ON BROWN'S PROBLEM, POINCARÉ MODELS FOR THE CLASSIFYING SPACES FOR PROPER ACTIONS AND NIELSEN REALIZATION

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Abstract. There is the problem, whether for a given virtually torsionfree discrete group Γ there exists a cocompact proper topological Γ-manifold, which is equivariantly homotopy equivalent to the classifying space for proper actions. It is related to Nielsen's Realization and to the problem of Brown, whether there is a d-dimensional model for the classifying space for proper actions, if the underlying group has virtually cohomological dimension d. Assuming that the expected manifold model has a zero-dimensional singular set, we solve the problem in the Poincaré category and obtain new results about Brown's problem under certain conditions concerning the underlying group, for instance if it is hyperbolic. In a sequel paper together with James Davis we will deal with this on the level of topological manifolds.

1. Introduction

1.1. **Manifold models for the classifying space for proper actions.** Let Γ be a discrete group. One can associate to it the *classifying space for proper* Γ*-actions* $\underline{E}\Gamma$. This is a Γ-CW-complex, whose isotropy groups are finite and whose H-fixed point set $\underline{E}\Gamma^H$ is contractible for every finite subgroup $H \subseteq \Gamma$. Two such models are Γ-homotopy equivalent. These Γ-CW-complexes $\underline{E}\Gamma$ are interesting in their own right and have many applications to group theory and equivariant homotopy theory and play a prominent role for the Baum-Connes Conjecture and the Farrell-Jones Conjecture. For a survey on $\underline{E}\Gamma$ we refer for instance to [25].

A natural question is, whether there are nice models for $\underline{E}\Gamma$. One may ask for finiteness properties up to Γ-homotopy equivalence such as being finite or finite dimensional. One may also try to address the much harder question, whether there is a *cocompact manifold model* for $\underline{E}\Gamma$, i.e., whether there is a cocompact proper Γ-manifold without boundary, which is homotopy equivalent to $\underline{E}\Gamma$. If the answer is yes, one of course encounters the problem, in which sense such a manifold model is unique. In general it makes a difference, whether one is asking this question in the smooth or in the topological category. We will mainly deal with the topological category.

In general there is no cocompact manifold model. There are some well-known prominent examples, where cocompact manifold models exist. For instance, if Γ acts properly, isometrically and cocompactly on a Riemannian manifold M with non-positive sectional curvature, then M is a cocompact model for $\underline{E}\Gamma$. Let L be a connected Lie group. Then L contains a maximal compact subgroup K, which is unique up to conjugation. If $\Gamma \subseteq L$ is a discrete subgroup such that L/Γ is compact, then $M = L/K$ is a smooth manifold with smooth Γ-action and a cocompact model for $\underline{E}\Gamma$.

Date: January, 2022.
2020 *Mathematics Subject Classification.* 55R35,57P10.
Key words and phrases. Poincaré complexes, Brown's Problem, classifying spaces for proper actions.

Our long term goal is to answer the question about a manifold model in a specific but interesting situation. Namely, we will assume throughout this paper that Γ is virtually torsionfree, in other words that there is a group extension

$$1 \to \pi \xrightarrow{i} \Gamma \xrightarrow{p} G \to 1 \tag{1.1}$$

such that π is torsionfree and G is finite. Constructing a manifold model M for $\underline{E}\Gamma$ is very hard, since one has to deal with equivariant surgery. We will confine ourselves to the case, where M is *pseudo-free*, i.e., the singular set $M^{>1} = \{x \in M \mid \Gamma_x \neq \{1\}\}$ is discrete, or, equivalently, the Γ-space $M^{>1}$ is the disjoint union of its Γ-orbits. Here and in the sequel Γ_x denotes the isotropy group $\{\gamma \in \Gamma \mid \gamma \cdot x = x\}$ of $x \in M$.

Let d be the dimension of M. Moreover, we will assume that $M^{>1}$ nicely embeds into M, namely, for every $x \in M^{>1}$ we assume the existence of a d-dimensional disk D_x with Γ_x-action and an Γ_x-equivariant embedding $D_x \subseteq M$ such that the origin of D_x is mapped to x and the Γ_x-action on D_x is free outside the origin. Let S_x be the boundary of D_x, which is a $(d-1)$-dimensional sphere with free Γ_x-action. We will assume that S_x carries a Γ_x-CW-structure, which is automatically the case, if $d \neq 5$, just apply [12, Section 9.2] or [16, Theorem 2.2 in III.2 on page 107] to S_x/Γ_x. Note that this implies that (D_x, S_x) is a finite Γ_x-CW-pair, since D_x is the cone over S_x.

Let $I \subset M^{>1}$ be a set-theoretic transversal of the projection $M^{>1} \to M^{>1}/\Gamma$. Put

$$\partial X := \coprod_{x \in I} \Gamma \times_{\Gamma_x} S_x; \tag{1.2}$$

$$C(\partial X) := \coprod_{x \in I} \Gamma \times_{\Gamma_x} D_x. \tag{1.3}$$

Then we get a free cocompact proper Γ-manifold X with boundary ∂X, if we put

$$X := M \setminus \left(\coprod_{x \in I} \Gamma \times_{\Gamma_x} (D_x \setminus S_x) \right) \tag{1.4}$$

and M becomes the Γ-pushout

$$\begin{tikzcd} \partial X = \coprod_{x \in I} \Gamma \times_{\Gamma_x} S_x \arrow[r] \arrow[d] & X \arrow[d] \\ C(\partial X) := \coprod_{x \in I} \Gamma \times_{\Gamma_x} D_x \arrow[r] & M. \end{tikzcd} \tag{1.5}$$

Note that the existence of the disks D_x and the existence of the diagram (1.5) is guaranteed, if M is a smooth Γ-manifold with discrete $M^{>1}$, since we can choose $C(\partial X)$ to be a tubular neighbourhood with boundary ∂X of the zero-dimensional Γ-invariant smooth submanifold $M^{>1}$ of M. Our main concern is

Problem 1.6 (Main Problem)**.** Does there exist a proper cocompact Γ-manifold M of the shape described in (1.5), which is Γ-homotopy equivalent to $\underline{E}\Gamma$?

If yes, are two such Γ-manifolds Γ-homeomorphic?

The collection $\{S_x \mid x \in I\}$ is an example of a so called *free d-dimensional slice system*, and this notion will be analyzed in Section 3. The pair $(X, \partial X)$ is an example of a so called *slice complement model for $\underline{E}\Gamma$*, and this notion will be investigated in Section 4.

1.2. **Some necessary conditions.** Next we figure out some necessary conditions for the existence of a Γ-manifold M, which is Γ-homotopy equivalent to $\underline{E}\Gamma$ and is of the shape described in (1.5).

Firstly we prove that any closed topological manifold M of the shape described in (1.5) has the Γ-homotopy type of a Γ-CW-complex.

The pair $(C(\partial X), \partial X)$ is a finite proper Γ-CW-pair of dimension d. The pair $(X/\Gamma, \partial X/\Gamma)$ is a topological compact manifold with boundary and hence is homotopy equivalent relative ∂X to a finite relative CW-complex $(Y, \partial X/\Gamma)$ of relative dimension d see [12, Section 9.2] or [16, Theorem 2.2 in III.2 on page 107]. Hence we can find a finite d-dimensional Γ-CW-pair $(\overline{Y}, \partial X)$, which is relatively free and is Γ-homotopy equivalent relative ∂X to $(X, \partial X)$. Take $Z = C(\partial X) \cup_{\partial X} \overline{Y}$. Then Z is a finite proper d-dimensional Γ-CW-complex with $Z^{>1} = M^{>1}$ such that M and Z are Γ-homotopy equivalent.

Note that this implies that M is Γ-homotopy equivalent to $\underline{E}\Gamma$, if and only if M is contractible.

Notation 1.7. Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal finite subgroups of Γ. Put

$$\begin{aligned}
\partial E\Gamma &:= \coprod_{F \in \mathcal{M}} \Gamma \times_F EF; \\
\partial \underline{E}\Gamma &:= \coprod_{F \in \mathcal{M}} \Gamma/F; \\
\partial B\Gamma &:= \coprod_{F \in \mathcal{M}} BF; \\
\underline{B}\Gamma &:= \underline{E}\Gamma/\Gamma.
\end{aligned}$$

In the sequel $H_*(Z) = H_*(Z, \mathbb{Z})$ for a space or CW-complex Z denotes its singular or cellular homology with coefficients in $\mathbb{Z}$.

We will consider the following conditions for the group Γ, where $H_d(B\Gamma, \partial B\Gamma)$ is the homology of the canonical map $\partial B\Gamma \to B\Gamma$.

Notation 1.8.

(M) Every non-trivial finite subgroup of Γ is contained in a unique maximal finite subgroup;

(NM) If F is a maximal finite subgroup, then $N_\Gamma F = F$;

(H) For the homomorphism $w\colon \Gamma \to \{\pm 1\}$ of Notation 6.7 the composite

$$H_d^\Gamma(E\Gamma, \partial E\Gamma; \mathbb{Z}^w) \xrightarrow{\partial} H_{d-1}^\Gamma(\partial E\Gamma; \mathbb{Z}^w) \xrightarrow{\cong} \bigoplus_{F \in \mathcal{M}} H_{d-1}^F(EF; \mathbb{Z}^{w|_F}) \to H_{d-1}^F(EF; \mathbb{Z}^{w|_F})$$

of the boundary map, the inverse of the obvious isomorphism, and the projection to the summand of $F \in \mathcal{M}$ is surjective for all $F \in \mathcal{M}$.

For simplicity we will assume in the remainder of this Subsection 1.2 that Γ-acts orientation preservingly on M, or, equivalently, that the homomorphism $w\colon \Gamma \to \{\pm 1\}$ of Notation 6.7 is trivial, and that $d = \dim(M)$ is even. These condition will be dropped in the main body of the paper, for instance the case, where d is odd and $d \geq 3$ is discussed in Section 10. Note that then the composite appearing in condition (H) above reduces to

$$H_d(B\Gamma, \partial B\Gamma) \xrightarrow{\partial} H_{d-1}(\partial B\Gamma) \xrightarrow{\cong} \bigoplus_{F \in \mathcal{M}} H_{d-1}(BF) \to H_{d-1}(BF).$$

Lemma 1.9. *Suppose that the topological Γ-manifold M of the shape described in* (1.5) *is Γ-homotopy equivalent to $\underline{E}\Gamma$ and Γ acts orientation preserving on M. Suppose that $d = \dim(M)$ is even and $d \geq 4$.*

Then the following conditions are satisfied:

1. *There is a finite d-dimensional Γ-CW-model for $\underline{E}\Gamma$ such that $\underline{E}\Gamma^{>1}$ is 0-dimensional;*
2. *The group Γ satisfies (M) and (NM). Moreover, we get a well-defined bijection*
$$I \xrightarrow{\cong} \mathcal{M}$$
by sending $x \in I$ to the element $F \in \mathcal{M}$, which is conjugated to Γ_x;
3. *The quotient M/π is an orientable aspherical closed d-dimensional manifold. In particular there is a finite CW-model for $B\pi$ and $H_d(B\pi)$ is infinite cyclic;*
4. *The group $H_d(B\Gamma, \partial B\Gamma)$ is infinite cyclic. For each $F \in \mathcal{M}$ the Tate cohomology of F is d-periodic and $H_{d-1}(BF)$ is finite cyclic of order $|F|$;*
5. *The group Γ-satisfies (H).*

Proof. (1) This follows from the consideration in the beginning of this Subsection 1.2.

