (X, A) a natural isomorphism $\mathcal{H}_*^G(X, A) \cong \mathcal{H}_*^{\{1\}}(X/G, A/G)$, using the induction structure.

Given a map $f\colon X \to Y$ of G-CW-complexes, one can define a $\mathbb{Z}$-graded abelian group $\mathcal{H}_*^G(f)$, which fits into a long exact sequence

$$\cdots \to \mathcal{H}_n^G(X) \xrightarrow{f_n} \mathcal{H}_n^G(Y) \to \mathcal{H}_n^G(f) \xrightarrow{\partial_n} \mathcal{H}_{n-1}^G(X)$$
$$\xrightarrow{f_{n-1}} \mathcal{H}_{n-1}^G(Y) \to \mathcal{H}_{n-1}^G(f) \xrightarrow{\partial_{n-1}} \cdots .$$

Given a commutative square of G-CW-complexes

$$\Phi = \begin{tikzcd} X_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & X_1 \arrow[d, "g_1"] \\ X_2 \arrow[r, "g_2"'] & X \end{tikzcd}$$

one obtains a $\mathbb{Z}$-graded abelian group $\mathcal{H}_*^G(\Phi)$, which fits into an exact sequence

$$\cdots \to \mathcal{H}_n^G(f_2) \to \mathcal{H}_n^G(g_1) \to \mathcal{H}_n^G(\Phi) \xrightarrow{\partial_n} \mathcal{H}_{n-1}^G(f_2)$$
$$\to \mathcal{H}_{n-1}^G(g_1) \to \mathcal{H}_{n-1}^G(\Phi) \xrightarrow{\partial_{n-1}} \cdots .$$

If Φ is a G-homotopy pushout, e.g., Φ is a G-pushout and f_1 or f_2 is an inclusion of G-CW-complexes, then $\mathcal{H}_n^G(\Phi) = 0$ for all $n \in \mathbb{Z}$.

3.2. **Equivariant homology theories from spectra over groupoids.** Let $\mathsf{Groupoids}$ be the category of small connected groupoids. A $\mathsf{Groupoids}$-*spectrum* $\mathbf{E}$ is a functor from $\mathsf{Groupoids}$ to the category of spectra $\mathsf{Spectra}$. Any $\mathsf{Groupoids}$-spectrum $\mathbf{E}$ gives rise to an equivariant homology theory $H_*^?(-;\mathbf{E})$ in the sense of [40, Chapter 1], see for instance [44, Proposition 157 on page 796] which is based on [24]. Thus for any group G, we get a G-homology theory $H_*^G(-;\mathbf{E})$ such that for any subgroup H we have

$$H_n^G(G/H;\mathbf{E}) \cong H_n^H(\{\bullet\},\mathbf{E}) \cong \pi_n(\mathbf{E}(\widehat{H})),$$

where $\widehat{H}$ is the groupoid with one object and H as its automorphism group.

Let $\mathbf{E}\colon \mathsf{Groupoids} \to \mathsf{Spectra}$ be a $\mathsf{Groupoids}$-spectrum. Denote by $\mathbf{E}\langle 1\rangle$ the $\mathsf{Groupoids}$-spectrum obtained from $\mathbf{E}$ by passing to the 1-connected covering. There is a morphism $\mathbf{E}\langle 1\rangle \to \mathbf{E}$ of $\mathsf{Groupoids}$-spectra such that for every groupoid $\mathcal{G}$ the map $\pi_q(\mathbf{E}\langle 1\rangle(\mathcal{G})) \to \pi_q(\mathbf{E}(\mathcal{G}))$ is an isomorphism for $q \geq 1$ and $\pi_q(\mathbf{E}\langle 1\rangle(\mathcal{G})) = 0$ for $q \leq 0$. Define a sequence of $\mathsf{Groupoids}$-spectra $\mathbf{E}\langle 1\rangle \to \mathbf{E} \to \overline{\mathbf{E}}$ such that its evaluation at any groupoid is a cofibration sequence of spectra. For any groupoid, $\pi_q(\overline{\mathbf{E}}(\mathcal{G})) = 0$ for $q \geq 1$ and $\pi_q(\mathbf{E}(\mathcal{G})) \xrightarrow{\cong} \pi_q(\overline{\mathbf{E}}(\mathcal{G}))$ is an isomorphism for $q \leq 0$. For any square Φ of Γ-CW-complexes, there is a long exact sequence, natural in Φ,

$$\cdots \to H_{n+1}^\Gamma(\Phi;\mathbf{E}\langle 1\rangle) \to H_{n+1}^\Gamma(\Phi;\mathbf{E}) \to H_{n+1}^\Gamma(\Phi;\overline{\mathbf{E}})$$
$$\to H_n^\Gamma(\Phi;\mathbf{E}\langle 1\rangle) \to H_n^\Gamma(\Phi;\mathbf{E}) \to H_n^\Gamma(\phi;\overline{\mathbf{E}}) \to \cdots . \tag{3.1}$$

3.3. **Some basics about K-and L-theory of groups rings.** Let $K_n(RG)$ denote the *n-th algebraic K-group* of the group ring RG in the sense of Quillen for $n \geq 0$ and in the sense of Bass for $n \leq -1$. Let $NK_n(R)$ denote the *Bass-Nil-groups* of R, which are defined as the cokernel of the map $K_n(R) \to K_n(R[x])$. Recall that the Bass-Heller-Swan decomposition says

$$K_n(R\mathbb{Z}) \quad\cong\quad K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R). \tag{3.2}$$

If R is a regular ring, then $NK_n(R) = 0$ for every $n \in \mathbb{Z}$, see for instance [52, Theorems 3.3.3 and 5.3.30].

For a ring with involution R, for a integer n, and for $j \in \{1,0,-1,-2,\ldots\} \amalg \{-\infty\}$, one defines the Wall-Ranicki algebraic L-group $L_n^{\langle j\rangle}(R)$, combine [49, Section 13] with [50, Section 17]. These groups are 4-periodic in n. The index j is called the decoration. The group $L_n^{\langle -\infty\rangle}(R)$ is called the *ultimate lower quadratic L-group.* The L-groups are given as the homotopy groups of a 4-periodic spectrum $\mathbf{L}^{\langle j\rangle}(R)$ (see [49, Section 13]). When $R = \mathbb{Z}$, the L-groups and L-spectra are constant in j, that is, they are independent of the decoration. Often $L^{\langle j\rangle}$ and $L^{\langle 1\rangle}$ are denote by L^p and L^h respectively. For groups, RG, one also includes $j = 2$, which is also denoted by $L^s(RG)$ (see [48, page 105]).

For $j \in \{2,1,0,-1,\ldots\} \amalg \{-\infty\}$, there are $\mathsf{Groupoids}$-spectra, see [24, Section 2],

$$\mathbf{K}_R\colon \mathsf{Groupoids} \;\to\; \mathsf{Spectra};$$
$$\mathbf{L}_R^{\langle j\rangle}\colon \mathsf{Groupoids} \;\to\; \mathsf{Spectra},$$

coming with natural identifications

$$\begin{aligned}\pi_n(\mathbf{K}_R(\widehat{G})) &= K_n(RG);\\ \pi_n(\mathbf{L}_R^{\langle j\rangle}(\widehat{G})) &= L_n^{\langle j\rangle}(RG).\end{aligned}$$

If G is a group with an orientation character, then some modifications to the above theory must be made, see [2]. An *orientation character* is a homomorphism $w\colon G \to \{\pm 1\}$. This determines the *w-twisted involution on RG* sending $\sum_{g\in G}\lambda_g\cdot g$ to $\sum_{g\in G}\overline{\lambda}_g\cdot w(g)\cdot g^{-1}$. The corresponding L-groups of this ring with involution are denoted $L_n^{\langle j\rangle}(RG,w)$. To deal with non-trivial orientation characters, one needs functors

$$\mathbf{L}_{R,w}^{\langle j\rangle}\colon \mathsf{Groupoids}\downarrow\{\pm 1\} \to \mathsf{Spectra}$$

coming with a natural identification

$$\pi_n(\mathbf{L}_{R,w}^{\langle j\rangle}(\widehat{G})) = L_n^{\langle j\rangle}(RG,w).$$

A group with orientation character (G,w) determines a G-homology theory denoted by $H_*^G(-;\mathbf{L}_{R,w}^{\langle j\rangle})$. There is an isomorphism $H_n^G(G/H;\mathbf{L}_{R,w}^{\langle j\rangle}) \cong L_n^{\langle j\rangle}(RH;w|_H)$.

For a group with orientation character (G,w) and for a free G-CW-complex X, define the *periodic n-th structure group with decoration $\langle j\rangle$* to be

$$\mathcal{S}_n^{\mathrm{per},\langle j\rangle}(X/G) \quad := \quad H_n^G(X\to\{\bullet\};\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}).$$

It the orientation character is trivial, these groups fit into the periodic version of the algebraic surgery exact sequence with decoration $\langle j\rangle$,

$$\cdots \to H_n(X/G;\mathbf{L}(\mathbb{Z})) \to L_n^{\langle j\rangle}(\mathbb{Z}G) \to \mathcal{S}_n^{\mathrm{per},\langle j\rangle}(X/G) \\ \to H_{n-1}(X/G;\mathbf{L}(\mathbb{Z})) \to L_{n-1}^{\langle j\rangle}(\mathbb{Z}G) \to \cdots .$$

Here we have identified $H_*^G(X;\mathbf{L}_{\mathbb{Z}}^{\langle j\rangle})$ with $H_*(X/G;\mathbf{L}(\mathbb{Z}))$ using homotopy invariance and the induction structure. This periodic surgery sequence appears in the classification of ANR-homology manifolds in Bryant-Ferry-Mio-Weinberger [9, Main Theorem]. It is related to the algebraic surgery exact sequence and thus to the classical surgery sequence, see Ranicki [49, Section 18].

For $n\in\mathbb{Z}$, the abelian group $H_n^{D_\infty}(\underline{E}D_\infty\to\{\bullet\};\mathbf{L}_R^{\langle-\infty\rangle})$ can be identified with the $\mathrm{UNil}(R;R,R)$-groups of Cappell [11] see Remark 5.23. If $w: D_\infty = \langle a,b \mid a^2 = b^2 = 1\rangle \to \{\pm 1\}$ is given by $w(a)=w(b)=(-1)^n$, then $H_n^{D_\infty}(\underline{E}D_\infty\to\{\bullet\};\mathbf{L}_{\mathbb{Z},w}^{\langle-\infty\rangle})$ agrees with the group $\mathrm{UNil}_n(\mathbb{Z};\mathbb{Z}^{(-1)^n},\mathbb{Z}^{(-1)^n})$ appearing in Theorem 1.16 and Theorem 1.21.

3.4. **The Full Farrell-Jones Conjecture.** For the precise formulation of the Full-Farrell-Jones Conjecture and its current status we refer to [43, Sections 13.6 and 16.2]. It is the most general version of the Farrell-Jones Conjecture. We call a group G a *Farrell-Jones group*, if it satisfies the Full Farrell-Jones Conjecture. The class of Farrell-Jones groups contains CAT(0)-groups, lattices in locally compact second countable Hausdorff groups, solvable groups, and fundamental group of manifolds of dimension ≤ 3. It is closed under taking subgroups, passing to overgroups of finite index, and colimits over directed systems of groups with not necessarily injective structure maps. For our purposes it suffices to know that the Full-Farrell-Jones Conjecture implies that the projection $\underline{\underline{E}}G\to\{\bullet\}$ induces for every $n\in\mathbb{Z}$ and every ring R (with involution) isomorphisms

$$\begin{aligned}H_n^G(\underline{\underline{E}}G;\mathbf{K}_R) &\xrightarrow{\cong} H_n^G(\{\bullet\};\mathbf{K}_R) = K_n(RG);\\ H_n^G(\underline{\underline{E}}G;\mathbf{L}_{R,w}^{\langle-\infty\rangle}) &\xrightarrow{\cong} H_n^G(\{\bullet\};\mathbf{L}_{R,w}^{\langle-\infty\rangle}) = L_n^{\langle-\infty\rangle}(RG,w),\end{aligned}$$

where $\underline{\underline{E}}G$ is the classifying space for the family of virtually cyclic subgroups of G.

4. Computing the K-theory

4.1. **The definition of Whitehead groups.** Define for a group G and a ring R the *n-th Whitehead group* $\mathrm{Wh}_n(G;R)$ to be $H_n^G(EG \to \{\bullet\};\mathbf{K}_R)$. The long exact sequence of the map $EG \to \{\bullet\}$ yields a long exact sequence

$$\begin{aligned}\cdots \to H_n(BG;\mathbf{K}(R)) \to K_n(RG) \to \mathrm{Wh}_n(G;R)&\\ \to H_{n-1}(BG;\mathbf{K}(R)) \to K_{n-1}(RG) \to \cdots,&\end{aligned}$$

where $H_*(-;\mathbf{K}(R))$ is the generalized (non-equivariant) homology theory associated to the non-connective K-theory spectrum $\mathbf{K}(R)$ of R. Suppose that R is regular. Then $K_n(R) = 0$ for $n \leq -1$. Hence the canonical map $K_n(RG) \to \mathrm{Wh}_n(G;R)$ is bijective for $n \leq -1$, we have the split short exact sequence $0 \to K_0(R) \to K_0(RG) \to \mathrm{Wh}_0(G;R) \to 0$, and the short exact sequence $H_1(BG,\mathbf{K}(R)) \to K_1(RG) \to \mathrm{Wh}_1(G;R) \to 0$. If R is regular and the canonical map $K_0(\mathbb{Z}) \to K_0(R)$ is bijective, then we get an isomorphism $\widetilde{K}_0(RG) \xrightarrow{\cong} \mathrm{Wh}_0(G;R)$ and a split short exact sequence $0 \to K_1(R) \oplus G/[G,G] \to K_1(RG) \to \mathrm{Wh}_1(G;R) \to 0$. If $R = \mathbb{Z}$, then $\mathrm{Wh}_1(G;\mathbb{Z})$ agrees with the classical Whitehead group $\mathrm{Wh}(G)$, $\widetilde{K}_0(\mathbb{Z}G) \cong \mathrm{Wh}_0(G;\mathbb{Z})$, and $K_n(\mathbb{Z}G) \cong \mathrm{Wh}_n(G,\mathbb{Z})$ for $n \leq -1$.

Whitehead groups arise naturally when studying h-cobordisms, pseudoisotopy, and Waldhausen's A-theory. Their geometric significance is reviewed, for example, in Dwyer-Weiss-Williams [31, Section 9] and Lück-Reich [44, Section 1.4.1], where additional references can also be found. When $G = \mathbb{Z}$, it follows from (3.2) and the fact that $H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_R) \cong K_n(R) \oplus K_{n-1}(R)$ that there is an identification

$$H_n^{\mathbb{Z}}(E\mathbb{Z} \to \{\bullet\};\mathbf{K}_R) \quad\cong\quad NK_n(R) \oplus NK_n(R). \tag{4.1}$$

4.2. **Computing Whitehead groups.** We will later need the following result to apply the Farrell-Jones Conjecture.

Theorem 4.2.

(1) *Suppose that Γ satisfies (M) and (NM). Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal finite subgroups $F \subseteq \Gamma$. Consider the cellular Γ-pushout*

$$\begin{tikzcd} \coprod_{F\in\mathcal{M}} \Gamma \times_F EF \arrow[r, "i"] \arrow[d, "\coprod_{F\in\mathcal{M}} p_F"'] & E\Gamma \arrow[d] \\ \coprod_{F\in\mathcal{M}} \Gamma/F \arrow[r] & X \end{tikzcd}$$

where the map p_F comes from the projection $EF \to \{\bullet\}$, and i is an inclusion of Γ-CW-complexes.

Then X is a model for $\underline{E}\Gamma$;

(2) *Assume that Γ satisfies conditions (V) and (NV). Let $\mathcal{V}$ be a complete system of representatives of the conjugacy classes of maximal infinite virtually cyclic subgroups $V \subseteq \Gamma$.*

Consider the cellular Γ-pushout

$$\begin{tikzcd} \coprod_{V\in\mathcal{V}} \Gamma \times_V \underline{E}V \arrow[r, "i"] \arrow[d, "\coprod_{V\in\mathcal{V}} p_V"'] & \underline{E}\Gamma \arrow[d] \\ \coprod_{V\in\mathcal{V}} \Gamma/V \arrow[r] & Y \end{tikzcd}$$

where the map p_V comes from the projection $\underline{E}V \to \{\bullet\}$, and i is an inclusion of Γ-CW-complexes.

Then Y is a model for $\underline{\underline{E}}V$.

(3) *Assume that Γ satisfies conditions ($\mathrm{V_{II}}$) and ($\mathrm{NV_{II}}$). Let $\mathcal{V}_{II}$ be a complete system of representatives of the conjugacy classes of maximal infinite virtually cyclic subgroups of type II. Let $E_{\mathcal{VCY}_\mathrm{I}}(V)$ be the classifying space for the family $\mathcal{VCY}_\mathrm{I}$ of G, which consists of finite subgroups of infinite virtually cyclic subgroups of type I.*

Consider the cellular Γ-pushout

$$\begin{array}{ccc} \coprod_{V\in\mathcal{V}_\mathrm{II}} \Gamma\times_V E_{\mathcal{VCY}_\mathrm{I}}(V) & \xrightarrow{i} & E_{\mathcal{VCY}_\mathrm{I}}(\Gamma) \\ {\scriptstyle \coprod_{V\in\mathcal{V}_{II}} p_V}\downarrow & & \downarrow \\ \coprod_{V\in\mathcal{V}_\mathrm{II}} \Gamma/V & \longrightarrow & Z \end{array}$$

where the map p_V comes from the projection $E_{\mathcal{VCY}_I}(V) \to \{\bullet\}$, and i is an inclusion of Γ-CW-complexes.

