

ON THE ALGEBRAIC  $K$ -THEORY OF HECKE ALGEBRAS

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ABSTRACT. Consider a totally disconnected group  $G$ , which is covirtually cyclic, i.e., contains a normal compact open subgroup  $L$  such that  $G/L$  is infinite cyclic. We establish a Wang sequence, which computes the algebraic  $K$ -groups of the Hecke algebra of  $G$  in terms of the one of  $L$ , and show that all negative  $K$ -groups vanish. This confirms the  $K$ -theoretic Farrell-Jones Conjecture for the Hecke algebra of  $G$  in this special case. Our ultimate long term goal is to prove it for any closed subgroup of any reductive  $p$ -adic group. The results of this paper will play a role in the final proof.

## 1. INTRODUCTION

Let  $G$  be a td-group, i.e., a locally compact second countable totally disconnected topological Hausdorff group. Our ultimate goal is to compute the algebraic  $K$ -groups and in particular the projective class group of the Hecke algebra  $\mathcal{H}(G)$  of  $G$ , which is defined in terms of locally constant functions with compact support from  $G$  to the real or complex numbers and the convolution product. We want to show that the canonical map

$$(1.1) \quad \operatorname{colim}_{K \in \operatorname{Sub}_{\operatorname{cop}}(G)} K_0(\mathcal{H}(K)) \xrightarrow{\cong} K_0(\mathcal{H}(G))$$

is bijective. Here  $\operatorname{Sub}_{\operatorname{cop}}(G)$  is the following category. Objects are compact open subgroups  $K$  of  $G$ , a morphism  $f: K \rightarrow K'$  is a group homomorphism, for which there exists  $g \in G$  satisfying  $f(k) = gkg^{-1}$  for all  $k \in K$ , and we identify two such group homomorphisms  $f: K \rightarrow K'$  and  $f': K \rightarrow K'$ , if they differ by an inner automorphism of  $K'$ . In particular the obvious map

$$(1.2) \quad \bigoplus_K K_0(\mathcal{H}(K)) \rightarrow K_0(\mathcal{H}(G))$$

is surjective, where  $K$  runs through the compact open subgroups of  $G$ .

Dat [7, Theorem 1.6 and Corollary 4.22] showed following ideas of Bernstein that the map (1.2) is rationally surjective for a reductive  $p$ -adic group  $G$ . He used for the proof the Hattori-Stallings rank and input from the representation theory of reductive  $p$ -adic groups. Dat also asked the question, whether the map (1.2) is surjective without rationalizing, see the sentence after [8, Proposition 1.10] and the formulation of the weaker conjecture [8, Conjecture 1.11].

The projective class group  $K_0(\mathcal{H}(G))$  is interesting for the study of smooth  $G$ -representations, since every finitely generated smooth  $G$ -representation has a finite projective resolution and hence define elements in it, see for instance [5, Theorem 29 on page 97 and Proposition 32 on page 60], [16], [17], [18], [19].

If  $G$  is discrete, the family  $\operatorname{Cop}$  of compact open subgroups reduces to the family  $\operatorname{Fin}$  of finite subgroups of  $G$  and the bijectivity of the map (1.1) reduces to the bijectivity of the canonical map

$$(1.3) \quad \operatorname{colim}_{F \in \operatorname{Sub}_{\operatorname{fin}}(G)} K_0(\mathbb{C}F) \xrightarrow{\cong} K_0(\mathbb{C}G),$$

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which follows from the  $K$ -theoretic Farrell-Jones Conjecture for  $\mathbb{C}G$ .

Our ultimate and long term goal is to prove the version of the  $K$ -theoretic Farrell-Jones Conjecture for the Hecke algebra of  $td$ -groups for any closed subgroup  $G$  of any reductive  $p$ -adic group. It predicts the bijectivity of the assembly map

$$(1.4) \quad H_n^G(E_{\text{Cop}}(G); \mathbf{K}_{\mathcal{H}}^\infty) \xrightarrow{\cong} H_n^G(G/G; \mathbf{K}_{\mathcal{H}}^\infty) = K_n(\mathcal{H}(G))$$

for every  $n \in \mathbb{Z}$ . Here the source is a smooth  $G$ -homology theory, which digests smooth  $G$ -CW-complexes and satisfies  $H_n^G(G/H; \mathbf{K}_{\mathcal{H}}^\infty) = K_n(\mathcal{H}(H))$  for open subgroups  $H \subseteq G$ , and the smooth  $G$ -CW-complex  $E_{\text{Cop}}(G)$  is a model for the classifying space of the family of compact open subgroups, or, equivalently the classifying space for smooth proper  $G$ -actions in the realm of  $G$ -CW-complexes. This map will be constructed in [3], where a formulation of the  $K$ -theoretic Farrell-Jones Conjecture is given for Hecke categories, which generalize the notion of a Hecke algebra.

We will not prove the  $K$ -theoretic Farrell-Jones Conjecture for Hecke categories in this paper. At least we present a direct proof of it in the special case that  $G$  is covirtually infinite cyclic, i.e.,  $G$  contains a normal compact open subgroup  $L$  such that the quotient  $G/L$  is the discrete group  $\mathbb{Z}$ . Then the conjecture boils down to Theorem 9.1 which says that there is a Wang sequence, infinite to the left,

$$\begin{aligned} \dots \xrightarrow{K_2(i)} K_2(\mathcal{H}(G)) \xrightarrow{\partial_2} K_1(\mathcal{H}(L)) \xrightarrow{\text{id}-K_1(\phi)} K_1(\mathcal{H}(L)) \\ \xrightarrow{K_1(i)} K_1(\mathcal{H}(G)) \xrightarrow{\partial_1} K_0(\mathcal{H}(L)) \xrightarrow{\text{id}-K_0(\phi)} K_0(\mathcal{H}(L)) \\ \xrightarrow{K_0(i)} K_0(\mathcal{H}(G)) \rightarrow 0, \end{aligned}$$

where  $\phi: L \rightarrow L$  is the automorphism given by conjugation with some preimage of the generator of the infinite cyclic group  $G/L$  under the projection  $G \rightarrow G/L$  and  $i: L \rightarrow G$  is the inclusion, and that we have

$$K_n(\mathcal{H}(G)) = 0 \quad \text{for } n \leq -1.$$

So in this paper we can confirm the Farrell-Jones Conjecture for covirtually infinite cyclic  $td$ -groups. One may say that this paper plays the same role for the Farrell-Jones Conjecture for Hecke algebras as the papers by Farrell-Hsiang [9] and Pimsner-Voiculescu [14] did for the Farrell-Jones Conjecture for discrete groups and the Baum-Connes Conjecture. To our knowledge this paper presents the first instance of a version of the Farrell-Jones Conjecture for non-discrete groups.

One application of this paper will be that the bijectivity of (1.4) implies the bijectivity of (1.1). Moreover, Theorem 7.2 and Theorem 10.1 will be key ingredients in the part of the forthcoming proof of the Farrell-Jones Conjecture, where we will reduce the family  $\mathcal{C}_{\text{vcy}}$  of (not necessarily open) covirtually cyclic subgroups to the family  $\mathcal{C}_{\text{op}}$ .

We mention that we will look at more complicated Hecke algebras than the standard ones. We will allow other rings than  $\mathbb{R}$  or  $\mathbb{C}$ . Moreover, we take a  $G$ -action on  $R$  by ring automorphisms and a normal character, which is an obvious generalization of a central character, into account. In the sequel papers we will replace the Hecke algebras by the more general notion of a Hecke category, since allowing more general coefficients will ensure the desirable inheritance to closed subgroups of the Farrell-Jones Conjecture. This is interesting in the case of reductive  $p$ -adic groups, since important subgroups such as the Borel subgroup are in general not open.

One ingredient for the main results of this paper is the Bass-Heller-Swan decompositions for additive categories and the presentation of criteria for the vanishing of the Nil-term, see Section 6, and [2, 12]. The second is the analysis of the filtration

of the Hecke algebra of a compact td-groups in terms of approximate units, see Section 7.

### 1.A. Conventions and notations.

- A td-group is a locally compact second countable totally disconnected topological Hausdorff group;
- A subgroup is always assumed to be closed;
- A group homomorphism has closed image and is an identification onto it;
- We denote by  $R$  an associative ring, which is not necessarily commutative and not necessarily has a unit. If a ring has a unit, it is called a *unital ring*. In almost all cases we will require for a unital ring  $R$  that  $\mathbb{Q} \subseteq R$  holds, i.e., for every integer  $n \geq 1$  the element  $n \cdot 1 = 1 + 1 + \cdots + 1$  has a multiplicative inverse in  $R$ ;
- In a ring the unit is denoted by 1. In a group the unit is denoted by  $e$ ;
- For an epimorphism  $p: S \rightarrow S'$  of sets, a *transversal*  $T$  is a subset  $T \subseteq S$  such that the restriction of  $p$  to  $T$  yields a bijection  $p|_T: T \xrightarrow{\cong} S'$ . If  $S$  is a group, we always assume that the unit is in  $T$ ;

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I wish Catriona all the best for the many years to come.

## 2. HECKE ALGEBRAS

In this section we slightly generalize the notions of a Hecke algebra by implementing a normal character.

**2.A. Normal characters.** Let  $R$  be a (not necessarily commutative) associative unital ring with  $\mathbb{Q} \subseteq R$ . Let  $G$  be a td-group with a normal (not necessarily open or central) subgroup  $N \subseteq G$ . Put  $Q = G/N$ . Then we obtain an extension of td-groups  $1 \rightarrow N \rightarrow G \xrightarrow{\text{pr}} Q \rightarrow 1$ .

Consider a group homomorphism  $\rho: G \rightarrow \text{aut}(R)$ , where  $\text{aut}(R)$  is the group of automorphism of the unital ring  $R$ . We will assume throughout the paper that the kernel of  $\rho$  is open, in other words,  $G$  acts smoothly on  $R$ .

We write  $gr = \rho(g)(r)$  for  $g \in G$  and  $r \in R$ . With this notation we get  $er = r$ ,  $g1 = 1$ ,  $(g_1g_2)r = g_1(g_2r)$ ,  $g(r_1r_2) = (gr_1)(gr_2)$  and  $g(r_1 + r_2) = gr_1 + gr_2$  for  $g, g_1, g_2 \in G$ ,  $r, r_1, r_2 \in R$ , and the units  $e \in G$  and  $1 \in R$ .

A *normal character* is a locally constant group homomorphism

$$\omega: N \rightarrow \text{cent}(R)^\times$$

to the multiplicative group of central units of  $R$  satisfying

$$(2.1) \quad \omega(gng^{-1}) = \omega(n)$$

for all  $n \in N$  and  $g \in G$ . Note that  $\ker(\omega)$  is an open subgroup of  $N$  and a normal subgroup of  $G$ . We will need the following compatibility condition between the normal character and the  $G$ -action  $\rho$  on  $R$ , namely for  $n \in N$ ,  $g \in G$ , and  $r \in R$

$$(2.2) \quad g\omega(n) = \omega(n);$$

$$(2.3) \quad n \cdot r = r.$$

**2.B. The construction of the Hecke algebra.** Let  $\mu$  be a  $\mathbb{Q}$ -valued Haar measure on  $Q$ , i.e., a Haar measure  $\mu$  on  $G$  such that for any compact open subgroup  $K \subseteq Q$  we have  $\mu(K) \in \mathbb{Q}^{>0}$ . Given any Haar measure  $\mu$  on  $G$ , we can normalize it to a  $\mathbb{Q}$ -valued Haar measure by choosing a compact open subgroup  $L_0 \subseteq G$  and defining  $\mu' = \frac{1}{\mu(L_0)} \cdot \mu$ .

An element  $s$  in the Hecke algebra  $\mathcal{H}(G; R, \rho, \omega)_\mu$  is given by a map  $s: G \rightarrow R$  with the following properties

- The map  $s: G \rightarrow R$  is locally constant;
- The image of its support  $\text{supp}(s) := \{g \in G \mid s(g) \neq 0\} \subseteq G$  under  $\text{pr}: G \rightarrow Q$  is a compact subset of  $Q$ ;
- For  $n \in N$  and  $g \in G$  we have

$$(2.4) \quad s(ng) = \omega(n) \cdot s(g);$$

$$(2.5) \quad s(gn) = s(g) \cdot \omega(n).$$

**Definition 2.6.** Let  $P_{\rho, \omega}$  the subset of compact open subgroups  $K \subseteq G$  satisfying

$$(2.7) \quad kr = r \quad \text{for } k \in K, r \in R.$$

$$(2.8) \quad \omega(n) = 1 \quad \text{for } n \in N \cap K;$$

We abbreviate  $P = P_{\rho, \omega}$  if  $\rho$  and  $\omega$  are clear from the context.

We call an element  $K \in P$  *admissible* for  $s: G \rightarrow R$ , if for all  $g \in G$  and  $k \in K$  we have

$$(2.9) \quad s(kg) = s(g);$$

$$(2.10) \quad s(gk) = s(g).$$

Note that the existence of an admissible element  $K \in P$  is equivalent to the condition that  $s$  is locally constant, since we assume that  $s$  has compact support. Moreover, for  $K \in P$ , which is admissible for  $s$ , every open subgroup  $K' \subseteq K$  is also admissible.

**Remark 2.11 (Redundancy).** Note that condition (2.5) follows from conditions (2.1) and (2.4) by the following calculation

$$s(gn) = s(gng^{-1}g) \stackrel{(2.4)}{=} \omega(gng^{-1}) \cdot s(g) \stackrel{(2.1)}{=} \omega(n) \cdot s(g) \stackrel{\omega(n) \in \text{cent}(R)}{=} s(g) \cdot \omega(n).$$

Analogously condition (2.4) follows from conditions (2.1) and (2.5).