(2) Consider a non-trivial finite subgroup $H \subseteq \Gamma$. Then H is contained in some maximal finite subgroup, since Γ is virtually torsionfree.

Let F_0 and F_1 be two maximal finite subgroups with $H \subsetneq F_0$ and $H \subseteq F_1$. Since $M^{F_0} \subseteq M^H$ and $M^{F_1} \subseteq M^H$ holds and M^H, M^{F_0} and M^{F_1} are contractible zero-dimensional manifolds and hence consist of precisely one point x, we get $M^{F_0} = M^H = M^{F_0} = \{x\}$ and hence $F_0 \subseteq \Gamma_x$ and $F_1 \subseteq \Gamma_x$. Since F_0 and F_1 are maximal finite and Γ_x is finite, we conclude $F_0 = F_1 = \Gamma_x$. Hence (M) is satisfied, the isotropy group of any point in M is either trivial or maximal finite, and any maximal finite subgroup occurs as isotropy group Γ_x of some element x in $M^{>1}$. Hence we get the desired bijection $I \xrightarrow{\cong} \mathcal{M}$, which we will use in the sequel as an identification.

Consider $F \in \mathcal{M}$. Then M^F consists of precisely one point x. Since $N_\Gamma F$ leaves M^F invariant, we conclude $N_\Gamma F \subseteq \Gamma_x = F$. Hence (NM) is satisfied.

(3) Since π is torsionfree and has finite index in Γ, it acts freely, properly, and cocompactly on M. Hence M/π is a closed orientable manifold and a finite CW-complex. This implies that $H_d(B\pi)$ is infinite cyclic. Since M is contractible, M/π is a model for $B\pi$.

(4) and (5) There is the following commutative diagram

$$\begin{tikzcd}
H_d(M/\Gamma, \partial M/\Gamma) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_d(\partial M/\Gamma) & \bigoplus_{F\in\mathcal{M}} H_{d-1}(S_F/F) \arrow[l, "\cong"'] \arrow[d] \\
H_d(B\Gamma, \partial B\Gamma) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_{d-1}(\partial B\Gamma) & \bigoplus_{F\in\mathcal{M}} H_{d-1}(BF) \arrow[l, "\cong"'] \\
H_d(\underline{B}\Gamma, \partial \underline{B}\Gamma) & & \\
H_d(\underline{B}\Gamma), \arrow[u, "\cong"] & &
\end{tikzcd}$$

whose maps are given by the obvious maps on space level, boundary homomorphisms, or the classifying maps $M/\Gamma \to B\Gamma$ and $S_F/F \to BF$. We will show using a transfer argument and conditions (M) and (NM) that the left vertical arrows are all bijective and that the inclusion $i\colon \pi \to \Gamma$ induces an injection of infinite cyclic groups $H_d(B\pi) \to H_d(\underline{B}\Gamma)$, see (6.4) and Lemma 6.21.

Since F acts freely on the $(d-1)$-dimensional sphere S_F, the finite group F has periodic cohomology, see [5, page 154]. Moreover $H_{d-1}(F)$ is cyclic of order $|F|$, see [5, Theorem 9.1 in Section VI.9 on page 154]. Each map $H_{d-1}(S_F/F) \to H_{d-1}(BF)$ is surjective, as S_F is $(d-2)$-connected. Since $(M/\Gamma, \partial M/\Gamma)$ is an orientable connected compact manifold, $H_d(M/\Gamma, \partial M/\Gamma)$ is infinite cyclic and the image of the fundamental class in $H_d(M/\Gamma, \partial M/\Gamma)$ under

$$H_d(M/\Gamma, \partial M/\Gamma) \xrightarrow{\partial_d} H_{d-1}(\partial M/\Gamma) \cong \bigoplus_{C \in \pi_0(\partial M/\Gamma)} H_{d-1}(C)$$

is given by the collection of the fundamental classes in $H_{d-1}(C)$ for each path component C of ∂M. Hence the composite

$$H_d(M/\Gamma, \partial M/\Gamma) \xrightarrow{\partial_d} H_{d-1}(\partial M/\Gamma) \xrightarrow{\cong} \bigoplus_{F \in \mathcal{M}} H_{d-1}(S_F/F) \to H_{d-1}(S_F/F)$$

of the boundary map, the inverse of the obvious isomorphism, and the projection to the summand of $F \in \mathcal{M}$ is surjective for all $F \in \mathcal{M}$. This finishes the proof of Lemma 1.9. □

Example 1.10. The following example is due to Dominik Kirstein and Christian Kremer. If G is cyclic of order p for a prime number p, and $H_k(B\pi;\mathbb{Z})_{(p)}$ vanishes for $1 \le k \le (d-1)$, then (H) is automatically satisfied. Its proof is left to the reader.

1.3. **Brown's problem.** The condition appearing in Theorem 1.9 (1) is hard to check and looks very restrictive. In particular we have to find a finite d-dimensional Γ-CW-complex model for $\underline{E}\Gamma$. This is a necessary condition, which is not at all obvious. In view of Theorem 1.9 (3) the assumption that there is a finite d-dimensional model for $B\pi$ is reasonable and will be made. So we are dealing with a special case of the following problem due to Brown [4, page 32].

Problem 1.11 (Brown's problem)**.** For which discrete groups Γ, which contain a torsionfree subgroup π of finite index and have virtual cohomological dimension $\le d$, does there exist a d-dimensional Γ-CW-model for $\underline{E}\Gamma$?

Meanwhile there are examples, where the answer is negative for Brown's problem 1.11 see [18], [19]. So in order to prove that the condition appearing in Theorem 1.9 (1) is satisfied, we must give a proof of a positive answer to Brown's problem 1.11 in our special case, and actually much more. We will show

Theorem 1.12 (Models for the classifying space for proper Γ-actions)**.** *Assume that the following conditions are satisfied:*

1. *The natural number d satisfies $d \ge 3$;*
2. *The group Γ satisfies conditions (M) and (NM), see Notation 1.8;*
3. *The group Γ satisfies one of the following conditions:*
   - (a) *There exists a finite Γ-CW-model for $\underline{E}\Gamma$;*
   - (b) *The group Γ is hyperbolic;*
   - (c) *The group Γ acts cocompactly properly and isometrically on a proper* CAT(0)*-space;*
4. *There is a finite CW-complex model of dimension d for $B\pi$.*

Then there exists a finite Γ-CW-model X for $\underline{E}\Gamma$ of dimension d such that its singular Γ-subspace $X^{>1}$ is $\coprod_{F \in \mathcal{M}} \Gamma/F$.

The proof of Theorem 1.12 will be given in Section 2.

Remark 1.13. The conditions (2), (3a), and (4) appearing in Theorem 1.12 are necessary. This follows for (2), since the argument appearing in the proof of Lemma 2 carries over directly, and is obvious for (3a) and and (4).

Remark 1.14. Suppose that assumptions (2) and (4) appearing in Theorem 1.12 hold and $\mathcal{M}$ is finite. Then we do get a finitely dominated Γ-CW-model for $\underline{E}\Gamma$ by the following argument. We obtain some finite-dimensional Γ-CW-model for $\underline{E}\Gamma$ from [24, Theorem 2.4]. We get a Γ-CW-model of finite type for $\underline{E}\Gamma$ from [24, Theorem 4.2], since $W_\Gamma H$ is finite for every non-trivial finite subgroup of Γ and there are only finitely many conjugacy classes of finite subgroups in Γ by condition (M) and (NM), see [9, Lemma 2.1], and $B\Gamma$ has a CW-model of finite type by [23, Lemma 7.2] applied to the fibration $B\pi \to B\Gamma \to BG$. Hence we get a finitely dominated Γ-CW-model for $\underline{E}\Gamma$ by [22, Proposition 14.9 on page 282].

If we want to turn this model into a finite Γ-CW-model, we have to compute its equivariant finiteness obstruction, see [22, Section 11]. Since there exists a Γ-CW-model for $\underline{E}\Gamma$ with finite $\underline{E}\Gamma^{>1}$, see Proposition 2.1, only the top component of the equivariant finiteness obstruction associated to the trivial subgroup may be non-trivial. It takes values in $\widetilde{K}_0(\mathbb{Z}\Gamma)$. Hence there exists a finite Γ-CW-model for $\underline{E}\Gamma$ if assumptions (2) and (4) hold, $\mathcal{M}$ is finite, and $\widetilde{K}_0(\mathbb{Z}\Gamma)$ vanishes. Suppose that π satisfies the Full Farrell-Jones Conjecture. Then the canoncial map $\bigoplus_{F\in\mathcal{M}} \widetilde{K}_0(\mathbb{Z}F) \xrightarrow{\cong} \widetilde{K}_0(\mathbb{Z}\Gamma)$ is an isomorphism, see [8, Theorem 5.1 (d)] or [9, Theorem 5.1]. Hence $\widetilde{K}_0(\mathbb{Z}\Gamma)$ vanishes, if and only if $\widetilde{K}_0(\mathbb{Z}F)$ vanishes for all $F \in \mathcal{M}$. If F is cyclic of order n, then $\widetilde{K}_0(\mathbb{Z}F)$ vanishes, if and only if $n \leq 11$ or $n \in \{13, 14, 17, 19\}$.

Remark 1.15 (The torsionfree case)**.** Now suppose that Γ is torsionfree, there is a finite model for $B\pi$, and that π satisfies the Full Farrell-Jones Conjecture in the sense of [26, Section 12.5]. Remark 1.14 implies that there is a finitely dominated CW-complex model for $B\Gamma$. Since $\widetilde{K}_0(\mathbb{Z}\Gamma)$ vanishes by the Full Farrell-Jones Conjecture, there is a finite CW-model for $B\Gamma$. Moreover, we conclude from [17, Theorem H] that there is a finite Poincaré complex homotopy equivalent to $B\Gamma$, if and only if there is a finite Poincaré complex homotopy equivalent to $B\pi$.

In view of Remark 1.15 we we will often and tacitly assume throughout the remainder of this paper that Γ is not torsionfree.

1.4. **Poincaré models.** Recall that we want to construct the desired Γ-manifold M described in (1.5) such that M is Γ-homotopy equivalent to $\underline{E}\Gamma$, or, equivalently such that M is contractible. For this purpose it suffices to find a free proper cocompact d-dimensional Γ-manifold X with boundary ∂X such that the space M defined in (1.5) is contractible. If we divide out the Γ-action, we get a compact manifold X/Γ with boundary $\partial X/\Gamma$. Hence it suffices to construct a compact d-dimensional manifold Y with fundamental group Γ and boundary $\partial X/\Gamma$ such that $Y \cup_{\partial X/\Gamma} C(\partial X)/\Gamma$ is aspherical, since then we can define X to be the universal covering of Y. Recall that any compact manifold with boundary is a finite Poincaré pair. Hence our first task is to construct a finite d-dimensional Poincaré pair $(Y, \partial X/\Gamma)$ such that the fundamental group of Y is Γ and $Y \cup_{\partial X/\Gamma} C(\partial X)/\Gamma$ is aspherical.

The next result will be a direct consequence of Theorem 1.12 and Theorem 7.12.