Then Z is a model for $\underline{\underline{E}}V$.

Proof. This follows from [45, Corollary 2.11] for assertions (1) and (2). The proof for assertion (3) is analogous, just apply [45, Corollary 2.8]. □

The next result has already been proved for $R = \mathbb{Z}$ in [25, Theorem 5.1 (d)].

Theorem 4.3. *Let R be a regular ring and let Γ be a Farrell-Jones group satisfying conditions (M) and (NM). Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal finite subgroups. Then the canonical map*

$$\bigoplus_{F\in\mathcal{M}} \mathrm{Wh}_n(F;R) \xrightarrow{\cong} \mathrm{Wh}_n(\Gamma;R)$$

is bijective for all $n\in\mathbb{Z}$.

Proof. Since Γ is a Farrell-Jones group,

$$H_n^\Gamma(\underline{\underline{E}}\Gamma;\mathbf{K}_R) \xrightarrow{\cong} H_n^\Gamma(\{\bullet\};\mathbf{K}_R)$$

is an isomorphism for all $n\in\mathbb{Z}$. The relative assembly map

$$H_n^\Gamma(E_{\mathcal{VCY}_I}(\Gamma);\mathbf{K}_R) \xrightarrow{\cong} H_n^\Gamma(\underline{\underline{E}}\Gamma;\mathbf{K}_R)$$

is an isomorphism for all $n\in\mathbb{Z}$ by [28, Remark 1.6]. Every element $V\in\mathcal{VCY}_I$, which is infinite, is an infinite cyclic group by Lemma 2.1. If V is infinite cyclic, we get an isomorphism $H_n^V(\underline{E}V \to \{\bullet\};\mathbf{K}_R) \cong NK_n(R)\oplus NK_n(R)$ from the Bass-Heller-Swan decomposition. Since R is regular, $NK_n(R)$ vanishes, see for instance [52, Theorems 3.3.3 and 5.3.30]. Hence we conclude that the assembly map

$$H_n^V(EV;\mathbf{K}_R) \to H_n^V(\{\bullet\};\mathbf{K}_R)$$

is bijective for all $n\in\mathbb{Z}$ and $V\in\mathcal{VCY}_I\setminus\mathcal{FIN}$. We conclude from the Transitivity Principle, see for instance [44, Theorem 65 on page 742], that the map

$$H_n^\Gamma(\underline{E}\Gamma;\mathbf{K}_R) \xrightarrow{\cong} H_n^\Gamma(E_{\mathcal{VCY}_I}(\Gamma);\mathbf{K}_R)$$

is an isomorphism for all $n\in\mathbb{Z}$. Hence the map induced by the projection $\underline{E}\Gamma\to\{\bullet\}$

$$H_n^\Gamma(\underline{E}\Gamma;\mathbf{K}_R) \xrightarrow{\cong} H_n^\Gamma(\{\bullet\};\mathbf{K}_R)$$

is an isomorphism for all $n\in\mathbb{Z}$. This implies that the map

$$H_n^\Gamma(E\Gamma\to\underline{E}\Gamma,\mathbf{K}_R) \to H_n^\Gamma(E\Gamma\to\{\bullet\},\mathbf{K}_R) = \mathrm{Wh}_n(G;R)$$

is bijective for all $n \in \mathbb{Z}$. Since Γ satisfies (M) and (NM), we get from excision and Theorem 4.2 (1) isomorphisms

$$\bigoplus_{F \in \mathcal{M}} H_n^F(EF \to \{\bullet\}; \mathbf{K}_R) \xrightarrow{\cong} H_n^\Gamma(E\Gamma \to \underline{E}\Gamma, \mathbf{K}_R)$$

for $n \in \mathbb{Z}$. Since $H_n^F(EF \to \{\bullet\}; \mathbf{K}_R) = \mathrm{Wh}(F; R)$, the proof of Theorem 4.3 is finished. □

If Γ satisfies (M), (NM), (V) and (NV) and the Farrell-Jones Conjecture, one can also compute the Whitehead group $\mathrm{Wh}_n(\Gamma; R)$ for arbitrary R. Namely, if we denote by $\mathcal{V}_{\mathrm{I}}$ and $\mathcal{V}_{\mathrm{II}}$ the subset of $\mathcal{V}$ consisting virtually cyclic subgroups of type I and of type II respectively, then

$$\begin{aligned}
\mathrm{Wh}_n(\Gamma; R) &\cong H_n^\Gamma(E\Gamma \to \{\bullet\}; \mathbf{K}_R) \\
&\cong H_n^\Gamma(E\Gamma \to \underline{\underline{E}}\Gamma; \mathbf{K}_R) \\
&\cong H_n^\Gamma(E\Gamma \to \underline{E}\Gamma; \mathbf{K}_R) \oplus H_n^\Gamma(\underline{E}\Gamma \to \underline{\underline{E}}\Gamma; \mathbf{K}_R) \\
&\cong \bigoplus_{F \in \mathcal{M}} H_n^F(EF \to \{\bullet\}; \mathbf{K}_R) \oplus \bigoplus_{V \in \mathcal{V}} H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{K}_R) \\
&\cong \bigoplus_{F \in \mathcal{M}} \mathrm{Wh}_n(F; R) \oplus \bigoplus_{V \in \mathcal{V}_{\mathrm{I}}} NK_n(R) \oplus NK_n(R) \oplus \bigoplus_{V \in \mathcal{V}_{\mathrm{II}}} NK_n(R).
\end{aligned}$$

The first isomorphism is by definition, the second by the Farrell-Jones Conjecture, the third by [4], the fourth by Theorem 4.2 (1) and (2), and the last by the Bass-Heller-Swan decomposition if $V \in \mathcal{V}_{\mathrm{I}}$ and by [23, Corollary 3.27] if $V \in \mathcal{V}_{\mathrm{II}}$.

5. Computing the L-theory

5.1. **Some basics about K- and L-theory for additive categories with involution.** Although we are only interested in the L-groups of group rings, we need some input from the L-theory for additive categories $\mathcal{A}$ with involution, see Remark 5.8. Ranicki defined decorated L-groups $L_n^{\langle j \rangle}(\mathcal{A})$ for $n \in \mathbb{Z}$ and $j \in \{1, 0, -1, -2, \ldots\} \amalg \{-\infty\}$ in [50, Section 13 and 17]. By convention $L_n^{\langle 1 \rangle}(\mathcal{A})$ agrees with the standard L-theory $L_n(\mathcal{A})$ of $\mathcal{A}$ and $L_n^{\langle 0 \rangle}(\mathcal{A})$ is the standard L-theory $L_n(\mathrm{Idem}(\mathcal{A}))$ of the idempotent completion $\mathrm{Idem}(\mathcal{A})$. There is a Shaneson splitting and there are Rothenberg sequences, see [50, Theorem 17.2] or (5.5), (5.6), and (5.7).

Given a ring S, let $\mathcal{F}(S)$ be the following small additive category. The set of objects is $\{[n] \mid n \in \mathbb{Z}, n \geq 0\}$. A morphism $A\colon [n] \to [m]$ for $m, n \geq 1$ is given by a m-by-n matrix with entries in S. The set of morphisms $[n] \to [m]$ is defined to be $\{0\}$, if the source or target is $[0]$. Composition is given by matrix multiplication. This category is equivalent to the category of finitely generated free S-modules. We define the small additive category $\mathcal{P}(S)$ to be the idempotent completion of $\mathcal{F}(S)$. One easily checks that $\mathcal{P}(S)$ is equivalent to the additive category of finitely generated projective S-modules. If S is a ring with involution, then $\mathcal{F}(S)$ and $\mathcal{P}(S)$ become additive categories with involution. One defines $L_n^{\langle j \rangle}(S) := L_n^{\langle j \rangle}(\mathcal{F}(S))$. With these conventions $L_n^{\langle 1 \rangle}(S) = L_n^h(S) = L_n(S)$ and $L_n^{\langle 0 \rangle}(S) = L_n^p(S)$.

One reason why it is better to work with additive categories with involutions instead of rings is the compatibility with direct sums and direct products. Namely, for a set of additive categories $\{\mathcal{A}_i \mid i \in I\}$ for arbitrary I, for $n \in \mathbb{Z}$, and $j \in$

$\{1, 0, -1, -2, \ldots\}$, the canonical map given by the projection yields isomorphisms

$$K_n(\prod_{i \in I} \mathcal{A}_i) \xrightarrow{\cong} \prod_{i \in I} K_n(\mathcal{A}_i); \tag{5.1}$$

$$L_n^{\langle j \rangle}(\prod_{i \in I} \mathcal{A}_i) \xrightarrow{\cong} \prod_{i \in I} L_n^{\langle j \rangle}(\mathcal{A}_i). \tag{5.2}$$

This is not true for the decoration $j = -\infty$ in general, unless I is finite or there exists $j_0 \in \mathbb{Z}$ such that $K_j(\mathcal{A}_i) = 0$ for all $i \in I$ and $j \leq j_0$, see [12], [13], [56].

For $n \in \mathbb{Z}$ and $j \in \{1, 0, -1, -2, \ldots\} \amalg \{-\infty\}$, the canonical maps given by the inclusions induce isomorphisms

$$\bigoplus_{i \in I} K_n(\mathcal{A}_i) \xrightarrow{\cong} K_n(\bigoplus_{i \in I} \mathcal{A}_i); \tag{5.3}$$

$$\bigoplus_{i \in I} L_n^{\langle j \rangle}(\mathcal{A}_i) \xrightarrow{\cong} L_n^{\langle j \rangle}(\bigoplus_{i \in I} \mathcal{A}_i). \tag{5.4}$$

This follows for finite I from (5.1) and (5.2) and for general I from the fact that K-theory and the L-theory with decoration $\langle j \rangle$ commute with colimits over directed systems of additive categories.

5.2. **Some basics about L-theory for rings with involution.** Let R be a ring satisfying $\widetilde{K}_n(R) = 0$ for $n < 0$ and $K_0(\mathbb{Z}) \xrightarrow{\cong} K_0(R)$, e.g., a principal ideal domain R. Let G be a group. Recall that $\widetilde{K}_n(R)$ is defined to be the cokernel of $K_n(\mathbb{Z}) \to K_n(R)$. Consider $S = RG$ equipped with the w-twisted involution for a fixed orientation character $w\colon G \to \{\pm 1\}$. Define $L^{\langle 2 \rangle}(RG, w)$ to be the X-decorated n-th L-group of $\mathcal{F}(S)$, where X is the image of the assembly map $H_1(BG; \mathbf{K}(R)) \to K_1(RG)$. We define $L^{\langle j \rangle}(RG, w)$ for $j \in \{1, 0, -1, \ldots\} \amalg \{-\infty\}$ by $L_n^{\langle j \rangle}(\mathcal{F}(S))$. One can define decorated L-groups for arbitrary rings with involutions. However, we made the assumption on R essentially in order to guarantee the following facts. We have

$$\begin{aligned}
H_n(BG; \mathbf{K}(R)) &= \{0\} \quad \text{for } n \leq -1; \\
H_0(BG; \mathbf{K}(R)) &= K_0(R) \cong \mathbb{Z}; \\
H_1(BG; \mathbf{K}(R)) &\cong G/[G, G] \times K_1(R), \\
\mathrm{Wh}_j(G; R) &= K_j(RG) \quad \text{for } n \leq -1; \\
\mathrm{Wh}_0(G; R) &= \widetilde{K}_0(RG)
\end{aligned}$$

and a split short exact sequence

$$0 \to H_1(BG; \mathbf{K}(R)) \to K_1(RG) \to \mathrm{Wh}_1(G; R) \to 0.$$

There are Rothenberg sequences for $j \in \{2, 1, 0, -1, \ldots\}$

$$\begin{aligned}
\cdots \to L_n^{\langle j+1 \rangle}(RG, w) \to L_n^{\langle j \rangle}(RG, w) \to \widehat{H}^n(\mathbb{Z}/2, \mathrm{Wh}_j(G; R)) \\
\to L_{n-1}^{\langle j+1 \rangle}(RG, w) \to L_{n-1}^{\langle j \rangle}(RG, w) \to \cdots .
\end{aligned} \tag{5.5}$$

For a $\mathbb{Z}[\mathbb{Z}/2]$-module A with $\mathbb{Z}/2$-action $a \mapsto \overline{a}$, we define the Tate cohomology

$$\widehat{H}^n(\mathbb{Z}/2; A) = \frac{\{a \in A \mid \overline{a} = (-1)^n a\}}{\{a + (-1)^n \overline{a} \mid a \in A\}}$$

Moreover, the Shaneson splitting gives for $j \in \{2, 1, 0, -1, \ldots\}$ and $n \in \mathbb{Z}$

$$L_n^{\langle j \rangle}(R[G \times \mathbb{Z}], \mathrm{pr}^* w) \cong L_n^{\langle j \rangle}(RG, w) \oplus L_{n-1}^{\langle j-1 \rangle}(RG, w) \tag{5.6}$$

for $\mathrm{pr}^* w\colon G \times \mathbb{Z} \xrightarrow{\mathrm{pr}} G \xrightarrow{w} \{\pm 1\}$. One defines

$$L_n^{\langle -\infty \rangle}(RG, w) := \mathrm{colim}_{j \to -\infty} L_n^{\langle j \rangle}(RG, w).$$

We have

$$L_n^{\langle -\infty\rangle}(R[G\times\mathbb{Z}], \mathrm{pr}^* w) \cong L_n^{\langle -\infty\rangle}(RG, w) \oplus L_{n-1}^{\langle -\infty\rangle}(RG, w). \tag{5.7}$$

If $R = \mathbb{Z}$, then $\mathrm{Wh}_1(G; R)$ is the classical Whitehead group $\mathrm{Wh}(G)$. Moreover, $L^{\langle j\rangle}(\mathbb{Z}G, w)$ agrees with the classical decorated L-groups $L_n^s(\mathbb{Z}G, w)$, $L_n^h(\mathbb{Z}G, w)$, and $L_n^p(\mathbb{Z}G, w)$ for $j = 2, 1, 0$.

Remark 5.8 (Decorated L-theory is not compatible with finite products of rings)**.** Note that decorated L-theory is compatible with finite products of rings only for $j = p$, (or, equivalently, $j = 0$) but in general not for the other decorations. One can see the problem for example for the decoration $j = h$, (or, equivalently, $j = 1$) from the Rothenberg sequences, since the canonical map $\widetilde{K}_0(S_1\times S_2) \to \widetilde{K}_0(S_1)\times\widetilde{K}_0(S_2)$ for two rings S_1 and S_2 is not bijective in general. All of this is due to the facts that for two rings S_1 and S_2 the canonical functor $\mathcal{P}(S_1)\times\mathcal{P}(S_2) \to \mathcal{P}(S_1\times S_2)$ is an equivalence of additive categories, where $\mathcal{F}(S_1)\times\mathcal{F}(S_2) \to \mathcal{F}(S_1\times S_2)$ is *not* an equivalence of additive categories, since $S_1\times\{0\}$ is not a free $S_1\times S_2$-module.

5.3. **A construction of Ranicki.** We need the following result of Ranicki [48, Proposition 2.5.1 on page 166], which is stated there only for rings but carries over to additive categories with involutions.

