A Panorama of L^2-Invariants

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Some motivation

Theorem (Euler characteristic of amenable groups, Cheeger-Gromov)

Let G be a group which contains a normal infinite amenable subgroup. Suppose that there is a finite model for BG.

Then its Euler characteristic

$$\chi(BG) := \sum_{n \geq 0} (-1)^n \dim_{\mathbb{C}}(H_n(BG;\mathbb{C})).$$

vanishes.

Definition (Deficiency)

Let G be a finitely presented group. Define its deficiency

$$\operatorname{defi}(G) := \max\{g(P) - r(P)\}$$

where P runs over all presentations P of G and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation P.

- The deficiency is an important invariant in group theory and low-dimensional topology.
- Lower bounds can be obtained by investigating specific presentations. The hard part is to find upper bounds.
- Often the deficiency is not realized by the “obvious” presentation.

Example

- The group

$$(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$$

has the obvious presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = t_0^2 = [s_0, t_0] = s_1^3 = t_1^3 = [s_1, t_1] = 1 \rangle$$

- One may think that its deficiency is -2.

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- One may think that its deficiency is -2.
- However, it turns out that its deficiency is -1 realized by the following presentation
$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = 1, [s_0, t_0] = t_0^2, s_1^3 = 1, [s_1, t_1] = t_1^3, t_0^2 = t_1^3 \rangle.$$

Theorem (Deficiency and group extensions, Lück)

Let $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ *be an exact sequence of infinite groups. Suppose that* G *is finitely presented and* H *is finitely generated. Then:*

$$\operatorname{defi}(G) \leq 1.$$

- An important invariant of a closed oriented $4k$-dimensional manifold M is its signature

$$\operatorname{sign}(M) \in \mathbb{Z}$$

which is the signature of its intersection pairing.

- We have the relation $\operatorname{sign}(M) \equiv \chi(M) \mod 2$.

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Theorem (Signatures of 4-manifolds and group extensions, Lück)

Let M be a closed oriented 4-manifold. Suppose that $\pi_1(M)$ contains an infinite normal finitely generated subgroup of infinite index.

Then

$$|\operatorname{sign}(M)| \leq \chi(M).$$

- Let R be a ring and let G be a group.
- An element x in the group ring RG is a formal sum $\sum_{g \in G} r_g \cdot g$ such that only finitely many of the coefficients $r_g \in R$ are different from zero.
- The multiplication comes from the tautological formula $g \cdot h = g \cdot h$, more precisely from the convolution product

$$\left(\sum_{g \in G} r_g \cdot g\right) \cdot \left(\sum_{g \in G} s_g \cdot g\right) := \sum_{g \in G} \left(\sum_{h,k \in G, hk = g} r_h s_k\right) \cdot g.$$

- Group rings arise in algebra, representation theory, and topology in a natural way and are from the ring theoretic point of view very complicated rings.

Conjecture (Idempotent Conjecture, (Kaplansky))

Let G be a torsionfree group. Then all idempotents of $\mathbb{C}G$ are trivial, i.e., equal to 0 or 1.

Conjecture (Zero-divisor Conjecture, (Kaplansky))

Let G be a torsionfree group. Then $\mathbb{C}G$ has no zero-divisors.

Conjecture (Embedding Conjecture, (Kaplansky))

Let G be a torsionfree group. Then $\mathbb{C}G$ embeds into a skew-field.

- Embedding Conjecture $\implies$ Zero-divisor Conjecture $\implies$ Idempotent Conjecture.

- The notion of the dimension $\dim_{\mathcal{N}(G)}$ has several applications to algebraic K-theory. We mention one example.

Theorem (Lück-Rœrdam)

Let G be a group and $H \subseteq G$ be a normal finite subgroup. Then the canonical map

$$\mathbb{Z} \otimes_{\mathbb{Z}G} \mathrm{Wh}(H) \to \mathrm{Wh}(G)$$

is rationally injective.

Conjecture (Euler characteristic and sectional curvature, Hopf)

Let M be a closed Riemannian manifold of even dimension $2n$. Then:

- *If its sectional curvature satisfied* $\sec(M) \leq 0$*, then* $(-1)^n \cdot \chi(M) \geq 0$*;*
- *If its sectional curvature satisfied* $\sec(M) < 0$*, then* $(-1)^n \cdot \chi(M) > 0$*.*

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Theorem (S^1-actions and hyperbolic manifolds)

Any S^1-action on a hyperbolic closed manifold is trivial.