Theorem 1.16 (Poincaré models)**.** *Suppose that the following conditions are satisfied:*

- *The natural number d is even and satisfies $d \geq 4$;*
- *The group Γ satisfies conditions (M), (NM), and (H), see Notation 1.8;*

- *The homomorphism* $w\colon \Gamma \to \{\pm 1\}$ *of Notation 6.7 has the property that* $w|_F$ *is trivial for every* $F \in \mathcal{M}$*;*
- *One of the following assertions holds:*
  - *There exists a finite* Γ*-*CW*-model for* $\underline{E}\Gamma$*;*
  - *The group* π *is hyperbolic;*
  - *The group* Γ *acts cocompactly, properly, and isometrically on a proper* $\mathrm{CAT}(0)$*-space;*
- *There is a finite* d*-dimensional Poincaré complex, which is homotopy equivalent to* $B\pi$*;*
- *There exists an oriented free* d*-dimensional slice system* $\mathcal{S}$*, see Definition 3.1, satisfying condition (S), see Definition 7.9.*

Put $\partial X = \coprod_{F \in \mathcal{M}} \Gamma \times_F S_F$ *and* $C(\partial X) = \coprod_{F \in \mathcal{M}} \Gamma \times_F D_F$ *for* D_F *the cone over* S_F*.*

Then there exists a finite free Γ*-*CW*-pair* $(X, \partial X)$ *such that* $X \cup_{\partial X} C(\partial X)$ *is a model for* $\underline{E}\Gamma$ *and* $(X/\Gamma, \partial X/\Gamma)$ *is a finite* d*-dimensional Poincaré pair.*

We will also explain that for every $F \in \mathcal{M}$ there is only one choice for S_F up to F-homotopy and how we can determine this choice from Γ, see Section 7.1, and prove that the free Γ-CW-pair $(X, \partial X)$ is unique up to Γ-homotopy equivalence, see Theorem 8.9. Uniqueness up to simple Γ-homotopy equivalence is investigated in Section 9.

1.5. **Surgery Theory.** The second step is to promote the finite d-dimensional Poincaré pair $(Y, \partial X/\Gamma)$ of Subsection 1.4 up to homotopy to a compact manifold with boundary. This will be presented in a different paper, namely in [9]. In that paper surgery theory will come in, whereas in this paper our main techniques stem from algebraic topology and equivariant homotopy theory. In general further surgery obstructions, notably splitting obstructions taking values in UNil-groups, will appear.

As an illustration we mention that our ultimate result in [9] will imply the following result:

Theorem 1.17 (Manifold models)**.** *Suppose that the following conditions are satisfied:*

- *The natural number* d *is even and satisfies* $d \geq 6$*;*
- *The group* Γ *satisfies conditions (M), (NM), and (H), see Notation 1.8;*
- *The group* π *is hyperbolic;*
- *There exists a closed* d*-dimensional manifold, which is homotopy equivalent to* $B\pi$*;*
- *The group* F *is cyclic of odd order for every* $F \in \mathcal{M}$*.*

Then there exists a proper cocompact d*-dimensional topological* Γ*-manifold* M *of the shape described in* (1.5)*, where each* S_x *is homeomorphic to* S^{d-1}*, such that* M *is* Γ*-homotopy equivalent to* $\underline{E}\Gamma$*.*

In the paper [9] we will also discuss the uniqueness problem extending the results of [6].

1.6. **Nielsen Realization.** One motivation for searching for manifold models for $\underline{E}\Gamma$ comes from the following classical

Question 1.18 (Nielsen Realization Problem)**.** Let M be an aspherical closed manifold with fundamental group π. Let $j\colon G \to \mathrm{Out}(\pi)$ be an embedding of a finite group G into the outer automorphism group of π.

Is there an effective G-action $\rho\colon G \to \mathrm{Aut}(M)$ of G on M such that the composite of ρ with the canonical map $\nu\colon\ \mathrm{Aut}(M) \to \mathrm{Out}(\pi)$ is j?

In the smooth category it is easy to give examples using exotic spheres, where the answer is negative to Question 1.18, see for instance [2, page 22]. Therefore we will only consider the topological category.

Remark 1.19 (Original version)**.** Question 1.18 was originally formulated for closed orientable hyperbolic surfaces of genus ≥ 1 by Nielsen and was proved by Kerckhoff [15]. Subsequently, Tromba [33], Gabai [13], and Wolpert [36] gave new proofs.

Remark 1.20 (Counterexamples)**.** There do exist examples, where the answer to Question 1.18 is negative. A necessary condition is that there exists an extension $1 \to \pi \to \Gamma \to G \to 1$ such that the conjugation action of G on π is the given map j, see for instance [22, Theorem 8.1 on page 138]. This condition is automatically satisfied, if π is centerless, see [5, Corollary 6.8 in Chapter IV page 106], but not in general, see [31]. Even for centerless π there are examples, where the answer is negative for the Question 1.18, see [2, Theorem 1.5 and Theorem 1.6].

Some positive results about Question 1.18 have been obtained by Farrell-Jones, see for instance [10, page 282ff].

Remark 1.21 (Nielsen Realization for torsionfree Γ)**.** Suppose that the group Γ appearing in the extension (1.1) is torsionfree, $\dim(M) \neq 3, 4$, and π satisfies the Full Farrell-Jones Conjecture in the sense of [26, Section 12.5]. This means that the K-and L-theoretic Farrell-Jones Conjecture with coefficients in additive π-categories and with finite wreath products is satisfied for π. Examples for such π are hyperbolic groups or CAT(0)-groups. Then also Γ satisfies the Full Farrell-Jones Conjecture. Supppose that $B\pi$ is homotopy equivalent to a closed manifold M. Since Γ is torsionfree, $B\Gamma$ can be realized as a finite Poincaré complex by Remark 1.15. We conclude from [1, Theorem 1.2], that $B\Gamma$ has the homotopy type of a closed ANR-homology manifold N with fundamental group Γ. We have the finite covering $\overline{N} \to N$ associated to $1 \to \pi \to \Gamma \to G \to 1$. Note that $\overline{N}$ comes with a G-action. It suffices to show that $\overline{N}$ is homeomorphic to M. Since $\overline{N}$ and M have the same Quinn obstruction by [11, Theorem 5.28] and M is manifold, the Quinn obstruction of $\overline{N}$ is trivial and hence $\overline{N}$ is an aspherical closed manifold, whose fundamental group is identified with $\pi = \pi_1(M)$. Thanks to the Borel Conjecture, which follows from the Farrell-Jones Conjecture, $\overline{N}$ is homeomorphic to M.

To the authors' knowledge, there is no counterexample to the following

Conjecture 1.22 (Nielsen Realization for aspherical closed manifolds with hyperbolic fundamental group)**.** *Let M be an aspherical closed manifold with hyperbolic fundamental group π. Let $j\colon G \to \operatorname{Out}(\pi)$ be an embedding of a finite group G into the outer automorphism group of π.*

Then there is an effective topological G-action $\rho\colon G \to \operatorname{Aut}(M)$ of G on M such that the composite of ρ with the canonical map $\nu\colon \operatorname{Aut}(M) \to \operatorname{Out}(\pi)$ is j.

Remark 1.23. If in Conjecture 1.22 the dimension of M is greater or equal to 3, then $\operatorname{Out}(\pi)$ is known to be finite, see [14, § 5, 5.4.A], and hence one can take and it suffices to consider $G = \operatorname{Out}(\pi)$.

Remark 1.24. The proper cocompact Γ-manifold M appearing in Problem 1.6 yields a solution to the Nielsen Realization Problem for $1 \to \pi \to \Gamma \to G \to 1$, provided that the Borel Conjecture holds for π and M is contractible. Namely, M/π is aspherical with fundamental group π and hence any closed aspherical manifold with π as fundamental group admits a homeomorphism to M/π inducing the identity on the fundamental groups and G acts on M/π in the obvious way. Hence the Nielsen Realization problem has a positive answer, if the conditions appearing in Theorem 1.17 are satisfied.

1.7. **Acknowledgments.** The paper is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – GZ 2047/1, Projekt-ID 390685813, Hausdorff Center for Mathematics at Bonn.

The author is grateful especially to James Davis and also to Christian Kremer and Kevin Li for fruitful discussions and useful comments. James Davis has informed the author that he has independent solutions to Brown's Problem and to the Nielsen Realization Problem in the case that all finite subgroups of Γ are of order two, which have not appeared as preprint so far.

The paper is organized as follows:

Contents

2. Some results about the classifying space for proper actions

This section is devoted to the proof of Theorem 1.12

Proposition 2.1. *Suppose that Γ satisfies (M) and (NM). Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal finite subgroups $F \subseteq$*

Γ. *Consider the cellular Γ-pushout*

$$\begin{tikzcd}
\partial E\Gamma = \coprod_{F\in\mathcal{M}} \Gamma \times_F EF \arrow[r, "i"] \arrow[d, "\coprod_{F\in\mathcal{M}} p_F"'] & E\Gamma \arrow[d] \\
\partial \underline{E}\Gamma = \coprod_{F\in\mathcal{M}} \Gamma/F \arrow[r] & Z
\end{tikzcd}$$

where the map p_F comes from the projection $EF \to \{\bullet\}$, and i is an inclusion of Γ-CW-complexes.

Then Z is a model for $\underline{E}\Gamma$.

Proof. This follows from [27, Corollary 2.11]. □

Proposition 2.2. *Let Γ be a hyperbolic group. Then there is a finite Γ-CW-model for the classifying space for proper actions $\underline{E}\Gamma$.*

Proof. See [28]. □

Proposition 2.3. *Suppose that Γ acts cocompactly, properly, and isometrically on a proper* CAT(0)*-space X Then there is a finite Γ-CW-model for $\underline{E}\Gamma$.*

Proof. By [3, Corollary II.2.8 on page 179] the H-fixed point set X^H of X is a non-empty convex subset of X and hence contractible for any finite subgroup $H \subset G$. Since the action is proper, all isotropy groups are finite. We conclude from [30, Proposition A] that X is Γ-homotopy equivalent to a finite Γ-CW-complex Y, since for a proper Γ-action each Γ-orbit is discrete. Hence Y is a finite Γ-CW-model for $\underline{E}\Gamma$. □

Lemma 2.4. *Let Γ be a group and d be a natural number satisfying $d \geq 3$. Let X be a Γ-CW-complex of finite type. Suppose that X is Γ-homotopy equivalent to some (not necessarily finite) d-dimensional Γ-CW-complex and to some finite Γ-CW-complex (of arbitrary dimension). Then there exists a Γ-CW-complex $\widehat{X}$ satisfying*

(1) *$\widehat{X}$ is finite and d-dimensional;*
(2) *The $(d-2)$-skeleton of X and the $(d-2)$-skeleton of $\widehat{X}$ agree;*
(3) *If $\widehat{X}$ has an equivariant cell of the type $\Gamma/H \times D^{d-1}$, then X contains an equivariant cell of the type $\Gamma/H \times D^k$ for some $k \in \{d-1, d\}$. If $\widehat{X}$ has an equivariant cell of the type $\Gamma/H \times D^d$, then X contains an equivariant cell of the type $\Gamma/H \times D^d$. In particular $Y^{>1} = X^{>1}$, if $X^{>1}$ is contained in the $(d-2)$-skeleton of X;*
(4) *The Γ-CW-complexes X and $\widehat{X}$ are Γ-homotopy equivalent.*