The sum of two elements  $s, s'$  in  $\mathcal{H}(G; R, \rho, \omega)_\mu$  is defined by

$$(2.12) \quad (s + s')(g) := s(g) + s'(g) \quad \text{for } g \in G.$$

Consider  $K \in P$  which is admissible for  $s$  and admissible for  $s'$ , and a transversal  $T$  for the projection  $p: G \rightarrow G/NK$ , where  $NK$  is the subgroup of  $G$  given by  $\{nk \mid n \in N, k \in K\}$ . Define the product  $s \cdot s'$  by

$$(2.13) \quad (s \cdot s')(g) := \mu(\text{pr}(K)) \cdot \sum_{g' \in T} s(gg') \cdot gg's'(g'^{-1}).$$

Note that  $K$  may depend on  $s$ , but not on  $g$ , whereas  $T$  can depend on both  $s$  and  $g$ . The independence of the transversal follows from the following computation for  $g \in G, g' \in G', n \in N$  and  $k \in K$

$$\begin{aligned} s(g(g'nk)) \cdot g(g'nk)s'((g'nk)^{-1}) &= s((gg'n)k) \cdot (gg'n)ks'(k^{-1}n^{-1}g'^{-1}) \\ &\stackrel{(2.9), (2.10)}{=} s(gg'n) \cdot (gg'n)ks'(n^{-1}g'^{-1}) \stackrel{(2.7)}{=} s(gg'n) \cdot gg'ns'(n^{-1}g'^{-1}) \\ &\stackrel{(2.4), (2.5)}{=} s(gg') \cdot \omega(n) \cdot gg'n(\omega(n^{-1}) \cdot s'(g'^{-1})) = s(gg') \cdot \omega(n) \cdot gg'n\omega(n^{-1}) \cdot gg'ns'(g'^{-1}) \\ &\stackrel{(2.2), (2.3)}{=} s(gg') \cdot \omega(n) \cdot \omega(n^{-1}) \cdot gg's'(g'^{-1}) = s(gg') \cdot \omega(n \cdot n^{-1}) \cdot gg's'(g'^{-1}) \\ &= s(gg') \cdot \omega(e) \cdot gg's'(g'^{-1}) = s(gg') \cdot gg's'(g'^{-1}). \end{aligned}$$

We leave the elementary proof to the reader that the definition of the product (2.13) is independent of the choice of  $K$  and that we do get the structure of a (non-unital) ring on  $\mathcal{H}(G; R, \rho, \omega)_\mu$ . A more general setting including all proofs will be presented in details in [3]. Moreover, one easily checks

**Lemma 2.14.** *Consider two elements  $s, s' \in \mathcal{H}(G; R, \rho, \omega)_\mu$  and compact open subgroups  $K, K'$  of  $G$ . Suppose that  $K$  admissible for  $s$  and  $K'$  is admissible for  $s'$ .*

*Then  $K \cap K'$  is admissible for the product  $s' \cdot s$ .*

**2.c. Functoriality in  $Q$ .** Let  $G, N, Q, R, \rho, \omega$ , and  $\mu$  be as in Subsection 2.A. In particular we can consider the Hecke algebra  $\mathcal{H}(G; R, \rho, \omega)_\mu$  see Subsection 2.B.

Consider a (not necessarily injective or surjective) open group homomorphism  $\phi: G' \rightarrow G$  of td-groups. Let  $N' \subseteq G'$  be a normal subgroup satisfying

$$(2.15) \quad \phi(N') = N.$$

Denote by  $\text{pr}' : G' \rightarrow Q' := G'/N'$  the projection. Let  $\bar{\phi} : Q' \rightarrow Q$  be the open group homomorphism induced by  $\phi$ . Define a group homomorphism  $\rho' : G' \rightarrow \text{aut}(R)$  and a normal character  $\omega' : N' \rightarrow \text{cent}(R)^\times$  by

$$(2.16) \quad \rho' = \rho \circ \phi;$$

$$(2.17) \quad \omega'(n') = \omega(\phi(n')) \quad \text{for } n' \in N'.$$

Choose a  $\mathbb{Q}$ -valued Haar measure on  $\mu'$  on  $Q'$ . Then we can consider the Hecke algebra  $\mathcal{H}(G'; R, \rho', \omega')_{\mu'}$ . Next we want to construct a homomorphism of rings

$$(2.18) \quad \phi_* : \mathcal{H}(G'; R, \rho', \omega')_{\mu'} \rightarrow \mathcal{H}(G; R, \rho, \omega)_\mu.$$

Consider an element  $s' : G' \rightarrow R$  in  $\mathcal{H}(G'; R, \rho', \omega')_{\mu'}$ . Choose  $K' \in P_{\rho', \omega'}$ , which is admissible for  $s'$ . Then  $\phi(K') \in P_{\rho, \omega}$ . Fix  $g \in G$ . Consider  $g' \in \phi^{-1}(g\phi(N'K'))$ . Then  $\phi(g')^{-1}g$  belongs to  $\phi(N'K')$ . Choose  $n' \in N'$  and  $k' \in K'$  with  $\phi(g'n'k') = g$ . Put

$$(2.19) \quad \tilde{s}'(g', g) := s'(g') \cdot \omega(\phi(n')) \in R.$$

One easily checks that this definition independent of the choice of  $n' \in N'$  and  $k' \in K'$ . Obviously we have  $\tilde{s}'(g', \phi(g')) = s'(g')$  for  $g' \in G'$ . Choose a transversal  $T'$  of the projection  $G' \rightarrow G'/N'K'$ , which is allowed to depend on  $s'$ . Put  $T'(g) = T' \cap \phi^{-1}(g\phi(N'K'))$ . Then we define

$$(2.20) \quad \phi_*(s')(g) = \frac{\mu'(\text{pr}'(K'))}{\mu(\text{pr}(\phi(K')))} \cdot \sum_{g' \in T'(g)} \tilde{s}'(g', g).$$

This is a well-defined element in  $\mathcal{H}(G; R, \rho, \omega)_\mu$ , which is independent of the choice of  $T$  and  $K'$ . One easily checks

**Lemma 2.21.**

- (i) We have  $\text{supp}(\phi_*(s')) \subseteq \phi(\text{supp}(s'))$ ;
- (ii) If  $K' \in P'_{\rho', \omega'}$  is admissible for  $s'$ , then  $\phi(K')$  is admissible for  $\phi_*(s')$ ;
- (iii) Suppose that  $\phi$  is injective. Then we get

$$\phi_*(s')(g) = \begin{cases} \frac{\mu'(\text{pr}'(K'))}{\mu(\text{pr}(\phi(K')))} \cdot s(g') & \text{if } \phi(g') = g \text{ for some } g' \in G' \\ 0 & g' \notin \text{im}(\phi). \end{cases}$$

and

$$\text{supp}_G(\phi_*(s')) = \phi(\text{supp}_{G'}(s'));$$

- (iv) The map  $\phi_*: \mathcal{H}(G'; R, \rho', \omega')_{\mu'} \rightarrow \mathcal{H}(G; R, \rho, \omega)_\mu$  is a homomorphism of (non-unital) rings.

**2.D. Approximate units.**

**Definition 2.22** (Rings with approximate units). An *approximate unit* for a ring  $R$  is a subset  $\{e_i \mid i \in I\}$  of elements  $e_i \in R$  indexed by some directed set  $I$  such that  $e_i \cdot e_j = e_i = e_j \cdot e_i$  holds for  $i \leq j$  and for every element  $r \in R$  there exists an index  $i \in I$  with  $e_i \cdot r = r = r \cdot e_i$ .

The ring  $R$  has an approximate unit, if and only if there is a directed system of subrings  $\{R_i \mid i \in I\}$  indexed by inclusion such that each  $R_i$  is unital and  $R = \bigcup_{i \in I} R_i$ . Obviously a unital ring has an approximate unit.

Note that the ring  $\mathcal{H}(G; R, \rho, \omega)_\mu$  has a unit, if and only if  $G$  is discrete. If  $G$  is not discrete,  $\mathcal{H}(G; R, \rho, \omega)_\mu$  has at least an approximate unit by the following construction.

Lemma 2.14 implies for  $K \in P$  that the subset

$$(2.23) \quad \mathcal{H}(G//K; R, \rho, \omega)_\mu \subseteq \mathcal{H}(G; R, \rho, \omega)_\mu$$

consisting of those elements, for which  $K$  is admissible, is closed under addition and multiplication and hence is a subring. Define an element  $1_K$  in  $\mathcal{H}(G//K; R, \rho, \omega)_\mu$  by

$$(2.24) \quad 1_K(g) = \begin{cases} \frac{1}{\mu(\text{pr}(K))} \cdot \omega(n) & \text{if } g \in NK, g = nk \text{ for } n \in N, k \in K; \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 2.25.** *The element  $1_K$  is a unit in  $\mathcal{H}(G//K; R, \rho, \omega)_\mu$ . Moreover*

$$\begin{aligned} \mathcal{H}(G//K; R, \rho, \omega)_\mu &\subseteq \mathcal{H}(G//K'; R, \rho, \omega)_\mu \quad \text{if } K' \subseteq K; \\ \mathcal{H}(G; R, \rho, \omega)_\mu &= \bigcup_K \mathcal{H}(G//K; R, \rho, \omega)_\mu, \end{aligned}$$

where  $K$  and  $K'$  run through the elements of  $P$ .

**2.E. Discarding  $\mu$ .** In the sequel we omit the subscript  $\mu$  in the notation of the Hecke algebra, since for two  $\mathbb{Q}$ -valued Haar measures  $\mu$  and  $\mu'$  on  $G/N$  there is precisely one rational number  $r$  satisfying  $r > 0$  and  $\mu' = r \cdot \mu$ , and the map

$$\mathcal{H}(G; R, \rho, \omega)_{\mu'} \xrightarrow{\cong} \mathcal{H}(G; R, \rho, \omega)_\mu, \quad s \mapsto r \cdot s.$$

is an isomorphism of rings.

3.  $\mathbb{Z}$ -CATEGORIES, ADDITIVE CATEGORIES AND IDEMPOTENT COMPLETIONS

A  $\mathbb{Z}$ -category is a category  $\mathcal{A}$  such that for every two objects  $A$  and  $A'$  in  $\mathcal{A}$  the set of morphisms  $\text{mor}_{\mathcal{A}}(A, A')$  has the structure of a  $\mathbb{Z}$ -module and composition is  $\mathbb{Z}$ -bilinear. If  $G$  is a group, a  $G$ - $\mathbb{Z}$ -category is a  $\mathbb{Z}$ -category with a left  $G$ -action by automorphisms of  $\mathbb{Z}$ -categories. Note that we do not require that  $\mathcal{A}$  has identity morphisms. Given a ring  $R$ , we denote by  $\underline{R}$  the  $\mathbb{Z}$ -category with precisely one object, whose  $\mathbb{Z}$ -module of endomorphisms is given by  $R$  with its additive structure and composition is given by the multiplication in  $R$ . Obviously  $\underline{R}$  is unital, if and only if  $R$  is unital.

An *additive category* is a  $\mathbb{Z}$ -category with finite direct sums. Given a ring  $R$ , the category  $R\text{-MOD}_{\text{fgf}}$  of finitely generated free  $R$ -modules carries an obvious structure of an additive category. Note that we do not require that  $\mathcal{A}$  has identity morphisms. If it does, we call it unital.

Given a  $\mathbb{Z}$ -category  $\mathcal{A}$ , let  $\mathcal{A}_{\oplus}$  be the associated additive category, whose objects are finite tuples of objects in  $\mathcal{A}$  and whose morphisms are given by matrices of morphisms in  $\mathcal{A}$  (of the right size) and the direct sum is given by concatenation of tuples and the block sum of matrices, see for instance [12, Section 1.3]. If  $\mathcal{A}$  is unital,  $\mathcal{A}_{\oplus}$  is unital.

Let  $R$  be a unital ring. Then the obvious inclusion of unital additive categories

$$(3.1) \quad \underline{R}_{\oplus} \xrightarrow{\cong} R\text{-MOD}_{\text{fgf}}$$

is an equivalence of unital additive categories.

Given an additive category  $\mathcal{A}$ , its *idempotent completion*  $\text{Idem}(\mathcal{A})$  is defined to be the following additive category. Objects are morphisms  $p: A \rightarrow A$  in  $\mathcal{A}$  satisfying  $p \circ p = p$ . A morphism  $f$  from  $p_1: A_1 \rightarrow A_1$  to  $p_2: A_2 \rightarrow A_2$  is a morphism  $f: A_1 \rightarrow A_2$  in  $\mathcal{A}$  satisfying  $p_2 \circ f \circ p_1 = f$ . Note that  $\text{Idem}(\mathcal{A})$  is always unital, regardless whether  $\mathcal{A}$  is unital or not. The identity of an object  $(A, p)$  is given by the morphism  $p: (A, p) \rightarrow (A, p)$ .

If  $\mathcal{A}$  is unital, then there is a obvious embedding

$$\eta(\mathcal{A}): \mathcal{A} \rightarrow \text{Idem}(\mathcal{A})$$

sending an object  $A$  to  $\text{id}_A: A \rightarrow A$  and a morphism  $f: A \rightarrow B$  to the morphism given by  $f$  again. A unital additive category  $\mathcal{A}$  is called *idempotent complete*, if  $\eta(\mathcal{A}): \mathcal{A} \rightarrow \text{Idem}(\mathcal{A})$  is an equivalence of unital additive categories, or, equivalently, if for every idempotent  $p: A \rightarrow A$  in  $\mathcal{A}$  there are objects  $B$  and  $C$  and an isomorphism  $f: A \xrightarrow{\cong} B \oplus C$  in  $\mathcal{A}$  such that  $f \circ p \circ f^{-1}: B \oplus C \rightarrow B \oplus C$  is given by  $\begin{pmatrix} \text{id}_B & 0 \\ 0 & 0 \end{pmatrix}$ . The idempotent completion  $\text{Idem}(\mathcal{A})$  of a unital additive category  $\mathcal{A}$  is idempotent complete.

Let  $R$  be unital ring. Let  $R\text{-MOD}_{\text{fgp}}$  be the unital additive category of finitely generated projective  $R$ -modules. We obtain an equivalence of unital additive categories  $\text{Idem}(R\text{-MOD}_{\text{fgf}}) \xrightarrow{\cong} R\text{-MOD}_{\text{fgp}}$  by sending an object  $(F, p)$  to  $\text{im}(p)$ . It and the functor of (3.1) induce an equivalence of unital additive categories

$$(3.2) \quad \theta_R: \text{Idem}(\underline{R}_{\oplus}) \xrightarrow{\cong} R\text{-MOD}_{\text{fgp}}.$$

Let  $\mathcal{A}$  be an additive category. Let  $\Phi: \mathcal{A} \rightarrow \mathcal{A}$  be an automorphism of additive categories. Define the the additive category  $\mathcal{A}_{\Phi}[t, t^{-1}]$  called  $\Phi$ -*twisted finite Laurent category* as follows. It has the same objects as  $\mathcal{A}$ . Given two objects  $A$  and  $B$ , a morphism  $f: A \rightarrow B$  in  $\mathcal{A}_{\Phi}[t, t^{-1}]$  is a formal sum  $f = \sum_{i \in \mathbb{Z}} f_i \cdot t^i$ , where  $f_i: \Phi^i(A) \rightarrow B$  is a morphism in  $\mathcal{A}$  from  $\Phi^i(A)$  to  $B$  and only finitely many of the morphisms  $f_i$  are non-trivial. If  $g = \sum_{j \in \mathbb{Z}} g_j \cdot t^j$  is a morphism in  $\mathcal{A}_{\Phi}[t, t^{-1}]$  from

$B$  to  $C$ , we define the composite  $g \circ f: A \rightarrow C$  by

$$g \circ f := \sum_{k \in \mathbb{Z}} \left( \sum_{\substack{i, j \in \mathbb{Z}, \\ i+j=k}} g_j \circ \Phi^j(f_i) \right) \cdot t^k.$$

If  $\mathcal{A}$  is unital, then  $\mathcal{A}_\Phi[t, t^{-1}]$  is unital again.