Theorem 5.9. *Let $U\colon \mathcal{A}\to\mathcal{B}$ be a functor of additive categories with involution. Consider $j \in \{1, 0, -1, \ldots\}$. Then one can construct a commutative diagram with long exact rows and columns*

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & L_n^{\langle j+1\rangle}(\mathcal{A}) \arrow[r, "U_*"] \arrow[d] & L_n^{\langle j+1\rangle}(\mathcal{B}) \arrow[r] \arrow[d] & L_n^{\langle j+1\rangle}(U) \arrow[r] \arrow[d] & L_{n-1}^{\langle j+1\rangle}(\mathcal{A}) \arrow[r, "U_*"] \arrow[d] & \cdots \\
\cdots \arrow[r] & L_n^{\langle j\rangle}(\mathcal{A}) \arrow[r, "U_*"] \arrow[d] & L_n^{\langle j\rangle}(\mathcal{B}) \arrow[r] \arrow[d] & L_n^{\langle j\rangle}(U) \arrow[r] \arrow[d] & L_{n-1}^{\langle j\rangle}(\mathcal{A}) \arrow[r, "U_*"] \arrow[d] & \cdots \\
\cdots \arrow[r] & \widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_j(\mathcal{A})) \arrow[r, "U_*"] \arrow[d] & \widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_j(\mathcal{B})) \arrow[r] \arrow[d] & \widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_j(U)) \arrow[r] \arrow[d] & \widehat{H}^{n-1}(\mathbb{Z}/2; \widetilde{K}_j(\mathcal{A})) \arrow[r, "U_*"] \arrow[d] & \cdots \\
\cdots \arrow[r] & L_{n-1}^{\langle j+1\rangle}(\mathcal{A}) \arrow[r, "U_*"] \arrow[d] & L_{n-1}^{\langle j+1\rangle}(\mathcal{B}) \arrow[r] \arrow[d] & L_{n-1}^{\langle j+1\rangle}(U) \arrow[r] \arrow[d] & L_{n-2}^{\langle j+1\rangle}(\mathcal{A}) \arrow[r, "U_*"] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

5.4. **A relative L-theory spectrum.** Given $j \in \{2, 1, 0, , -1\ldots\}$, define the $(\mathsf{Groupoids}\downarrow\{\pm 1\})$-spectrum $\mathbf{L}_{R,w}^{\langle j+1,j\rangle}$ to be the cofiber of the map of $(\mathsf{Groupoids}\downarrow\{\pm 1\})$-spectra $\mathbf{L}_{R,w}^{\langle j+1\rangle} \to \mathbf{L}_{R,w}^{\langle j\rangle}$. Then we get for any group with orientation character (G, w) and any morphism $f\colon X\to Y$ of G-CW-complexes a commutative

diagram with long exact rows and columns
(5.10)

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots \to H_n^G(X, \mathbf{L}_{R,w}^{\langle j+1\rangle}) \arrow[r, "f_*"] \arrow[d] & H_n^G(Y, \mathbf{L}_{R,w}^{\langle j+1\rangle}) \arrow[r] \arrow[d] & H_n^G(f, \mathbf{L}_{R,w}^{\langle j+1\rangle}) \arrow[r] \arrow[d] & H_{n-1}^G(X, \mathbf{L}_{R,w}^{\langle j+1\rangle}) \xrightarrow{f_*} \cdots \arrow[d] \\
\cdots \to H_n^G(X, \mathbf{L}_{R,w}^{\langle j\rangle}) \arrow[r, "f_*"] \arrow[d] & H_n^G(Y, \mathbf{L}_{R,w}^{\langle j\rangle}) \arrow[r] \arrow[d] & H_n^G(f, \mathbf{L}_{R,w}^{\langle j\rangle}) \arrow[r] \arrow[d] & H_{n-1}^G(X, \mathbf{L}_{R,w}^{\langle j\rangle}) \xrightarrow{f_*} \cdots \arrow[d] \\
\cdots \to H_n^G(X, \mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \arrow[r, "f_*"] \arrow[d] & H_n^G(Y, \mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \arrow[r] \arrow[d] & H_n^G(f, \mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \arrow[r] \arrow[d] & H_{n-1}^G(X, \mathbf{L}_{R,w}^{\langle j+1,,j\rangle}) \xrightarrow{f_*} \cdots \arrow[d] \\
\cdots \to H_{n-1}^G(X, \mathbf{L}_{R,w}^{\langle j+1\rangle}) \arrow[r, "f_*"] \arrow[d] & H_{n-1}^G(Y, \mathbf{L}_{R,w}^{\langle j+1\rangle}) \arrow[r] \arrow[d] & H_{n-1}^G(f, \mathbf{L}_{R,w}^{\langle j+1\rangle}) \arrow[r] \arrow[d] & H_{n-2}^G(X, \mathbf{L}_{R,w}^{\langle j+1\rangle}) \xrightarrow{f_*} \cdots \arrow[d] \\
\cdots \to H_{n-1}^G(X, \mathbf{L}_{R,w}^{\langle j\rangle}) \arrow[r, "f_*"] \arrow[d] & H_{n-1}^G(Y, \mathbf{L}_{R,w}^{\langle j\rangle}) \arrow[r] \arrow[d] & H_{n-1}^G(f, \mathbf{L}_{R,w}^{\langle j\rangle}) \arrow[r] \arrow[d] & H_{n-2}^G(X, \mathbf{L}_{R,w}^{\langle j\rangle}) \xrightarrow{f_*} \cdots \arrow[d] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

In particular we get long exact sequences

$$\begin{aligned}
(5.11)\quad \cdots \to H_n^G(\{\bullet\}; \mathbf{L}_{R,w}^{\langle j+1\rangle}) = L_n^{\langle j+1\rangle}(RG, w) \to H_n^G(\{\bullet\}; \mathbf{L}_{R,w}^{\langle j\rangle}) = L_n^{\langle j\rangle}(RG, w) \\
\to H_n^G(\{\bullet\}; \mathbf{L}_{R,w}^{\langle j+1,j\rangle}) = \pi_n(\mathbf{L}_{R,w}^{\langle j+1,j\rangle}(\widehat{G})) \\
\to H_{n-1}^G(\{\bullet\}; \mathbf{L}_{R,w}^{\langle j+1\rangle}) = L_{n-1}^{\langle j+1\rangle}(RG, w) \to H_{n-1}^G(\{\bullet\}; \mathbf{L}_{R,w}^{\langle j\rangle}) = L_{n-1}^{\langle j\rangle}(RG, w) \to \cdots,
\end{aligned}$$

where $\widehat{G}$ is the one-object groupoid associated to G. In view of the Rothenberg sequence (5.5) this leads to the very reasonable conjecture that there is a natural identification[1]

$$(5.12)\qquad H_n^G(\{\bullet\}; \mathbf{L}_{R,w}^{\langle j+1,j\rangle}) = \widehat{H}^n(\mathbb{Z}/2; \mathrm{Wh}_j(G; R)).$$

If we would know this claim, this would make the exposition easier and more transparent. Actually, this claim will be proven in [10], where also complete constructions of the spectra $\mathbf{L}_{R,w}^{\langle j\rangle}$ will be presented and the spectra $\mathbf{L}_{R,w}^{\langle j+1,j\rangle}$ will be identified with the corresponding Tate spectra. (This is private communication with Markus Land.)

Instead of using the unpublished work above, we take a shortcut based on Theorem 5.9. The same attitude is taken in the proof of [19, Corollary 5.6]. There only rings are considered, which is problematic in view of the failure of decorated L-groups to be compatible with finite products of rings, see Remark 5.8. We want to explain here that this problem can be solved by passing to additive categories with involution as explained in Subsection 5.1.

Lemma 5.13.

(1) The following assertions are equivalent for $j \in \{1, 0, -1, -2, \ldots\}$:

[1]Christian Kremer gives a proof of this equality (5.12) in his master thesis [38].

(a) *The abelian group* $\widehat{H}^n(\mathbb{Z}/2; \mathrm{Wh}_j(G;R))$ *vanishes for all* $n \in \mathbb{Z}$*;*

(b) *The abelian group* $H_n^G(\{\bullet\}; \mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \cong \pi_n(\mathbf{L}_{R,w}^{\langle j+1,j\rangle}(\widehat{G}))$ *vanishes for all* $n \in \mathbb{Z}$*;*

(2) *If* X *is a free* G*-CW-complex, then* $H_n^G(X; \mathbf{L}_{R,w}^{\langle j+1,j\rangle}) = 0$ *holds for all* $n \in \mathbb{Z}$.

Proof. (1) This follows from exact sequences (5.5) and (5.11), since both statements are equivalent to the assertion that the map $L_n^{\langle j+1\rangle}(RG,w) \to L_n^{\langle j\rangle}(RG,w)$ is bijective for all $n \in \mathbb{Z}$.

(2) We have $\mathrm{Wh}_j(\{1\}, R) = 0$ for $j \le 1$. Hence $\widehat{H}^n(\mathbb{Z}/2; \mathrm{Wh}_j(\{1\}; R)) = 0$ for all $n \in \mathbb{Z}$. By assertion (1) we get $H_n^G(G/\{1\}; \mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \cong \pi_n(\mathbf{L}_{R,w}^{\langle j+1,j\rangle}(\{1\})) = 0$ for all $n \in \mathbb{Z}$. This implies by the equivariant Atiyah-Hirzebruch spectral sequence [24, Theorem 4.7] that $H_n^G(X; \mathbf{L}_{R,w}^{\langle j+1,j\rangle}) = 0$ holds for all $n \in \mathbb{Z}$, if X is a free G-CW-complex. $\square$

5.5. **Computing L-groups.**

Theorem 5.14. *Let* R *be a ring with involution. Let* Γ *be a group coming with a group homomorphism* $w\colon \Gamma \to \{\pm 1\}$*. Let* $\mathcal{M}$ *be a complete system of representatives of the conjugacy classes of maximal finite subgroups and let* $\mathcal{V}_{\mathrm{II}}$ *be a complete system of representatives of the conjugacy classes of maximal virtually cyclic subgroups of type II. Suppose that the following conditions are satisfied:*

- *The group* Γ *satisfies conditions (M), (NM), and* (V_{II})*, see Definition 1.3;*
- *The group* Γ *is a Farrell-Jones group;*
- *There exists* $j_0 \in \mathbb{Z}$ *such that* $\mathrm{Wh}_j(H;R) = 0$ *holds for every finite subgroup* $H \subseteq \Gamma$ *and every* $j \le j_0$*;*
- *The ring* R *is regular,* $K_n(R) = 0$ *for* $n < 0$*, and* $K_0(\mathbb{Z}) \xrightarrow{\cong} K_0(R)$*, e.g.,* R *is a principal ideal domain;*

Consider any $j \in \{2,1,0,-1,\ldots\} \amalg \{-\infty\}$*. Then:*

(1) *The map induced by the projection* $\underline{\underline{E}}\Gamma \to \{\bullet\}$ *induces an isomorphism for every* $n \in \mathbb{Z}$

$$H_n^\Gamma(\underline{\underline{E}}\Gamma; \mathbf{L}_{R,w}^{\langle j\rangle}) \xrightarrow{\cong} H_n^\Gamma(\{\bullet\}; \mathbf{L}_{R,w}^{\langle j\rangle}) = L_n^{\langle j\rangle}(R\Gamma, w);$$

(2) *For every* $n \in \mathbb{Z}$ *we have the short split exact sequence*

$$0 \to H_n^\Gamma(\underline{E}\Gamma; \mathbf{L}_{R,w}^{\langle -\infty\rangle}) \to H_n^\Gamma(\underline{\underline{E}}\Gamma; \mathbf{L}_{R,w}^{\langle -\infty\rangle}) \to H_n^\Gamma(\underline{E}\Gamma \to \underline{\underline{E}}\Gamma; \mathbf{L}_{R,w}^{\langle -\infty\rangle}) \to 0;$$

(3) *We obtain an isomorphism for any* $n \in \mathbb{Z}$

$$\bigoplus_{F \in \mathcal{M}} H_n^F(EF \to \{\bullet\}; \mathbf{L}_{R,w|_F}^{\langle j\rangle}) \xrightarrow{\cong} H_n^\Gamma(E\Gamma \to \underline{E}\Gamma; \mathbf{L}_{R,w}^{\langle j\rangle});$$

$$\bigoplus_{V \in \mathcal{V}_{\mathrm{II}}} H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}_{R,w|_V}^{\langle j\rangle}) \xrightarrow{\cong} H_n^\Gamma(\underline{E}\Gamma \to \underline{\underline{E}}\Gamma; \mathbf{L}_{R,w}^{\langle j\rangle});$$

For every $V \in \mathcal{V}_{\mathrm{II}}$ *and* $n \in \mathbb{Z}$ *the canonical map*

$$H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}_{R,w|_V}^{\langle j\rangle}) \xrightarrow{\cong} H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}_{R,w|_V}^{\langle -\infty\rangle})$$

is bijective.

Proof. (1) Since Γ is a Farrell-Jones group, the map

$$H_n^\Gamma(\underline{\underline{E}}\Gamma; \mathbf{L}_{R,w}^{\langle -\infty\rangle}) \xrightarrow{\cong} H_n^\Gamma(\{\bullet\}; \mathbf{L}_{R,w}^{\langle -\infty\rangle}) = L_n^{\langle -\infty\rangle}(R\Gamma, w)$$

is bijective for all $n \in \mathbb{Z}$. This takes care of the case $j = -\infty$.

We can assume without loss of generality that $j_0 \le -1$, otherwise replace j_0 by -1. Next we prove assertions (1) for $j \in \{j_0, j_0-1, j_0-2, \ldots\}$.

By assumption $\mathrm{Wh}_j(H;R)$ vanishes for every $j \le j_0$ and every finite subgroup $H \subseteq \Gamma$. We conclude from Lemma 2.1 that every infinite virtually cyclic subgroup of Γ is either infinite cyclic or isomorphic to D_∞. If we take $\Gamma = \mathbb{Z}$ or D_∞, the assumptions of Theorem 4.3 are satisfied and therefore $\mathrm{Wh}_j(W;R)$ vanishes for every $j \le j_0$ and every virtually cyclic subgroup W of Γ. We conclude from the Rothenberg sequence (5.5) that the canonical map $L_n^{\langle j+1\rangle}(RW, w|_W) \to L_n^{\langle j\rangle}(RW, w|_W)$ is bijective for every $n \in \mathbb{Z}$, $j \le j_0$ and every virtually cyclic subgroup W of Γ. Hence the canonical map of spectra $\mathbf{L}_{R,w}^{\langle j\rangle}(\Gamma/W) \to \mathbf{L}_{R,w}^{\langle -\infty\rangle}(\Gamma/W)$ is a weak homotopy equivalence for all virtually cyclic subgroups W of Γ and all $j \le j_0$. Since all isotropy groups of $\underline{\underline{E}}\Gamma$ are virtually cyclic, the canonical map

$$H_n^\Gamma(\underline{\underline{E}}\Gamma, \mathbf{L}_{R,w}^{\langle j\rangle}) \xrightarrow{\cong} H_n^\Gamma(\underline{\underline{E}}\Gamma, \mathbf{L}_{R,w}^{\langle -\infty\rangle})$$

is bijective for $n \in \mathbb{Z}$ and $j \le j_0$. Since R is regular by assumption and hence $\mathrm{Wh}_j(\Gamma;R)$ vanishes for $j \le j_0$ by Theorem 4.3, the canonical map

$$H_n^\Gamma(\{\bullet\}, \mathbf{L}_{R,w}^{\langle j\rangle}) = L_n^{\langle j\rangle}(R\Gamma, w) \xrightarrow{\cong} H_n^\Gamma(\{\bullet\}, \mathbf{L}_{R,w}^{\langle -\infty\rangle}) = L_n^{\langle -\infty\rangle}(R\Gamma, w)$$

is bijective for $n \in \mathbb{Z}$ by the Rothenberg sequence (5.5). We conclude that assertion (1) holds for $j \in \{j_0, j_0-1, j_0-2, \ldots\}$, since we have already proved it for $j = -\infty$.

It remains to show for $j \in \mathbb{Z}$ with $j \le 1$ that assertion (1) holds for $j+1$, if it holds for j. This is done as follows.

We get from excision and Theorem 4.2 (1) the isomorphism

$$\bigoplus_{F \in \mathcal{M}} H_n^\Gamma(\Gamma \times_F EF \to \Gamma/F, \mathbf{L}_{R,w|_F}^{\langle j+1,j\rangle}) \xrightarrow{\cong} H_n^\Gamma(E\Gamma \to \underline{E}\Gamma, \mathbf{L}_{R,w|_F}^{\langle j+1,j\rangle}).$$

The groups $H_n^\Gamma(E\Gamma, \mathbf{L}_{R,w|_F}^{\langle j+1,j\rangle})$ and $H_n^\Gamma(\Gamma \times_F EF, \mathbf{L}_{R,w|_F}^{\langle j+1,j\rangle})$ vanish for $n \in \mathbb{Z}$ by Lemma 5.13 (2), as Γ acts freely on $E\Gamma$ and on $\Gamma \times_F EF$. This implies that the canonical maps

$$H_n^\Gamma(\Gamma/F, \mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \xrightarrow{\cong} H_n^\Gamma(\Gamma \times_F EF \to \Gamma/F, \mathbf{L}_{R,w}^{\langle j+1,j\rangle})$$

and

$$H_n^\Gamma(\underline{E}\Gamma, \mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \xrightarrow{\cong} H_n^\Gamma(E\Gamma \to \underline{E}\Gamma, \mathbf{L}_{R,w}^{\langle j+1,j\rangle})$$

are isomorphisms. Hence we get isomorphisms

$$\bigoplus_{F \in \mathcal{M}} H_n^\Gamma(\Gamma/F, \mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \xrightarrow{\cong} H_n^\Gamma(\underline{E}\Gamma, \mathbf{L}_{R,w}^{\langle j+1,j\rangle}).$$

This shows that the up to Γ-homotopy unique map $f\colon \coprod_{F\in\mathcal{M}} \Gamma/F \to \underline{E}\Gamma$ induces for $n \in \mathbb{Z}$ an isomorphism

$$H_n^\Gamma(f; \mathbf{L}_{R,w}^{\langle j+1,j\rangle})\colon H_n^\Gamma(\coprod_{F\in\mathcal{M}} \Gamma/F; \mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \xrightarrow{\cong} H_n^\Gamma(\underline{E}\Gamma; \mathbf{L}_{R,w}^{\langle j+1,j\rangle}). \tag{5.15}$$

Let $p\colon \coprod_{F\in\mathcal{M}} \Gamma/F \to \{\bullet\}$ be the projection. Next we want to show that it induces for all $n \in \mathbb{Z}$ an isomorphism

$$p_n\colon H_n^\Gamma(\coprod_{F\in\mathcal{M}} \Gamma/F; \mathbf{L}_{R,w|_F}^{\langle j+1,j\rangle}) \xrightarrow{\cong} H_n^\Gamma(\{\bullet\}; \mathbf{L}_{R,w}^{\langle j+1,j\rangle}). \tag{5.16}$$