Theorem (Kähler manifolds and projective algebraic varieties, Gromov)

Let M be a closed Kähler manifold, i.e., a complex manifold which comes with a Kähler Hermitian metric and Kähler 2-form. Suppose that it admits some Riemannian metric with negative sectional curvature, or, more generally, that $\pi_1(M)$ is hyperbolic (in the sense of Gromov) and $\pi_2(M)$ is trivial.

Then M is a projective algebraic variety.

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- The point is that the proofs of the results above or of the conjectures in certain special cases do rely on L^2-methods. The use of L^2-methods made a lot of progress possible although on the first glance they seem to be unrelated to the results and conjectures mentioned above.

- Next we give a very brief introduction to the L^2-setting.

Group von Neumann algebras

- Denote by $L^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$.

Definition (Group von Neumann algebra and its trace)

- Define the group von Neumann algebra

$$\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G))^G = \overline{\mathbb{C}G}^{\text{weak}}$$

to be the algebra of bounded G-equivariant operators $L^2(G) \to L^2(G)$.

- The von Neumann trace is defined by

$$\operatorname{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$

Definition (L^2-homology and L^2-Betti numbers)

- Let X be a connected CW-complex of finite type. Let $\widetilde{X}$ be its universal covering and $\pi = \pi_1(M)$. Denote by $C_*(\widetilde{X})$ its cellular $\mathbb{Z}\pi$-chain complex.
- Define its cellular L^2-chain complex to be the Hilbert $\mathcal{N}(\pi)$-chain complex

$$C_*^{(2)}(\widetilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) = \overline{C_*(\widetilde{X})}.$$

- Define its n-th L^2-homology to be the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_n^{(2)}(\widetilde{X}) := \ker(c_n^{(2)})/\overline{\operatorname{im}(c_{n+1}^{(2)})}.$$

- Define its n-th L^2-Betti number

$$b_n^{(2)}(\widetilde{X}) := \dim_{\mathcal{N}(\pi)}\left(H_n^{(2)}(\widetilde{X})\right) \quad \in \mathbb{R}^{\geq 0}.$$

Theorem (Main properties of Betti numbers)

Let X and Y be connected CW-complexes of finite type.

- *Homotopy invariance*

  If X and Y are homotopy equivalent, then

  $$b_n(X) = b_n(Y);$$

- *Euler-Poincaré formula*

  We have

  $$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n(X);$$

- *Poincaré duality*

  Let M be a closed manifold of dimension d. Then

  $$b_n(M) = b_{d-n}(M);$$

Theorem (Main properties of L^2-Betti numbers)

Let X and Y be connected CW-complexes of finite type.

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Theorem (Continued)

- *Künneth formula*

$$b_n(X \times Y) = \sum_{p+q=n} b_p(X) \cdot b_q(Y);$$

- *Zero-th L^2-Betti number*

  We have

$$b_0(X) = 1;$$

Theorem (Continued)

- *Künneth formula*

$$b_n^{(2)}(\widetilde{X \times Y}) = \sum_{p+q=n} b_p^{(2)}(\widetilde{X}) \cdot b_q^{(2)}(\widetilde{Y});$$

- *Zero-th L^2-Betti number*

  We have

$$b_0^{(2)}(\widetilde{X}) = \frac{1}{|\pi|};$$

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- *Künneth formula*

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- *Zero-th L^2-Betti number*

  We have

$$b_0^{(2)}(\widetilde{X}) = \frac{1}{|\pi|};$$

- *Finite coverings*

  If $X \to Y$ is a finite covering with d sheets, then

$$b_n^{(2)}(\widetilde{X}) = d \cdot b_n^{(2)}(\widetilde{Y}).$$

Example ($\pi = \mathbb{Z}^d$)

- Let X be a connected CW-complex of finite type with fundamental group $\mathbb{Z}^d$.
- Let $\mathbb{C}[\mathbb{Z}^d]^{(0)}$ be the quotient field of the commutative integral domain $\mathbb{C}[\mathbb{Z}^d]$.
- Then

$$b_n^{(2)}(\widetilde{X}) = \dim_{\mathbb{C}[\mathbb{Z}^d]^{(0)}} \left(\mathbb{C}[\mathbb{Z}^d]^{(0)} \otimes_{\mathbb{Z}[\mathbb{Z}^d]} H_n(\widetilde{X})\right).$$

- Obviously this implies

$$b_n^{(2)}(\widetilde{X}) \in \mathbb{Z}.$$

Some computations and results

Theorem (S^1-actions on aspherical manifolds, Lück)

Let M be an aspherical closed manifold with non-trivial S^1-action. Then

1. *The action has no fixed points;*
2. $b_n^{(2)}(\widetilde{M}) = 0$ *for* $n \geq 0$ *and* $\chi(M) = 0$.