Let  $R$  be a (not necessarily unital) ring with an automorphism  $\phi: R \xrightarrow{\cong} R$  of rings. Let  $R_\phi[t, t^{-1}]$  be the ring of  $\phi$ -twisted finite Laurent series with coefficients in  $R$ . We obtain from  $\phi$  an automorphism  $\Phi: \underline{R} \xrightarrow{\cong} \underline{R}$  of  $\mathbb{Z}$ -categories. There is an obvious isomorphism of  $\mathbb{Z}$ -categories

$$(3.3) \quad \underline{R}_\Phi[t, t^{-1}] \xrightarrow{\cong} \underline{R}_\phi[t, t^{-1}].$$

If  $R$  is unital, then we obtain equivalences of unital additive categories

$$(3.4) \quad \begin{aligned} (\underline{R}_\oplus)_\Phi[t, t^{-1}] &\xrightarrow{\cong} R_\phi[t, t^{-1}]\text{-MOD}_{\text{fgf}}; \\ \text{Idem}((\underline{R}_\oplus)_\Phi[t, t^{-1}]) &\xrightarrow{\cong} R_\phi[t, t^{-1}]\text{-MOD}_{\text{fgp}}. \end{aligned}$$

#### 4. THE ALGEBRAIC $K$ -THEORY OF $\mathbb{Z}$ -CATEGORIES

Let  $\mathcal{A}$  be a unital additive category. A construction of the *non-connective  $K$ -theory spectrum*  $\mathbf{K}^\infty(\mathcal{A})$  of a unital additive category can be found for instance in [10] or [13]. We get from the canonical embedding  $\eta(\mathcal{A}): \mathcal{A} \rightarrow \text{Idem}(\mathcal{A})$  a weak homotopy equivalence  $\mathbf{K}^\infty(\eta(\mathcal{A})): \mathbf{K}^\infty(\mathcal{A}) \rightarrow \mathbf{K}^\infty(\text{Idem}(\mathcal{A}))$  on the non-connective  $K$ -theory, see for instance [2, Lemma 3.3 (ii)].

**Definition 4.1** (Algebraic  $K$ -theory of (not necessarily unital)  $\mathbb{Z}$ -categories). We will define the *algebraic  $K$ -theory spectrum*  $\mathbf{K}^\infty(\mathcal{A})$  of the (not necessarily unital)  $\mathbb{Z}$ -category  $\mathcal{A}$  to be the non-connective algebraic  $K$ -theory spectrum of the unital additive category  $\text{Idem}(\mathcal{A}_\oplus)$ . Define for  $n \in \mathbb{Z}$

$$K_n(\mathcal{A}) := \pi_n(\mathbf{K}^\infty(\mathcal{A})).$$

Note that Definition 4.1 extends the definition of the non-connective  $K$ -theory spectrum of unital additive categories to not necessarily unital  $\mathbb{Z}$ -categories.

A functor  $F: \mathcal{A} \rightarrow \mathcal{A}'$  of (not necessarily unital)  $\mathbb{Z}$ -categories induces a map of spectra

$$(4.2) \quad \mathbf{K}^\infty(F): \mathbf{K}^\infty(\mathcal{A}) \rightarrow \mathbf{K}^\infty(\mathcal{A}').$$

If the (not necessarily unital)  $\mathbb{Z}$ -category  $\mathcal{A}$  is the directed union of (not necessarily unital)  $\mathbb{Z}$ -subcategories  $\mathcal{A}_i$ , then the canonical map

$$(4.3) \quad \text{hocolim}_{i \in I} \mathbf{K}^\infty(\mathcal{A}_i) \xrightarrow{\cong} \mathbf{K}^\infty(\mathcal{A})$$

is a weak homotopy equivalence and for every  $n \in \mathbb{Z}$  the canonical map

$$(4.4) \quad \text{colim}_{i \in I} K_n(\mathcal{A}_i) \xrightarrow{\cong} K_n(\mathcal{A})$$

is a bijection. We conclude (4.3) and (4.4) for instance from [10, Corollary 7.2].

If  $R$  is an associative ring (not necessarily with a unit), we define the non-connective  $K$ -theory spectrum  $\mathbf{K}^\infty(R)$  to be  $\mathbf{K}^\infty(\underline{R})$  and  $K_n(R) := \pi_n(\mathbf{K}^\infty(R))$  for  $n \in \mathbb{Z}$ . If  $R$  has an approximate unit, then our definition of  $K_n(R)$  agrees with the usual definition of  $K_n(R)$  for a ring without unit by the kernel of the map  $K_n(R_+) \rightarrow K_n(\mathbb{Z})$ , where  $R_+$  is the ring with unit associated to  $R$ . Because of Lemma 2.25 this applies to the Hecke algebra  $\mathcal{H}(G; R, \rho, \omega)$ .

5. COVIRTUALLY  $\mathbb{Z}$  GROUPS

Let  $G, N, Q, R, \rho, P, \omega$ , and  $\mu$  as in Subsection 2.A. In particular we can consider the Hecke algebra  $\mathcal{H}(G; R, \rho, \omega)$ , see Subsection 2.B. Assume furthermore, that we have a normal open subgroup  $L \subseteq G$  satisfying:

- $G/L$  is isomorphic to  $\mathbb{Z}$ ;
- $N \subseteq L$ ;
- $M := L/N$  is compact;

Note that we get exact sequences of td-groups  $1 \rightarrow L \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$  and  $1 \rightarrow M \rightarrow Q \rightarrow \mathbb{Z} \rightarrow 1$ , where  $\mathbb{Z}$  is considered as discrete group and  $M$  is compact.

Let  $g_0 \in G$  be any element which represents in  $G/L$  a generator. Let  $\phi: L \rightarrow L$  be the automorphism of  $L$  given by conjugation with  $g_0$ . Denote by  $L \rtimes_{c_{g_0}} \mathbb{Z}$  the td-group given by the semi-direct product of  $L$  with the discrete group  $\mathbb{Z}$  with respect to  $c_{g_0}$ . Then we get an isomorphism of td-groups

$$\alpha: L \rtimes_{c_{g_0}} \mathbb{Z} \xrightarrow{\cong} G; \quad lt^n \mapsto lg_0^n,$$

if  $t \in \mathbb{Z}$  is a fixed generator. It induces also an isomorphism  $\beta: M \rtimes_{c_{g_0}} \mathbb{Z} \xrightarrow{\cong} Q$ , if we put  $q_0 = \text{pr}(g_0)$ . In the sequel we identify  $G = L \rtimes_{c_{g_0N}} \mathbb{Z}$  and  $g_0$  with  $e_L t$  for  $e_L \in L$  the unit and  $Q = M \rtimes_{c_{g_0}} \mathbb{Z}$  and  $g_0 N$  with  $e_Q t$  for  $e_Q \in Q$  the unit.

Since  $L \subseteq G$  is open, the  $\mathbb{Q}$ -valued measure  $\mu$  on  $G$  defines a  $\mathbb{Q}$ -valued measure on  $L$  by restriction, which we will denote by  $\mu$  again. Note that we can consider the Hecke algebra  $\mathcal{H}(L; R, \rho|_L, \omega)$ .

Next we check that the automorphism  $c_{g_0}: L \rightarrow L$  induces an automorphism of rings

$$(5.1) \quad \phi: \mathcal{H}(L; R, \rho|_L, \omega) \xrightarrow{\cong} \mathcal{H}(L; R, \rho|_L, \omega)$$

by sending  $s \in \mathcal{H}(L; R, \rho|_L, \omega)$  given by a function  $s: L \rightarrow R$  to the element given by the function  $\phi(s): L \rightarrow R, l \mapsto ts(t^{-1}lt)$ . Note that this is not just (2.18) applied to  $c_{g_0}$ , condition (2.16)  $c_{g_0}$  is not satisfied for  $c_{g_0}$ . So we have to check that  $\phi(s)$  defines an element in  $\mathcal{H}(L; R, \rho|_L, \omega)$ .

Obviously the image of the support of  $\phi(s)$  under  $L \rightarrow L/N$  is compact, since this is true for  $\text{supp}(s)$  and  $\text{supp}(\phi(s)) = t \text{supp}(s) t^{-1}$ .

Suppose that  $K \in P$  is admissible for  $s$ . Then  $tKt^{-1}$  is admissible for  $\phi(s)$  by the following calculation for  $l \in L$  and  $k' \in tKt^{-1}$ , if we write  $k' = tkt^{-1}$  for  $k \in K$

$$\phi(s)(k'l) = ts(t^{-1}k'lt) = ts(t^{-1}tkt^{-1}lt) = ts(kt^{-1}lt) \stackrel{(2.9)}{=} ts(t^{-1}lt) = \phi(s)(l),$$

and

$$\phi(s)(lk') = ts(t^{-1}lk't) = ts(t^{-1}lkt^{-1}t) = ts(t^{-1}ltk) \stackrel{(2.10)}{=} ts(t^{-1}lt) = \phi(s)(l).$$

The following calculation shows that condition (2.4) is satisfied.

$$\begin{aligned} \phi(s)(nl) &= ts(t^{-1}nlt) = ts(t^{-1}ntt^{-1}lt) \stackrel{(2.4)}{=} t(\omega(t^{-1}nt) \cdot s(t^{-1}lt)) \\ &= t\omega(t^{-1}nt) \cdot ts(t^{-1}lt) \stackrel{(2.1), (2.2)}{=} \omega(n) \cdot ts(t^{-1}lt) = \omega(n) \cdot \phi(s)(l). \end{aligned}$$

Recall that the condition (2.5) holds automatically, see Remark 2.11. Hence  $\phi$  is well-defined.

It is obviously compatible with the addition. It is compatible with the multiplication by the following calculation for two elements  $s, s' \in \mathcal{H}(L; R, \rho|_L, \omega)$  and  $l \in L$ , where  $K \in P$  is admissible for both  $s$  and  $s'$ , and  $T$  is a transversal for the projection  $L \rightarrow L/NK$ , and  $\text{pr}: L \rightarrow M = L/N$  is the projection. We will use the

fact that  $tTt^{-1}$  is a transversal for the projection  $L \rightarrow L/NtKt^{-1}$  and  $tKt^{-1}$  is admissible for  $\phi(s)$  and  $\phi(s')$ . Moreover, we have

$$(5.2) \quad [M : \text{pr}(K)] = [tMt^{-1} : t \text{pr}(K)t^{-1}] = [M : \text{pr}(tKt^{-1})].$$

We compute

$$\begin{aligned} \phi(s \cdot s')(l) &= t(s \cdot s')(t^{-1}lt) \\ &\stackrel{(2.13)}{=} t \left( \mu(\text{pr}(K)) \cdot \sum_{g' \in T} s(t^{-1}ltg') \cdot t^{-1}ltg's'(g'^{-1}) \right) \\ &= \mu(\text{pr}(K)) \cdot \sum_{g' \in T} ts(t^{-1}ltg') \cdot ltg's'(g'^{-1}) \\ &= \frac{\mu(M)}{[M : \text{pr}(K)]} \cdot \sum_{g' \in T} ts(t^{-1}ltg't^{-1}t) \cdot ltg't^{-1}ts'(t^{-1}tg'^{-1}t^{-1}t) \\ &\stackrel{(5.2)}{=} \frac{\mu(M)}{[M : \text{pr}(tKt^{-1})]} \cdot \sum_{g'' \in tTt^{-1}} ts(t^{-1}lg''t) \cdot lg''ts'(t^{-1}g''^{-1}t) \\ &= \mu(\text{pr}(tKt^{-1})) \cdot \sum_{g'' \in tTt^{-1}} \phi(s)(lg'') \cdot lg''\phi(s')(g''^{-1}) \\ &\stackrel{(2.13)}{=} (\phi(s) \cdot \phi(s'))(l). \end{aligned}$$

**Lemma 5.3.** *There is a natural isomorphism of (non-unital) rings*

$$\Xi: \mathcal{H}(L; R, \rho|_L, \omega)_\phi[t, t^{-1}] \xrightarrow{\cong} \mathcal{H}(G; R, \rho, \omega).$$

*Proof.* Consider an element  $s \in \mathcal{H}(L; R, \rho|_L, \omega)$  and an element  $n \in \mathbb{Z}$ . Then  $\Xi(st^n)$  is defined to be the element in  $\mathcal{H}(G; R, \rho, \omega)$  given by

$$(5.4) \quad G \rightarrow R, \quad (lt^m) \mapsto \begin{cases} s(l) & \text{if } m = n; \\ 0 & \text{otherwise.} \end{cases}$$

Obviously the image of the support of  $\Xi(st^n)$  under  $\text{pr}: G \rightarrow Q$  is compact, as it is a closed subset of  $t^nM$  and  $M \subseteq Q$  is compact. Suppose that the compact open subgroup  $K \subseteq L$  is admissible for  $s$ . Then  $K \cap t^{-n}Kt^n \subseteq L \subseteq G$  is admissible for  $\Xi(st^n)$  by the following calculation for  $l \in L$  and  $k \in K \cap t^{-n}Kt^n$

$$\Xi(st^n)(klt^n) \stackrel{(5.4)}{=} s(kl) \stackrel{(2.9)}{=} s(l) \stackrel{(5.4)}{=} \Xi(st^n)(lt^n),$$

and

$$\Xi(st^n)(lt^n k) = \Xi(st^n)(lt^n kt^{-n}t^n) \stackrel{(5.4)}{=} s(lt^n kt^{-n}) \stackrel{(2.10)}{=} s(l) \stackrel{(5.4)}{=} \Xi(st^n)(lt^n)$$

and the observation that we have  $\Xi(st^n)(lt^m k) = \Xi(st^n)(klt^m) = \Xi(st^n)(lt^m) = 0$  for  $m \in \mathbb{Z}$  with  $m \neq n$ . Next we verify condition (2.4). We get for  $z \in N$  and  $m \in \mathbb{Z}$  with  $m \neq n$  that  $\Xi(st^n)(zlt^m) = 0 = \Xi(st^n)(nlt^m)$  and

$$\Xi(st^n)(zlt^n) \stackrel{(5.4)}{=} s(zl) \stackrel{(2.4)}{=} \omega(z) \cdot s(l) \stackrel{(5.4)}{=} \omega(z) \cdot \Xi(st^n)(lt^n)$$

hold. Recall that the condition (2.5) holds automatically, see Remark 2.11. Thus we have shown that  $\Xi(st^n)$  is a well-defined element in  $\mathcal{H}(G; R, \rho, \omega)$ .