Put $X = \coprod_{F\in\mathcal{M}} \Gamma/F$. The following commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H_n^\Gamma(X;\mathbf{L}_{R,w}^{\langle j+1\rangle}) \arrow[r, "p_*"] \arrow[d] & H_n^\Gamma(\{\bullet\};\mathbf{L}_{R,w}^{\langle j+1\rangle}) \arrow[r] \arrow[d] & H_n^\Gamma(p;\mathbf{L}_{R,w}^{\langle j+1\rangle}) \arrow[r] \arrow[d] & H_{n-1}^\Gamma(X;\mathbf{L}_{R,w}^{\langle j+1\rangle}) \arrow[r, "p_*"] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_n^\Gamma(X;\mathbf{L}_{R,w}^{\langle j\rangle}) \arrow[r, "p_*"] & H_n^\Gamma(\{\bullet\};\mathbf{L}_{R,w}^{\langle j\rangle}) \arrow[r] & H_n^\Gamma(p;\mathbf{L}_{R,w}^{\langle j\rangle}) \arrow[r] & H_{n-1}^\Gamma(X;\mathbf{L}_{R,w}^{\langle j\rangle}) \arrow[r, "p_*"] & \cdots
\end{tikzcd}$$

can be identified with the first two rows in the commutative diagram with exact rows and columns of Theorem 5.9, if we take $\mathcal{A} = \bigoplus_{F\in\mathcal{M}} \mathcal{F}(RF)$, $\mathcal{B} = \mathcal{F}(R\Gamma)$, and $U\colon \mathcal{A} \to \mathcal{B}$ to be $\bigoplus_{F\in\mathcal{M}} \mathcal{F}(i_F)\colon \bigoplus_{F\in\mathcal{M}} \mathcal{F}(RF) \to \mathcal{F}(R\Gamma)$ for $i_F\colon RF \to R\Gamma$ the ring homomorphism induced by the inclusion $F \to \Gamma$. This identification uses (5.4). The third row in the commutative diagram with exact rows and columns of Theorem 5.9 can be written as

$$\cdot \to \widehat{H}^n(\mathbb{Z}/2; \bigoplus_{F\in\mathcal{M}} \mathrm{Wh}_j(F;R)) \to \widehat{H}^n(\mathbb{Z}/2; \mathrm{Wh}_j(\Gamma;R)) \to \widehat{H}^n(\mathbb{Z}/2;U)$$
$$\to \widehat{H}^{n-1}(\mathbb{Z}/2; \bigoplus_{F\in\mathcal{M}} \mathrm{Wh}_j(F;R)) \to \widehat{H}^{n-1}(\mathbb{Z}/2; \mathrm{Wh}_j(\Gamma;R)) \to \cdots .$$

This identification uses (5.3). Since $\bigoplus_{F\in\mathcal{M}} \mathrm{Wh}_j(F;R) \to \mathrm{Wh}_j(\Gamma;R)$ is an isomorphism for all $n \in \mathbb{Z}$ by Theorem 4.3, the first and the third arrow appearing in the long exact sequence above is bijective for all $n \in \mathbb{Z}$. Hence $\widehat{H}^n(\mathbb{Z}/2;U)$ vanishes for all $n \in \mathbb{Z}$. We conclude that the map $H_n^\Gamma(p;\mathbf{L}_{R,w}^{\langle j+1\rangle}) \xrightarrow{\cong} H_n^\Gamma(p;\mathbf{L}_{R,w}^{\langle j\rangle})$ is an isomorphism for all $n \in \mathbb{Z}$. This implies that $H_n^\Gamma(p;\mathbf{L}_{R,w}^{\langle j+1,j\rangle})$ vanishes for all $n \in \mathbb{Z}$. Hence (5.16) is an isomorphism. Note that the proof of the bijectivity of (5.16) would be rather easy, if we would know (5.12).

We conclude from (5.15) and (5.16) that the map

$$H_n^\Gamma(\underline{E}\Gamma;\mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \xrightarrow{\cong} H_n^\Gamma(\{\bullet\};\mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \tag{5.17}$$

is bijective for all $n \in \mathbb{Z}$.

Next we show

$$H_n^V(\underline{E}V \to \{\bullet\};\mathbf{L}_{R,w|_V}^{\langle j+1,j\rangle}) = 0 \quad \text{for } n \in \mathbb{Z}, V \in \mathcal{V}. \tag{5.18}$$

Recall from Lemma 2.1 that any virtually cyclic subgroup of Γ is infinite cyclic or isomorphic to D_∞. Hence V satisfies the assumptions of Theorem 5.14, if we take $\Gamma = V$. Now (5.18) follows from the isomorphism (5.17), which we have already established.

Next we show that

$$H_n^\Gamma(\underline{E}\Gamma;\mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \xrightarrow{\cong} H_n^\Gamma(\underline{\underline{E}}\Gamma;\mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \tag{5.19}$$

is an isomorphism for all $n \in \mathbb{Z}$. We get from Theorem 4.2 (2) an isomorphism

$$\bigoplus_{V\in\mathcal{V}} H_n^\Gamma(\Gamma \times_V \underline{E}V \to \Gamma \times_V \{\bullet\};\mathbf{L}_{R,w|_V}^{\langle j+1,j\rangle}) \xrightarrow{\cong} H_n^\Gamma(\underline{E}\Gamma \to \underline{\underline{E}}\Gamma;\mathbf{L}_{R,w}^{\langle j+1,j\rangle}).$$

Using the induction structure of the equivariant homology theory $\mathcal{H}_*^?(-;\mathbf{L}_{R,w}^{\langle j+1,j\rangle})$, we get isomorphisms

$$H_n^V(\underline{E}V \to \{\bullet\};\mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \xrightarrow{\cong} H_n^\Gamma(\Gamma \times_V \underline{E}V \to \Gamma \times_V \{\bullet\};\mathbf{L}_{R,w}^{\langle j+1,j\rangle})$$

for every $V \in \mathcal{V}$. Hence (5.18) implies that (5.19) is bijective.

We conclude from (5.17) and (5.19) that the map

$$H_n^\Gamma(\underline{\underline{E}}\Gamma;\mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \xrightarrow{\cong} H_n^\Gamma(\{\bullet\};\mathbf{L}_{R,w}^{\langle j+1,j\rangle}) \tag{5.20}$$

is bijective for all $n \in \mathbb{Z}$.

By the induction hypothesis the map

$$H_n^\Gamma(\underline{\underline{E}}\Gamma;\mathbf{L}_{R,w}^{\langle j\rangle}) \xrightarrow{\cong} H_n^\Gamma(\{\bullet\};\mathbf{L}_{R,w}^{\langle j\rangle})$$

is bijective for all $n \in \mathbb{Z}$. Now we conclude from (5.20), the Five-Lemma and the long exact sequence given by the first two columns in (5.10) applied to the projection $f\colon X = \underline{\underline{E}}\Gamma \to Y = \{\bullet\}$ that also the map

$$H_n^\Gamma(\underline{\underline{E}}\Gamma;\mathbf{L}_{R,w}^{\langle j+1\rangle}) \xrightarrow{\cong} H_n^\Gamma(\{\bullet\};\mathbf{L}_{R,w}^{\langle j+1\rangle})$$

is bijective for all $n \in \mathbb{Z}$. This finishes the proof of the induction step and hence of assertion (1).

(2) We conclude from [23, Corollary 3.27] and the Bass-Heller-Swan decomposition that $\mathrm{Wh}_j(D_\infty; R) = 0$ and $\mathrm{Wh}_j(\mathbb{Z}; R) = 0$ holds for $j \le j_0$. Since any infinite cyclic subgroup of Γ is infinite cyclic or isomorphic to D_∞ by Lemma 2.1, we get $\mathrm{Wh}_j(W; R) = 0$ for $j \le j_0 \le -1$ for any virtually cyclic subgroup W of Γ. Since R is regular, $K_j(R) = 0$ for $j \le j_0$. This implies $H_j^W(EW; \mathbf{K}_R) = 0$ for $j \le j_0$ by a spectral sequence argument. Hence $K_j(RW) = H_j^W(\{\bullet\}; \mathbf{K}_R) \cong H_j^W(EW \to \{\bullet\}; \mathbf{K}_R) \cong \mathrm{Wh}_j(W; R) = 0$ holds for $j \le j_0$. Therefore we obtain from [4, Section 1] the desired short split exact sequence.

(3) We get from excision and Theorem 4.2 (1) the first desired isomorphism

$$\bigoplus_{F \in \mathcal{M}} H_n^F(EF \to \{\bullet\}; \mathbf{L}^{\langle j \rangle}_{R, w|_F}) \xrightarrow{\cong} H_n^\Gamma(E\Gamma \to \underline{E}\Gamma; \mathbf{L}^{\langle j \rangle}_{R,w}).$$

Next we show

$$H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}^{\langle j \rangle}_{R,w|_V}) = 0 \quad \text{for } n \in \mathbb{Z}, V \in \mathcal{V}_{\mathrm{I}}. \tag{5.21}$$

This is obvious, if V is finite. We conclude from Lemma 2.1 that every $V \in \mathcal{V}_{\mathrm{I}}$ is infinite cyclic. Hence it suffices to treat the case where V is infinite cyclic.

The assembly map $H_n^V(\underline{E}V; \mathbf{L}^{\langle -\infty \rangle}_{R,w|_V}) \to H_n^V(\{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}_{R,w|_V})$ is bijective for all $n \in \mathbb{Z}$. This follows from [43, Theorem 13.56] which is in this case essentially the Shaneson-splitting. Hence $H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}^{\langle -\infty \rangle}_{R,w|_V})$ vanishes for $n \in \mathbb{Z}$.

Next we show that $H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}^{\langle j \rangle}_{R,w|_V}) = 0$ holds for $j \in \{2, 1, 0, -1, \ldots\}$ and $n \in \mathbb{Z}$. For this purpose it suffices to show that $H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}^{\langle j+1, j \rangle}_{R,w|_V}) = 0$ holds for $j \in \{1, 0, -1, \ldots\}$, since we have already proved the claim for $j = -\infty$. Since V acts freely on $\underline{E}V$, we conclude from Lemma 5.13 (2)

$$H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}^{\langle j+1, j \rangle}_{R,w|_V}) \cong H_n^V(\{\bullet\}; \mathbf{L}^{\langle j+1, j \rangle}_{R,w|_V}).$$

Because of Lemma 5.13 (1) it suffices to show $\widehat{H}^n(\mathbb{Z}/2, \mathrm{Wh}_j(V; R)) = 0$. This follows from the conclusion of Theorem 4.3 that $\mathrm{Wh}_j(V; R)$ vanishes. This finishes the proof of (5.21).

Next we prove

$$H_n^\Gamma(\underline{E}\Gamma \to E_{\mathcal{VCY}_I}(\Gamma); \mathbf{L}^{\langle j \rangle}_{R,w|_V}) = 0 \quad \text{for } n \in \mathbb{Z}. \tag{5.22}$$

For a Γ-CW-complex Z let $\mathrm{pr}_Z \colon \underline{E}\Gamma \times Z \to Z$ be the projection. Since the projection $\underline{E}\Gamma \times E_{\mathcal{VCY}_I}(\Gamma) \to \underline{E}\Gamma$ is a Γ-homotopy equivalence, it suffices to show $H_n^\Gamma(\mathrm{pr}_{E_{\mathcal{VCY}_I}(\Gamma)}; \mathbf{L}^{\langle j \rangle}_{R,w|_V}) = 0$ for all $n \in \mathbb{Z}$. We will show more generally for any Γ-CW-complex Z, whose isotropy groups belong to $\mathcal{VCY}_I$, that $H_n^V(\mathrm{pr}_Z; \mathbf{L}^{\langle j \rangle}_{R,w|_V})$ vanishes for all $n \in \mathbb{Z}$. By a colimit argument and induction over the skeletons this claim can be reduced to the special case $Z = \Gamma/V$ for $V \in \mathcal{VCY}_I$. Then $\underline{E}\Gamma \times \Gamma/V$ is Γ-homeomorphic to $\Gamma \times_V \underline{E}V$ and pr_{Γ_V} is the induction with the inclusion $V \to \Gamma$ applied to $\underline{E}V \to \{\bullet\}$. By the induction structure it remains to show $H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}^{\langle j \rangle}_{R,w|_V}) = 0$ for all $n \in \mathbb{Z}$ and all $V \in \mathcal{VCY}_I$, what we have already done, see (5.21). This finishes the proof of (5.22).

From (5.22) we obtain an isomorphism

$$H_n^\Gamma(\underline{E}\Gamma \to \underline{\underline{E}}\Gamma; \mathbf{L}^{\langle j \rangle}_{R,w}) \xrightarrow{\cong} H_n^\Gamma(E_{\mathcal{VCY}_I}(\Gamma) \to \underline{\underline{E}}\Gamma; \mathbf{L}^{\langle j \rangle}_{R,w}).$$

We conclude from Lemma 2.1 that Γ satisfies ($\mathrm{V_{II}}$) and ($\mathrm{NV_{II}}$). We get from excision and Theorem 4.2 (3) isomorphisms

$$\bigoplus_{V \in \mathcal{V}_{\mathrm{II}}} H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}_{R,w|_V}^{\langle j \rangle}) \xrightarrow{\cong} H_n^\Gamma(E_{\mathcal{VCY}_I}(\Gamma) \to \underline{\underline{E}}\Gamma; \mathbf{L}_{R,w}^{\langle j \rangle}).$$

Hence we obtain for every $n \in \mathbb{Z}$ the second desired isomorphism

$$\bigoplus_{V \in \mathcal{V}_{\mathrm{II}}} H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}_{R,w|_V}^{\langle j \rangle}) \xrightarrow{\cong} H_n^\Gamma(\underline{E}\Gamma \to \underline{\underline{E}}\Gamma; \mathbf{L}_{R,w|_V}^{\langle j \rangle}).$$

For any $V \in \mathcal{V}_{\mathrm{II}}$ the canonical map

$$H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}_{R,w|_V}^{\langle j \rangle}) \xrightarrow{\cong} H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}_{R,w|_V}^{\langle -\infty \rangle})$$

is bijective, since we have already shown that $H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}_{R,w|_V}^{\langle j+1,j \rangle})$ vanishes for all $n \in \mathbb{Z}$ and $j \in \{1, 0, -1, \ldots\}$ in (5.19). This finishes the proof of Theorem 5.14. □

Remark 5.23 (Identification with UNil-groups)**.** The groups $H_n^V(\underline{E}V \to \{\bullet\}; \mathbf{L}_{R,w}^{\langle j \rangle})$ appearing in assertion (3) of Theorem 5.14 for $V \in \mathcal{V}_{\mathrm{II}}$, which has to be isomorphic to D_∞, are independent of the decoration j, and can be identified with Cappell's UNil-groups $\mathrm{UNil}_n(R; R^{\pm 1}, R^{\pm 1})$, where the signs ± 1 come from the orientation character $w|V$. If both signs are $+1$, they will be omitted from the notation. In the case $R = \mathbb{Z}$ the groups $\mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}, \mathbb{Z})$ have been computed by Banagl, Connolly, Davis, Kozniewski, and Ranicki, see [21, 1, 20, 22],

$$\mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \cong \begin{cases} \{0\} & n \equiv 0 \mod (4); \\ \{0\} & n \equiv 1 \mod (4); \\ (\mathbb{Z}/2)^\infty & n \equiv 2 \mod (4); \\ (\mathbb{Z}/2 \oplus \mathbb{Z}/4)^\infty & n \equiv 3 \mod (4). \end{cases}$$

There is an isomorphism $\mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}, \mathbb{Z}) \xrightarrow{\cong} \mathrm{UNil}_{n+2}(\mathbb{Z}; \mathbb{Z}^{-1}, \mathbb{Z}^{-1})$ for all n (see [11, page 1118]).

Remark 5.24. We do not know, whether assertion (2) in Theorem 5.14 does hold also for other decorations than $-\infty$.

Remark 5.25. One interesting feature of Theorem 5.14 (1) is that it holds for all decorations. Note that the L-theoretic Farrell-Jones Conjecture is formulated for the decoration $\langle -\infty \rangle$ only. Indeed, there are counterexamples for the decorations s, h and p, see [33].

Remark 5.26. Assume in the sequel that the orientation homomorphism is trivial. Then the computation of $L_n^{\langle j \rangle}(R\Gamma, w)$ boils down by Theorem 5.14 (1) to the computation of the terms $H_n^\Gamma(\underline{E}\Gamma; \mathbf{L}_R^{\langle j \rangle})$ and $H_n^\Gamma(\underline{E}\Gamma \to \underline{\underline{E}}\Gamma; \mathbf{L}_R^{\langle j \rangle})$, modulo a possible extension problem unless $j = -\infty$, see Theorem 5.14 (2) and Remark 5.24. The computation of $H_n^\Gamma(\underline{E}\Gamma \to \underline{\underline{E}}\Gamma; \mathbf{L}_R^{\langle j \rangle})$ is complete by Theorem 5.14 (3) and Remark 5.23, if $R = \mathbb{Z}$. Some information about $H_n^\Gamma(\underline{E}\Gamma; \mathbf{L}_R^{\langle j \rangle})$ is given in Theorem 5.14 (3). One can do a little better than this. Namely, there exists the following long exact sequence (see [26, Lemma 7.2(ii)])

$$\cdots \to H_{n+1}(\underline{B}\Gamma; \mathbf{L}(R)) \to \bigoplus_{F \in \mathcal{M}} \widetilde{L}_n^{\langle j \rangle}(RF) \to H_n^\Gamma(\underline{E}\Gamma; \mathbf{L}_R^{\langle j \rangle})$$
$$\to H_n(\underline{B}\Gamma; \mathbf{L}^{\langle j \rangle}(R)) \to \bigoplus_{F \in \mathcal{M}} \widetilde{L}_n^{\langle j \rangle}(RF) \to \cdots.$$

Here $\underline{B}\Gamma$ is $\Gamma\backslash\underline{E}\Gamma$ and $H_n(\underline{B}\Gamma;\mathbf{L}^{\langle j\rangle}(R))$ is its homology with respect to the L-theoretic ring spectrum with decoration $\langle j\rangle$ of the ring R. The group $\widetilde{L}_n^{\langle j\rangle}(RF)$ is defined to be the kernel of the map $L_n^{\langle j\rangle}(RF)\to L_n^{\langle j\rangle}(R)$ coming from the group homomorphism $F\to\{1\}$. The composite

$$\bigoplus_{F\in\mathcal{M}}\widetilde{L}_n^{\langle j\rangle}(RF)\to H_n^{\Gamma}(\underline{E}\Gamma;\mathbf{L}_R^{\langle j\rangle})\to H_n^{\Gamma}(\{\bullet\};\mathbf{L}_R^{\langle j\rangle})=L_n^{\langle j\rangle}(R\Gamma)$$

is the direct sum of the maps induced by the various inclusions $F\to\Gamma$. The map $H_n^{\Gamma}(\underline{E}\Gamma;\mathbf{L}_R^{\langle j\rangle})\to H_n(\underline{B}\Gamma;\mathbf{L}^{\langle j\rangle}(R))$ comes from the induction structure applied to the group homomorphism $\Gamma\to\{1\}$. Note that this map has a section, if one inverts the orders of all finite subgroups of Γ. Essentially the sequence above reduces the computation of $L_n^{\langle j\rangle}(R\Gamma)$ to that of $H_n(\underline{B}\Gamma;\mathbf{L}^{\langle j\rangle}(R))$, which can be done in special cases, when one understands the structure of $\underline{B}\Gamma$. (See [26, Theorem 10.1].)