Proof. We use the machinery and notation developed in [22]. During this proof we abbreviate $\mathcal{C} := \Pi(\Gamma, X)$, where $\Pi(\Gamma, X)$ is the fundamental category, see [22, Definition 8.15 on page 144]. Let $C_*^c(X)$ be the cellular $\mathbb{Z}\mathcal{C}$-chain complex. It is a finitely generated free $\mathbb{Z}\mathcal{C}$-chain complex, as X is of finite type. Since X is homotopy equivalent to a d-dimensional Γ-CW-complex, $C_*^c(X)$ is $\mathbb{Z}\mathcal{C}$-chain homotopy equivalent to a free d-dimensional $\mathbb{Z}\mathcal{C}$-chain complex. We conclude from [22, Proposition 11.10 on page 221] that we can find a d-dimensional finitely generated projective $\mathbb{Z}\mathcal{C}$-subchain complex D_* of $C_*^c(X)$ such that D_* and $C_*^c(X)$ agree in dimensions $\leq (d-1)$, D_d is a direct summand of $C_*^c(X)$, and the inclusion $j_* \colon D_* \to C_*^c(X)$ is a $\mathbb{Z}\mathcal{C}$-chain homotopy equivalence. Hence the equivariant finiteness obstruction $\widetilde{o}^G(X) \in \widetilde{K}_0(\mathbb{Z}\mathcal{C})$ is the class of the finitely generated projective $\mathbb{Z}\mathcal{C}$-module D_d. Since X is by assumption Γ-homotopy equivalent to a finite Γ-CW-complex, the equivariant finiteness obstruction $\widetilde{o}^G(X) \in \widetilde{K}_0(\mathbb{Z}\mathcal{C})$ vanishes, see [22, Definition 14.4 and Theorem 14.6 on page 278]. This implies that D_d is stably free. We will need a stronger statement about D_d stated and proved below.

There is a $\mathbb{Z}\mathcal{C}$-isomorphism

$$C^c_d(X) \quad = \quad \bigoplus_{i=1}^m \mathbb{Z}\operatorname{mor}_{\mathcal{C}}(?,x_i)^{a_i}$$

for integers $m \geq 0$ and $a_i \geq 1$ and for a finite set $\{x_i \colon \Gamma/H_i \to X \mid i = 1,2,\ldots,m\}$ of objects in $\mathcal{C}$, whose elements are mutually not isomorphic in $\mathcal{C}$. Note that X must contain at least one equivariant cell of type $\Gamma/H_i \times D^d$ for each $i \in \{1,2,\ldots,m\}$. Let S_x be the splitting functor and E_x be the extension functor associated to an object $x \colon \Gamma/H \to X$, see [22, (9.26) and (9.28) on page 170]. If $S_x(C^c_d(X)) \neq 0$ holds, then x is isomorphic to precisely one of the x_i-s and in particular H is conjugated to H_i. Since D_d is a direct summand of $C_d(X)$ and S_x is compatible with direct sums, the analogous statement holds for D_d. Hence D_d is $\mathbb{Z}\mathcal{C}$-isomorphic to $\bigoplus_{i=1}^m E_{x_i} \circ S_{x_i}(D_d)$ by [22, Corollary 9.40 page 176]. We conclude from [22, Theorem 10.34 page 196] and the vanishing of the equivariant finiteness obstruction $\widetilde{o}^G(X) \in \widetilde{K}_0(\mathbb{Z}\mathcal{C})$ that for every $i \in \{1,2,\ldots,m\}$ the class of the finitely generated projective $\mathbb{Z}[\operatorname{aut}_{\mathcal{C}}(x_)]$-module $S_{x_i}D_d$ in $\widetilde{K}_0(\mathbb{Z}[\operatorname{aut}_{\mathcal{C}}(x)])$ vanishes. Hence we can find for every $i \in \{1,2,\ldots,m\}$ integers k_i, l_i with $0 \leq k_i \leq l_i$ such that $S_{x_i}D_d \oplus \mathbb{Z}[\operatorname{aut}_{\mathcal{C}}(x_i)]^{k_i}$ is $\mathbb{Z}[\operatorname{aut}_{\mathcal{C}}(x_i)]$ isomorphic to $\mathbb{Z}[\operatorname{aut}_{\mathcal{C}}(x_i)]^{l_i}$. This implies the existence of an isomorphism of $\mathbb{Z}\mathcal{C}$-modules

$$D_d \oplus \bigoplus_{i=1}^m \mathbb{Z}\operatorname{mor}_{\mathcal{C}}(?,x_i)^{k_i} \cong \bigoplus_{i=1}^n \mathbb{Z}\operatorname{mor}_{\mathcal{C}}(?,x_i)^{l_i}.$$

Hence we can add to D_* an elementary $\mathbb{Z}\mathcal{C}$-chain complex E_* concentrated in dimensions d and $(d-1)$ and associated to the finitely generated free $\mathbb{Z}\mathcal{C}$-module $\bigoplus_{i=1}^n \mathbb{Z}\operatorname{mor}_{\mathcal{C}}(?,x_i)^{k_i}$. Denote the result by $D'_* := D_* \oplus E_*$. We obtain a $\mathbb{Z}\mathcal{C}$-chain homotopy equivalence $j_* \oplus \operatorname{id}_{E_*} \colon D_* \oplus E_* \to C_*(X) \oplus E_*$. By attaching equivariant cells of the shape $\Gamma/H_i \times D^{d-1}$ and $\Gamma/H_i \times D^d$ for $i \in \{1,2,\ldots m\}$ to X, we get a finite Γ-CW-complex X' such that the inclusion $X \to X'$ is a Γ-homotopy equivalence and there is an identification of $\mathbb{Z}\mathcal{C}$-chain complexes $C_*(X') = C_*(X) \oplus E_*$. Then D'_* is a $\mathbb{Z}\mathcal{C}$-subchain complex of X' such that $C_*(X')$ and D'_* agree in dimensions $\leq (d-1)$, the inclusion $j_* \colon D'_* \to C_*(X')$ is a $\mathbb{Z}\mathcal{C}$-chain homotopy equivalence, and $D'_d = \bigoplus_{i=1}^n \mathbb{Z}\operatorname{mor}_{\mathcal{C}}(?,x_i)^{l_i}$. Note that X and X' have the same $(d-2)$-skeleton. Moreover, if X' has an equivariant cell of the type $\Gamma/H \times D^{d-1}$, then X contains an equivariant cell of the type $\Gamma/H \times D^k$ for some $k \in \{d-1,d\}$ and if X' has an equivariant cell of the type $\Gamma/H \times D^d$, then X contains an equivariant cell of the type $\Gamma/H \times D^d$. Now apply the Equivariant Realization Theorem, see [22, Theorem 13.19 on page 268], in the case $r = (d-1)$, $A = B = \emptyset$, $Z = X'_{d-1}$, $Y = X'$, and h the inclusion $X_{d-1} \to X$, and $C = D'_*$. The resulting finite Γ-CW-complex $\widehat{X}$ has the desired properties, since by construction its $(d-1)$-skeleton is X'_{d-1} and we have $C^c_*(\widehat{X}) = D'_*$. □

Lemma 2.5. *Let d be a natural number with $d \geq 3$. Let Γ be a group with a torsionfree subgroup π of finite index. Suppose that there is a Γ-CW-model $\underline{E}\Gamma$ such that $\dim(\underline{E}\Gamma^{>1}) \leq (d-1)$ holds for the Γ-CW-subcomplex $\underline{E}\Gamma^{>1}$ of $\underline{E}\Gamma$ consisting of points with non-trivial isotropy group. Then the following assertions are equivalent:*

1. *There is a model $\underline{E}\Gamma$ with $\dim(\underline{E}\Gamma) \leq d$;*
2. *There is a model $E\pi$ with $\dim(E\pi) \leq d$.*

Proof. Let $i \colon \pi \to \Gamma$ be the inclusion. Since the restriction $i^*\underline{E}\Gamma$ of $\underline{E}\Gamma$ with i is a model for $E\pi$, the implication (1) $\implies$ (2) is true. The hard part is to show the implication (2) $\implies$ (1) what we do next.

We show by induction for $m = d \cdot [\Gamma : \pi], d \cdot [\Gamma : \pi] - 1, d \cdot [\Gamma : \pi] - 2, \ldots, d$ that there is a model for $\underline{E}\Gamma$ with $\dim(\underline{E}\Gamma) \le m$.

The induction beginning is done as follows. Let $i \colon \pi \to \Gamma$ be the inclusion. Then the coinduction $i_! E\pi$ is a Γ-CW-model for $\underline{E}\Gamma$ of dimension $\dim(E\pi) \cdot [\Gamma : \pi]$, see [24, Theorem 2.4].

The induction step from $m+1$ to m for $d \le m \le [\Gamma : \pi] - 1$ is described next. Let X be a Γ-CW-complex such that all isotopy groups are finite, X^H is contractible for every finite subgroup $H \subseteq \Gamma$, and for every Γ-cell $\Gamma/H \times D^k$ with $H \ne \{1\}$ we have $k \le (d-1)$. Note that X is a model for $\underline{E}\Gamma$. We have to show that X is Γ-homotopy equivalent to a Γ-CW-complex of dimension $\le m$, provided that X is Γ-homotopy equivalent to a Γ-CW-complex of dimension $\le (m+1)$ and $m \ge d$.

Because of the Equivariant Realization Theorem, see [22, Theorem 13.19 on page 268], and [22, Proposition 11.10 on page 221], it suffices to show that the cellular $\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$-chain complex $C_*^c(X)$ is $\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$-chain homotopy equivalent to an m-dimensional $\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$-chain complex. Note that we can confine ourselves in this special case to the $\mathcal{FIN}$-restricted orbit category $\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$, since all isotropy groups are finite and X^H is non-empty and simply connected for every finite subgroup $H \subseteq \Gamma$. Namely, the latter implies that the fundamental category $\Pi(\Gamma, X)$ appearing in [22, Definition 8.15 on page 144] reduces to $\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$ and the cellular $\mathbb{Z}\Pi(\Gamma, X)$-chain complex of X appearing in [22, Definition 8.37 on page 152] reduces to the corresponding cellular chain complex $C_*^c(X)$ over $\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$, which sends an object Γ/H to the cellular chain complex of X^H. Recall that $\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$ has as objects homogeneous Γ-spaces Γ/H with $|H| < \infty$ as objects and Γ-maps between them as morphisms.

Define the $\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(G)$-module B_m to be the image of the $(m+1)$th-differential $c_{m+1} \colon C_{m+1}^c(X) \to C_m^c(X)$. Because of [22, Proposition 11.10 on page 221] it suffices to show that $B_m \subseteq C_m^c(X)$ is a direct summand in $C_m^c(X)$. The following argument shows that for this purpose it suffices to show that $H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_*^c(X), B_m)$ is trivial. Namely, from $H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_*^c(X); B_m) = 0$ we get the exact sequence

$$\hom_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_m^c(X), B_m) \xrightarrow{c_{m+1}^*} \hom_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_{m+1}^c(X), B_m) \xrightarrow{c_{m+2}^*} \hom_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_{m+2}^c(X), B_m).$$

The element $c_{m+1} \colon C_{m+1}^c(X) \to B_m$ in $\hom_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_{m+1}^c(X), B_m)$ is sent under the right arrow to $c_{m+1} \circ c_{m+2} = 0$ and hence has a preimage $r \colon C_m^c(X) \to B_m$ under the first arrow. This implies $r \circ c_{m+1} = c_{m+1}$ and hence $r|_{B_m} = \mathrm{id}_{B_m}$.