Define the image under  $\Xi$  of an arbitrary element in  $\mathcal{H}(L; R, \rho|_L, \omega)_\phi[t, t^{-1}]$  given by a finite sum  $\sum_{n \in \mathbb{Z}} s_n t^n$  to be the element  $\sum_{n \in \mathbb{Z}} \Xi(s_n t^n)$  in  $\mathcal{H}(G; R, \rho, \omega)$ . Obviously  $\Xi$  is compatible with the addition. In order to show that  $\Xi$  is compatible with the multiplication, it suffices to show for  $s, s' \in \mathcal{H}(L; R, \rho|_L, \omega)$ ,  $l \in L$ , and  $m', n, n' \in \mathbb{Z}$

$$(\Xi(st^n) \cdot \Xi(s't^{n'}))(lt^m) = \Xi(st^n \cdot s't^{n'})(lt^m).$$

Fix a compact open subgroup  $K \subseteq G$  such that  $K$  is admissible for both  $\Xi(st^n)$  and  $\Xi(s't^{n'})$  and  $t^n K t^{-n}$  is admissible for both  $s$  and  $\phi^n(s)$ . Consider a transversal  $T'$  for the projection  $L \rightarrow L/NK$ . Then  $T = \{t^{m'}l' \mid m' \in \mathbb{Z}, l' \in T'\}$  is a transversal for the projection  $G \rightarrow G/NK$  and the map  $\mathbb{Z} \times T' \xrightarrow{\cong} T$  sending  $(m', l')$  to  $t^{m'}l'$  is a bijection. Moreover,  $t^n T' t^{-n}$  is a transversal for the projection  $L \rightarrow L/Nt^n K t^{-n}$ . We have

$$(5.5) \quad \begin{aligned} \mu(\text{pr}(t^n K t^{-n})) &= \mu(t^n \text{pr}(K) t^{-n}) \\ &= \frac{\mu(M)}{[M : t^n \text{pr}(K) t^{-n}]} \stackrel{(5.2)}{=} \frac{\mu(M)}{[M : \text{pr}(K)]} = \mu(\text{pr}(K)). \end{aligned}$$

We compute

$$\begin{aligned} &(\Xi(st^n) \cdot \Xi(s't^{n'}))(lt^m) \\ &\stackrel{(2.13)}{=} \mu(\text{pr}(K)) \cdot \sum_{g' \in T} \Xi(st^n)(lt^m g') \cdot lt^m g' \Xi(s't^{n'})(g'^{-1}) \\ &= \mu(\text{pr}(K)) \cdot \sum_{l' \in T'} \sum_{m' \in \mathbb{Z}} \Xi(st^n)(lt^m t^{m'} l') \cdot lt^m t^{m'} l' \Xi(s't^{n'})((t^{m'} l')^{-1}) \\ &= \mu(\text{pr}(K)) \cdot \sum_{l' \in T'} \sum_{m' \in \mathbb{Z}} \Xi(st^n)(lt^{m+m'} l' t^{-m-m'} t^{m+m'}) \cdot lt^{m+m'} l' \Xi(s't^{n'})(l'^{-1} t^{-m'}) \\ &\stackrel{(5.4)}{=} \mu(\text{pr}(K)) \cdot \sum_{l' \in T'} \sum_{\substack{m+m'=n, -m'=n'}} s(lt^{m+m'} l' t^{-m-m'}) \cdot lt^{m+m'} l' s'(l'^{-1}) \\ &= \begin{cases} \mu(\text{pr}(K)) \cdot \sum_{l' \in T'} s(lt^n l' t^{-n}) \cdot lt^n l' s'(l'^{-1}) & m = n + n' \\ 0 & m \neq n + n' \end{cases} \\ &= \begin{cases} \mu(\text{pr}(K)) \cdot \sum_{l' \in T'} s(lt^n l' t^{-n}) \cdot lt^n l' t^{-n} t^n s'(t^{-n} t^n l'^{-1} t^{-n} t^n) & m = n + n' \\ 0 & m \neq n + n' \end{cases} \\ &= \begin{cases} \mu(\text{pr}(K)) \cdot \sum_{l' \in T'} s(lt^n l' t^{-n}) \cdot lt^n l' t^{-n} \phi^n(s')(t^n l'^{-1} t^{-n}) & m = n + n' \\ 0 & m \neq n + n' \end{cases} \\ &\stackrel{(5.5)}{=} \begin{cases} \mu(t^n K t^{-n}) \cdot \sum_{l'' \in t^n T' t^{-n}} s(ll'') \cdot ll'' \phi^n(s')(l''^{-1}) & m = n + n' \\ 0 & m \neq n + n' \end{cases} \\ &\stackrel{(2.13)}{=} \begin{cases} (s \cdot \phi^n(s'))(l) & m = n + n' \\ 0 & m \neq n + n' \end{cases} \\ &\stackrel{(5.4)}{=} \Xi(s \cdot \phi^n(s') \cdot t^{n+n'})(lt^m) \\ &= \Xi(st^n \cdot s't^{n'})(lt^m). \end{aligned}$$

Obviously  $\Xi$  is injective. It remains to show that  $\Xi$  is surjective. Any element in  $\mathcal{H}(G; R, \rho, \omega)$  can be written as a sum of elements  $s'$  for which the support is contained in  $Lt^n$  for some  $n \in \mathbb{Z}$ . Hence it suffices to show that such  $s'$  is in the image. Define  $s: L \rightarrow R$  by  $s(l) = s'(lt^n)$ . Choose  $K \in P$  such that both  $K$  and  $t^{-n} K t^n$  are admissible for  $s'$ . Obviously  $K \subseteq L$  and  $t^{-n} K t^n \subseteq L$ . We have for  $l \in L$  and  $k \in K$  the equality  $s'(klt^n) \stackrel{(2.9)}{=} s'(lt^n)$ , which implies  $s(kl) = s(l)$ . We also have  $s'(lkt^n) = s'(lt^n t^{-n} k t^n) \stackrel{(2.10)}{=} s'(lt^n)$  which implies  $s(lk) = s(l)$ . Hence  $K$  is admissible for  $s$ . Condition (2.4) follows from the calculation for  $z \in N$ .

$$s(zl) = s'(zlt^n) \stackrel{(2.4)}{=} \omega(z) \cdot s'(lt^n) = \omega(z) \cdot s(l).$$

Recall that the condition (2.5) holds automatically, see Remark 2.11. We conclude that  $s$  defines an element in  $\mathcal{H}(L; R, \rho|_L, \omega)$  with  $\Xi(st^n) = s'$ . This finishes the proof of Lemma 5.3.  $\square$

**Lemma 5.6.** *Let  $\mathcal{A}$  be a (not necessarily unital) additive category, which is the directed union  $\mathcal{A} = \bigcup_{i \in I} \mathcal{A}_i$  of unital additive categories. Let  $\Phi: \mathcal{A} \xrightarrow{\cong} \mathcal{A}$  be an automorphism of (non-unital) additive categories.*

*There is an equivalence of unital additive categories*

$$F: \text{Idem}(\text{Idem}(\mathcal{A})_{\text{Idem}(\Phi)}[t, t^{-1}]) \xrightarrow{\cong} \text{Idem}(\mathcal{A}_\Phi[t, t^{-1}]).$$

*Proof.* Recall that an object in  $\text{Idem}(\mathcal{A})$  is given by a pair  $(A, p)$ , where  $A$  is an object in  $\mathcal{A}$  and  $p: A \rightarrow A$  is a morphism in  $\mathcal{A}$  with  $p \circ p = p$ . Moreover, a morphism  $f: (A, p) \rightarrow (A', p')$  in  $\text{Idem}(\mathcal{A})_{\text{Idem}(\Phi)}[t, t^{-1}]$  is given by a finite sum  $f = \sum_{j \in \mathbb{Z}} f_j \cdot t^j$ , where  $f_j: \text{Idem}(\Phi)^j(A, p) := (\Phi^j(A), \Phi^j(p)) \rightarrow (A', p')$  is a morphism in  $\text{Idem}(\mathcal{A})$ . Hence each  $f_j$  is given by a morphism  $f_j: \Phi^j(A) \rightarrow A'$  satisfying  $f_j = p' \circ f_j \circ \Phi^j(p)$ . We conclude that the morphism  $f: (A, p) \rightarrow (A', p')$  in  $\text{Idem}(\mathcal{A})_{\text{Idem}(\Phi)}[t, t^{-1}]$  is the same as a morphism  $f: A \rightarrow A'$  in  $\mathcal{A}_\Phi[t, t^{-1}]$  satisfying  $(p' \cdot t^0) \circ f \circ (p \cdot t^0) = f$ , since we get in  $\mathcal{A}_\Phi[t, t^{-1}]$

$$(p' \cdot t^0) \circ f \circ (p \cdot t^0) = \sum_{j \in \mathbb{Z}} (p' \cdot t^0) \circ f_j \cdot t^j \circ (p \cdot t^0) = \sum_{j \in \mathbb{Z}} (p' \circ f_j \cdot \Phi^j(p)) \cdot t^j.$$

Now an object in  $\text{Idem}(\text{Idem}(\mathcal{A})_{\text{Idem}(\Phi)}[t, t^{-1}])$  is given by  $((A, p), q)$ , where  $A$  is an object in  $\mathcal{A}$ ,  $p: A \rightarrow A$  is a morphism in  $\mathcal{A}$  with  $p \circ p = p$ , and  $q: (A, p) \rightarrow (A, p)$  is a morphism in  $\text{Idem}(\mathcal{A})_{\text{Idem}(\Phi)}[t, t^{-1}]$  satisfying  $q \circ q = q$ . The morphism  $q$  is the same as a morphism  $q: A \rightarrow A$  in  $\mathcal{A}_\Phi[t, t^{-1}]$  satisfying  $(p \cdot t^0) \circ q \circ (p \cdot t^0) = q$  and  $q \circ q = q$ . Hence we can define  $F$  on objects by

$$F((A, p), q) = (A, q).$$

Consider two objects  $((A, p), q)$  and  $((A', p'), q')$ . A morphism  $f: ((A, p), q) \rightarrow ((A', p'), q')$  in  $\text{Idem}(\text{Idem}(\mathcal{A})_{\text{Idem}(\Phi)}[t, t^{-1}])$  is the same as a morphism  $f: (A, p) \rightarrow (A', p')$  in  $\text{Idem}(\mathcal{A})_{\text{Idem}(\Phi)}[t, t^{-1}]$  satisfying  $q' \circ f \circ q = f$  and therefore the same as a morphism  $f: A \rightarrow A'$  in  $\mathcal{A}_\Phi[t, t^{-1}]$  satisfying  $(p' \cdot t^0) \circ f \circ (p \cdot t^0) = f$  and  $q' \circ f \circ q = f$ .

Hence we can define  $F$  on morphisms by sending the morphism  $f: ((A, p), q) \rightarrow ((A', p'), q')$  in  $\text{Idem}(\text{Idem}(\mathcal{A})_{\text{Idem}(\Phi)}[t, t^{-1}])$  to the morphism  $(A, q) \rightarrow (A', q')$  in  $\text{Idem}(\mathcal{A}_\Phi[t, t^{-1}])$  given by the morphism  $f: A \rightarrow A'$  in  $\mathcal{A}_\Phi[t, t^{-1}]$ . One easily checks that  $F$  is compatible with composition and sends identity morphisms to identity morphisms.

Next we show that the map induced by  $F$

$$\begin{aligned} \text{mor}_{\text{Idem}(\text{Idem}(\mathcal{A})_{\text{Idem}(\Phi)}[t, t^{-1}])} & \left( ((A, p), q), ((A', p'), q') \right) \\ & \rightarrow \text{mor}_{\text{Idem}(\mathcal{A}_\Phi[t, t^{-1}])} \left( (A, q), (A', q') \right) \end{aligned}$$

is bijective. Obviously it is injective. In order to show surjectivity, we have to show for a morphism  $f: (A, q) \rightarrow (A', q')$  in  $\text{Idem}(\mathcal{A})_{\text{Idem}(\Phi)}[t, t^{-1}]$  satisfying  $q' \circ f \circ q = f$  that  $(p' \cdot t^0) \circ f \circ (p \cdot t^0) = f$  holds. This follows from the following computation using  $(p \cdot t^0) \circ q \circ (p \cdot t^0) = q$ ,  $q \circ q = q$ ,  $p \circ p = p$ ,  $(p' \cdot t^0) \circ q' \circ (p' \cdot t^0) = q'$ ,  $q' \circ q' = q'$ , and  $p' \circ p' = p'$ ,

$$\begin{aligned} (p' \cdot t^0) \circ f \circ (p \cdot t^0) &= (p' \cdot t^0) \circ q' \circ f \circ q \circ (p \cdot t^0) \\ &= (p' \cdot t^0) \circ (p' \cdot t^0) \circ q' \circ (p' \cdot t^0) \circ f \circ (p \cdot t^0) \circ q \circ (p \cdot t^0) \circ (p \cdot t^0) \\ &= (p' \cdot t^0) \circ q' \circ (p' \cdot t^0) \circ f \circ (p \cdot t^0) \circ q \circ (p \cdot t^0) = q' \circ f \circ q = f. \end{aligned}$$

Finally we show that  $F$  is surjective on objects. Consider any object  $(A, q)$  in  $\text{Idem}(\mathcal{A}_\Phi[t, t^{-1}])$ . In order to show that  $(A, q)$  is in the image of  $F$ , we have to construct a morphism  $p: A \rightarrow A$  in  $\mathcal{A}$  such that  $p \circ p = p$  holds in  $\mathcal{A}$  and  $(p \cdot t^0) \circ q \circ (p \cdot t^0) = q$  holds in  $\mathcal{A}_\Phi[t, t^{-1}]$ .

We can write  $q$  as a finite sum  $q = \sum_{j \in \mathbb{Z}} q_j \cdot t^j$  for morphism  $q_j: \Phi^j(A) \rightarrow A$  in  $\mathcal{A}$ . Since  $\mathcal{A}$  is the directed union  $\bigcup_{i \in I} \mathcal{A}_i$  of the unital subcategories  $\mathcal{A}_i$ , we can find an index  $i_0 \in I$  such that for each  $j \in \mathbb{Z}$  with  $q_j \neq 0$  and hence for all  $j \in \mathbb{Z}$  the morphisms  $q_j$  and  $\Phi^{-j}(q_j)$  belong to  $\mathcal{A}_{i_0}$ . Let  $p \in \mathcal{A}_{i_0}$  be the identity morphism of the object  $A$  in  $\mathcal{A}_{i_0}$ . Then we get  $p \circ p = p$ ,  $p \circ q_j = q_j$ , and  $\Phi^{-j}(q_j) \circ p = \Phi^{-j}(q_j)$  in  $\mathcal{A}$  for all  $j \in \mathbb{Z}$ . Now we compute

$$\begin{aligned} (p \cdot t^0) \circ q \circ (p \cdot t^0) &= (p \cdot t^0) \circ \left( \sum_{j \in \mathbb{Z}} q_j \cdot t^j \right) \circ (p \cdot t^0) \\ &= \sum_{j \in \mathbb{Z}} (p \cdot t^0) \circ (q_j \cdot t^j) \circ (p \cdot t^0) = \sum_{j \in \mathbb{Z}} (p \circ q_j \cdot \Phi^j(p)) \cdot t^j = \sum_{j \in \mathbb{Z}} (q_j \cdot \Phi^j(p)) \cdot t^j \\ &= \sum_{j \in \mathbb{Z}} \Phi^j(\Phi^{-j}(q_j) \cdot p) \cdot t^j = \sum_{j \in \mathbb{Z}} \Phi^j(\Phi^{-j}(q_j)) \cdot t^j = \sum_{j \in \mathbb{Z}} q_j \cdot t^j = q. \end{aligned}$$

This finishes the proof of Lemma 5.6  $\square$

The next lemma allows to reduced the computation of the algebraic  $K$ -theory of the non-unital ring  $\mathcal{H}(G; R, \rho, \omega)$  to the calculation of the algebraic  $K$ -theory of a unital additive category given by the twisted finite Laurent category of an automorphism of a unital additive category. The main advantage will be that for such a category Bass-Heller-Swan decompositions will be available.