Theorem (Hodge - de Rham Theorem)

Let M be a closed Riemannian manifold. Put

$$\mathcal{H}^n(M) = \{\omega \in \Omega^n(M) \mid \Delta_n(\omega) = 0\}.$$

Then integration defines an isomorphism of real vector spaces

$$\mathcal{H}^n(M) \xrightarrow{\cong} H^n(M;\mathbb{R}).$$

Corollary (Betti numbers and heat kernels)

$$b_n(M) = \lim_{t\to\infty} \int_M \mathrm{tr}_{\mathbb{R}}(e^{-t\Delta_n}(x,x)) \; d\mathrm{vol}\,.$$

where $e^{-t\Delta_n}(x,y)$ is the heat kernel on M.

Theorem (L^2-Hodge - de Rham Theorem, Dodziuk)

Let M be a closed Riemannian manifold. Put

$$\mathcal{H}^n_{(2)}(\widetilde{M}) = \{\widetilde{\omega} \in \Omega^n(\widetilde{M}) \mid \widetilde{\Delta}_n(\widetilde{\omega}) = 0,\ ||\widetilde{\omega}||_{L^2} < \infty\}.$$

Then integration defines an isomorphism of finitely generated Hilbert $\mathcal{N}(\pi)$-modules

$$\mathcal{H}^n_{(2)}(\widetilde{M}) \xrightarrow{\cong} H^n_{(2)}(\widetilde{M}).$$

Corollary (L^2-Betti numbers and heat kernels)

$$b_n^{(2)}(\widetilde{M}) = \lim_{t\to\infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{R}}(e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{x}))\, dvol\,.$$

where $e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{y})$ is the heat kernel on $\widetilde{M}$ and $\mathcal{F}$ is a fundamental domain for the π-action.

Theorem (Hyperbolic manifolds, Dodziuk)

Let M be a hyperbolic closed Riemannian manifold of dimension d. Then:

$$b_n^{(2)}(\widetilde{M}) = \begin{cases} = 0 & , \text{if } 2n \neq d; \\ > 0 & , \text{if } 2n = d. \end{cases}$$

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Corollary

Let M be a hyperbolic closed manifold of dimension d. Then

1. *If $d = 2m$ is even, then*

$$(-1)^m \cdot \chi(M) > 0;$$

2. *M carries no non-trivial S^1-action.*

Theorem (3-manifolds, Lott-Lück)

Let the 3-manifold M be the connected sum $M_1 \sharp \ldots \sharp M_r$ of (compact connected orientable) prime 3-manifolds M_j. Assume that $\pi_1(M)$ is infinite. Then

$$
\begin{aligned}
b_1^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|} - \chi(M) \\
&\quad + \left|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\right|; \\
b_2^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|} \\
&\quad + \left|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\right|; \\
b_n^{(2)}(\widetilde{M}) &= 0 \quad \text{for } n \neq 1, 2.
\end{aligned}
$$

Corollary

Let M be a compact n-manifold such that $n \leq 3$ and its fundamental group is torsionfree.

Then all its L^2-Betti numbers are integers.

Theorem (Mapping tori, Lück)

Let $f \colon X \to X$ be a cellular self-homotopy equivalence of a connected CW-complex X of finite type. Let T_f be the mapping torus. Then

$$b_n^{(2)}(\widetilde{T_f}) = 0 \quad \textit{for } n \geq 0.$$

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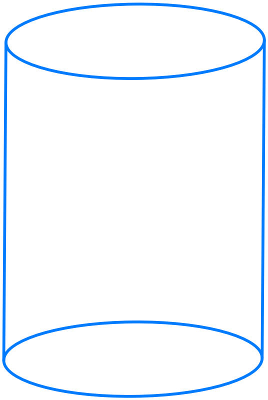
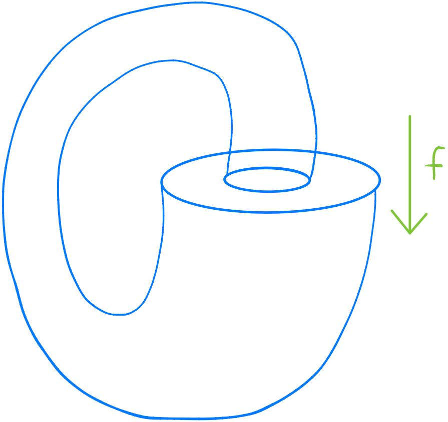
Proof.