We have the short exact sequence $0 \to Z_{m+1} \to C_{m+1}^c(X) \xrightarrow{c_{m+1}} B_m \to 0$ of $\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$-modules, where Z_{m+1} is the kernel of c_{m+1}. In the sequel we abbreviate $F = C_{m+1}^c(X)$. It yields a long exact sequence

$$\cdots \to H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_*^c(X); Z_{m+1}) \to H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_*^c(X); F) \to H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_*^c(X); B_m) \to H^{m+2}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_*^c(X); Z_{m+1}) \to \cdots .$$

Since by the induction hypothesis X is Γ-homotopy equivalent to an $(m+1)$-dimensional Γ-CW-complex, $H^{m+2}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_*^c(X); Z_{m+1})$ vanishes and hence the map $H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_*^c(X); F) \to H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_*^c(X); B_m)$ is surjective. Therefore it suffices to show that $H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C_*^c(X); F)$ vanishes.

We have the short exact sequence of $\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$-chain complexes

$$0 \to C_*^c(\underline{E}\Gamma^{>1}) \to C_*^c(\underline{E}\Gamma) \to C_*^c(\underline{E}\Gamma, \underline{E}\Gamma^{>1}) \to 0. \tag{2.6}$$

It yields a long cohomology sequence

$$\cdots \to H^m_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C^c_*(\underline{E}\Gamma^{>1});F) \to \ H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C^c_*(\underline{E}\Gamma,\underline{E}\Gamma^{>1});F)$$
$$\to H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C^c_*(\underline{E}\Gamma);F) \to H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C^c_*(\underline{E}\Gamma^{>1});F) \to \cdots .$$

Since $\dim(\underline{E}\Gamma^{>1}) \le (d-1) \le (m-1)$ holds, we get $H^l_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C^c_*(\underline{E}\Gamma^{>1});F) = 0$ for $l \ge m$. Hence we get an isomorphism

$$H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C^c_*(\underline{E}\Gamma,\underline{E}\Gamma^{>1});F) \xrightarrow{\cong} H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C^c_*(\underline{E}\Gamma);F).$$

Note that $C^c_*(\underline{E}\Gamma,\underline{E}\Gamma^{>1})$ for every $l \ge 0$ and F are $\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$-modules, which are direct sums of $\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$-modules of the shape $\mathbb{Z}\operatorname{mor}_{\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(?,\Gamma/1)$. Let $\mathcal{TR}$ be the family of subgroups of Γ consisting of one elements, namely the trivial subgroup. We have the inclusion $j\colon \mathsf{Or}_{\mathcal{TR}}(\Gamma) \to \mathsf{Or}_{\mathcal{FIN}}(\Gamma)$ where $\mathsf{Or}_{\mathcal{TR}}(\Gamma)$ is the full subcategory of $\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$ consisting of one object $\Gamma/1$. Note that the category of $\mathbb{Z}\mathsf{Or}_{\mathcal{TR}}(\Gamma)$-modules can be identified with the category of $\mathbb{Z}\Gamma$-modules. Hence $j_*j^*C^c_*(\underline{E}\Gamma,\underline{E}\Gamma^{>1})$ and $C^c_*(\underline{E}\Gamma,\underline{E}\Gamma^{>1})$ are isomorphic $\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$-chain complexes and j_*j^*F and F are isomorphic $\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$-modules. From the adjunction (j_*,j^*), we obtain an isomorphism

$$H^{m+1}_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)}(C^c_*(\underline{E}\Gamma,\underline{E}\Gamma^{>1});F) \cong H^{m+1}_{\mathbb{Z}\Gamma}(j^*C^c_*(\underline{E}\Gamma,\underline{E}\Gamma^{>1});j^*F)$$

Obviously j^*F is a free $\mathbb{Z}\Gamma$-module. Hence there exists a free $\mathbb{Z}\pi$-module F' with $i_*F' \cong_{\mathbb{Z}\Gamma} j^*F$. We conclude from [5, Proposition 5.9 in III.5 on page 70] that the coinduction $i_!F'$ with i of F' is $\mathbb{Z}\Gamma$-isomorphic to j^*F. From the adjunction $(i^*,i_!)$, we obtain an isomorphism

$$H^{m+1}_{\mathbb{Z}\Gamma}(j^*C^c_*(\underline{E}\Gamma,\underline{E}\Gamma^{>1});j^*F) \cong H^{m+1}_{\mathbb{Z}\pi}(i^*j^*C^c_*(\underline{E}\Gamma,\underline{E}\Gamma^{>1});F')$$

Hence it suffices to show $H^{m+1}_{\mathbb{Z}\pi}(i^*j^*C^c_*(\underline{E}\Gamma,\underline{E}\Gamma^{>1});F') = 0$.

If we apply $i^* \circ j^*$ to the short exact sequence (2.6) of $\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(\Gamma)$-chain complexes, we obtain an exact sequence of $\mathbb{Z}\pi$-chain complexes and hence a long exact sequence

$$\cdots \to H^m_{\mathbb{Z}\pi}(i^*j^*C^c_*(\underline{E}\Gamma^{>1});F') \to H^{m+1}_{\mathbb{Z}\pi}(i^*j^*C^c_*(j^*\underline{E}\Gamma,\underline{E}\Gamma^{>1});F')$$
$$\to H^{m+1}_{\mathbb{Z}\pi}(i^*j^*C^c_*(\underline{E}\Gamma);F') \to H^{m+1}_{\mathbb{Z}\pi}(i^*j^*C^c_*(\underline{E}\Gamma^{>1});F') \to \cdots .$$

Since $\dim(\underline{E}\Gamma^{>1}) \le (d-1) \le (m-1)$ holds, we get $H^k_{\mathbb{Z}\pi}(C^c_*(i^*\underline{E}\Gamma^{>1});F') = 0$ for $k \ge m$. Therefore we obtain an isomorphism

$$H^{m+1}_{\mathbb{Z}\pi}(i^*j^*C^c_*(\underline{E}\Gamma,\underline{E}\Gamma^{>1});F') \xrightarrow{\cong} H^{m+1}_{\mathbb{Z}\pi}(i^*j^*C^c_*(\underline{E}\Gamma);F').$$

Hence it remains to show $H^{m+1}_{\mathbb{Z}\pi}(i^*j^*C^c_*(\underline{E}\Gamma);F') = 0$. This follows from the fact that $i^*\underline{E}\Gamma$ is a model for $E\pi$ and up to π-homotopy there is a π-CW-model for $\underline{E}\pi$ with $\dim(\underline{E}\pi) \le d \le m$. $\square$

Now we are ready to give the proof of Theorem 1.12.

Proof of Theorem 1.12. We conclude from Proposition 2.2 and Proposition 2.3 that we can assume without loss of generality that the condition (3a) appearing under (3) in Theorem 1.12 is satisfied, namely, that there is a finite Γ-CW-model for $\underline{E}\Gamma$.

This implies that there is a Γ-CW-model for $E\Gamma$ of finite type, see [24, Lemma 4.1]. Moreover, we can arrange in the Γ-pushout appearing in Proposition 2.1 that the Γ-CW-complexes $\coprod_{F\in\mathcal{M}} \Gamma \times_{N_\Gamma F} EF$ and $E\Gamma$ are of finite type. Hence we get from Proposition 2.1 a Γ-CW-model X of finite type for $\underline{E}\Gamma$ such that $X^{>1}$ is $\coprod_{F\in\mathcal{M}} \Gamma/F$.

Let π be the subgroup of finite index appearing in condition (4) of Theorem 1.12. Since Γ is finitely generated, there exists a normal subgroup $\pi' \subseteq \Gamma$ of finite index with $\pi' \subseteq \pi$. Hence we can assume without loss of generality that π is normal, otherwise replace π by π'.

There is a d-dimensional Γ-CW-model for $\underline{E}\Gamma$ by Lemma 2.5.

Now Theorem 1.12 follows from Lemma 2.4. □

3. Free slice systems

Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal finite subgroups of Γ.

Definition 3.1. A *d-dimensional free slice system* $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$, or shortly *slice system*, consists of a free $(d-1)$-dimensional CW-complex S_F for every $F \in \mathcal{M}$ such that S_F is after forgetting the F-action homotopy equivalent to the $(d-1)$-dimensional standard sphere S^{d-1}.

We call $\mathcal{S}$ *oriented*, if we have chosen a generator $[S_F]$, called fundamental class, for the infinite cyclic group $H_{d-1}(S_F)$ for every $F \in \mathcal{M}$.

We denote by D_F the cone over S_F. Obviously (D_F, S_F) is F-CW-pair and (D_F, S_F) is after forgetting the F-action homotopy equivalent to (D^d, S^{d-1}).

Let $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$ and $\mathcal{S}' = \{S'_F \mid F \in \mathcal{M}\}$ be d-dimensional free slice systems. Note that this implies that each F in $\mathcal{M}$ has periodic cohomology, or, equivalently, that any Sylow subgroup of F is cyclic or a generalized quaternion group, see [5, Section VI.9 on pages 153–160].

Assume that F acts orientation preserving on S_F. Then $H_{d-1}(S_F/F)$ is infinite cyclic and comes with a preferred generator $[S_F/F]$, which is uniquely determined by the property that the map $H_{d-1}(S_F) \to H_{d-1}(S_F/F)$ induced by the projection $S_F \to S_F/F$ sends $[S_F]$ to $|F| \cdot [S_F/F]$. Let $c(S_F) \colon S_F \to EF$ be a classifying map. Denote by $c_F \colon S_F/F \to BF$ the map $c(S_F)/F$. If $\mathcal{S}$ is oriented, we can define

$$d(S_F) \in H_{d-1}(BF) \tag{3.2}$$

to be the class given by the image of $[S_F/F] \in H_{d-1}(S_F/F)$ under the homomorphism $H_{d-1}(c_F) \colon H_{d-1}(S_F/F) \to H_{d-1}(BF)$. Note that $H_{d-1}(S_F/F)$ is infinite cyclic. Since $c(S_F) \colon S_F \to EG$ and hence $c_F \colon S_F/F \to BF$ are $(d-1)$-connected, the homomorphism $H_{d-1}(c_F) \colon H_{d-1}(S_F/F) \to H_{d-1}(BF)$ is surjective. We conclude that $H_{d-1}(c_F)([S_F/F])$ is a generator of the cyclic group $H_{d-1}(BF)$. The order of $H_{d-1}(BF)$ is $|F|$, see [5, Theorem 9.1 in Chapter VI on page 154].

Lemma 3.3. *Let $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$ and $\mathcal{S}' = \{S'_F \mid F \in \mathcal{M}\}$ be slice systems.*

1. *If d is odd, then $F \cong \mathbb{Z}/2$ for all $F \in \mathcal{M}$;*
2. *If d is even, then the F-action on S_F is orientation preserving. If d is odd, then the F-action on S_F is orientation reversing;*
3. *Suppose that F acts orientation preserving on S_F and S'_F and $\mathcal{S}$ and $\mathcal{S}'$ are oriented. Then there exists an orientation preserving F-homotopy equivalence $S_F \xrightarrow{\simeq} S'_F$, if and only if $d(S_F) = d(S'_F)$ holds in $H_{d-1}(BF)$;*
4. *If $|F| \geq 3$, then any F-selfhomotopy equivalence $S_F \to S_F$ is F-homotopic to the identity.*
5. *Any F-selfhomotopy equivalence $S_F \to S_F$ is simple*

Proof. (1) Since F acts freely, we have $1 + (-1)^{d-1} = \chi(S_F) \equiv 0 \mod |F|$. This implies that $|F| = 2$, if d is odd.