**Lemma 5.7.** *There is a weak equivalence*

$$\mathbf{K}^\infty(\text{Idem}(\underline{\mathcal{H}(L; R, \rho|_L, \omega)}_{\oplus})_{\text{Idem}(\underline{\phi}_{\oplus})}[t, t^{-1}]) \xrightarrow{\simeq} \mathbf{K}^\infty(\mathcal{H}(G; R, \rho, \omega)).$$

*Proof.* Recall that for a unital additive category  $\mathcal{B}$  the obvious map  $\mathbf{K}^\infty(\mathcal{B}) \rightarrow \mathbf{K}^\infty(\text{Idem}(\mathcal{B}))$  is a weak homotopy equivalence. We can apply Lemma 5.6 to  $\mathcal{A} = \underline{\mathcal{H}(L; R, \rho|_L, \omega)}_{\oplus}$  and the automorphism  $\underline{\phi}_{\oplus}$  because of Lemma 2.25. Hence we obtain a weak equivalence

$$\begin{aligned} \mathbf{K}^\infty(\text{Idem}(\underline{\mathcal{H}(L; R, \rho|_L, \omega)}_{\oplus})_{\text{Idem}(\underline{\phi}_{\oplus})}[t, t^{-1}]) \\ \xrightarrow{\simeq} \mathbf{K}^\infty(\text{Idem}(\underline{(\mathcal{H}(L; R, \rho|_L, \omega))_{\oplus}}_{\underline{\phi}_{\oplus}}}[t, t^{-1}])). \end{aligned}$$

The (non-unital) additive category  $(\underline{\mathcal{H}(L; R, \rho|_L, \omega)}_{\oplus})_{\underline{\phi}_{\oplus}}[t, t^{-1}]$  is isomorphic to the (non-unital) additive category  $(\underline{\mathcal{H}(L; R, \rho|_L, \omega)}_{\oplus})_{\underline{\phi}_{\oplus}}[t, t^{-1}]_{\oplus}$  by (3.3), and hence by Lemma 5.3 to the (non-unital) additive category  $\underline{\mathcal{H}(G; R, \rho, \omega)}_{\oplus}$ . Hence we obtain a weak homotopy equivalence

$$\mathbf{K}^\infty(\text{Idem}(\underline{(\mathcal{H}(L; R, \rho|_L, \omega))_{\oplus}}_{\underline{\phi}_{\oplus}}}[t, t^{-1}])) \xrightarrow{\simeq} \mathbf{K}^\infty(\text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)}_{\oplus})).$$

$\square$

## 6. A REVIEW OF THE TWISTED BASS-HELLER-SWAN DECOMPOSITION FOR UNITAL ADDITIVE CATEGORIES

In this section additive category means always a small unital additive category and functors are assumed to respect identity morphisms. The same is true for rings.

The following definitions are taken from [2, Definition 6.1].

**Definition 6.1** (Regularity properties of rings). Let  $l$  be a natural number.

- (i) We call  $R$  *Noetherian*, if any  $R$ -submodule of a finitely generated  $R$ -module is again finitely generated;
- (ii) We call  $R$  *regular coherent*, if every finitely presented  $R$ -module  $M$  is of type FP;
- (iii) We call  $R$   *$l$ -uniformly regular coherent*, if every finitely presented  $R$ -module  $M$  admits an  $l$ -dimensional finite projective resolution, i.e., there exist an exact sequence  $0 \rightarrow P_l \rightarrow P_{l-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$  such that each  $P_i$  is finitely generated projective;
- (iv) We call  $R$  *regular*, if it is Noetherian and regular coherent;
- (v) We call  $R$   *$l$ -uniformly regular*, if it is Noetherian and  $l$ -uniformly regular coherent.

These notions are generalized to additive categories in [2, Section 6] in such a way that they reduce in the special case  $\mathcal{A} = \underline{R}$  to the ones appearing in Definition 6.1. Therefore the precise definitions for additive categories are not needed to comprehend the material of this paper.

The following result follows from [2, Theorem 7.8 and Theorem 10.1].

**Theorem 6.2** (The non-connective  $K$ -theory of additive categories). *Let  $\mathcal{A}$  be an additive category. Suppose that  $\mathcal{A}$  is regular. Consider any automorphism  $\Phi: \mathcal{A} \xrightarrow{\cong} \mathcal{A}$  of additive categories.*

*Then we get a weak homotopy equivalence of non-connective spectra*

$$\mathbf{a}^\infty: \mathbf{T}_{\mathbf{K}^\infty(\Phi^{-1})} \xrightarrow{\cong} \mathbf{K}^\infty(\mathcal{A}_\Phi[t, t^{-1}]),$$

where  $\mathbf{T}_{\mathbf{K}^\infty(\Phi^{-1})}$  is the mapping torus of the map of spectra  $\mathbf{K}^\infty(\Phi): \mathbf{K}^\infty(\mathcal{A}) \rightarrow \mathbf{K}^\infty(\mathcal{A})$  induced by  $\Phi$ .

## 7. HECKE ALGEBRAS OVER COMPACT TD-GROUPS AND CROSSED PRODUCT RINGS

Let  $G, N, Q := G/N$ ,  $\text{pr}: G \rightarrow Q$ ,  $R, P, \rho, \omega$ , and  $\mu$  be as in Subsection 2.A and denote by  $\mathcal{H}(G; R, \rho, \omega)$  the Hecke algebra, which we have introduced in Subsection 2.B. Our main assumption in this section will be that  $Q$  is compact.

**Definition 7.1.** We call a subgroup  $N \subseteq G$  *locally central*, if the centralizer  $C_G N$  of  $N$  in  $G$  is an open subgroup.

The main result of this section is

**Theorem 7.2.** *Suppose that  $Q$  is compact and  $N$  is locally central. Let  $l$  be a natural number. Let  $R$  be a unital ring with  $\mathbb{Q} \subseteq R$  such that  $R$  is  $l$ -uniformly regular or regular respectively.*

*Then the additive category  $\text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)}_{\oplus}[\mathbb{Z}^m])$  is  $(l + 2m)$ -uniformly regular or regular respectively for all  $m \geq 0$ .*

For the purpose of this paper we need Theorem 7.2 only for the property regular, but for later applications it will be crucial to consider the property  $l$ -uniformly regular as well. The point will be that the property  $l$ -uniformly regular is compatible with infinite products of additive categories, in contrast to the property regular.

### 7.A. Existence of normal $K \in P$ .

**Lemma 7.3.** *Suppose that  $Q$  is compact and  $N$  is locally central.*

*Then for every compact open subgroup  $K \subseteq G$  there exists a compact open subgroup  $K' \subseteq G$  such that  $K' \subseteq K$ ,  $K' \subseteq C_G N$ , and  $K'$  is normal in  $G$ .*

*Proof.* Put  $L = K \cap C_G N$ . Then  $L$  is a compact open subgroup of  $G$  satisfying  $L \subseteq K$  and  $L \subseteq C_G N$ . Choose a transversal  $T$  of the projection  $G \rightarrow G/NL =$

$Q/\text{pr}(L)$ . Define  $K' = \bigcap_{t \in t} tLt^{-1}$ . Since  $Q/\text{pr}(L)$  is compact and discrete, the set  $T$  is finite. Hence  $K' \subseteq G$  is again compact open. We get for  $n \in N$  and  $l \in L$

$$(tnl)L(tnl)^{-1} = tnlLl^{-1}n^{-1}t^{-1} = tnLn^{-1}t^{-1} \stackrel{L \subseteq C_G N}{=} tLt^{-1}.$$

This implies  $K' = \bigcap_{g \in G} gLg^{-1}$ . Hence  $K' \subseteq G$  is a compact open normal subgroup and obviously satisfies  $K' \subseteq K$  and  $K' \subseteq C_G N$ .  $\square$

**7.B. Crossed products of finite groups and regularity.** Let  $R$  be a unital ring and  $D$  be a (discrete) group. Recall that a *crossed product ring*  $R * D$  is a unital ring, which is a free left  $R$ -module with an  $R$ -basis  $\{b_d \mid d \in D\}$  indexed by the elements in  $D$  such that  $b_e$  is the unit in  $R * D$ , for  $d_1, d_2 \in D$  there is a unit  $w(d_1, d_2) \in R^\times$  satisfying  $b_{d_1 d_2} = w(d_1, d_2) \cdot b_{d_1} \cdot b_{d_2}$ , and for  $r \in R$  and  $d \in D$  there exists  $c_d(r) \in R$  with  $c_d(r) \cdot b_d = b_d \cdot (r \cdot b_e)$ , where  $c_e(r) = r$  is required for  $r \in R$ . In particular each element  $b_d$  has an inverse  $b_d^{-1}$  in  $R * D$ , (which is *not* given by  $b_{d^{-1}}$  in general,) and there is an inclusion of rings  $R \rightarrow R * D$  sending  $r$  to  $r \cdot b_e$ .

The notion of crossed product ring is a generalization of the notion of a twisted group ring, which is the special case, where  $w$  is trivial. For more details we refer for instance to [1, Section 4] or [4, Section 6].

**Lemma 7.4.** *Let  $R$  be a ring with  $\mathbb{Q} \subseteq R$  and  $D$  be a finite group. Let  $R * D$  be a crossed product ring.*

(i) *Let  $M$  be any  $R * D$ -module. Let  $j: R \rightarrow R * D$  be the canonical inclusion of rings. Then we obtain  $R * D$ -homomorphisms*

$$\begin{aligned} i: M &\rightarrow R * D \otimes_R j^* M, & x &\mapsto \sum_{d \in D} \frac{1}{|D|} \cdot b_d \otimes b_d^{-1} \cdot x; \\ p: R * D \otimes_R j^* M &\rightarrow M, & u \otimes y &\mapsto u \cdot y, \end{aligned}$$

*satisfying  $p \circ i = \text{id}_M$ , where  $b_d^{-1}$  denotes the inverse of  $b_d$  in  $R * D$ ;*

(ii) *If  $R$  is regular, then  $R * D$  is regular;*

(iii) *If  $R$  is  $l$ -uniformly regular, then  $R * D$  is  $l$ -uniformly regular;*

(iv) *If  $R$  is semi-simple, then  $R * D$  is semi-simple.*

*Proof.* (i) We check that  $i$  is  $R * D$ -linear. Obviously  $i$  is compatible with addition, it remains to treat multiplication. Consider  $r \in R$  and  $d_0 \in D$ . Note for the sequel that the element  $b_d^{-1} \cdot r \cdot b_{d_0} \cdot b_{d_0^{-1}d}$  in  $R * D$  belongs to  $R$ . Hence we get for  $x \in M$ ,  $r \in R$  and  $d_0 \in D$

$$\begin{aligned} i(r \cdot b_{d_0} \cdot x) &= \sum_{d \in D} \frac{1}{|D|} \cdot b_d \otimes b_d^{-1} \cdot (r \cdot b_{d_0} \cdot x) = \sum_{d \in D} \frac{1}{|D|} \cdot b_d \otimes (b_d^{-1} \cdot r \cdot b_{d_0} \cdot b_{d_0^{-1}d}) \cdot (b_{d_0^{-1}d})^{-1} \cdot x \\ &= \sum_{d \in D} \frac{1}{|D|} \cdot b_d \cdot (b_d^{-1} \cdot r \cdot b_{d_0} \cdot b_{d_0^{-1}d}) \otimes (b_{d_0^{-1}d})^{-1} \cdot x = \sum_{d \in D} \frac{1}{|D|} \cdot r \cdot b_{d_0} \cdot b_{d_0^{-1}d} \otimes (b_{d_0^{-1}d})^{-1} \cdot x \\ &= r \cdot b_{d_0} \cdot \frac{1}{|D|} \cdot \sum_{d \in D} b_{d_0^{-1}d} \otimes (b_{d_0^{-1}d})^{-1} \cdot x = r \cdot b_{d_0} \cdot \frac{1}{|D|} \cdot \sum_{d' \in D} b_{d'} \otimes (b_{d'})^{-1} \cdot x = r \cdot b_{d_0} \cdot i(x). \end{aligned}$$

Obviously  $p$  is a well-defined  $R * D$ -homomorphism satisfying  $p \circ i = \text{id}_M$ .

(ii) Since  $R$  is regular,  $R$  is in particular Noetherian. Since  $R * D$  is a finitely generated  $R$ -module,  $R * D$  is Noetherian as well.

It remains to show that a finitely presented  $R * D$ -module  $M$  is of type FP. Since  $R$  is regular and the  $R$ -module  $i^* M$  is finitely presented,  $i^* M$  is of type FP. Since  $R * D$  is free as  $R$ -module and hence the functor sending an  $R$ -module  $N$  to the  $R * D$ -module  $R * D \otimes_R N$  is flat and sends finitely generated projective  $R$ -modules to finitely generated projective  $R * D$ -modules, the  $R * D$ -module  $R * D \otimes_R i^* M$  is of type FP. Since a direct summand in a module of type FP is of type FP again,

the  $R * D$ -module  $M$  is of type FP.

(iii) The proof is analogous to assertion (ii), since all the statements about finite-dimension remain true, if one inserts  $l$ -dimensional everywhere.

(iv) This follows from assertion (i). This finishes the proof of Lemma 7.4.  $\square$

**7.C. The Hecke algebra and crossed products.** In this subsection we will assume that  $Q$  is compact.

Consider a compact open normal subgroup  $K$  of  $G$  satisfying  $K \in P$ . Since both  $K$  and  $N$  are normal in  $G$ , the subgroup  $NK$  of  $G$  is also normal. Put

$$(7.5) \quad D := G/NK = Q/\text{pr}(K).$$

Note that  $D$  is a finite discrete group.

Next we show that  $\mathcal{H}(G; R, \rho, \omega)$  is a left  $R$ -module. Namely, define for  $s \in \mathcal{H}(G; R, \rho, \omega)$  the new element  $rs$  by  $rs(g) := r \cdot s(g)$ . One easily checks that  $rs$  satisfies (2.4), (2.5), (2.9), and (2.10).