The construction of this sequence is analogous to the one appearing in [25, Theorem 5.1 (b)] and left to the reader.

6. The (periodic) structure group of a pair

.

Let $(A,\partial A)$ be a CW-pair. Suppose for simplicity that A is connected. We do not assume that ∂A is connected. Let $\Gamma=\pi_1(A)$ be its fundamental group and $\widetilde{A}$ its universal cover. Let $w\colon\pi_1A\to\{\pm1\}$ be the orientation character. Let $\Pi(A)$ and $\Pi(\partial A)$ be the fundamental groupoids.

There is a long exact sequence of abelian groups, called the *periodic algebraic surgery exact sequence*

$$\begin{aligned}(6.1)\quad \cdots\to H_n^{\Gamma}(\widetilde{A},\partial\widetilde{A};\mathbf{L}^s_{\mathbb{Z},w})\to L_n^s(\mathbb{Z}\Pi(\partial A)\to\mathbb{Z}\Pi(A),w)\to\mathcal{S}_n^{\mathrm{per},s}(A,\partial A)\\ \to H_{n-1}^{\Gamma}(\widetilde{A},\partial\widetilde{A};\mathbf{L}^s_{\mathbb{Z},w})\to L_{n-1}^s(\mathbb{Z}\Pi(\partial A)\to\mathbb{Z}\Pi(A),w)\to\cdots.\end{aligned}$$

If we replace $\mathbf{L}^s_{\mathbb{Z},w}$ by its 1-connective cover, see Subsection 3.2, we obtain a long exact sequence of abelian groups, called the *algebraic surgery exact sequence*

$$\begin{aligned}(6.2)\quad \cdots\to H_n^{\Gamma}(\widetilde{A},\partial\widetilde{A};\mathbf{L}^s_{\mathbb{Z},w}\langle1\rangle)\to L_n^s(\mathbb{Z}\Pi(\partial A)\to\mathbb{Z}\Pi(A),w)\to\mathcal{S}_n^{\mathrm{per},s}(A,\partial A)\\ \to H_{n-1}^{\Gamma}(\widetilde{A},\partial\widetilde{A};\mathbf{L}^s_{\mathbb{Z},w}\langle1\rangle)\to L_{n-1}^s(\mathbb{Z}\Pi(\partial A)\to\mathbb{Z}\Pi(A),w)\to\cdots.\end{aligned}$$

These sequences can be constructed by taking homotopy groups of certain fibrations of spectra, see for instance [49, Definition 14.6 on page 148].

Suppose that $(A,\partial A)$ is a $(n-1)$-dimensional finite Poincaré pair. Then the algebraic surgery sequence 6.2 can be identified with the geometric surgery exact sequence due to Sullivan and Wall, see for instance [49, Theorem 18.5 on page 198]. So the algebraic structure groups $\mathcal{S}_n^s(A,\partial A)$ are relevant for the application of surgery theory for topological manifolds, whereas the periodic version $\mathcal{S}_n^{\mathrm{per},s}(A,\partial A)$ is for us a very good approximation of $\mathcal{S}_n^s(A,\partial A)$.

The definition of the algebraic periodic structure group $\mathcal{S}_n^{\mathrm{per},\langle j\rangle}(A,\partial A)$ and its non-periodic companion $\mathcal{S}_n^{\langle j\rangle}(A,\partial A)$ make sense for any $j\in\{2,1,0,-1,\ldots\}\amalg\{-\infty\}$. This includes the exact sequences (6.1) and (6.2).

The next result will be crucial for the proof of our main theorems. A Γ-homotopy equivalence of free cocompact Γ-CW-complexes $(F,f)\colon(X,A)\to(Y,B)$ is called simple, if the Whitehead torsions $\tau(f)$, $\tau(\partial f)$ and $\tau(f,\partial f)$ vanish in $\mathrm{Wh}(\Gamma)$. Additivity for the Whitehead torsion implies that (F,f) is simple, if two of the three elements $\tau(f)$, $\tau(\partial f)$ and $\tau(f,\partial f)$ in $\mathrm{Wh}(\Gamma)$ vanish.

Theorem 6.3. *Let π be a finite index subgroup of a group Γ. Let $(Z, \partial Z)$ be a free cocompact Γ-CW-pair such that Z is simply connected and $(Z/\Gamma, \partial Z/\Gamma)$ is a simple finite Poincaré pair of dimension $d \geq 6$.*

(1) Then $(Z/\pi, \partial Z/\pi)$ is a finite simple Poincaré pair of dimension d. Suppose that the transfer map

$$p^* \colon \mathcal{S}^s_d(Z/\Gamma, \partial Z/\Gamma) \to \mathcal{S}^s_d(Z/\pi, \partial Z/\pi)$$

is injective. Then the following assertions are equivalent:

(a) There is a free cocompact π-manifold N with boundary ∂N and a simple π-homotopy equivalence of pairs $(N, \partial N) \to (Z, \partial Z)$;

(b) There is a free cocompact Γ-manifold M with boundary ∂M and a simple Γ-homotopy equivalence of pairs $(M, \partial M) \to (Z, \partial Z)$;

(2) Let $(f_i, \partial f_i) \colon (M_i, \partial M_i) \to (Z, \partial Z)$ be simple Γ-homotopy equivalences with free compact Γ-manifolds M_i with boundary for $i = 0, 1$. Suppose that $\mathcal{S}^s_{d+1}(Z/\Gamma, \partial Z/\Gamma)$ vanishes.

Then there is a Γ-homeomorphism $(g, \partial g) \colon (M_0, \partial M_0) \xrightarrow{\cong} (M_1, \partial M_1)$ such that $(f_1, \partial f_1) \circ (g, \partial g)$ is Γ-homotopic as map of Γ-pairs to $(f_0, \partial f_0)$.

(3) There are also versions of assertions (1) *and* (2) *for the decoration h. One has to drop simple everywhere, change the decoration from s to h everywhere, and weaken the conclusion in assertion* (2) *to the following statement: There is a Γ-h-cobordism $(W, \partial W)$ with a Γ-homotopy equivalence of pairs $(F, \partial F) \colon (W, \partial W) \to (Z \times [0,1], \partial(Z \times [0,1]))$ from $(f_0, \partial f_0) \colon (M_0, \partial M_0) \to (Z, \partial Z)$ to $(f_1, \partial f_1) \colon (M_1, \partial M_1) \to (Z, \partial Z)$.*

Proof. (1) The implication (1b) $\implies$ (1a) is obvious. The implication (1a) $\implies$ (1b) follows from Ranicki's theory of the total surgery obstruction, which is explained for closed manifolds and Poincaré complexes in [49, Definition 17.1 and Proposition 17.2 on page 190] and extends to pairs, see [49, page 207–208]. More information can be found in [39]. The total surgery obstruction assigns to $(Z/\pi, \partial Z/\pi)$ an element in $\mathcal{S}^s_d(Z\pi, \partial Z/\pi)$, which vanishes, if and only if assertion (1a) is true. The total surgery obstruction assigns to $(Z/\Gamma, \partial Z\Gamma)$ an element in $\mathcal{S}^s_d(Z/\Gamma, \partial Z/\Gamma)$, which vanishes, if and only if assertion (1b) is true. The map $p^* \colon \mathcal{S}^s_d(Z\Gamma, \partial Z/\Gamma) \to \mathcal{S}^s_d(Z\pi, \partial Z/\pi)$ sends the total surgery obstruction of $(Z/\Gamma, \partial Z/\Gamma)$ to that of $(Z/\pi, \partial Z/\pi)$. Since p^* is assumed to be injective, the result follows.

(2),(3) This follows from surgery theory and the identification of the geometric and the algebraic structure set, see [49, Theorem 18.5 on page 198], which also makes sense for Poincaré pairs. See also [39]. □

For the reader's convenience we spell out what a Γ-h-cobordism $(W, \partial W)$ with a Γ-homotopy equivalence of Γ-pairs $(F, \partial F) \colon (W, \partial W) \to (Z \times [0,1], \partial(Z \times [0,1]))$ is. Namely, we have a decomposition $\partial W = \partial_0 W \cup \partial_1 W \cup \partial_2 W$ into Γ-submanifolds of codimension zero such that $\partial\partial_2 W = \partial\partial_0 W \cup \partial\partial_1 W$ and $\partial_0 W \cap \partial_1 W = \emptyset$ hold, ∂F induces Γ-homotopy equivalences of pairs

$$\begin{aligned}
\partial_0 F \colon (\partial_0 W, \partial\partial_0 W) &\xrightarrow{\simeq} (Z, \partial Z) \times \{0\};\\
\partial_1 F \colon (\partial_1 W, \partial\partial_1 W) &\xrightarrow{\simeq} (Z, \partial Z) \times \{1\};\\
\partial_2 F \colon (\partial_2 W, \partial\partial_2 W) &\xrightarrow{\simeq} (\partial Z \times [0,1], \partial Z \times \{0,1\}),
\end{aligned}$$

and there are identifications of $(M_i, \partial M_i)$ with $(\partial_i W, \partial\partial_i W)$ compatible with the Γ-maps to Z for $i = 0, 1$. In particular $\partial_2 F \colon \partial_2 W \to \partial Z \times [0,1]$ yields an Γ-h-cobordism from $\partial f_0 \colon \partial_0 M \to \partial Z$ to $\partial f_1 \colon \partial_1 M \to \partial Z$.

7. Computing the periodic structure group

Let Γ be the group appearing in the extension (1.1).

The next theorem follows directly from [42, Theorem 1.12, Theorem 7.12, and Theorem 10.2]. It solves the existence question on the level of Poincaré pairs and thus opens the door to apply surgery theory to replace up to (simple) homotopy a Poincaré pair by a manifold with boundary.

Recall that $\mathcal{M}$ is a complete system of representatives of the conjugacy classes of maximal finite subgroups and $\mathcal{V}_{\mathrm{II}}$ is a complete system of representatives of the conjugacy classes of maximal virtually cyclic subgroups of type II.

Theorem 7.1 (Poincaré models)**.** *Suppose that the following conditions are satisfied:*

- *The natural number d satisfies $d \geq 3$;*
- *There is a finite d-dimensional Poincaré complex, which is homotopy equivalent to $B\pi$. Fix a generator $[B\pi]$ of the infinite cyclic group $H_d^{\pi}(E\pi;\mathbb{Z}^{w|_\pi})$;*
- *For every $F \in \mathcal{M}$ the restriction of the homomorphism w of (1.13) to F is trivial, if d is even, and is non-trivial, if d is odd;*
- *Γ satisfies conditions (M), (NM), and (H), see Definitions 1.3 and 1.14;*
- *There exists a finite Γ-CW-model for $\underline{E}\Gamma$;*
- *There exists an oriented free d-dimensional slice system $\mathcal{S}$ in the sense of Definition 1.6, which satisfies condition (S). Fix such a choice.*

Put $\partial X = \coprod_{F\in\mathcal{M}} \Gamma \times_F S_F$ and $C(\partial X) = \coprod_{F\in\mathcal{M}} \Gamma \times_F D_F$ for D_F the cone over S_F.

Then there exists a finite free Γ-CW-pair $(X, \partial X)$ such that $X \cup_{\partial X} C(\partial X)$ is a model for $\underline{E}\Gamma$ and $(X/\Gamma, \partial X/\Gamma)$ is a finite d-dimensional Poincaré pair.

Recall that the first Stiefel-Whitney class of the finite d-dimensional Poincaré pair $(X/\Gamma, \partial X/\Gamma)$ is automatically the homomorphism $w\colon \Gamma \to \{\pm 1\}$ of (1.13).

Theorem 7.2. *Suppose:*

- *The natural number d satisfies $d \geq 3$;*
- *There is a finite d-dimensional Poincaré complex, which is homotopy equivalent to $B\pi$. Fix a generator $[B\pi]$ of the infinite cyclic group $H_d^{\pi}(E\pi;\mathbb{Z}^{w|_\pi})$;*
- *For every $F \in \mathcal{M}$ the restriction of the homomorphism w of (1.13) to F is trivial, if d is even, and is non-trivial, if d is odd;*
- *Γ satisfies (M), (NM), (H), and (V_{II});*
- *There exists a finite Γ-CW-model for $\underline{E}\Gamma$;*
- *There exists an oriented free d-dimensional slice system $\mathcal{S}$ in the sense of Definition 1.6, which satisfies condition (S). Fix such a choice;*
- *The group π is a Farrell-Jones group.*

Let $(X, \partial X)$ be a finite free Γ-CW-pair such that $\partial X = \coprod_{F\in\mathcal{M}} \Gamma \times_F S_F$ holds, $X \cup_{\partial X} C(\partial X)$ is a model for $\underline{E}\Gamma$, and $(X/\Gamma, \partial X/\Gamma)$ is a finite d-dimensional Poincaré pair. (It exists by Theorem 7.1.)

Consider any decoration $j \in \{2, 1, 0, -1, \ldots\} \amalg \{-\infty\}$. In the sequel everything has to be understood with respect to the orientation homomorphism $w\colon \Gamma \to \{\pm 1\}$ of (1.13).

Then:

1. *We get for any $n \in \mathbb{Z}$ an isomorphism*
$$H_n^{\Gamma}(\underline{E}\Gamma \to \underline{\underline{E}}\Gamma; \mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}) \xrightarrow{\cong} H_n^{\Gamma}(\underline{E}\Gamma \to \{\bullet\}; \mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}) \xrightarrow{\cong} \mathcal{S}_n^{\mathrm{per},\langle j\rangle}(X/\Gamma, \partial X/\Gamma);$$
2. *For every $n \in \mathbb{Z}$, there is an isomorphism*
$$\bigoplus_{V\in\mathcal{V}_{\mathrm{II}}} \mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}^{(-1)^d}, \mathbb{Z}^{(-1)^d}) \xrightarrow{\cong} \mathcal{S}_n^{\mathrm{per},\langle j\rangle}(X/\Gamma, \partial X/\Gamma);$$

(3) *Fix* $n \in \mathbb{Z}$. *Then* $H_n^\Gamma(\underline{E}\Gamma \to \{\bullet\}; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})$ *vanishes, if and only if we have* $\mathrm{UNil}_n(\mathbb{Z}; \mathbb{Z}^{(-1)^d}, \mathbb{Z}^{(-1)^d}) = 0$ *for every* $V \in \mathcal{V}_{\mathrm{II}}$ *or* $\mathcal{V}_{\mathrm{II}}$ *is empty.*

Proof. (1) Consider the following sequence of squares Φ_1, Φ_2, and Φ_3,

$$\begin{tikzcd}
\partial X = \bigcup_{F \in \mathcal{M}} \Gamma \times_F S_F \arrow[r] \arrow[d] & X \arrow[d] \\
\coprod_{F \in \mathcal{M}} \Gamma/F \arrow[r] \arrow[d, "\mathrm{id}"'] & \underline{E}\Gamma \arrow[d] \\
\coprod_{F \in \mathcal{M}} \Gamma/F \arrow[r] \arrow[d, "\mathrm{id}"'] & \underline{\underline{E}}\Gamma \arrow[d, "\mathrm{id}"'] \\
\coprod_{F \in \mathcal{M}} \Gamma/F \arrow[r] & \{\bullet\}.
\end{tikzcd}$$

Let $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$ be the square given by the composition of the three squares above. This implies by inspecting the definition of the algebraic structure group of a pair

$$\mathcal{S}_n^{\mathrm{per},\langle j \rangle}(X/\Gamma, \partial X/\Gamma) = H_n^\Gamma(\Phi; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}). \tag{7.3}$$

Since Φ_1 is a Γ-pushout, we conclude $H_n^\Gamma(\Phi_1; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}) = \{0\}$ for all $n \in \mathbb{Z}$ from excision. The ring $\mathbb{Z}$ is a principal ideal domain and $\mathrm{Wh}_j(H; \mathbb{Z}) = 0$ holds for every finite group H and $j \leq -2$, see [15]. Hence Theorem 5.14 (1) applies and we get $H_n^\Gamma(\Phi_3; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}) = \{0\}$ for all $n \in \mathbb{Z}$. Therefore we get canonical isomorphisms

$$H_n^\Gamma(\Phi_2; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}) \xrightarrow{\cong} H_n^\Gamma(\Phi_3 \circ \Phi_2; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}) \xleftarrow{\cong} H_n^\Gamma(\Phi; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}) \tag{7.4}$$

for all $n \in \mathbb{Z}$. There are also canoncial isomorphisms

$$H_n^\Gamma(\underline{E}\Gamma \to \underline{\underline{E}}\Gamma; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}) \xrightarrow{\cong} H_n^\Gamma(\Phi_2; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}); \tag{7.5}$$

$$H_n^\Gamma(\underline{E}\Gamma \to \{\bullet\}; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}) \xrightarrow{\cong} H_n^\Gamma(\Phi_3 \circ \Phi_2; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}). \tag{7.6}$$

Now put the isomorphisms (7.3), (7.4), (7.5), and (7.6) together.