- As $T_{f^d} \to T_f$ is up to homotopy a d-sheeted covering, we get

$$b_n^{(2)}(\widetilde{T_f}) = \frac{b_n^{(2)}(\widetilde{T_{f^d}})}{d}.$$

□





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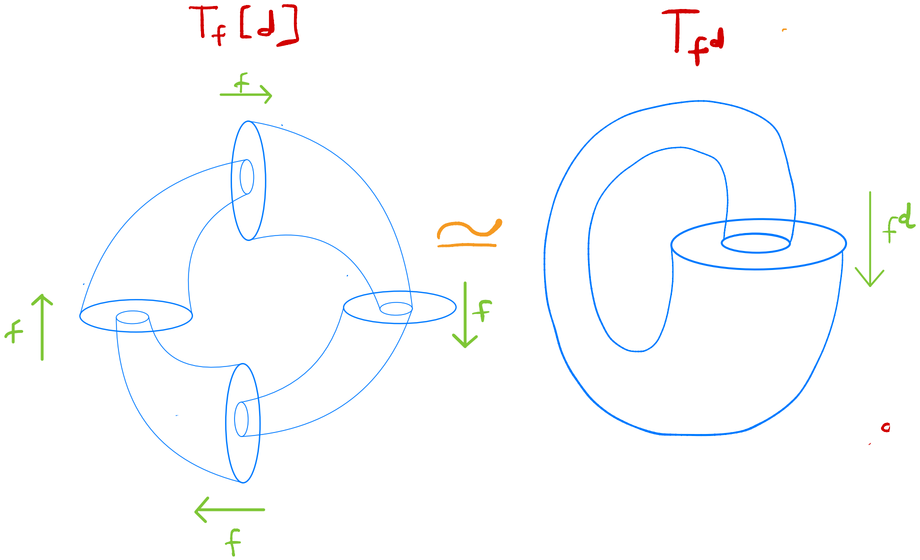

$T_f[d]$
f
f
f
f
$\simeq$
T_{f^d}
f^d

Proof continued.

- There is up to homotopy equivalence a CW-structure on T_{f^d} with $\beta_n(T_{f^d}) = \beta_n(X) + \beta_{n-1}(X)$, where $\beta_n(X)$ is the number of n-cells. We have
$$b_n^{(2)}(\widetilde{T_{f^d}}) \leq \beta_n(T_{f^d}).$$
- This implies for all $d \geq 1$
$$b_n^{(2)}(\widetilde{T_f}) \leq \frac{\beta_n(X) + \beta_{n-1}(X)}{d}.$$
- Taking the limit for $d \to \infty$ yields the claim.

□

The fundamental square and the Atiyah Conjecture

Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)

Let G be a torsionfree finitely presented group. We say that G satisfies the Atiyah Conjecture if for any closed Riemannian manifold M with $\pi_1(M) \cong G$ we have for every $n \geq 0$

$$b_n^{(2)}(\widetilde{M}) \in \mathbb{Z}.$$

- All computations presented above support the Atiyah Conjecture.

- The fundamental square is given by the following inclusions of rings

$$\begin{array}{ccc} \mathbb{C}G & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G) \end{array}$$

- $\mathcal{U}(G)$ is the algebra of affiliated operators.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{C}G$ in $\mathcal{U}(G)$,

Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- Obviously the Atiyah Conjecture implies the Embedding Conjecture and hence the Zero-divisor Conjecture and the Idempotent Conjecture due to Kaplansky.

- There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.

- However, there exist closed Riemannian manifolds whose universal coverings have an L^2-Betti number which is irrational, see Austin, Grabowski.

Theorem (Linnell, Schick)

If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.

- A group is called locally indicable if every non-trivial finitely generated subgroup admits an epimorphism onto $\mathbb{Z}$. Examples are one-relator-groups.

Theorem (Jaikin-Zapirain & Lopez-Alvarez)

If G is locally indicable, then it satisfies the Atiyah Conjecture.

Approximation

- In general there are no relations between the Betti numbers $b_n(X)$ and the L^2-Betti numbers $b_n^{(2)}(\widetilde{X})$ for a connected CW-complex X of finite type except for the Euler Poincaré formula

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{X}) = \sum_{n \geq 0} (-1)^n \cdot b_n(X).$$

Theorem (Approximation Theorem, Lück)

Let X be a connected CW-complex of finite type. Suppose that π is residually finite, i.e., there is a nested sequence

$$\pi = G_0 \supset G_1 \supset G_2 \supset \ldots$$

of normal subgroups of finite index with $\cap_{i \geq 1} G_i = \{1\}$. Let X_i be the finite $[\pi : G_i]$-sheeted covering of X associated to G_i.