Fix a set-theoretic section  $\sigma: D \rightarrow G$  of the projection  $p: G \rightarrow D = G/NK$  satisfying  $\sigma(e_D) = e_G$ . In the sequel we denote by  $T$  the transversal of  $p$  given by  $T := \{\sigma(d)^{-1} \mid d \in D\}$ . For  $d \in D$  define  $b_d \in \mathcal{H}(G//K; R, \rho, \omega)$  by the function

$$(7.6) \quad b_d: G \rightarrow R, \\ g \mapsto \begin{cases} \frac{1}{\mu(\text{pr}(K))} \cdot \omega(n) & \text{if } p(g) = d \text{ and } g = nk\sigma(d) \text{ for } n \in N, k \in K; \\ 0 & p(g) \neq d. \end{cases}$$

This is independent of the choice of  $n \in N$  and  $k \in K$ , since for  $n_0, n_1 \in N$  and  $k_0, k_1 \in K$  with  $n_0k_0 = n_1k_1$  we have  $n_1^{-1}n_0 = k_1k_0^{-1} \in N \cap K$  and we compute

$$(7.7) \quad \omega(n_1) = \omega(n_1) \cdot \omega(n_1^{-1}n_0) \stackrel{(2.8)}{=} \omega(n_1) \cdot 1 = \omega(n_0).$$

We have to check that the required transformation formulas (2.9) and (2.10) for  $g \in G$  and  $k \in K$  are satisfied. If  $p(g) \neq d$ , then  $b_d(kg) = b_d(g) = b_d(gk) = 0$  and the formulas hold. It remains to treat the case  $p(g) = d$ . This follows from the calculations for  $g = nk\sigma(d)$  for  $n \in N$ ,  $k \in K$  and  $k' \in K$  using  $\sigma(d)k'\sigma(d)^{-1} \in K$

$$b_d(k'g) = b_d(k'nk\sigma(d)) = b_d((k'nk'^{-1})(k'k)\sigma(d)) \\ \stackrel{(7.6)}{=} \frac{1}{\mu(\text{pr}(K))} \cdot \omega(k'nk'^{-1}) \stackrel{(2.1)}{=} \frac{1}{\mu(\text{pr}(K))} \cdot \omega(n) \stackrel{(7.6)}{=} b_d(g),$$

and

$$b_d(gk') = b_d(nk\sigma(d)k') = b_d(n(k\sigma(d)k'\sigma(d)^{-1})\sigma(d)) \\ \stackrel{(7.6)}{=} \frac{1}{\mu(\text{pr}(K))} \cdot \omega(n) \stackrel{(7.6)}{=} b_d(g).$$

The verification of (2.4) and (2.5) is left to the reader. This finishes the proof that  $b_d$  is a well-defined element in  $\mathcal{H}(G//K; R, \rho, \omega)$ .

Consider any element  $s \in \mathcal{H}(G//K; R, \rho, \omega)$ . Then we get

$$(7.8) \quad s = \sum_{d \in D} \mu(\text{pr}(K)) \cdot s(\sigma(d)) \cdot b_d$$

by the following calculation for  $g \in G$  with  $g = nk\sigma(d)$  for  $n \in N$  and  $k \in K$

$$s(g) = s(nk\sigma(d)) \stackrel{(2.4), (2.9)}{=} \omega(n) \cdot s(\sigma(d)) \\ \stackrel{\omega(n) \in \text{cent}(R)}{=} \mu(\text{pr}(K)) \cdot s(\sigma(d)) \cdot \left( \frac{1}{\mu(\text{pr}(K))} \cdot \omega(n) \right) \stackrel{(7.6)}{=} \mu(\text{pr}(K)) \cdot s(\sigma(d)) \cdot b_d(g).$$

We conclude from (7.8) that  $\{b_d \mid d \in D\}$  is an  $R$ -basis for the left  $R$ -module  $\mathcal{H}(G//K; R, \rho, \omega)$ .

For  $d_1, d_2$  in  $D$ , define an element

$$(7.9) \quad w(d_1, d_2) := \omega(n) \in R^\times,$$

if  $\sigma(d_1 d_2) \sigma(d_2)^{-1} \sigma(d_1)^{-1} = nk$  for  $n \in N$  and  $k \in K$ . This is independent of the choice of  $n \in N$  and  $k \in K$  by (7.7). Next we want to show

$$(7.10) \quad b_{d_1} \cdot b_{d_2} = w(d_1, d_2) \cdot b_{d_1 d_2}.$$

Consider  $d_1, d_2 \in D$  and  $g \in G$ . Choose elements  $n \in N$  and  $k \in K$  satisfying  $\sigma(d_1 d_2) \sigma(d_2)^{-1} \sigma(d_1)^{-1} = nk$ . If  $p(g) = d_1 d_2$ , we fix  $n_0 \in N$  and  $k_0 \in K$  satisfying  $g = n_0 k_0 \sigma(d_1) \sigma(d_2)$ . We compute

$$(7.11) \quad \begin{aligned} & (b_{d_1} \cdot b_{d_2})(g) \\ & \stackrel{(2.13)}{=} \mu(\text{pr}(K)) \cdot \sum_{d \in D} b_{d_1}(g\sigma(d)^{-1}) \cdot g\sigma(d)^{-1} b_{d_2}(\sigma(d)) \\ & \stackrel{(7.6)}{=} \mu(\text{pr}(K)) \cdot \sum_{d \in D, p(\sigma(d))=d_2} b_{d_1}(g\sigma(d)^{-1}) \cdot g\sigma(d)^{-1} b_{d_2}(\sigma(d)) \\ & = \mu(\text{pr}(K)) \cdot b_{d_1}(g\sigma(d_2)^{-1}) \cdot g\sigma(d_2)^{-1} b_{d_2}(\sigma(d_2)) \\ & \stackrel{(7.6)}{=} \mu(\text{pr}(K)) \cdot b_{d_1}(g\sigma(d_2)^{-1}) \cdot g\sigma(d_2)^{-1} \cdot \left( \frac{1}{\mu(\text{pr}(K))} \cdot \omega(e) \right) \\ & = b_{d_1}(g\sigma(d_2)^{-1}) \cdot (g\sigma(d_2)^{-1} \cdot 1) \\ & = b_{d_1}(g\sigma(d_2)^{-1}) \\ & \stackrel{(7.6)}{=} \begin{cases} \frac{1}{\mu(\text{pr}(K))} \cdot \omega(n_0) & \text{if } p(g) = d_1 d_2; \\ = 0 & \text{if } p(g) \neq d_1 d_2. \end{cases} \end{aligned}$$

Suppose for  $g \in G$  that  $p(g) = d_1 d_2$ . We can write

$$g = n_0 k_0 \sigma(d_1) \sigma(d_2) = (n_0 k_0 k^{-1} n^{-1} (k_0 k^{-1})^{-1}) (k_0 k^{-1}) \sigma(d_1 d_2)$$

and have  $n_0 k_0 k^{-1} n^{-1} (k_0 k^{-1})^{-1} \in N$  and  $k_0 k^{-1} \in K$ . We compute

$$(7.12) \quad \begin{aligned} & w(d_1, d_2) \cdot b_{d_1 d_2}(g) \\ & \stackrel{(7.9)}{=} \omega(n) \cdot b_{d_1 d_2}(g) \\ & \stackrel{(7.6)}{=} \omega(n) \cdot \frac{1}{\mu(\text{pr}(K))} \cdot \omega(n_0 k_0 k^{-1} n^{-1} (k_0 k^{-1})^{-1}) \\ & = \omega(n) \cdot \frac{1}{\mu(\text{pr}(K))} \cdot \omega(n_0) \cdot \omega(k_0 k^{-1} n^{-1} (k_0 k^{-1})^{-1}) \\ & \stackrel{(2.1)}{=} \omega(n) \cdot \frac{1}{\mu(\text{pr}(K))} \cdot \omega(n_0) \cdot \omega(n^{-1}) \\ (7.13) \quad & \stackrel{\omega(n_0) \in \text{cent}(R)}{=} \frac{1}{\mu(\text{pr}(K))} \cdot \omega(n) \cdot \omega(n^{-1}) \cdot \omega(n_0) \\ & = \frac{1}{\mu(\text{pr}(K))} \cdot \omega(n_0). \end{aligned}$$

Since  $w(d_1, d_2) \cdot b_{d_1 d_2}(g) = 0$ , if  $p(g) \neq d_1 d_2$ , we conclude (7.10) from (7.11) and (7.12).

We compute for  $d \in D, r \in R$  and  $d' \in D$  using the fact that  $\{\sigma(d'')^{-1} \mid d'' \in D\}$  is a transversal for  $G \rightarrow G/NK = D$  and  $\sigma(e_D) = e_G$

$$\begin{aligned}
& (b_d \cdot (r \cdot b_{e_D}))(\sigma(d')) \\
& \stackrel{(2.13)}{=} \mu(\text{pr}(K)) \cdot \sum_{d'' \in D} b_d(\sigma(d')\sigma(d'')^{-1}) \cdot \sigma(d')\sigma(d'')^{-1}((r \cdot b_{e_D})(\sigma(d''))) \\
& \stackrel{(7.6)}{=} \mu(\text{pr}(K)) \cdot \sum_{d'' \in \{e_Q\}} b_d(\sigma(d')\sigma(d'')^{-1}) \cdot \sigma(d')\sigma(d'')^{-1}((r \cdot b_{e_D})(\sigma(d''))) \\
& = \mu(\text{pr}(K)) \cdot b_d(\sigma(d')e_G^{-1}) \cdot \sigma(d')e_G^{-1}((r \cdot b_{e_D})(e_G)) \\
& \stackrel{(7.6)}{=} \mu(\text{pr}(K)) \cdot b_d(\sigma(d')) \cdot \sigma(d') \left( \frac{1}{\mu(\text{pr}(K))} \cdot r \cdot 1 \right) \\
& = b_d(\sigma(d')) \cdot \sigma(d')r \\
& \stackrel{(7.6)}{=} \begin{cases} \frac{1}{\mu(\text{pr}(K))} \cdot \sigma(d)r & \text{if } d' = d; \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

This implies for  $d \in D$ ,  $r \in R$ , and  $g \in G$

$$\begin{aligned}
(7.14) \quad (b_d \cdot (r \cdot b_{e_D})) & \stackrel{(7.8)}{=} \sum_{d' \in D} \mu(\text{pr}(K)) \cdot b_d \cdot (r \cdot b_{e_D})(\sigma(d'))b_{d'} \\
& = \mu(\text{pr}(K)) \cdot \left( \frac{1}{\mu(\text{pr}(K))} \cdot \sigma(d)r \right) \cdot b_d = \sigma(d)r \cdot b_d.
\end{aligned}$$

Recall from Lemma 2.24 that  $\mathcal{H}(G//K; R, \rho, \omega)$  has a unit, namely  $b_{e_D}$ .

We conclude from (7.10) and (7.14)

**Lemma 7.15.** *Suppose that  $Q$  is compact. Consider a compact open normal subgroup  $K$  of  $G$  satisfying  $K \in P$ .*

*Then the unital ring  $\mathcal{H}(G//K; R, \rho, \omega)$  is the crossed product  $R * D$  associated to  $(w, c)$  for  $w$  defined in (7.9) and  $c_d(r) := (\rho \circ \sigma(d))(r)$ .*

**7.D. Filtering the Hecke algebra of a compact group by normal compact open subgroups.** Consider a sequence  $G = K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$  of normal compact open subgroups of  $G$  with  $\bigcap_{n \geq 0} K_n = \{1\}$  such that  $K_n \in P$  holds for  $n \in \mathbb{N}$ . It exists by Lemma 7.3 as we assume throughout this section that  $Q$  is compact and  $N$  is locally central. Let  $1_{K_n}$  be the element in  $\mathcal{H}(G; R, \rho, \omega)$  defined in (2.24). Then  $1_{K_n}$  is central in  $\mathcal{H}(G; R, \rho, \omega)$ , since  $K_n$  is normal in  $G$ . We have  $1_{K_n} \cdot 1_{K_m} = 1_{K_n} = 1_{K_m} \cdot 1_{K_n}$  for  $m \leq n$ . For every  $s \in \mathcal{H}(G)$  there exists a natural number  $n \in \mathbb{N}$  satisfying  $1_{K_n} \cdot s = s = s \cdot 1_{K_n}$ . In the sequel we sometimes abbreviate  $1_n = 1_{K_n}$ ,  $\mathcal{H}(G) = \mathcal{H}(G; R, \rho, \omega)$  and  $\mathcal{H}(G//K_n) = \mathcal{H}(G//K_n; R, \rho, \omega)$  and put  $1_{-1} = 0$ . The elementary proof of the next lemma is left to the reader.

**Lemma 7.16.** *We have the subrings rings  $1_n \mathcal{H}(G) 1_n = \mathcal{H}(G//K_n)$  and  $(1_n - 1_{n-1}) \mathcal{H}(G) (1_n - 1_{n-1})$  of  $\mathcal{H}(G)$ , which have  $1_n$  and  $(1_n - 1_{n-1})$  as unit. We get an obvious identification of rings (without unit)*

$$\bigoplus_{m \geq 0} (1_m - 1_{m-1}) \mathcal{H}(G) (1_m - 1_{m-1}) = \mathcal{H}(G),$$

and for  $n \geq 0$  of rings with unit

$$\bigoplus_{m=0}^n (1_m - 1_{m-1}) \mathcal{H}(G) (1_m - 1_{m-1}) = 1_n \mathcal{H}(G) 1_n.$$

Recall that a sequence  $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2$  in an additive category  $\mathcal{A}$  is called *exact* at  $A_1$ , if  $f_1 \circ f_0 = 0$  and for every object  $A$  and morphism  $g: A \rightarrow A_1$  with  $f_1 \circ g = 0$  there exists a morphism  $\bar{g}: A \rightarrow A_0$  with  $f_0 \circ \bar{g} = g$ . For information how this notion

is related by the Yoneda embedding to the usually notion of exactness for modules we refer to [2, Lemma 5.10 and Lemma 6.3]. A functor  $F: \mathcal{A} \rightarrow \mathcal{A}'$  of additive categories is called *faithfully flat*, provided that a sequence  $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2$  in  $\mathcal{A}$  is exact, if and only if the sequence  $F(A_0) \xrightarrow{F(f_0)} F(A_1) \xrightarrow{F(f_1)} F(A_2)$  in  $\mathcal{A}'$  is exact.