(2) Consider a nontrivial element $g \in F$. Multiplication with g induces a map $l_g \colon S_F \to S_F$, which has no fixed points. Hence its Lefschetz number $\Lambda(l_g)$ vanishes. We have $\Lambda(l_g) = 1 + (-1)^{d-1} \cdot \deg(l_g)$, where $\deg(l_g)$ is the degree. If d is even, $\deg(l_g) = 1$, and, if d is odd, $\deg(l_g) = -1$;

(3) Let $f \colon S_F \to S'_F$ be an orientation preserving F-homotopy equivalence. Since

the F-maps $c(S'_F) \circ f$ and $c(S_F)$ are F-homotopic, $H_{d-1}(c(S_F))([S_F/F])$ agrees with $H_{d-1}(c(S'_F))([S'_F/F])$. This implies $d(S_F) = d(S'_F)$.

Now suppose $d(S_F) = d(S'_F)$. By elementary obstruction theory one can find an F-map $f\colon S_F \to S'_F$. We get $H_{d-1}(f/F)([S_F/F]) = \deg(f) \cdot [S'_F/F]$, if $\deg(f)$ is the degree of f. Since $c(S'_F) \circ f$ is F-homotopic to $c(S_F)$, we conclude

$$H_{d-1}(c_F)([S_F]) = \deg(f) \cdot H_{d-1}(c(S'_F)/F)([S'_F]).$$

Since $H_{d-1}(BF)$ is a finite cyclic of order $|F|$ and both elements $H_{d-1}(c_F)([S_F])$ and $H_{d-1}(c(S'_F)/F)([S'_F])$ are generators, we conclude $\deg(f) \equiv 1 \mod |F|$ from $d(S_F) = d(S'_F)$. Since for any integer $m \in \mathbb{Z}$ with $m \equiv \deg(f) \mod |F|$ one can find an F-map $f'\colon S_F \to S'_F$ with $\deg(f') = m$, see [32, Theorem 4.11 on page 126] or [21, Theorem 3.5 on page 139], there exists an F-map $f'\colon S_F \to S'_F$ of degree 1. This implies that f' is an orientation preserving F-homotopy equivalence.

(4) If $f\colon S_F \to S_F$ is an F-map, then its degree satisfies $\deg(f) \equiv 1 \mod |F|$, see [32, Theorem 4.11 on page 126] or [21, Theorem 3.5 on page 139]. Consider an F-self homotopy equivalence $f\colon S_F \to S_F$. Then $\deg(f) \in \{\pm 1\}$. Since $|F| \geq 3$ holds by assumption, $\deg(f) = 1$. Since two F-maps $S_F \to S_F$ are F-homotopic, if and only if their degrees agree, see [32, Theorem 4.11 on page 126] or [21, Theorem 3.5 on page 139], f is F-homotopic to the identity.

(5) If $F \geq 3$, this follows from assertion (4). If $|F| \leq 2$, this follows from $\mathrm{Wh}(F) = \{0\}$. □

Lemma 3.4. *Let $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$ and $\mathcal{S}' = \{S'_F \mid F \in \mathcal{M}\}$ be slice systems. Let*

$$v\colon \coprod_{F \in \mathcal{M}} \Gamma \times_F S_F \to \coprod_{F \in \mathcal{M}} \Gamma \times_F S'_F$$

be a Γ-map. Suppose that conditions (M) and (NM) are satisfied. Then:

1. *The map v induces for each $F \in \mathcal{M}$ an F-map $v_F\colon S_F \to S'_F$ such that $v = \coprod_{F \in \mathcal{M}} \mathrm{id}_\Gamma \times_F v_F$ holds. Moreover, the collection of the F-homotopy classes of the F-maps v_F for $F \in \mathcal{M}$ is determined by the Γ-homotopy class of v and vice versa;*
2. *There exists a Γ-map $V\colon \coprod_{F \in \mathcal{M}} \Gamma \times_F D_F \to \coprod_{F \in \mathcal{M}} \Gamma \times_F D'_F$ extending v;*

Proof. We obtain a commutative diagram of Γ-sets induced by v

$$\begin{tikzcd}
\pi_0\big(\coprod_{F \in \mathcal{M}} \Gamma \times_F S_F\big) \arrow[r, "\pi_0(v)"] \arrow[d, "\cong"] & \pi_0\big(\coprod_{F \in \mathcal{M}} \Gamma \times_F S'_F\big) \arrow[d, "\cong"] \\
\coprod_{F \in \mathcal{M}} \Gamma/F \arrow[r, "\overline{v}"] & \coprod_{F \in \mathcal{M}} \Gamma/F
\end{tikzcd}$$

whose vertical arrows are the obvious bijections. Consider $F \in \mathcal{M}$. Since $\overline{v}$ is a Γ-map, there exists $F' \in F$ such that $\overline{v}$ induces a Γ-map $\overline{v}_F\colon \Gamma/F \to \Gamma/F'$. There exists $\gamma \in \Gamma$ such that $\overline{v}_F$ sends eF to $\gamma F'$ for $e \in \Gamma$ the unit. Then $\gamma^{-1} F \gamma \subseteq F'$. Since F and F' are maximal finite, we obtain $\gamma^{-1} F \gamma = F'$. Since two elements in $\mathcal{M}$, which are conjugated, are automatically equal, we get $F = \gamma^{-1} F \gamma = F'$. Hence we get $\gamma \in N_\Gamma F = F = F'$ because of (NM). This implies $F = F'$ and $\overline{v}_F = \mathrm{id}_{\Gamma/F}$. Hence $\overline{v}$ is the identity. This implies $v(S_F) \subseteq S'_F$, where we identify S_F with the subspace $\{(\gamma, x) \mid \gamma \in F, x \in S_F\}$ of $\Gamma \times_F S_F$ and analogously for S'_F. This shows assertion (1). Assertion (2) follows from assertion (1). □

Remark 3.5. Note that the existence of a d-dimensional free slice system depends only on Γ, actually only on $\mathcal{M}$. If for instance Γ contains a normal torsionfree subgroup π such that $G := \Gamma/\pi$ is finite cyclic and d is even, then obviously one can find a d-dimensional free slice system, since G acts freely on S^{d-1}.

4. Slice complement models

Notation 4.1. Given a space Z, let $C(Z)$ be its *path componentwise cone*, i.e, $C(Z) := \coprod_{C \in \pi_0(Z)} \operatorname{cone}(C)$.

One may describe $C(Z)$ also by the pushout

$$\begin{tikzcd} Z \arrow[r, "p"] \arrow[d, "i_0"'] & \pi_0(Z) \arrow[d] \\ Z \times [0,1] \arrow[r] & C(Z) \end{tikzcd}$$

where $i_0\colon Z \to Z \times [0,1]$ sends z to $(z,0)$, $p\colon Z \to \pi_0(Z)$ is the projection, and $\pi_0(Z)$ is equipped with the discrete topology. If Z is a Γ-CW-complex, then $C(Z)$ inherits a Γ-CW-structure. If $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$ is a free d-dimension slice system, we get an identification of Γ-CW-complexes

$$C(\coprod_{F \in \mathcal{M}} \Gamma \times_F S_F) = \coprod_{F \in \mathcal{M}} \Gamma \times_F D_F.$$

Definition 4.2 (Slice complement model)**.** We call a free Γ-CW-pair $(X, \partial X)$ a *slice complement model for* $\underline{E}\Gamma$, or shortly *slice complement model*, with respect to the slice system $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$, if $\partial X = \coprod_{F \in \mathcal{M}} \Gamma \times_F S_F$ and $X \cup_{\partial X} C(\partial X)$ is a model for $\underline{E}\Gamma$.

We will frequently use that a slice complement model comes with a canonical homotopy Γ-pushout

(4.3)
$$\begin{tikzcd} \partial X \arrow[r] \arrow[d] & X \arrow[d] \\ \partial \underline{E}\Gamma \arrow[r] & \underline{E}\Gamma. \end{tikzcd}$$

Lemma 4.4. *Consider a free d-dimensional Γ-CW-pair $(X, \partial X)$ and a slice system $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$ such that $\partial X = \coprod_{F \in \mathcal{M}} \Gamma \times_F S_F$. Suppose $d \geq 3$ and that conditions (M) and (NM) hold, see Notation 1.8. Fix $k \in \{2, 3, \ldots, (d-1)\}$.*

Then the following assertions are equivalent:

1. *$(X, \partial X)$ is a slice complement model for $\underline{E}\Gamma$;*
2. *$X \cup_{\partial X} C(\partial X)$ is contractible (after forgetting the Γ-action);*
3. *The space X is $(k-1)$-connected and $H_n(X, \partial X)$ vanishes for $k \leq n \leq d$.*

Proof. (1) $\iff$ (2) For every finite non-trivial subgroup $H \subseteq G$, there exists precisely one element $F \in \mathcal{M}$ such that for some $\gamma \in \Gamma$ we have $\gamma H \gamma^{-1} \subseteq F$ and hence

$$X \cup_{\partial X} C(\partial X)^H \cong X \cup_{\partial X} C(\partial X)^{\gamma H \gamma} = X \cup_{\partial X} C(\partial X)^F = \{\bullet\}$$

holds. Hence $X \cup_{\partial X} C(\partial M)$ is a model for $\underline{E}\Gamma$, if and only if $X \cup_{\partial X} C(\partial X)$ is contractible.

(2) $\iff$ (3) Since the map $\partial X \to C(\partial X)$ is $(d-1)$-connected, the inclusion $X \to X \cup_{\partial X} C(\partial X)$ is $(d-1)$-connected. In particular X is $(k-1)$-connected if and only if $X \cup_{\partial X} C(\partial X)$ is $(k-1)$ connected. We have $H_n(C(\partial X)) = 0$ for $n \geq 1$. Hence we get from the long exact sequence of the pair $(X \cup_{\partial X} C(\partial X), C(\partial X))$ and by excision isomorphisms for $n \geq 2$

$$H_n(X, \partial X) \xrightarrow{\cong} H_n(X \cup_{\partial X} C(\partial X), C(\partial X)) \xleftarrow{\cong} H_n(X \cup_{\partial X} C(\partial X)).$$

By the Hurewicz Theorem $X \cup_{\partial X} C(\partial X)$ is contractible, if and only if $X \cup_{\partial X} C(\partial X)$ is $(k-1)$-connected and $H_n(X \cup_{\partial X} C(\partial X))$ vanishes for $k \leq n$. As $X \cup_{\partial X} C(\partial X)$ is

d-dimensional $H_n(X \cup_{\partial X} C(\partial X))$ vanishes for $n \geq (d+1)$. This finishes the proof of Lemma 4.4. □

Lemma 4.5. *Consider two free d-dimensional slice systems $\mathcal{S} = \{S_F \mid F \in \mathcal{M}\}$ and $\mathcal{S}' = \{S'_F \mid F \in \mathcal{M}\}$. Put $\partial X = \coprod_{F\in\mathcal{M}} \Gamma \times_F S_F$ and $\partial X' = \coprod_{F\in\mathcal{M}} \Gamma \times_F S'_F$. Let $\partial u \colon \partial X \to \partial X'$ be any cellular Γ-map. Consider the Γ-pushout*

$$\begin{tikzcd} \partial X \arrow[r] \arrow[d, "\partial u"'] & X \arrow[d, "u"] \\ \partial X' \arrow[r] & X' \end{tikzcd}$$

Suppose that $(X, \partial X)$ is a finite free d-dimensional Γ-CW-pair. Equip $(X', \partial X')$ with the induced structure of a finite free d-dimensional Γ-CW-pair.