Then for any such sequence $(G_i)_{i \geq 1}$

$$b_n^{(2)}(\widetilde{X}) = \lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}.$$

Applications to deficiency and signature

Lemma

Let G be a finitely presented group. Then

$$\operatorname{defi}(G) \;\le\; 1 - |G|^{-1} + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Proof.

We have to show for any presentation P

$$g(P) - r(P) \;\leq\; 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Let X be a CW-complex realizing P. Then

$$\chi(X) = 1 - g(P) + r(P) = b_0^{(2)}(\widetilde{X}) + b_1^{(2)}(\widetilde{X}) - b_2^{(2)}(\widetilde{X}).$$

Since the classifying map $X \to BG$ is 2-connected, we get

$$\begin{array}{lcll} b_n^{(2)}(\widetilde{X}) & = & b_n^{(2)}(G) & \quad \text{for } n = 0, 1; \\ b_2^{(2)}(\widetilde{X}) & \geq & b_2^{(2)}(G). & \end{array}$$

□

Theorem (Deficiency and extensions, Lück)

Let $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ *be an exact sequence of infinite groups. Suppose that* G *is finitely presented and* H *is finitely generated. Then:*

1. $b_1^{(2)}(G) = 0$;
2. $\operatorname{defi}(G) \leq 1$;
3. *Let* M *be a closed oriented* 4*-manifold with* G *as fundamental group. Then*
$$|\operatorname{sign}(M)| \leq \chi(M).$$

Conjecture (Singer Conjecture)

If M is an aspherical closed manifold, then

$$b_n^{(2)}(\widetilde{M}) = 0 \quad \textit{if } 2n \neq \dim(M).$$

If M is a closed Riemannian manifold with negative sectional curvature, then

$$b_n^{(2)}(\widetilde{M}) \begin{cases} = 0 & \textit{if } 2n \neq \dim(M); \\ > 0 & \textit{if } 2n = \dim(M). \end{cases}$$

- The computations presented above do support the Singer Conjecture.

- Under certain negative pinching conditions the Singer Conjecture has been proved by Ballmann-Brüning, Donnelly-Xavier, Jost-Xin.

- Because of the Euler-Poincaré formula

$$\chi(M) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{M})$$

the Singer Conjecture implies in the case $\dim(M) = 2n$

$$(-1)^n \cdot \chi(M) = b_n^{(2)}(\widetilde{M})$$

and hence the Hopf Conjecture.

- The Singer Conjecture gives also evidence for the Atiyah Conjecture.

Outlook

- Unfortunately, there are a lot of very interesting aspects and very deep results by many people, which we have not covered. At least we want to mention some highlights.
- Gaboriau showed that the L^2-Betti numbers are (up to scaling) invariants of the measure equivalences class.
- Using L^2-Betti numbers and Gaboriau's ideas Popa solved some prominent outstanding problems about von Neumann algebras.

- Connes-Shlyakhtenko have defined L^2-Betti numbers for finite von Neumann algebras using Hochschild homology and the generalized dimension function of Lück. If one can show that their definition applied to $\mathcal{N}(G)$ agrees with the L^2-Betti numbers of G, this would lead to a positive solution to the outstanding problem whether two finitely generated free groups are isomorphic if and only if their group von Neumann algebras are isomorphic. This is important for free probability theory.

- There is the notion of L^2-torsion due to Lück-Rothenberg in the topological and to Lott, Mathai in the analytic setting. These are the L^2-analogues of Reidemeister torsion and analytic Ray-Singer torsion. Burghelea-Friedlander-Kappeller-McDonald proved that these two notions agree.

- There is conjecture due to Bergeron-Venkatesh which is an analogue to the Approximation Theorem for L^2-Betti numbers for the L^2-torsion in terms of torsion homology growth.

- Another question is whether the Approximation for L^2-Betti numbers or L^2-torsion holds in prime characteristic. New results have recently been obtained by Avramidi-Okun-Schreve.

- L^2-torsion and Fuglede-Kadison determinants have been linked to entropy by Deninger and Li-Thom.

- Universal L^2-torsion has been defined by Friedl-Lück and related to the Thurston polytope for 3-manifolds. Applications of it to BNS-invariants have been established by Kielak.

- There is the conjecture that for an aspherical closed manifold with vanishing simplicial volume in the sense of Gromov & Thurston all its L^2-invariants vanish.