**Lemma 7.17.** *Let  $S$  and  $T$  be unital rings. Let  $\text{pr}: S \times T \rightarrow T$  be the projection, which is a homomorphism of unital rings. Let  $i: S \rightarrow S \times T$  be the inclusion sending  $s$  to  $(s, 0)$ , which is a homomorphism of rings (without units). Then*

- (i) *There exists a diagram of unital additive categories commuting up to natural equivalence of unital additive categories*

$$\begin{array}{ccc} \text{Idem}(\underline{S}_{\oplus}) & \xrightarrow{\text{Idem}(i_{\oplus})} & \text{Idem}(\underline{S \times T}_{\oplus}) \\ \Theta_S \downarrow \simeq & & \simeq \downarrow \Theta_{S \times T} \\ S\text{-MOD}_{\text{fgp}} & \xrightarrow{\text{pr}^*} & S \times T\text{-MOD}_{\text{fgp}} \end{array}$$

where the vertical arrows are the equivalences of unital additive categories of (3.2) and  $\text{pr}^*$  is restriction with  $\text{pr}$ ;

- (ii) *The functor  $\text{Idem}(i_{\oplus}): \text{Idem}(\underline{S}_{\oplus}) \rightarrow \text{Idem}(\underline{S \times T}_{\oplus})$  has retraction, namely  $\text{Idem}(\text{pr}_{\oplus}): \text{Idem}(\underline{S \times T}_{\oplus}) \rightarrow \text{Idem}(\underline{S}_{\oplus})$ ;*

- (iii) *The functor  $\text{Idem}(i_{\oplus}): \text{Idem}(\underline{S}_{\oplus}) \rightarrow \text{Idem}(\underline{S \times T}_{\oplus})$  is faithfully flat.*

*Proof.* (i) Next we construct for every object  $([l], p)$  in  $\text{Idem}(\underline{S}_{\oplus})$  an isomorphism in  $S \times T\text{-MOD}_{\text{fgp}}$

$$T([l], p): \text{pr}^* \circ \Theta_S([l], p) \xrightarrow{\cong} \Theta_{S \times T} \circ \text{Idem}(i_{\oplus})([l], p).$$

Let  $A$  be the  $(l, l)$ -matrix over  $S$ , for which  $p: [l] \rightarrow [l]$  is given by  $A$ . If  $i(A)$  is the  $(l, l)$ -matrix over  $S \times T$  given by applying  $i$  to each element in  $A$ , then  $\Theta_{S \times T} \circ i_{\oplus}(p)$  is the  $S \times T$ -homomorphism  $r_{i(A)}: (S \times T)^l \rightarrow (S \times T)^l$  given by right multiplication with  $i(A)$ . Let  $i^l: S^l \rightarrow (S \times T)^l$  be the map sending  $(x_1, x_2, \dots, x_l)$  to  $(i(x_1), i(x_2), \dots, i(x_l))$ . We obtain a commutative diagram of abelian groups

$$\begin{array}{ccc} S^l & \xrightarrow{i^l} & (S \times T)^l \\ r_A \downarrow & & \downarrow r_{i(A)} \\ S^l & \xrightarrow{i'^l} & (S \times T)^l. \end{array}$$

Now  $i'^l$  induce a homomorphism of abelian groups.

$$T([l], p): \text{im}(r_A) \rightarrow \text{im}(r_{i(A)}).$$

It is injective, since  $i$  and hence  $i^l$  is injective. Next we show that  $T([l], p)$  is bijective. Let  $y$  be an element of the image of  $r_{i(A)}$ . Choose  $x = ((s_1, t_1), \dots, (s_l, t_l))$  in  $(S \times T)^l$  with  $r_{i(A)}(x) = y$ . Define  $x' \in S$  by  $x' = (s_1, \dots, s_l)$ . Then  $r_{i(A)} \circ i^l(x') = r_{i(A)}(x) = y$ . Hence  $i^l$  sends  $r_A(x')$  to  $y$ . This finishes the proof that  $T([l], p)$  is an isomorphisms of abelian groups. One easily checks that it is an isomorphism of  $S \times T$ -modules.

We leave it to the reader to check that the collection of the isomorphisms  $T([l], p)$  defines a natural equivalence of functors  $\text{Idem}(\underline{S}_{\oplus}) \rightarrow S \times T\text{-MOD}_{\text{fgp}}$  from  $\text{pr}^* \circ \Theta_S$  to  $\Theta_{S \times T} \circ \text{Idem}(i_{\oplus})$ .

- (ii) This follows from  $\text{pr} \circ i = \text{id}_S$ .

- (iii) Since restriction is faithfully flat, the claim follows from assertion (i).  $\square$

We record for later purposes

**Lemma 7.18.** *Suppose that  $Q$  is compact. Consider normal compact open subgroups  $K$  and  $K'$  of  $G$  satisfying  $K' \subseteq K$  and  $K, K' \in P$ . Let*

$$i: \mathcal{H}(G//K; R, \rho, \omega) \rightarrow \mathcal{H}(G//K'; R, \rho, \omega)$$

be the inclusion of rings. Let  $m \geq 0$  be an integer. Denote by

$$i[\mathbb{Z}^m]: \mathcal{H}(G//K; R, \rho, \omega)[\mathbb{Z}^m] \rightarrow \mathcal{H}(G//K'; R, \rho, \omega)[\mathbb{Z}^m]$$

the inclusion of the (untwisted) group rings induced by  $i$ .

Then the functor

$$\begin{aligned} \text{Idem}(i[\mathbb{Z}^m]_{\oplus}): \text{Idem}(\mathcal{H}(G//K; R, \rho, \omega)[\mathbb{Z}^m]_{\oplus}) \\ \rightarrow \text{Idem}(\mathcal{H}(G//K'; R, \rho, \omega)[\mathbb{Z}^m]_{\oplus}) \end{aligned}$$

has a retraction and is faithfully flat.

*Proof.* This follows from Lemma 7.17 and the decomposition of unital rings

$$\mathcal{H}(G//K'; R, \rho, \omega) = \mathcal{H}(G//K; R, \rho, \omega) \oplus (1_{K'} - 1_K)\mathcal{H}(G//K'; R, \rho, \omega)(1_{K'} - 1_K),$$

cf. Lemma 7.16.  $\square$

### 7.E. Proof of Theorem 7.2.

**Lemma 7.19.** *Let  $\mathcal{A}_i$  be a collection of additive categories. Then  $\bigoplus_{i \in I} \mathcal{A}_i$  is  $l$ -uniformly regular or regular respectively, if and only if each  $\mathcal{A}_i$  is  $l$ -uniformly regular or regular respectively.*

*Proof.* This is a consequence of the observations following from [2, Lemma 5.3], that for an object  $A \in \bigoplus_{i \in I} \mathcal{A}_i$  there exists a finite subset  $J \subseteq I$  with  $A \in \bigoplus_{i \in J} \mathcal{A}_i$  and we have the identifications

$$\begin{aligned} \text{mor}_{\bigoplus_{i \in I} \mathcal{A}_i}(\cdot, A) &= \text{mor}_{\bigoplus_{i \in J} \mathcal{A}_i}(\cdot, A); \\ \mathbb{Z}(\bigoplus_{i \in J} \mathcal{A}_i)\text{-MOD} &= \prod_{i \in J} \mathbb{Z}\mathcal{A}_i\text{-MOD}. \end{aligned}$$

More details of the proof can be found in [2, Section 11].  $\square$

Consider a sequence  $G = K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$  of normal compact open subgroups of  $G$  with  $\bigcap_{n \geq 0} K_n = \{1\}$  such that  $K_n \in P$  holds for  $n \in \mathbb{N}$ . We get from Lemma 7.16 identifications of additive categories

$$\begin{aligned} \bigoplus_{m \geq 0} \text{Idem}(\mathcal{H}(G//K_m; R, \rho, \omega)[\mathbb{Z}^m]_{\oplus}) &= \text{Idem}(\mathcal{H}(G)_{\oplus}[\mathbb{Z}^m]); \\ \bigoplus_{m=0}^n \text{Idem}(\mathcal{H}(G//K_m; R, \rho, \omega)[\mathbb{Z}^m]_{\oplus}) &= \text{Idem}(\mathcal{H}(G//K_n)_{\oplus}[\mathbb{Z}^m]). \end{aligned}$$

Hence by Lemma 7.19 it suffices to show that  $\text{Idem}(\mathcal{H}(G//K_n)_{\oplus}[\mathbb{Z}^m])$  is  $(l + 2m)$ -uniformly regular or regular respectively for every  $n \in \mathbb{N}$ .

The unital ring  $\mathcal{H}(G//K_n)$  is  $l$ -uniformly regular or regular respectively, since  $R$  is  $l$ -uniformly regular or regular respectively by assumption and we have Lemma 7.4 and Lemma 7.15. Hence  $\text{Idem}(\mathcal{H}(G//K_n)_{\oplus}[\mathbb{Z}^m])$  is  $(l + 2m)$ -uniformly regular or regular respectively by [2, Corollary 6.5 and Theorem 10.1]. This finishes the proof of Theorem 7.2.

8. NEGATIVE  $K$ -GROUPS AND THE PROJECTIVE CLASS GROUP OF HECKE ALGEBRAS OVER COMPACT TD-GROUPS

Let  $G, N, Q := G/N$ ,  $\text{pr}: G \rightarrow Q$ ,  $R, \mathcal{P}, \rho, \omega$ , and  $\mu$  be as in Subsection 2.A and denote by  $\mathcal{H}(G; R, \rho, \omega)$  the Hecke algebra, which we have introduced in Subsection 2.B. Our main assumption in this section will be that  $Q$  is compact.

**Lemma 8.1.** *Suppose that  $Q$  is compact and  $N$  is locally central. Suppose that the unital ring  $R$  is regular and satisfies  $\mathbb{Q} \subseteq R$ . Then:*

(i) *Let  $\mathcal{K}$  be the set of compact open normal subgroups  $K \subseteq G$  with  $K \in \mathcal{P}$  directed by  $K \leq K' \iff K' \subseteq K$ .*

*Then we get for  $n \in \mathbb{Z}$*

$$K_n(\mathcal{H}(G; R, \rho, \omega)) = \text{colim}_{K \in \mathcal{K}} K_n(\mathcal{H}(G//K; R, \rho, \omega));$$

(ii) *We get*

$$K_n(\mathcal{H}(G; R, \rho, \omega)) = 0 \quad \text{for } n \leq -1.$$

*Proof.* (i) We conclude from Lemma 2.25 and Lemma 7.3

$$\mathcal{H}(G; R, \rho, \omega) = \bigcup_{K \in \mathcal{K}} \mathcal{H}(G//K; R, \rho, \omega).$$

Now apply (4.4).

(ii) For  $K \in \mathcal{K}$  the unital ring  $\mathcal{H}(G//K; R, \rho, \omega)$  is regular by Lemma 7.4 (ii) and Lemma 7.15. Hence  $K_n(\mathcal{H}(G//K; R, \rho, \omega)) = \{0\}$  for  $n \leq -1$ , see [15, page 154]. Now apply assertion (i).  $\square$

**Remark 8.2.** Suppose that  $Q$  is compact and  $N$  is locally central. Because of Lemma 7.3 we can choose a nested sequence of elements in  $\mathcal{K}$

$$K_0 \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$$

satisfying  $\bigcap_{i=0}^{\infty} K_i = \{1\}$ . Then for every  $K \in \mathcal{K}$  there is a natural number  $i$  with  $K_i \subseteq K$ . Abbreviate  $\mathcal{H}(G//K_i) = \mathcal{H}(G//K_i; R, \rho, \omega)$ . Then the inclusion  $\mathcal{H}(G//K_i) \rightarrow \mathcal{H}(G//K_{i+1})$  induces a split injection  $K_n(\mathcal{H}(G//K_i)) \rightarrow K_n(\mathcal{H}(G//K_{i+1}))$  for  $i \in \mathbb{N}$  and  $n \in \mathbb{Z}$  by Lemma 7.18. Lemma 8.1 (i) implies that there is an isomorphism

$$\begin{aligned} K_n(\mathcal{H}(G; R, \rho, \omega)) \\ \cong K_n(\mathcal{H}(G//K_0)) \oplus \bigoplus_{i \geq 0} \text{cok}(K_n(\mathcal{H}(G//K_i)) \rightarrow K_n(\mathcal{H}(G//K_{i+1}))) \end{aligned}$$

and  $\text{cok}(K_n(\mathcal{H}(G//K_i)) \rightarrow K_n(\mathcal{H}(G//K_{i+1})))$  is isomorphic to a direct summand of  $K_n(\mathcal{H}(G//K_{i+1}))$ .

Now suppose additionally that  $R$  is semisimple. Then  $\mathcal{H}(G//K_i)$  is semisimple and hence the abelian group  $K_0(\mathcal{H}(G//K_i))$  is finitely generated free for  $i \in \mathbb{N}$  by Lemma 7.4 (iv) and Lemma 7.15, Hence the abelian group  $K_0(\mathcal{H}(G; R, \rho, \omega))$  is free and in particular torsionfree.

9. ON THE ALGEBRAIC  $K$ -THEORY OF THE HECKE ALGEBRA OF A COVIRTUALLY  $\mathbb{Z}$  TOTALLY DISCONNECTED GROUP

Consider the setup of Section 5. In particular  $Q$  is covirtually cyclic. Denote by  $\mathbf{T}_{\mathbf{K}^\infty(\phi^{-1})}$  the mapping torus of the map

$$\mathbf{K}^\infty(\phi^{-1}): \mathbf{K}^\infty(\mathcal{H}(L; R, \rho|_L, \omega)) \rightarrow \mathbf{K}^\infty(\mathcal{H}(L; R, \rho|_L, \omega))$$

of non-connective  $K$ -theory spectra.

**Theorem 9.1** (Wang sequence). *Suppose that the unital ring  $R$  is regular and satisfies  $\mathbb{Q} \subseteq R$ . Assume that  $N$  is locally central. Then:*

(i) There is a weak homotopy equivalence of non-connective spectra

$$\mathbf{a}^\infty : \mathbf{T}_{\mathbf{K}^\infty(\phi^{-1})} \xrightarrow{\simeq} \mathbf{K}^\infty(\mathcal{H}(G; R, \rho, \omega));$$

(ii) We get a long exact sequence, infinite to the left

$$\begin{aligned} \dots &\xrightarrow{K_2(i)} K_2(\mathcal{H}(G; R, \rho, \omega)) \xrightarrow{\partial_2} K_1(\mathcal{H}(L; R, \rho|_L, \omega)) \\ &\xrightarrow{\text{id} - K_1(\phi^{-1})} K_1(\mathcal{H}(L; R, \rho|_L, \omega)) \xrightarrow{K_1(i)} K_1(\mathcal{H}(G; R, \rho, \omega)) \\ &\xrightarrow{\partial_1} K_0(\mathcal{H}(L; R, \rho|_L, \omega)) \xrightarrow{\text{id} - K_0(\phi^{-1})} K_0(\mathcal{H}(L; R, \rho|_L, \omega)) \\ &\xrightarrow{K_0(i)} K_0(\mathcal{H}(G; R, \rho, \omega)) \rightarrow 0; \end{aligned}$$

(iii) We get for  $n \leq 1$

$$K_n(\mathcal{H}(G; R, \rho, \omega)) = 0.$$

*Proof.* (i) This follows from Lemma 5.7 and Theorem 6.2 applied to the additive category  $\mathcal{A} = \text{Idem}(\underline{\mathcal{H}(L; R, \rho|_L, \omega)}_{\oplus})$  after we have shown that the additive category  $\text{Idem}(\underline{\mathcal{H}(L; R, \rho|_L, \omega)}_{\oplus})$  is regular. This has already been done in Theorem 7.2.