Then $(X', \partial X')$ is a slice complement model for $\underline{E}\Gamma$ if $(X, \partial X)$ is a slice complement model for $\underline{E}\Gamma$ and $d \geq 3$, and $(X, \partial X)$ is a slice complement model for $\underline{E}\Gamma$ if $(X', \partial X')$ is a slice complement model for $\underline{E}\Gamma$ and $d \geq 4$.

Proof. Since ∂u is $(d-2)$-connected, the map u is $(d-2)$-connected. We conclude from excision that the map $H_n(u, \partial u) \colon H_n(X, \partial X) \xrightarrow{\cong} H_n(X', \partial X')$ is bijective for all $n \in \mathbb{Z}$. Now the claim follows from Lemma 4.4. □

5. Poincaré pairs

Recall from the introduction that the main goal of this paper is to find a slice complement model $(X, \partial X)$ such that $(X/\partial X, \partial X/\Gamma)$ is a Poincaré pair. This is a necessary condition for finding a slice complement model $(X, \partial X)$ such that $(X/\partial X, \partial X/\Gamma)$ is a compact manifold, which we will finally prove in the sequel [9] to this paper. Next we give some information about Poincaré pairs in general and prove some results needed later.

5.1. **Review on Poincaré pairs.** We recall some basics about Poincaré pairs following [34] and prove Lemma 5.8, which is a mild generalization of [34, Theorem 2.1].

Given a CW-complex X, we denote by $p_X \colon \widetilde{X} \to X$ its universal covering. If π is the fundamental group of X, then $\widetilde{X}$ is a free π-CW-complex and p_X is the quotient map of this π-action on $\widetilde{X}$. If $A \subseteq X$ is a CW-subcomplex, we denote by $\overline{A} = p_X^{-1}(A)$. We get a free π-CW-pair $(\widetilde{X}, \overline{A})$. Note that $p|_{\overline{A}} \colon \overline{A} \to A$ is a π-covering. It is the universal covering of A, if and only if A is connected and the inclusion $A \to X$ induces a bijection on the fundamental groups. Given an element $w \in H^1(\pi, \mathbb{Z}/2)$, which is the same as a group homomorphism $w \colon \pi \to \{\pm 1\}$, we denote by $\mathbb{Z}^w$ the $\mathbb{Z}\pi$-module, whose underlying abelian group is $\mathbb{Z}$ and on which $\omega \in \pi$ acts by multiplication with $w(\omega)$. We denote by $H_*^\pi(\widetilde{X}, \overline{A}; \mathbb{Z}^w)$ the homology of the $\mathbb{Z}$-chain complex $\mathbb{Z}^w \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}, \overline{A})$. Consider an element $u \in H_d^\pi(\widetilde{X}, \overline{A}; \mathbb{Z}^w)$. Let $\partial \colon H_d^\pi(\widetilde{X}, \overline{A}; \mathbb{Z}^w) \to H_{d-1}^\pi(\overline{A}; \mathbb{Z}^w)$ be the boundary homomorphism. We obtain by the cap product $\mathbb{Z}\pi$-chain maps, unique up to $\mathbb{Z}\pi$-chain homotopy

$$-\cap u \colon C^{d-*}(\widetilde{X}, \overline{A}) \to C_*(\widetilde{X}); \tag{5.1}$$

$$-\cap u \colon C^{d-*}(\widetilde{X}) \to C_*(\widetilde{X}, \overline{A}); \tag{5.2}$$

$$-\cap \partial(u) \colon C^{d-1-*}(\overline{A}) \to C_*(\overline{A}). \tag{5.3}$$

Here the dual chain complexes are to be understood with respect to the w-twisted involution $\mathbb{Z}\pi \to \mathbb{Z}\pi$ sending $\sum_{\omega\in\pi} n_\omega \cdot \omega$ to $\sum_{\omega\in\pi} w(\omega) \cdot n_\omega \cdot \omega^{-1}$.

Definition 5.4 ((Simple) finite d-dimensional Poincaré pair)**.** We call a finite d-dimensional CW-pair $(X, \partial X)$ with connected X a *(simple) finite d-dimensional*

Poincaré pair with respect to the *orientation homomorphism* $w \in H^1(X,\mathbb{Z}/2) = \hom(\pi_1(X),\{\pm 1\})$ and *fundamental class* $[X,\partial X] \in H_d^{\pi}(\widetilde{X},\overline{\partial X};\mathbb{Z}^w)$, if the $\mathbb{Z}\pi$-chain maps (5.1), (5.2) and (5.3) for $u = [X,\partial X]$ are (simple) $\mathbb{Z}\pi$-chain homotopy equivalences.

Note that all three $\mathbb{Z}\pi$-chain maps (5.1), (5.2), and (5.3) are (simple) $\mathbb{Z}\pi$-chain homotopy equivalences, if two of them are.

Since the $\mathbb{Z}\pi$-chain map (5.1) is a $\mathbb{Z}\pi$-chain homotopy equivalence and induces an isomorphism $H_d^{\pi}(\widetilde{X},\overline{X};\mathbb{Z}^w) \xrightarrow{\cong} H_{\pi}^0(\widetilde{X})$ and $H_{\pi}^0(\widetilde{X}) \cong H^0(X) \cong \mathbb{Z}$, the group $H_d^{\pi}(\widetilde{X},\overline{X};\mathbb{Z}^w)$ is infinite cyclic and $[X,\partial X] \in H_d^{\pi_1(X)}(\widetilde{X},\overline{\partial X};\mathbb{Z}^w)$ is a generator.

Remark 5.5 (Orientation homomorphisms)**.** Part of the definition of a Poincaré pair $(X,\partial X)$ is the existence of an appropriate orientation homomorphism $w \in H^1(X;\mathbb{Z}/2)$. Note that w is uniquely determined by the mere fact that $(X,\partial X)$ together with the choice of w is a finite Poincaré pair. There is even a recipe, how to construct w from the underlying CW-pair $(X,\partial X)$, see [7, Lemma 5.46 on page 109], which we will recall next.

Let the untwisted dual $\mathbb{Z}\pi$-chain complex $C^{d-*}_{\mathrm{untw}}(\widetilde{X},\overline{\partial X})$ be defined with respect to the untwisted involution on $\mathbb{Z}\pi$ sending $\sum_{\omega\in\pi} n_\omega\cdot\omega$ to $\sum_{\omega\in\pi} n_\omega\cdot\omega^{-1}$. Note that $C^{d-*}_{\mathrm{untw}}(\widetilde{X},\overline{\partial X})$ depends only on the Γ-CW-complex X but not on w, and that $\mathbb{Z}^w\otimes_{\mathbb{Z}} C^{d-*}_{\mathrm{untw}}(\widetilde{X},\overline{\partial X})$ equipped with the diagonal π-action is $C^{d-*}(\widetilde{X},\overline{\partial X})$. Now w is given by the π-action on the infinite cyclic group $H_0(C^{d-*}_{\mathrm{untw}}(\widetilde{X},\overline{\partial X}))$, or, equivalently by the condition that the $\mathbb{Z}\Gamma$-module $H_0(C^{d-*}_{\mathrm{untw}}(\widetilde{X},\overline{\partial X}))$ is $\mathbb{Z}\Gamma$-isomorphic to $\mathbb{Z}^w$. This follows from the fact that the $\mathbb{Z}\pi$-chain map (5.1) induces a $\mathbb{Z}\pi$-isomorphism from $H_0(C^{d-*}(\widetilde{X},\overline{\partial X}))$ to $H_0^{\pi}(C_*(\widetilde{X}))$ and the π-action on $H_0^{\pi}(C_*(\widetilde{X})) \cong \mathbb{Z}$ is trivial.

There are only two possible choices for the fundamental class $[X/\pi,\partial X/\pi]$, since it has to be a generator of the infinite cyclic group $H_d^{\pi}(\widetilde{X},\overline{X};\mathbb{Z}^w)$.

This definition extends to not necessarily connected X as follows. We call a finite d-dimensional CW-pair $(X,\partial X)$ a finite d-dimensional Poincaré pair with respect to $w \in H^1(X;\mathbb{Z}/2)$ with fundamental class

$$[X,\partial X] \in \bigoplus_{C\in\pi_0(C)} H_d^{\pi_1(C)}(\widetilde{C},\overline{C\cap\partial X};\mathbb{Z}^{w|_C})$$

if for each path component C of X the pair $(C, C\cap\partial X)$ is a finite d-dimensional Poincaré complex with respect to $w|_C \in H^1(C;\mathbb{Z}/2)$ coming from w by restriction to C and the fundamental class $[C,(C\cap\partial X)]$ in $H_d^{\pi_1(C)}(\widetilde{C},\overline{C\cap\partial X};\mathbb{Z}^{w|_C})$, which is the component associated to $C\in\pi_0(X)$ of $[X,\partial X]$.

Note that ∂X inherits the structure of a finite $(d-1)$-dimensional Poincaré-complex, provided that $\pi_1(\partial X,x) \to \pi_1(X,x)$ is injective for every $x\in\partial X$. Moreover, ∂X inherits the structure of a simple finite $(d-1)$-dimensional Poincaré-complex, provided that $\pi_1(\partial X,x) \to \pi_1(X,x)$ is injective for every $x\in\partial X$ and $\mathrm{Wh}(\partial X) \to \mathrm{Wh}(X)$ is injective. The last condition is satisfied for instance, if the functor $\Pi(\partial X) \to \Pi(X)$ on the fundamental groupoids induced by the inclusion $\partial X \to X$ is an equivalence of categories. All these claims follow from the following subsection.

5.2. **Arbitrary coverings.** We have to deal with arbitrary coverings as well. Let (X,A) be finite CW-pair with connected X. Put $\pi = \pi_1(X)$. Let Γ be a group and $p\colon \widehat{X} \to X$ be a Γ-covering. Put $\widehat{A} = p^{-1}(A)$. Let $p_X\colon \widetilde{X} \to X$ be the universal covering. Put $\overline{A} = p_X^{-1}(\widetilde{X})$. Consider group homomorphisms $w\colon \pi \to \{\pm 1\}$ and $v\colon \Gamma \to \{\pm 1\}$. Let $\phi\colon \pi \to \Gamma$ be the group homomorphism, for which there is a

Γ-homeomorphism $F\colon \Gamma \times_\phi \widetilde{X} \xrightarrow{\cong} \widehat{X}$. Note that F induces a Γ-homeomorphism $f\colon \Gamma \times_\phi \overline{A} \xrightarrow{\cong} \widehat{A}$. Suppose that $v = w \circ \phi$.