(2) This follows from assertion (1), Theorem 5.14 (3), and Remark 5.23, since $V \cong D_\infty$ holds for every $V \in \mathcal{V}_{\mathrm{II}}$ by Lemma 2.1 and $F \cong \mathbb{Z}/2$ holds for every $F \in \mathcal{M}$, if d is odd, see [42, Lemma 3.3 (2)].

(3) This follows from assertion (2). □

8. Existence of manifold models

Let Γ be the group appearing in the extension (1.1). Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal finite subgroups.

Theorem 8.1 (Existence of manifold models)**.** *Suppose:*

- *The natural number d satisfies $d \geq 5$;*
- *There exists a closed manifold of dimension d, which is homotopy equivalent to $B\pi$. Fix a generator $[B\pi]$ of the infinite cyclic group $H_d^\pi(E\pi; \mathbb{Z}^{w|_\pi})$;*
- *For every $F \in \mathcal{M}$ the restriction of the homomorphism w of* (1.13) *to F is trivial, if d is even, and is non-trivial, if d is odd;*
- *The group Γ satisfies conditions (M), (NM), and (H), see Definitions 1.3 and 1.14;*
- *There exists a finite Γ-CW-model for $\underline{E}\Gamma$;*

- *There exists an oriented free d-dimensional slice system $\mathcal{S}$ in the sense of Definition 1.6, which satisfies condition (S). Fix such a choice;*
- *The group $H_m^\Gamma(\underline{E}\Gamma \to \{\bullet\}; \mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})$ vanishes for every $m \in \{d, d+1\}$.*

Let $(X, \partial X)$ be a finite free Γ-CW-pair such that $\partial X = \coprod_{F \in \mathcal{M}} \Gamma \times_F S_F$ holds, $X \cup_{\partial X} C(\partial X)$ is a model for $\underline{E}\Gamma$, and $(X/\Gamma, \partial X/\Gamma)$ is a finite d-dimensional Poincaré pair. (It exists by Theorem 7.1.) Then:

(1) *The structure groups $\mathcal{S}_d^{\langle j\rangle}(X/\Gamma, \partial X/\Gamma)$ and $\mathcal{S}_d^{\langle j\rangle}(X/\pi, \partial X/\pi)$ are infinite cyclic and the map induced by restriction with $i\colon \pi \to \Gamma$ induces an injection*
$$i^*\colon \mathcal{S}_d^{\langle j\rangle}(X/\Gamma, \partial X/\Gamma) \to \mathcal{S}_d^{\langle j\rangle}(X/\pi, \partial X/\pi);$$
(2) *There exists a cocompact free d-dimensional Γ-manifold N with boundary ∂N together with a Γ-homotopy equivalence $(f, \partial f)\colon (N, \partial N) \xrightarrow{\simeq} (X, \partial X)$ of Γ-pairs;*
(3) *Let M be $N \cup_{\partial N} C(\partial N)$. Then M is a slice manifold model for $\underline{E}\Gamma$ with N as a slice complement in the sense of Definition 1.11. If $\mathcal{S}' = \{S'_F \mid F \in \mathcal{M}\}$ is the underlying slice manifold system, then S_F and S'_F are F-homotopy equivalent for every $F \in \mathcal{M}$.*

Proof. (1) In the sequel $w\colon \Gamma \to \{\pm 1\}$ is the group homomorphism (1.13). Recall that it agrees with the orientation homomorphism $\Gamma \to \{\pm 1\}$ associated to the Poincaré pair $(X/\Gamma, \partial X/\Gamma)$. Note that we can choose a finite d-dimensional Γ-CW-complex model for $\underline{E}\Gamma$ such that $\underline{E}\Gamma^{>1} = \coprod_{F \in \mathcal{M}} \Gamma/F$ holds, see [42, Theorem 1.12].

Let $\mathbf{E}\colon \mathsf{Groupoids} \to \mathsf{Spectra}$ be a $\mathsf{Groupoids}$-spectrum. We have introduced in Subsection 3.2 the $\mathsf{Groupoids}$-spectrum $\mathbf{E}\langle 1\rangle$ obtained from $\mathbf{E}$ by passing to the 1-connective covering. Consider a Γ-map $f\colon X \to Y$. The equivariant Atiyah-Hirzebruch spectral sequence for the Γ-homology theory associated to $\overline{\mathbf{E}}$ defined in Subsection 3.2 has as E^2-term $E^2_{p,q}$ the Bredon homology $BH_p^\Gamma(f; \pi_q(\overline{\mathbf{E}}))$ and converges to $H_{p+q}^\Gamma(f; \overline{\mathbf{E}})$. An easy spectral sequence argument yields an isomorphism, natural in f,

$$H_{d+1}^\Gamma(f; \overline{\mathbf{E}}) \xrightarrow{\cong} BH_{d+1}^\Gamma(f; \pi_0(\mathbf{E})), \tag{8.2}$$

provided that $\dim(X) \le d$ and $\dim(Y) \le d+1$ holds. If we apply equation (8.2) to $\underline{E}\Gamma \to \{\bullet\}$ for $\mathbf{E} = \mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}$ and write $\overline{\mathbf{E}} = \overline{\mathbf{L}}_{\mathbb{Z},w}^{\langle j\rangle}$, we obtain from the long exact sequence of the map $\underline{E}\Gamma \to \{\bullet\}$ for Bredon homology an isomorphism

$$\partial_{d+1}\colon H_{d+1}^\Gamma(\underline{E}\Gamma \to \{\bullet\}; \overline{\mathbf{L}}_{\mathbb{Z},w}^{\langle j\rangle}) \xrightarrow{\cong} BH_d^\Gamma(\underline{E}\Gamma; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})). \tag{8.3}$$

We put

$$\Phi := \begin{array}{ccc} \partial X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \pi_0(\partial X) & \longrightarrow & \underline{E}\Gamma \end{array} \quad \text{and} \quad \Psi := \begin{array}{ccc} \partial X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \pi_0(\partial X) & \longrightarrow & \{\bullet\}. \end{array}$$

We have the following long exact sequence

$$\cdots \to H_{n+1}^\Gamma(\Phi; \mathbf{E}) \to H_{n+1}^\Gamma(\Psi; \mathbf{E}) \to H_{n+1}^\Gamma(\underline{E}\Gamma \to \{\bullet\}; \mathbf{E}) \\ \to H_n^\Gamma(\Phi; \mathbf{E}) \to H_n^\Gamma(\Psi; \mathbf{E}) \to \cdots.$$

Since Φ is a Γ-pushout and its upper horizontal arrow is a Γ-cofibration, $H_n^\Gamma(\Phi; \mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}\langle 1\rangle)$ vanishes for every $n \in \mathbb{Z}$. Hence we get for $n \in \mathbb{Z}$ an isomorphism

$$H_n^\Gamma(\Psi; \mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}\langle 1\rangle) \xrightarrow{\cong} H_n^\Gamma(\underline{E}\Gamma \to \{\bullet\}; \mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}\langle 1\rangle). \tag{8.4}$$

If we apply the long exact sequence (3.1) to $\mathbf{E} = \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}$ and $\underline{E}\Gamma \to \{\bullet\}$ and use the assumption that $H_m^{\Gamma}(\underline{E}\Gamma \to \{\bullet\}; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})$ vanishes for $m = d, d+1$, we get an isomorphism

$$\partial_{d+1} \colon H_{d+1}^{\Gamma}(\underline{E}\Gamma \to \{\bullet\}; \overline{\mathbf{L}}_{\mathbb{Z},w}^{\langle j \rangle}) \xrightarrow{\cong} H_d^{\Gamma}(\underline{E}\Gamma \to \{\bullet\}; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}\langle 1 \rangle). \tag{8.5}$$

Combining (8.3), (8.4) and (8.5) yields an isomorphism

$$H_d^{\Gamma}(\Psi; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}\langle 1 \rangle) \xrightarrow{\cong} BH_d^{\Gamma}(\underline{E}\Gamma; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})). \tag{8.6}$$

Since the image of the inclusion $i \colon \pi \to \Gamma$ has finite index, there are natural restrictions functors

$$\begin{array}{rcl} i^* \colon H_d^{\Gamma}(\Psi, \mathbf{L}_{\mathbb{Z},w}^{\langle -j \rangle}\langle 1 \rangle) & \to & H_d^{\pi}(i^*\Psi, \mathbf{L}_{\mathbb{Z},w}^{\langle -j \rangle}\langle 1 \rangle); \\ i^* \colon H_d^{\Gamma}(\Psi, \mathbf{L}_{\mathbb{Z},w}^{\langle -j \rangle}) & \to & H_d^{\pi}(i^*\Psi, \mathbf{L}_{\mathbb{Z},w}^{\langle -j \rangle}); \\ i^* \colon H_d^{\Gamma}(\Psi, \overline{\mathbf{L}}_{\mathbb{Z},w}^{\langle -j \rangle}) & \to & H_d^{\pi}(i^*\Psi, \overline{\mathbf{L}}_{\mathbb{Z},w}^{\langle -j \rangle}), \end{array}$$

and analogously for Bredon homology

$$i^* \colon BH_d^{\Gamma}(\Psi; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})) \to BH_d^{\pi}(i^*\Psi; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})).$$

These are compatible with the exact sequence (3.1) and the isomorphism (8.2) for $\mathbf{E} = \mathbf{L}_{\mathbb{Z},w}^{\langle -j \rangle}$. Moreover, they are compatible with the isomorphisms (8.3), (8.4) (8.5) and (8.6) as well. In particular, we get a commutative square whose vertical arrows are isomorphisms

$$\begin{tikzcd} H_d^{\Gamma}(\Psi; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}\langle 1 \rangle) \arrow[r, "i^*"] \arrow[d, "\cong"'] & H_d^{\pi}(i^*\Psi; \mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}\langle 1 \rangle) \arrow[d, "\cong"'] \\ BH_d^{\Gamma}(\underline{E}\Gamma; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})) \arrow[r, "i^*"'] & BH_d^{\pi}(i^*\underline{E}\Gamma; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})). \end{tikzcd} \tag{8.7}$$

Since $\underline{E}\Gamma$ is a finite Γ-CW- complex such that $\dim(\underline{E}\Gamma) \leq d$ and all it cells in dimension d have trivial isotropy group, $BH_d^{\Gamma}(\underline{E}\Gamma; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle}))$ is an abelian subgroup of a finite direct sum of copies of $L_0^{\langle j \rangle}(\mathbb{Z}) \cong L_0(\mathbb{Z}) \cong \mathbb{Z}$. We conclude that there is a natural number r satisfying

$$BH_d^{\Gamma}(\underline{E}\Gamma; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})) \cong \mathbb{Z}^r. \tag{8.8}$$

Consider the following commutative diagram

$$\begin{tikzcd} BH_d^{\Gamma}(\underline{E}\Gamma; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})) \arrow[r, "i^*"'] \arrow[d, "\cong"] & BH_d^{\pi}(i^*\underline{E}\Gamma; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})) \arrow[d, "\cong"'] \\ BH_d^{\Gamma}(\coprod_{F \in \mathcal{M}} \Gamma/F \to \underline{E}\Gamma; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})) \arrow[r, "i^*"'] & BH_d^{\pi}(\coprod_{F \in \mathcal{M}} i^*\Gamma/F \to i^*\underline{E}\Gamma; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})) \\ BH_d^{\Gamma}(\coprod_{F \in \mathcal{M}} \Gamma \times_F EF \to E\Gamma; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})) \arrow[r, "i^*"'] \arrow[u, "\cong"] & BH_d^{\pi}(\coprod_{F \in \mathcal{M}} i^*(\Gamma \times_F EF) \to i^*E\Gamma; \pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j \rangle})). \arrow[u, "\cong"] \end{tikzcd}$$

We have the long exact sequence

$$\cdots \to BH_d^{\Gamma}(\coprod_{F\in\mathcal{M}} \Gamma\times_F EF;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})) \to BH_d^{\Gamma}(E\Gamma;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})) \to$$
$$\to BH_d^{\Gamma}(\coprod_{F\in\mathcal{M}} \Gamma\times_F EF \to E\Gamma;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})) \to$$
$$\to BH_{d-1}^{\Gamma}(\coprod_{F\in\mathcal{M}} \Gamma\times_F EF;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})) \to BH_{d-1}^{\Gamma}(E\Gamma;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})) \to \cdots .$$

Since Γ acts freely on $\Gamma\times_F EF$ and $E\Gamma$ and $L_0^{\langle j\rangle}(\mathbb{Z})$ is independent of the decoration j, we get identifications

$$\begin{aligned} BH_k^{\Gamma}(\coprod_{F\in\mathcal{M}} \Gamma\times_F EF;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})) &\cong \bigoplus_{F\in\mathcal{M}} H_k^F(EF;L_0(\mathbb{Z})^{w|_F}) \\ BH_k^{\Gamma}(E\Gamma;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})) &\cong H_k^{\Gamma}(E\Gamma;L_0(\mathbb{Z})^w). \end{aligned}$$

Since F is a finite group, we get for $k\ge 1$

$$H_k^F(EF;L_0(\mathbb{Z})^{w|_F})_{(0)} = 0,$$

where for any abelian group A we denote by $A_{(0)} := \mathbb{Q}\otimes_{\mathbb{Z},w} A$ its rationalization. Hence we obtain an isomorphism

$$BH_d^{\Gamma}(\underline{E}\Gamma;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}))_{(0)} \cong H_d^{\Gamma}(E\Gamma;L_0(\mathbb{Z})^w)_{(0)}.$$

Since we have the exact sequence $1\to\pi\xrightarrow{i}\Gamma\to G\to 1$, we conclude from the Leray-Serre spectral sequence that restriction with i yields an isomorphism

$$i^*_{(0)}\colon H_d^{\Gamma}(E\Gamma;L_0(\mathbb{Z})^w)_{(0)} \xrightarrow{\cong} H_d^{\pi}(E\pi;L_0(\mathbb{Z})^{w|_\pi})_{(0)}^G.$$

By Poincaré duality we obtain an isomorphism

$$H_d^{\pi}(E\pi;L_0(\mathbb{Z})^{w|_\pi}) \cong H^0_{\pi}(E\pi;L_0(\mathbb{Z})) \cong H^0(B\pi;L_0(\mathbb{Z})) \cong L_0(\mathbb{Z}),$$

Moreover, the G-action on $H_d^{\pi}(E\pi;L_0(\mathbb{Z})^w)_{(0)}$ is trivial by a direct inspection, cf. [42, Proof of Lemma 6.15]. Hence we get an isomorphism

$$BH_d^{\Gamma}(\underline{E}\Gamma;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}))_{(0)} \xrightarrow{\cong} L_0(\mathbb{Z})_{(0)} \cong \mathbb{Q}. \tag{8.9}$$

Combining (8.8) and (8.9) yields

$$BH_d^{\Gamma}(\underline{E}\Gamma;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})) \cong \mathbb{Z}. \tag{8.10}$$

The following diagram commutes

$$\begin{tikzcd}
H_d^{\Gamma}(E\Gamma;L_0(\mathbb{Z})^w)_{(0)} \arrow[r, "i^*"] \arrow[d] & H_d^{\pi}(E\pi;L_0(\mathbb{Z})^{w|_\pi}) \arrow[d, "\cong"] \\
BH_d^{\Gamma}(\underline{E}\Gamma;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})) \arrow[r, "i^*"] & BH_d^{\pi}(i^*\underline{E}\Gamma;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})).
\end{tikzcd}$$

The upper horizontal arrow, the left vertical arrow, and the right vertical arrow are bijective after rationalizing. Hence the lower horizontal arrow is bijective after rationalizing. Since its source and target consists of infinite cyclic groups, the map

$$i^*\colon BH_d^{\Gamma}(\underline{E}\Gamma;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})) \to BH_d^{\pi}(i^*\underline{E}\Gamma;\pi_0(\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}))$$

is injective. We conclude from (8.7) that the map

$$i^*\colon H_d^{\Gamma}(\Psi;\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}\langle 1\rangle) \to H_d^{\pi}(i^*\Psi;\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}\langle 1\rangle)$$

is injective. This map can be identified with $i^*\colon \mathcal{S}_d^{\langle j\rangle}(X/\Gamma,\partial X/\Gamma) \to \mathcal{S}_d^{\langle j\rangle}(X/\pi,\partial X/\pi)$, c.f. (7.3). This finishes the proof of assertion(1).

(2) This follows from assertion (1) and Theorem 6.3. In dimension $d = 5$ we use that fact that on the boundary, which is 4-dimensional, all Γ-components are induced from the finite group F, which is good in the sense of Freedman.