(ii) and (iii) These follow from the Wang sequence associated to the left hand side of the weak homotopy equivalence appearing in assertion (i) and Lemma 8.1 (ii).  $\square$

## 10. SOME INPUT FOR THE FARRELL-JONES CONJECTURE

In forthcoming papers we will need for the proof and the application of the  $K$ -theoretic Farrell-Jones Conjecture for the Hecke algebra of a closed subgroup of a reductive  $p$ -adic group, which is our ultimate goal, Theorem 7.2 and the following Theorem 10.1.

Consider the setup of Subsection 2.A. For the remainder of this subsection we will assume that the td-group  $Q$  is compact and  $N$  is locally central. Let  $\bar{i}: Q' \rightarrow Q$  be the inclusion of a compact open subgroup of  $Q$ . Put  $G' = \text{pr}^{-1}(Q')$ . Let  $i: G' \rightarrow G$  be the inclusion. The construction in Subsection 2.C yields a ring homomorphism

$$\mathcal{H}(i): \mathcal{H}(G'; R, \rho', \omega) \rightarrow \mathcal{H}(G; R, \rho, \omega)$$

where  $\rho' = \rho \circ i$ ,  $\mu'$  is obtained from  $\mu$  by restriction with  $i$ , and we take  $N' = N$  and  $\omega' = \omega$ . The image  $\mathcal{H}(i)(s)$  of an element  $s \in \mathcal{H}(G'; R, \rho', \omega)$ , which is given by an appropriate function  $s: G' \rightarrow R$ , is specified by the function  $\mathcal{H}(i)(s): G \rightarrow R$  sending  $g$  to  $s(g)$ , if  $g \in G'$ , and to 0, if  $g \notin G'$ , see Lemma 2.21 (iii).

**Theorem 10.1.** *Suppose that  $Q$  is compact and  $N$  is locally central. Then the functor of unital additive categories*

$$\text{Idem}(\underline{\mathcal{H}(i)}_{\oplus}[\mathbb{Z}^m]): \text{Idem}(\underline{\mathcal{H}(G'; R, \rho', \omega)}_{\oplus}[\mathbb{Z}^m]) \rightarrow \text{Idem}(\underline{\mathcal{H}(G; R, \rho, \omega)}_{\oplus}[\mathbb{Z}^m])$$

*is faithfully flat.*

*Proof.* Let  $\mathcal{K}'$  be the directed set of normal compact open subgroups of  $Q$  which satisfy  $K \subseteq Q'$ , and  $K \in P$ , where we put  $K \leq K' \iff K' \subseteq K$ . Note that for any compact open subgroup  $L$  of  $Q$  there exists  $K \in \mathcal{K}'$  with  $K \subseteq L$  by Lemma 7.3.

In the sequel we abbreviate

$$\begin{aligned} \mathcal{H}(G) &:= \mathcal{H}(G; R, \rho, \omega); \\ \mathcal{H}(G//K) &:= \mathcal{H}(G//K; R, \rho, \omega), \end{aligned}$$

and analogously for  $G'$ . Next we want to show that the functor

$$\text{Idem}(\underline{j_K}_{\oplus}[\mathbb{Z}^m]): \text{Idem}(\underline{\mathcal{H}(G//K)}_{\oplus}[\mathbb{Z}^m]) \rightarrow \text{Idem}(\underline{\mathcal{H}(G)}_{\oplus}[\mathbb{Z}^m])$$

is faithfully flat for  $K \in \mathcal{K}'$ , where  $i_K: \mathcal{H}(G//K) \rightarrow \mathcal{H}(G)$  is the inclusion. Consider morphisms  $f_0: A_0 \rightarrow A_1$  and  $f_1: A_1 \rightarrow A_2$  in  $\text{Idem}(\underline{\mathcal{H}(G//K)}_{\oplus}[\mathbb{Z}^m])$  with  $f_1 \circ f_0 = 0$ . Note that we can consider them also as morphisms in  $\text{Idem}(\underline{\mathcal{H}(G)}_{\oplus}[\mathbb{Z}^m])$ . We have to show that it is exact in  $\text{Idem}(\underline{\mathcal{H}(G//K)}_{\oplus}[\mathbb{Z}^m])$ , if and only if it is exact in  $\text{Idem}(\underline{\mathcal{H}(G)}_{\oplus}[\mathbb{Z}^m])$ .

Suppose that  $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2$  is exact in  $\text{Idem}(\underline{\mathcal{H}(G//K)}_{\oplus}[\mathbb{Z}^m])$ . In order to show that it is exact in  $\text{Idem}(\underline{\mathcal{H}(G)}_{\oplus}[\mathbb{Z}^m])$ , we have to find for any object  $A$  and any morphism  $g: Q \rightarrow A_1$  in  $\text{Idem}(\underline{\mathcal{H}(G)}_{\oplus}[\mathbb{Z}^m])$  with  $f_1 \circ g = 0$  a morphism  $\bar{g}: A \rightarrow A_0$  in  $\text{Idem}(\underline{\mathcal{H}(G)}_{\oplus}[\mathbb{Z}^m])$  with  $f_0 \circ \bar{g} = g$ . We can choose an element  $K' \in \mathcal{K}'$  with  $K \leq K'$  such that  $A$  and  $g$  live already in  $\text{Idem}(\underline{\mathcal{H}(G//K')}_{\oplus}[\mathbb{Z}^m])$  by Lemma 2.25. Since the inclusion

$$\text{Idem}(\underline{\mathcal{H}(G//K)}_{\oplus}[\mathbb{Z}^m]) \rightarrow \text{Idem}(\underline{\mathcal{H}(G//K')}_{\oplus}[\mathbb{Z}^m])$$

is faithfully flat by Lemma 7.18, we can find  $\bar{g}: A \rightarrow P_0$  with  $f_0 \circ \bar{g} = g$  in  $\text{Idem}(\underline{\mathcal{H}(G//K')}_{\oplus}[\mathbb{Z}^m])$  and hence also in  $\text{Idem}(\underline{\mathcal{H}(G)}_{\oplus}[\mathbb{Z}^m])$ .

Suppose that  $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2$  is exact in  $\text{Idem}(\underline{\mathcal{H}(G)}_{\oplus}[\mathbb{Z}^m])$ . In order to show that it is exact in  $\text{Idem}(\underline{\mathcal{H}(G//K)}_{\oplus}[\mathbb{Z}^m])$  we have to find for any object  $A$  and any morphism  $g: Q \rightarrow A_1$  in  $\text{Idem}(\underline{\mathcal{H}(G//K)}_{\oplus}[\mathbb{Z}^m])$  with  $f_1 \circ g = 0$  a morphism  $\bar{g}: A \rightarrow A_0$  in  $\text{Idem}(\underline{\mathcal{H}(G//K)}_{\oplus}[\mathbb{Z}^m])$  with  $f_0 \circ \bar{g} = g$ . At any rate we can find such  $\bar{g}: A \rightarrow A_1$  in  $\text{Idem}(\underline{\mathcal{H}(G)}_{\oplus}[\mathbb{Z}^m])$ . We conclude from Lemma 2.25 that there exists  $K' \in \mathcal{K}'$  with  $K \leq K'$  such that  $\bar{g}: A \rightarrow A_1$  lies already in  $\text{Idem}(\underline{\mathcal{H}(G//K')}_{\oplus}[\mathbb{Z}^m])$ . Recall from Lemma 7.18 that there is a retraction of the inclusion

$$\text{Idem}(\underline{\mathcal{H}(G//K)}_{\oplus}[\mathbb{Z}^m]) \rightarrow \text{Idem}(\underline{\mathcal{H}(G//K')}_{\oplus}[\mathbb{Z}^m])$$

If we apply it to  $\bar{g}$ , we get a morphism  $\bar{g}': A \rightarrow A_1$  in  $\text{Idem}(\underline{\mathcal{H}(G//K)}_{\oplus}[\mathbb{Z}^m])$  satisfying  $f_1 \circ \bar{g}' = g$  in  $\text{Idem}(\underline{\mathcal{H}(G//K)}_{\oplus}[\mathbb{Z}^m])$ . This finishes the proof that functor  $\text{Idem}(j_{K\oplus}[\mathbb{Z}^m])$  is faithfully flat. Analogously one shows that the functor  $\text{Idem}(j'_{K\oplus}[\mathbb{Z}^m]): \text{Idem}(\underline{\mathcal{H}(G'//K)}_{\oplus}[\mathbb{Z}^m]) \rightarrow \text{Idem}(\underline{\mathcal{H}(G')}_{\oplus}[\mathbb{Z}^m])$  is faithfully flat for the inclusion  $j'_K: \mathcal{H}(G'//K) \rightarrow \mathcal{H}(G')$ .

We have the following commutative diagram of functors of additive categories

$$\begin{array}{ccc} \text{Idem}(\underline{\mathcal{H}(G')}_{\oplus}[\mathbb{Z}^m]) & \xrightarrow{\text{Idem}(\underline{\mathcal{H}(i)}_{\oplus}[\mathbb{Z}^m])} & \text{Idem}(\underline{\mathcal{H}(G)}_{\oplus}[\mathbb{Z}^m]) \\ \text{Idem}(j'_{K\oplus}[\mathbb{Z}^m]) \uparrow & & \uparrow \text{Idem}(j_{K\oplus}[\mathbb{Z}^m]) \\ \text{Idem}(\underline{\mathcal{H}(G'//K)}_{\oplus}[\mathbb{Z}^m]) & \xrightarrow{\text{Idem}(\underline{\mathcal{H}(i//K)}_{\oplus}[\mathbb{Z}^m])} & \text{Idem}(\underline{\mathcal{H}(G//K)}_{\oplus}[\mathbb{Z}^m]) \end{array}$$

whose two left vertical arrows are faithfully flat. We conclude from Lemma 2.25 that it suffices to show that the lower vertical arrow in the diagram above is faithfully flat.

We have identified  $\mathcal{H}(G//K)$  and  $\mathcal{H}(G'//K)$  respectively as a crossed product ring  $R * F$  and  $R * F'$  respectively for the finite group  $F = G/K$  and  $F' = G'/K$  respectively in Lemma 7.15. Moreover the inclusion  $\mathcal{H}(G//K)[\mathbb{Z}^m] \rightarrow \mathcal{H}(G'//K)[\mathbb{Z}^m]$  corresponds under these identifications to the inclusions  $R * F[\mathbb{Z}^m] \rightarrow R * F'[\mathbb{Z}^m]$  coming from the inclusion of finite groups  $F' \rightarrow F$ . The lower horizontal arrow  $\text{Idem}(\underline{\mathcal{H}(i//K)}_{\oplus}[\mathbb{Z}^m])$  becomes under the equivalences of categories of (3.2)

and (3.4) the functor

$$F: R * F'[\mathbb{Z}^m]\text{-MOD}_{\text{fgp}} \rightarrow R * F[\mathbb{Z}^m]\text{-MOD}_{\text{fgp}}, \quad P \mapsto R * F[\mathbb{Z}^m] \otimes_{R * F'[\mathbb{Z}^m]} P.$$

There is a commutative diagram

$$\begin{array}{ccc} R * F'[\mathbb{Z}^m]\text{-MOD}_{\text{fgp}} & \xrightarrow{F} & R * F[\mathbb{Z}^m]\text{-MOD}_{\text{fgp}} \\ \downarrow & & \downarrow \\ R[\mathbb{Z}^m]\text{-MOD}_{\text{fgp}} & \longrightarrow & R[\mathbb{Z}^m]\text{-MOD}_{\text{fgp}} \end{array}$$

whose vertical arrows are given by restriction from  $R * F[\mathbb{Z}^m]$  or  $R * F'[\mathbb{Z}^m]$  to  $R[\mathbb{Z}^m]$  and whose lower vertical arrow is given by  $P \mapsto \bigoplus_{i=1}^{[F:F']} P$ . Since the vertical arrows and the lower horizontal arrow are obviously faithfully flat, the upper vertical arrow is faithfully flat. This finishes the proof of Lemma 10.1.  $\square$

## 11. CHARACTERISTIC $p$

We have assumed  $\mathbb{Q} \subseteq R$ , or in other words that any natural number  $n \geq 1$  is invertible in  $R$ . One may wonder what happens, if one drops this condition, for instance, if  $R$  is a field of prime characteristic. The following condition appearing in [6, page 9] suffices to make sense of the Hecke algebra.

**Condition 11.1.** *There exists a compact open subgroup  $K$  in  $Q$  such that the index  $[K : K_0]$  of any open subgroup  $K_0$  of  $K$  is invertible in  $R$ .*

Let  $Q$  be a reductive  $p$ -adic group. Then Condition 11.1 is satisfied, if  $p$  is invertible in  $R$ , see [6, page 9].

However, this does not mean that assertion of the Farrell-Jones Conjecture or Theorem 9.1 remains true integrally. Our arguments would go through if for every compact open subgroup  $K$  in  $Q$  the index  $[K : K_0]$  of any open subgroup  $K_0$  of  $K$  is invertible in  $R$  which is stronger than Condition 11.1.

One may hope that under Condition 11.1 the Farrell-Jones Conjecture or Theorem 9.1 remain true rationally. Let us confine ourselves to the setup of Section 5 and Theorem 9.1. Then we get from [12, Theorem 0.1] a weak homotopy equivalence, where we abbreviate  $\mathcal{H}(G) := \mathcal{H}(G; R, \rho, \omega)$  and analogously for  $L$

$$\begin{aligned} \mathbf{TK}^\infty(\text{Idem}(\phi)^{-1}) \vee \mathbf{NK}^\infty(\text{Idem}(\underline{\mathcal{H}(L)}_{\oplus})_{\text{Idem}(\phi_{\oplus})}[t]) \vee \mathbf{NK}^\infty(\text{Idem}(\underline{\mathcal{H}(L)}_{\oplus})_{\text{Idem}(\phi_{\oplus})}[t^{-1}]) \\ \xrightarrow{\cong} \mathbf{K}^\infty(\mathcal{H}(G; R, \rho, \omega)). \end{aligned}$$

So we need to show that the homotopy groups of the Nil-terms all vanish rationally. If  $L$  is finite, this is known to be true, see [11, Theorem 0.3 and Theorem 9.4]. Under the strong condition that there is a sequence  $L \supseteq L_1 \supseteq L_2 \supseteq L_2 \cdots$  of in  $L$  normal compact open subgroups such that  $\bigcap_{i \geq 0} L_i = \{1\}$  and  $\phi(L_i) = L_i$  holds for  $i \geq 0$ , this implies that the homotopy groups of the Nil-terms all vanish rationally. Without this strong condition we do not have a proof.

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