(3) By assertion (2) we can choose a free cocompact d-dimensional Γ-manifold N with boundary ∂N together with a Γ-homotopy equivalence $(f, \partial f)\colon (N, \partial N) \xrightarrow{\simeq} (X, \partial X)$ of Γ-pairs. Because the Poincaré Conjecture is known to be true, there exists a d-dimensional free slice manifold system $\mathcal{S}' = \{S'_F \mid F \in \mathcal{M}\}$ with $\partial N = \coprod_{F\in\mathcal{M}} \Gamma \times_F S'_F$. Let D'_F be the cone of S'_F. Define the Γ-spaces M and $X \cup_{\partial X} C(\partial X)$ by the Γ-pushouts

$$\begin{array}{ccc} \partial N = \coprod_{F\in\mathcal{M}} \Gamma \times_F S'_F & \longrightarrow & N \\ \downarrow & & \downarrow \\ C(\partial N) = \coprod_{F\in\mathcal{M}} \Gamma \times_F D'_F & \longrightarrow & M \end{array}$$

and

$$\begin{array}{ccc} \partial X = \coprod_{F\in\mathcal{M}} \Gamma \times_F S_F & \longrightarrow & X \\ \downarrow & & \downarrow \\ C(\partial X) = \coprod_{F\in\mathcal{M}} \Gamma \times_F D_F & \longrightarrow & X \cup_{\partial X} C(\partial X). \end{array}$$

Obviously the Γ-homotopy equivalence $\partial f\colon \coprod_{F\in\mathcal{M}} \Gamma \times_F S'_F = \partial N \to \Gamma \times_F S_F = \partial X$ extends to a Γ-homotopy equivalence $\widehat{\partial f}\colon \coprod_{F\in\mathcal{M}} \Gamma \times_F D'_F \to \Gamma \times_F D_F$. Let $F\colon M \to X \cup_{\partial X} C(\partial X)$ be the Γ-map given by the Γ-pushout of the three Γ-homotopy equivalences $\widehat{\partial f}$, ∂f and f. Then F itself is a Γ-homotopy equivalence. The Γ-space $X \cup_{\partial X} C(\partial X)$ is a Γ-CW-model for $\underline{E}\Gamma$ by assumption. This finishes the proof of Theorem 8.1. ☐

9. Uniqueness of manifold models

Let Γ be the group appearing in the extension (1.1). Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal finite subgroups.

Theorem 9.1 (Uniqueness of manifold models)**.** *Suppose:*

- *The natural number d satisfies $d \geq 5$;*
- *Γ satisfies condition (V_{II}), see Definition 1.3;*
- *The group π is a Farrell-Jones group, see Subsection 3.4;*
- *The group $\mathrm{UNil}_{d+1}(\mathbb{Z}; \mathbb{Z}^{(-1)^d}, \mathbb{Z}^{(-1)^d})$ vanishes or Γ contains no subgroup isomorphic to D_∞.*

Let M and M' be two slice manifold models for $\underline{E}\Gamma$ with respect to the d-dimensional free slice manifold systems $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$ and $\mathcal{S}' = \{S'_F \mid F \in \mathcal{M}\}$ in the sense of Definition 1.11. Let N and N' be slice complements, i.e., cocompact proper proper free d-dimensional Γ-manifolds with boundary such that there are Γ-pushouts

$$\begin{array}{ccc} \partial N = \coprod_{F\in\mathcal{M}} \Gamma \times_F S_F & \longrightarrow & N \\ \downarrow & & \downarrow \\ C(\partial N) & \longrightarrow & M \end{array} \quad \text{and} \quad \begin{array}{ccc} \partial N' = \coprod_{F\in\mathcal{M}} \Gamma \times_F S'_F & \longrightarrow & N \\ \downarrow & & \downarrow \\ C(\partial N') & \longrightarrow & M' \end{array}$$

where we abbreviate $C(\partial N) := \coprod_{F\in\mathcal{M}} \Gamma \times_F D_F$ and $C(\partial N') := \coprod_{F\in\mathcal{M}} \Gamma \times_F D'_F$.

Then:

1. *The group $H^\Gamma_{d+1}(\underline{E}\Gamma \to \{\bullet\}; \mathbf{L}^{\langle j\rangle}_{\mathbb{Z},w})$ vanishes for every $j \in \{2, 1, 0, -1, \ldots\} \amalg \{-\infty\}$;*

(2) The structure group $\mathcal{S}^{\langle j\rangle}_{d+1}(N'/\Gamma, \partial N'/\Gamma)$ vanishes for $j \in \{2,1,0,-1,\dots\} \amalg \{-\infty\}$;

(3) We have

(a) For every $F \in F$ there exists an F-equivariant h-cobordism between S_F and S'_F;

(b) There exists a Γ-homeomorphism $f\colon M \to M'$;

(4) (a) Every Γ-homeomorphism of cocompact proper free Γ-manifolds with boundary $(N, \partial N) \to (N', \partial N')$ is a simple Γ-homotopy equivalence of Γ-CW-pairs;

(b) Every simple Γ-homotopy equivalence $(N, \partial N) \to (N', \partial N')$ of Γ-CW-pairs is Γ-homotopic to a Γ-homeomorphism $(N, \partial N) \to (N', \partial N')$ of cocompact proper free Γ-manifolds with boundary;

(5) The following assertions are equivalent, if we additionally assume that for all $F \in \mathcal{M}$ the 2-Sylow subgroup of F is cyclic;

(a) There exists a Γ-homeomorphism of cocompact proper free Γ-manifolds with boundary $(h, \partial h)\colon (N, \partial N) \xrightarrow{\cong} (N', \partial N')$ such that ∂h induces for each $F \in \mathcal{M}$ a F-homeomorphism $\partial h_F\colon S_F \to S'_F$ satisfying $\partial h = \coprod_{F \in \mathcal{M}} \mathrm{id}_\Gamma \times_F \partial h_F$;

(b) There exists a simple Γ-homotopy equivalence $(f, \partial f)\colon (N, \partial N) \xrightarrow{\simeq_s} (N', \partial N')$ of Γ-CW-complexes such that such that ∂f induces for each $F \in \mathcal{M}$ a simple F-homotopy equivalence $\partial f_F\colon S_F \to S'_F$ satisfying $\partial f = \coprod_{F \in \mathcal{M}} \mathrm{id}_\Gamma \times_F \partial f_F$;

(c) For every $F \in F$ there exists a F-homeomorphism of cocompact proper free F-manifolds $S_F \xrightarrow{\cong} S'_F$;

(d) For every $F \in F$ there exists a simple F-homotopy equivalence of finite free Γ-CW-complexes $S_F \xrightarrow{\simeq_s} S'_F$.

Proof. (1) The existence of a slice model M' implies the following facts. We conclude from [42, Lemma 1.9], which directly extends to the case, where w is non-trivial, that Γ satisfies conditions (M), (NM), and (H), see Definitions 1.3 and 1.14, and that there is a finite d-dimensional Γ-CW-complex model for $\underline{E}\Gamma$ such that $\underline{E}\Gamma^{>1} = \coprod_{F \in \mathcal{M}} \Gamma/F$ holds. Moreover, there is a closed manifold model for $B\pi$ and the orientation character $w\colon \Gamma \to \{\pm 1\}$ has for every $F \in \mathcal{M}$ the property that $w|_F$ is trivial, if d is even, and is non-trivial, if d is odd, see [42, Subsection 6.2]. Now apply Theorem 7.2 (3).

(2) We proceed as in the proof of Theorem 8.1 (1) An easy spectral sequence argument shows that we get from assertion (1) an isomorphism

$$H^\Gamma_{d+2}(\underline{E}\Gamma \to \{\bullet\}; \overline{\mathbf{L}}^{\langle j\rangle}_{\mathbb{Z},w}) = \{0\}. \tag{9.2}$$

We put

$$\Phi := \begin{array}{ccc} \partial N' & \longrightarrow & N' \\ \big\downarrow & & \big\downarrow \\ \pi_0(\partial N') & \longrightarrow & \underline{E}\Gamma \end{array} \qquad \text{and} \qquad \Psi := \begin{array}{ccc} \partial N' & \longrightarrow & N' \\ \big\downarrow & & \big\downarrow \\ \pi_0(\partial N') & \longrightarrow & \{\bullet\}. \end{array}$$

We have the following long exact sequence

$$\cdots \to H^\Gamma_{n+1}(\Phi; \mathbf{L}^{\langle j\rangle}_{\mathbb{Z},w}\langle 1\rangle) \to H^\Gamma_{n+1}(\Psi; \mathbf{L}^{\langle j\rangle}_{\mathbb{Z},w}\langle 1\rangle) \to H^\Gamma_{n+1}(\underline{E}\Gamma \to \{\bullet\}; \mathbf{L}^{\langle j\rangle}_{\mathbb{Z},w}\langle 1\rangle)$$
$$\to H^\Gamma_{n}(\Phi; \mathbf{L}^{\langle j\rangle}_{\mathbb{Z},w}\langle 1\rangle) \to H^\Gamma_{n}(\Psi; \mathbf{L}^{\langle j\rangle}_{\mathbb{Z},w}\langle 1\rangle) \to \cdots .$$

Since Φ is a Γ-pushout and its upper horizontal arrow is a Γ-cofibration, $H_n^\Gamma(\Phi;\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}\langle 1\rangle)$ vanishes for every $n \in \mathbb{Z}$. Hence we get for $n \in \mathbb{Z}$ an isomorphism

$$H_n^\Gamma(\Psi;\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}\langle 1\rangle) \xrightarrow{\cong} H_n^\Gamma(\underline{E}\Gamma \to \{\bullet\};\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}\langle 1\rangle). \tag{9.3}$$

If we apply the long exact sequence (3.1) to $\mathbf{E} = \mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}$ and $\underline{E}\Gamma \to \{\bullet\}$ and use that $H_{d+1}^\Gamma(\underline{E}\Gamma \to \{\bullet\};\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle})$ vanishes by assertion (1), we get an epimorphism

$$\partial_{d+2}\colon H_{d+2}^\Gamma(\underline{E}\Gamma \to \{\bullet\};\overline{\mathbf{L}}_{\mathbb{Z},w}^{\langle j\rangle}) \xrightarrow{\cong} H_{d+1}^\Gamma(\underline{E}\Gamma \to \{\bullet\};\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}\langle 1\rangle). \tag{9.4}$$

Combining (9.2), (9.3) and (9.4) yields

$$H_{d+1}^\Gamma(\Psi;\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}\langle 1\rangle) = \{0\}. \tag{9.5}$$

Note that we get an identification $H_{d+1}^\Gamma(\Psi;\mathbf{L}_{\mathbb{Z},w}^{\langle j\rangle}\langle 1\rangle) = \mathcal{S}_{d+1}^{\langle j\rangle}(N'/\Gamma,\partial N'/\Gamma)$ from the definition of the algebraic structure groups. Hence $\mathcal{S}_{d+1}^{\langle j\rangle}(N'/\Gamma,\partial N'/\Gamma)$ vanishes because of (9.5).

(3) The main arguments for the proof of assertion (3) have already been presented in [19, Proof of Lemma 4.3]. For the reader's convenience we give more details here.

We conclude from [42, Theorem 8.9 and Theorem 10.2] taking [42, Lemma 3.4] into account that there is a Γ-homotopy equivalence of Γ-pairs $(f,\partial f)\colon (N,\partial N) \xrightarrow{\simeq_\Gamma} (N',\partial N')$ such that ∂f induces F-homotopy equivalences $\partial f_F\colon S'_F \to S'_F$ for every $F \in \mathcal{M}$. We get $\mathcal{S}_{d+1}^{\langle h\rangle}(N'/\Gamma,\partial N'/\Gamma) = 0$ from assertion (2). We conclude from Theorem 6.3 (3) that there is an h-cobordism $(W,\partial W)$ with a Γ-homotopy equivalence of pairs $(F,\partial F)\colon (W,\partial W) \to (N' \times [0,1],\partial(N' \times [0,1]))$ from $(f,\partial f)\colon (N,\partial N) \to (N',\partial N')$ to $\mathrm{id}_{(N',\partial N')}\colon (N',\partial N') \to (N',\partial N')$. In dimension $d = 5$ we use that fact that on the boundary, which is 4-dimensional, all Γ-components are induced from finite groups F which are all good in the sense of Freedman. In particular we see that (3a) holds. We conclude from Theorem 4.3, the s-Cobordism Theorem for pairs, and basic properties of Whitehead torsion as for instance homotopy invariance and the sum formula that we can choose for $F \in \mathcal{M}$ an F-h-cobordism V_F between S_F and S'_F such that $W = \bigcup_{F\in\mathcal{M}} \Gamma \times_F V_F$ is a Γ-h-cobordism between ∂N and $\partial N'$ and a simple Γ-h-cobordism of pairs between $N \cup_{\partial N} W$ and N' relative $\partial N'$. The latter implies that there is a Γ-homeomorphism of pairs $(N \cup_{\partial N} W,\partial N') \xrightarrow{\cong} (N',\partial N')$ which is the identity on $\partial N'$. It induces a Γ-homeomorphism $N \cup_{\partial N} W \cup_{\partial N'} C(\partial N') \xrightarrow{\cong} M' = N' \cup_{\partial N'} C(\partial N')$. Hence it suffices to show that $N \cup_{\partial N} W \cup_{\partial N'} C(\partial N')$ and $M = N \cup_{\partial N} C(\partial N)$ are Γ-homeomorphic.

Let W^- be a Γ-h-cobordism between $\partial N'$ and ∂N satisfying $\tau(\partial N' \to W^-) = -\tau(\partial N \to W)$. Then $W \cup_{\partial N'} W^-$ is a trivial Γ-h-cobordism over ∂N because of

$$\begin{aligned}\tau(\partial N \to W \cup_{\partial N'} W^-) &= \tau(\partial N \to W) + \tau(\partial N' \to W^-)\\ &= \tau(\partial N \to W) - \tau(\partial N \to W) = 0.\end{aligned}$$

Analogously one shows that $W^- \cup_{\partial N} W$ is a trivial F-h-cobordism over $\partial N'$. Now one constructs by an Eilenberg swindle a Γ-homeomorphism relative to ∂N

$$W \cup_{\partial N'} \partial N' \times [0,\infty) \xrightarrow{\cong} \partial N \times [0,\infty).$$

It extends by passing to the one-point-compatification of the various path components to a Γ-homeomorphism relative ∂N.

$$W \cup_{\partial N'} C(\partial N') \xrightarrow{\cong} C(\partial N).$$

It together with id_N yields the desired Γ-homeomorphism $N \cup_\partial W \cup_{\partial N'} C(\partial N') \xrightarrow{\cong} N \cup_{\partial N} C(\partial N)$.

(4) Since Γ acts freely on $(N, \partial N)$ and $(N', \partial N')$, any Γ-homeomorphism $(N, \partial N) \xrightarrow{\cong} (N', \partial N')$ of pairs is a simple Γ-homotopy equivalence of Γ-CW-pairs by the topological invariance of Whitehead torsion, see [16, 17].

Consider a simple homotopy equivalence of Γ-pairs $(f, \partial f)\colon (N, \partial N) \xrightarrow{\cong} (N', \partial N')$, We get $\mathcal{S}^{\langle s\rangle}_{d+1}(N'/\Gamma, \partial N'/\Gamma) = 0$ from assertion(1). Now we conclude from Theorem 6.3 (2)that $(f, \partial f)$ is Γ-homotopic to a Γ-homeomorphism of pairs.

(5) We conclude (5a) $\Longleftrightarrow$ (5b) from assertion (4). Obviously (5b) $\Longrightarrow$ (5d) and (5a) $\Longrightarrow$ (5c) hold. We get (5c) $\Longrightarrow$ (5d) from the topological invariance of Whitehead torsion, see [16, 17]. Hence it remains to prove the implication (5d) $\Longrightarrow$ (5b) what we do next.

Any F-homotopy equivalence $S_F \xrightarrow{\simeq_s} S'_F$ is simple by [42, Lemma 3.3 (5)], since there exists one F-homotopy equivalence $S_F \xrightarrow{\simeq_s} S'_F$ by assumption. Now apply [42, Theorem 9.1 and Theorem 10.3] taking [42, Lemma 3.4] into account. The condition (S) appearing in [42, Definition 7.9] can be arranged to hold by [42, Lemma 7.10]. This finishes the proof of Theorem 9.1. $\square$

Remark 9.6. Note that from Theorem 9.1 we get a slice model system $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$ such that each element S_F is unique up to F-h-cobordism. It is unclear how we can determine $\mathcal{S}$ just from Γ, provided that all the assumptions appearing in Theorem 9.1 are satisfied. Note that we have at least a recipe to determine $\{\kappa_F \mid F \in \mathcal{M}\}$ from Γ and hence the F-homotopy type of each S_F, see Definition 1.15.

If F is finite cyclic of odd order, the simple structure set of S_F/F has completely been determined in terms of Reidemeister torsion and ρ-invariants by Wall [54, Theorem 14E.7 on page 224].

But there is a geometric case, where the passage from Γ to the slice manifold system is explicit. Let X be a closed negatively curved manifold and suppose that Γ is a discrete cocompact subgroup of $\operatorname{Isom}(\widetilde{X})$, that Γ is a finite extension of the deck transformations π, and that Γ satisfies condition (F). Then the action of Γ extends to the sphere S^{d-1}_∞ and any non-trivial finite subgroup F fixes a point in $\widetilde{X}$ and acts freely on the boundary sphere. The F-h-cobordism between the boundary of a F-tubular neighborhood of the fixed point and the sphere at infinity determines the slice manifold structure.

10. Comparisons

10.1. **Cocompact and d-dimensional models.** We discuss the relationship between cocompact manifold models and d-dimensional manifold models for $\underline{E}\Gamma$.

Assume that Γ is a finite extension of the fundamental group of a closed aspherical manifold X of dimension d. Note that $H_d(X; \mathbb{Z}/2) = \mathbb{Z}/2$ and $H_e(X; \mathbb{Z}/2) = 0$ for $e > d$. We claim that a cocompact manifold model for $\underline{E}\Gamma$ is a d-dimensional manifold model for $\underline{E}\Gamma$ and conversely. If M is a cocompact manifold model for $\underline{E}\Gamma$, then M/π is a closed manifold which has the homotopy type of X; hence its homology shows that it has dimension d. Conversely, if M is a d-dimensional manifold model for $\underline{E}\Gamma$, then M/π is a d-manifold having the homology of X so must be closed. Hence M/Γ is compact.

10.2. **(M),(NM), and (F).** Clearly conditions (M) and (NM) imply condition (F), but the converse is not true in general. If Γ is virtually torsionfree and a cocompact manifold model for $\underline{E}\Gamma$ exists, then conditions (M)+(NM) are equivalent to condition (F) by Lemma 2.1 (4). This provides an intriguing possibility for a negative answer to the Manifold Model Question: Construct a finite extension Γ of a fundamental group of a closed aspherical manifold which satisfies condition (F), but not (M)+(NM).

10.3. **Assumptions.** The assumptions of our uniqueness Theorem 1.21 and the assumptions of the uniqueness theorem in [19] are different and it is important to compare them so that we can use the results of both. The main difference is our assumption of a slice manifold model and their assumption of condition (C$'$i), which says that there exists a proper cocompact Γ-manifold X such that $(X \setminus X^{>1})/\Gamma$ has the Γ-homotopy type of a finite Γ-CW-complex and X is Γ-homotopy equivalent to $\underline{E}\Gamma$.

The following further conditions appear in [19]. The condition (C$'$ii) says that each infinite dihedral subgroup of Γ lies in a unique maximal infinite dihedral group, Condition (C$'$iii) says that Γ satisfies the K-and L-theoretic Farrell-Jones Conjecture with coefficients in the ring $\mathbb{Z}$. Note that condition (C$'$iii) is automatically satisfied, if π is a Farrell-Jones group. The condition (C) says that there is a contractible Riemannian manifold of non-positive sectional curvature with effective cocompact proper Γ-action by isometries. In [19, Remark 1.2] it is proved that the conditions (F) and (C) together imply conditions (C$'$i), (C$''$ii), and (C$'$iii), actually such Γ is a Farrell-Jones group.

It is obvious that the existence of a slice manifold model for $\underline{E}\Gamma$ implies condition (C$'$i). In [19, Lemma 4.2] it is shown that there exists a slice manifold model for $\underline{E}\Gamma$, provided that conditions (F), (C$'$i) and (C$'$ii) and (C$'$iii) are satisfied. The conditions (F) and (C$'$i) follow from conditions (M) and (NM) by Lemma 2.1. Hence condition (C$'$i) is equivalent to the existence of slice manifold model for $\underline{E}\Gamma$, provided that conditions (M) and (NM) hold and π is a Farrell-Jones group. Recall that Theorem 1.16 gives conditions for the the existence of a slice manifold model for $\underline{E}\Gamma$ and that most of the conditions appearing there are necessary, see Remark 1.20. Obviously the conditions appearing in Theorem 1.16 are more accessible than the condition (C'i).

10.4. **All models are slice manifold models.** Now suppose that there exists a slice manifold model X for $\underline{E}\Gamma$. Consider any cocompact proper Γ-manifold M such that M is Γ-homotopy equivalent to $\underline{E}\Gamma$. Recall that X is a Γ-CW-model for $\underline{E}\Gamma$, which is pseudo-free, i.e., $X^{>1}$ is zero-dimensional. Hence also M is pseudo-free by Lemma 2.1 (4). We conclude from [19, Lemma 3.2] that there is an isovariant Γ-homotopy equivalence $f\colon M \to X$. Now [19, Proposition 4.2] implies that M is a slice manifold model for $\underline{E}\Gamma$.

10.5. **NRQ and MMQ.** In the introduction we argued, that in fairly general circumstances, an affirmative answer to the Manifold Model Question gives an affirmative answer to the Nielsen Realization Question. In this subsection we argue, that under quite technical conditions, an affirmative answer to the Nielsen Realization Question implies an affirmative answer to the Manifold Model Question.

Proposition 10.1. *Suppose a finite group G acts on a closed aspherical manifold X with fundamental group π. Let*

$$1 \to \pi \to \Gamma \to G \to 1$$

be the associated exact sequence of groups. If Γ satisfies condition (F), namely that every non-trivial finite subgroup of Γ has finite normalizer, then there is a pseudo-free cocompact Γ-manifold M which has the Γ-homotopy type of a model for $\underline{E}\Gamma$.

Proof. Let $M = \widetilde{X}$ which is a manifold by with a Γ-action by hypothesis. It is cocompact since M/Γ is a quotient of the compact space X. Proposition 2.3 of [19] asserts that the action of Γ on M is pseudo-free, that the fixed-point set of any finite non-trivial subgroup is a point, and the fixed-point set of any infinite subgroup is empty. Finally, Proposition 2.5 of [19] asserts that M has the Γ-homotopy type of a Γ-CW-complex. □

The conclusion stops short of answering the Manifold Model Question, since it does not assert that M is a Γ-CW-complex. However, we suspect that using some algebraic K-theory together with the proof of Proposition 2.5 of [19], that one could prove that M is a compact manifold model for $\underline{E}\Gamma$ if G has order 2 or 3, in which case the lower Whitehead groups $\mathrm{Wh}_i(G)$ vanish for $i \leq 1$.

10.6. **Hyperbolic manifolds satisfies NRQ and MMQ.** We point out that in a special case, geometry gives answers to the Nielsen Realization Question and the Manifold Model Question. A *hyperbolic manifold* is a manifold with constant sectional curvature equal to -1. The only complete simply connected hyperbolic n-manifold is hyperbolic n-space $\mathbb{H}^n$.

Theorem 10.2. *Let X be a closed hyperbolic n-manifold with fundamental group π. Any group monomorphism $\phi : G \to \mathrm{Out}(\pi)$ can be realized by a map $G \to \mathrm{Isom}(X)$.*

Proof. When $n = 2$, this is a consequence of Kerckhoff's solution of the Nielsen Realization Problem [35], so we assume $n > 2$.

Since π contains no rank 2 free abelian subgroup, the center of π is trivial. Thus, as mentioned in the introduction, group cohomology shows that ϕ can be realized by a group extension

$$1 \to \pi \to \Gamma \to G \to 1,$$

unique up to isomorphism.

We now need to use the Milnor-Schwarz Theorem, the work of Tukia, and the Mostow Rigidity Theorem. We will follow the terminology of the book [30]. Recall that there is an equivalence relation on metric spaces called quasi-isometry, see [30, Definition 8.10]. A finite generating set S of a group G defines a metric on the Cayley graph of (G, S), hence on G, its set of vertices. The isometric inclusion of G into the Cayley graph is a quasi-isometry. Different finite generating sets for a group G give quasi-isometric metrics on G (see [30, Exercise 7.82]).

An action of a group G on a metric space X is *geometric* if it is properly discontinuous, isometric, and cobounded. The Milnor-Schwarz Theorem [30, Theorem 8.37] states that if a group G acts geometrically on a proper geodesic metric space X, then G is finitely generated and for any $x \in X$, the orbit map $G \to X,\ g \mapsto gx$ is a quasi-isometry.

Let X be a closed hyperbolic manifold with fundamental group π. Applying the Milnor-Schwarz Theorem to the inclusion of an orbit gives that π is quasi-isometric to $\widetilde{X} = \mathbb{H}^n$. Since $\pi \subset \Gamma$ is finite index, Γ has a finite generating set S. An application of the Milnor-Schwarz Theorem to the π-action on the Cayley graph of (Γ, S) gives that π and Γ are quasi-isometric (see [30, Corollary 8.47 (1)]).

Thus Γ is quasi-isometric to $\mathbb{H}^n$. A theorem of Tukia (see [30, Theorem 23.1]) asserts that Γ acts geometrically on $\mathbb{H}^n$. We now have two embeddings of π as a discrete cocompact subset of $\mathrm{Isom}(\mathbb{H}^n)$, one as the restriction of the action of Γ to π and the other as deck transformations associated to the cover $\widetilde{X} \to X$. The Mostow Rigidity Theorem (see [30, Theorem 24.15]) implies that these two embeddings are conjugate by an element α of $\mathrm{Isom}(\mathbb{H}^n)$ since $n > 2$. Thus conjugating the geometric action of Γ by α gives an isometric extension of the π-action on $\widetilde{X}$ by deck transformations. Passing to the π-orbit space gives the isometric G-action on X. □

Corollary 10.3. *If X is a closed hyperbolic manifold with fundamental group π, than the answers to both the Nielsen Realization Question for any group monomorphism $\phi : G \to \mathrm{Out}(\pi)$ and the Manifold Model Question for a finite normal extension $\pi \subset \Gamma$ are yes.*

Proof. The Nielsen Realization Question is an immediate consequence of the previous theorem. So is the Manifold Model Question, since the fixed set of a subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ is the intersection of hyperbolic subspaces, hence a hyperbolic subspace, hence contractible. Hence $\mathbb{H}^n$ is a cocompact manifold model for $\underline{E}\Gamma$. □

References

[1] M. Banagl and A. Ranicki. Generalized Arf invariants in algebraic L-theory. *Adv. Math.*, 199(2):542–668, 2006.

[2] A. Bartels and W. Lück. On crossed product rings with twisted involutions, their module categories and L-theory. In *Cohomology of groups and algebraic K-theory*, volume 12 of *Adv. Lect. Math. (ALM)*, pages 1–54. Int. Press, Somerville, MA, 2010.

[3] A. Bartels and W. Lück. The Borel conjecture for hyperbolic and CAT(0)-groups. *Ann. of Math. (2)*, 175:631–689, 2012.

[4] A. C. Bartels. On the domain of the assembly map in algebraic K-theory. *Algebr. Geom. Topol.*, 3:1037–1050 (electronic), 2003.

[5] J. Block and S. Weinberger. On the generalized Nielsen realization problem. *Comment. Math. Helv.*, 83(1):21–33, 2008.

[6] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*. Springer-Verlag, Berlin, 1999. Die Grundlehren der mathematischen Wissenschaften, Band 319.

[7] K. Brown. Groups of virtually finite dimension. In *Proceedings* "Homological group theory", *editor: Wall, C.T.C., LMS Lecture Notes Series 36*, pages 27–70. Cambridge University Press, 1979.

[8] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[9] J. Bryant, S. Ferry, W. Mio, and S. Weinberger. Topology of homology manifolds. *Ann. of Math. (2)*, 143(3):435–467, 1996.

[10] B. Calmés, E. Dotto, Y. Harpaz, F. Hebestreit, M. Land, K. Moi, D. Nardin, T. Nikolaus, and W. Steimle. Hermitian K-theory for stable ∞-categories IV: Poincaré motives and Karoubi-Grothendieck-Witt groups. in preparation, 2024.

[11] S. E. Cappell. Unitary nilpotent groups and Hermitian K-theory. I. *Bull. Amer. Math. Soc.*, 80:1117–1122, 1974.

[12] G. Carlsson. On the algebraic K-theory of infinite product categories. *K-theory*, 9:305–322, 1995.

[13] G. Carlsson and E. K. Pedersen. Controlled algebra and the Novikov conjectures for K- and L-theory. *Topology*, 34(3):731–758, 1995.

[14] H. Cartan and S. Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.

[15] D. W. Carter. Lower K-theory of finite groups. *Comm. Algebra*, 8(20):1927–1937, 1980.

[16] T. A. Chapman. Compact Hilbert cube manifolds and the invariance of Whitehead torsion. *Bull. Amer. Math. Soc.*, 79:52–56, 1973.

[17] T. A. Chapman. Topological invariance of Whitehead torsion. *Amer. J. Math.*, 96:488–497, 1974.

[18] F. Connolly, J. F. Davis, and Q. Khan. Topological rigidity and H_1-negative involutions on tori. *Geom. Topol.*, 18(3):1719–1768, 2014.

[19] F. Connolly, J. F. Davis, and Q. Khan. Topological rigidity and actions on contractible manifolds with discrete singular set. *Trans. Amer. Math. Soc. Ser. B*, 2:113–133, 2015.

[20] F. Connolly and A. Ranicki. On the calculation of UNil. *Adv. Math.*, 195(1):205–258, 2005.

[21] F. X. Connolly and J. F. Davis. The surgery obstruction groups of the infinite dihedral group. *Geom. Topol.*, 8:1043–1078 (electronic), 2004.

[22] F. X. Connolly and T. Koźniewski. Nil groups in K-theory and surgery theory. *Forum Math.*, 7(1):45–76, 1995.

[23] J. F. Davis, Q. Khan, and A. Ranicki. Algebraic K-theory over the infinite dihedral group: an algebraic approach. *Algebr. Geom. Topol.*, 11(4):2391–2436, 2011.

[24] J. F. Davis and W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory*, 15(3):201–252, 1998.

[25] J. F. Davis and W. Lück. The p-chain spectral sequence. *K-Theory*, 30(1):71–104, 2003. Special issue in honor of Hyman Bass on his seventieth birthday. Part I.

[26] J. F. Davis and W. Lück. The topological K-theory of certain crystallographic groups. *Journal of Non-Commutative Geometry*, 7:373–431, 2013.

[27] J. F. Davis and R. Milgram. A survey on the space form problem. *Math. reports*, 2:223–283, 1985.

[28] J. F. Davis, F. Quinn, and H. Reich. Algebraic K-theory over the infinite dihedral group: a controlled topology approach. *J. Topol.*, 4(3):505–528, 2011.

[29] M. W. Davis and I. J. Leary. Some examples of discrete group actions on aspherical manifolds. In *High-dimensional manifold topology*, pages 139–150. World Sci. Publ., River Edge, NJ, 2003.

[30] C. Druţu and M. Kapovich. *Geometric group theory*, volume 63 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2018. With an appendix by Bogdan Nica.

[31] W. Dwyer, M. Weiss, and B. Williams. A parametrized index theorem for the algebraic K-theory Euler class. *Acta Math.*, 190(1):1–104, 2003.

[32] F. T. Farrell and L. E. Jones. Rigidity for aspherical manifolds with $\pi_1 \subset GL_m(\mathbb{R})$. *Asian J. Math.*, 2(2):215–262, 1998.

[33] F. T. Farrell, L. E. Jones, and W. Lück. A caveat on the isomorphism conjecture in L-theory. *Forum Math.*, 14(3):413–418, 2002.

[34] M. H. Freedman and F. Quinn. *Topology of 4-manifolds*. Princeton University Press, Princeton, NJ, 1990.

[35] S. P. Kerckhoff. The Nielsen realization problem. *Ann. of Math. (2)*, 117(2):235–265, 1983.

[36] Q. Khan. Rigidity of pseudo-free group actions on contractible manifolds. In *Geometry of transformation groups and combinatorics*, RIMS Kôkyûroku Bessatsu, B39, pages 45–50. Res. Inst. Math. Sci. (RIMS), Kyoto, 2013. `https://repository.kulib.kyoto-u.ac.jp/dspace/bitstream/2433/207824/1/B39_005.pdf`.

[37] R. C. Kirby and L. C. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations*. Princeton University Press, Princeton, N.J., 1977. With notes by J. Milnor and M. F. Atiyah, Annals of Mathematics Studies, No. 88.

[38] C. Kremer. Manifold models for classifying spaces. Master thesis, 2022.

[39] P. Kühl, T. Macko, and A. Mole. The total surgery obstruction revisited. *Münster J. Math.*, 6:181–269, 2013.

[40] W. Lück. Chern characters for proper equivariant homology theories and applications to K- and L-theory. *J. Reine Angew. Math.*, 543:193–234, 2002.

[41] W. Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005.

[42] W. Lück. On Brown's problem, Poincaré models for the classifying spaces for proper action and Nielsen realization. preprint, arXiv:2201.10807 [math.AT], 2022.

[43] W. Lück. Isomorphism Conjectures in K- and L-theory. in preparation, see http://www.him.uni-bonn.de/lueck/data/ic.pdf, 2024.

[44] W. Lück and H. Reich. The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory. In *Handbook of K-theory. Vol. 1, 2*, pages 703–842. Springer, Berlin, 2005.

[45] W. Lück and M. Weiermann. On the classifying space of the family of virtually cyclic subgroups. *PAMQ*, 8(2):497–555, 2012.

[46] S. Mac Lane. *Homology*. Academic Press Inc., Publishers, New York, 1963.

[47] C. Manolescu. Triangulations of manifolds. *ICCM Not.*, 2(2):21–23, 2014.

[48] A. A. Ranicki. *Exact sequences in the algebraic theory of surgery*. Princeton University Press, Princeton, N.J., 1981.

[49] A. A. Ranicki. *Algebraic L-theory and topological manifolds*. Cambridge University Press, Cambridge, 1992.

[50] A. A. Ranicki. *Lower K- and L-theory*. Cambridge University Press, Cambridge, 1992.

[51] F. Raymond and L. L. Scott. Failure of Nielsen's theorem in higher dimensions. *Arch. Math. (Basel)*, 29(6):643–654, 1977.

[52] J. Rosenberg. *Algebraic K-theory and its applications*. Springer-Verlag, New York, 1994.

[53] R. G. Swan. Periodic resolutions for finite groups. *Ann. of Math. (2)*, 72:267–291, 1960.

[54] C. T. C. Wall. *Surgery on compact manifolds*, volume 69 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 1999. Edited and with a foreword by A. A. Ranicki.

[55] S. Weinberger. *Variations on a theme of Borel: an essay on the role of the fundamental group in rigidity*, volume 213 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2023.

[56] C. Winges. A note on the L-theory of infinite product categories. *Forum Math.*, 25(4):665–676, 2013.

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