Then the Γ-map (F, f) induces an isomorphism of $\mathbb{Z}\Gamma$-chain complexes

$$\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\phi} C_*(\widetilde{X}, \overline{A}) \xrightarrow{\cong} C_*(\widehat{X}, \widehat{A}).$$

In particular we get an isomorphism

$$\mu\colon H_d^\pi(\widetilde{X}, \overline{A}, \mathbb{Z}^w) \xrightarrow{\cong} H_d^\Gamma(\widehat{X}, \widehat{A}, \mathbb{Z}^v). \tag{5.6}$$

Given an element $u \in H_d^\pi(\widetilde{X}, \overline{A}, \mathbb{Z}^w)$, the $\mathbb{Z}\Gamma$-chain homotopy equivalence

$$- \cap \mu(u)\colon C^{d-*}(\widehat{X}, \widehat{A}) \to C_*(\widehat{X})$$

is obtained from the $\mathbb{Z}\pi$-chain map (5.1) by induction with $\phi\colon \pi \to \Gamma$. Note that $- \cap \mu(u)\colon C^{d-*}(\widehat{X}, \widehat{A}) \to C_*(\widehat{X})$ is a $\mathbb{Z}\Gamma$-chain homotopy equivalence, if the $\mathbb{Z}\pi$-chain map (5.1) is a $\mathbb{Z}\pi$-chain homotopy equivalence. The converse is true, provided that ϕ is injective. Moreover, $- \cap \mu(u)\colon C^{d-*}(\widehat{X}, \widehat{A}) \to C_*(\widehat{X})$ is a simple $\mathbb{Z}\Gamma$-chain homotopy equivalence, if the $\mathbb{Z}\pi$-chain map (5.1) is a simple $\mathbb{Z}\pi$-chain homotopy equivalence. The converse is true, provided that ϕ is injective and induces an injection $\mathrm{Wh}(\pi) \to \mathrm{Wh}(\Gamma)$.

The analogous statements are true for (5.2).

5.3. **Subtracting a Poincaré pair.** Let M be a closed manifold. Suppose that we have embedded a codimension zero manifold $(N, \partial N)$ into M. If we subtract $(N, \partial N)$ from M in the sense that we delete the interior of N from M, then we obtain a manifold with boundary ∂N. We want to prove the analogue for Poincaré pairs.

Consider a connected finite d-dimensional CW-complex Y with fundamental group π. Consider (not necessarily connected) CW-subcomplexes Y_1, Y_2, and Y_0 of Y satisfying $Y = Y_1 \cup Y_2$ and $Y_0 = Y_1 \cap Y_2$ such that $\dim(Y_1) = \dim(Y_2) = d$ and $\dim(Y_0) = d - 1$ hold. Let $p_Y\colon \widetilde{Y} \to Y$ be the universal covering of Y. Put $\overline{Y_i} = p_Y^{-1}(Y_i)$ for $i = 0, 1, 2$.

Consider elements $w \in H^1(Y; \mathbb{Z}/2)$ and $w_i \in H^1(Y_i; \mathbb{Z}/2)$ for $i = 1, 2$, such that $H^1(Y; \mathbb{Z}/2) \to H^1(Y_i; \mathbb{Z}/2)$ sends w to w_i for $i = 1, 2$. We conclude from (5.6) that there are isomorphisms

$$\mu_i\colon \bigoplus_{C \in \pi_0(Y_i)} H_d^{\pi_1(C)}(\widetilde{C}, \overline{C \cap Y_0}; \mathbb{Z}^{w_i|_C}) \to H_d^\pi(\overline{Y_i}, \overline{Y_0}; \mathbb{Z}^w) \tag{5.7}$$

for $i = 1, 2$.

By a Mayer-Vietoris argument we see that the map

$$H_d^\pi(\overline{Y_2}, \overline{Y_0}; \mathbb{Z}^w) \oplus H_d^\pi(\overline{Y_1}, \overline{Y_0}; \mathbb{Z}^w) \xrightarrow{\cong} H_d^\pi(\widetilde{Y}, \overline{Y_0}; \mathbb{Z}^w)$$

is bijective. Let $u \in H^\pi(\widetilde{Y}, \mathbb{Z}^w)$, $u_1 \in H_d^\pi(\overline{Y_1}, \overline{Y_0}; \mathbb{Z}^w)$, and $u_2 \in H_d^\pi(\overline{Y_2}, \overline{Y_0}; \mathbb{Z}^w)$ be elements such that the isomorphism above sends (u_1, u_2) to the image of u under the map $H_d^\pi(\widetilde{Y}; \mathbb{Z}^w) \to H_d^\pi(\widetilde{Y}, \overline{Y_0}; \mathbb{Z}^w)$. The next result is a variation of [34, Theorem 2.1].

Lemma 5.8. *(1) Suppose that for $i = 1, 2$ there are elements $[Y_i, Y_0]$ in $\bigoplus_{C \in \pi_0(Y_i)} H_d^{\pi_1(C)}(\widetilde{C}, \overline{C \cap Y_0}; \mathbb{Z}^{w_i|_C})$ such that $\mu_i([Y_i, Y_0]) = u_i$ for the isomorphism μ_i of (5.7) holds and (Y_i, Y_0) is a finite d-dimensional Poincaré complex with respect to w_i and $[Y_i, Y_0]$ as fundamental class;*

Then Y is a finite d-dimensional Poincaré complex with respect to w and u as fundamental class;

(2) *If we assume in assertion* (1) *additionally that* (Y_i, Y_0) *is a simple finite* d*-dimensional Poincaré complex for* $i = 1, 2$*, then* Y *is a simple finite* d*-dimensional Poincaré complex;*

(3) *Assume that the following conditions hold:*
- Y *is a finite* d*-dimensional Poincaré complex with respect to* w *and* u *as fundamental class;*
- *There is an element* $[Y_1, Y_0] \in \bigoplus_{C\in\pi_0(Y_1)}$ *in* $H_d^{\pi_1(C)}(\widetilde{C}, \overline{C \cap Y_0}; \mathbb{Z}^{w_1|_C})$ *such that* $\mu_1([Y_1, Y_0]) = u_1$ *holds and* (Y_1, Y_0) *is a finite* d*-dimensional Poincaré complex with respect to* w_1 *and* $[Y_1, Y_0]$ *as fundamental class;*
- *For every* $y_2 \in Y_2$ *the map* $\pi_1(Y_2, y_2) \to \pi_1(Y, y_2)$ *is injective.*

Then (Y_2, Y_0) *is a finite* d*-dimensional Poincaré pair with respect to* w_2 *and* $\mu_2^{-1}(u_2)$ *as fundamental class;*

(4) *If we assume in assertion* (3) *additionally that* (Y_1, Y_0) *and* Y *are simple and the map* $\mathrm{Wh}(Y_2) \to \mathrm{Wh}(Y)$ *is injective, then* (Y_2, Y_0) *is a simple finite* d*-dimensional Poincaré pair.*

Proof. Consider the following diagram of $\mathbb{Z}\pi$-chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C^{d-*}(\widetilde{Y}, \overline{Y_1}) \arrow[r] \arrow[d, "\cong"'] & C^{d-*}(\widetilde{Y}) \arrow[r] \arrow[dd, "-\cap u"] & C^{d-*}(\overline{Y_1}) \arrow[r] \arrow[d, "-\cap u_1"] & 0 \\
& C^{d-*}(\overline{Y_2}, \overline{Y_0}) \arrow[d, "-\cap u_2"] & & C_*(\overline{Y_1}, \overline{Y_0}) \arrow[d, "\cong"] & \\
0 \arrow[r] & C_*(\overline{Y_2}) \arrow[r] & C_*(\widetilde{Y}) \arrow[r] & C_*(\widetilde{Y}, \overline{Y_2}) \arrow[r] & 0
\end{tikzcd}$$

One can arrange the representatives of the $\mathbb{Z}\pi$-chain maps $-\cap u_1$, $-\cap u$, and $-\cap u_2$ such that the diagram commutes. The two rows are based exact sequences of pairs. The two arrows marked with $\cong$ are the base preserving isomorphisms given by excision. We conclude that all three $\mathbb{Z}\pi$-chain maps $-\cap u_1$, $-\cap u$, and $-\cap u_2$ are (simple) $\mathbb{Z}\pi$-chain homotopy equivalences, if and only if two of them are. Provided that for every $y_2 \in Y_2$ the map $\pi_1(Y_2, y_2) \to \pi_1(Y, y_2)$ is injective, then (Y_2, Y_0) is a Poincaré pair, if and only if $-\cap u_2 \colon C^{d-*}(\overline{Y_2}, \overline{Y_0}) \to C_*(\overline{Y_2})$ is a $\mathbb{Z}\pi$-chain homotopy equivalence

Now the claims follows from the conderations appearing in Subsection 5.2. $\square$

5.4. **Special Poincaré complexes.**

Definition 5.9 (Special Poincaré pair)**.** Let $(X, \partial X)$ be a finite (simple) Poincaré pair of dimension $n \geq 5$. It is called *special* if there exists

- An n-dimensional compact smooth manifold H with boundary ∂H such that the inclusion $i \colon \partial H \to H$ induces an epimorphism $\pi_0(\partial H) \to \pi_0(H)$, and for every component D of H an epimorphism $*_{\substack{C\in\pi_0(\partial H)\\ i(C)\subseteq D}} \pi_1(C) \to \pi_1(D)$;
- A finite CW-sub complex $\widehat{X}$ of X containing ∂X such that $\widehat{X} \setminus \partial X$ contains only cells of dimension $\leq (n-2)$;
- A cellular map $z \colon \partial H \to \widehat{X}$, which induces a bijection on π_0 and for every choice of base point in ∂H an epimorphism on π_1, where we consider $(H, \partial H)$ as a simple CW-pair by a smooth triangulation,

such that X is the pushout

$$\begin{tikzcd} \partial H \arrow[r, "z"] \arrow[d] & \widehat{X} \arrow[d] \\ H \arrow[r] & X = H \cup_z \widehat{X} \end{tikzcd}$$

Lemma 5.10. *Let $(X, \partial X)$ be an n-dimensional finite (simple) Poincaré pair of dimension $n \geq 5$. Then there exists a special Poincaré pair $(X', \partial X')$ with $\partial X = \partial X'$ together with a (simple) homotopy equivalence $g\colon X \to X'$ inducing the identity on ∂X.*

Proof. See [35, Lemma 2.8 on page 30 and the following paragraph]. □

6. Fixing the orientation homomorphisms and the fundamental classes

6.1. **Transfer.** In this subsection we recall the notion of a transfer. Consider a $\mathbb{Z}\Gamma$-chain complex C_* and a right $\mathbb{Z}\Gamma$-module M. We make no assumptions about C_* of the kind that its chain modules are projective or finitely generated. Let i^*C_* and i^*M be the $\mathbb{Z}\pi$-chain complex and $\mathbb{Z}\pi$-module obtained by restriction with the inclusion $i\colon \pi \to \Gamma$.

Define $\mathbb{Z}$-chain maps

$$i_*\colon i^*M \otimes_{\mathbb{Z}\pi} i^*C_* \to M \otimes_{\mathbb{Z}\Gamma} C_* \quad m \otimes x \mapsto m \otimes x \tag{6.1}$$

and

$$\operatorname{trf}_*\colon M \otimes_{\mathbb{Z}\Gamma} C_* \to i^*M \otimes_{\mathbb{Z}\pi} i^*C_*, \quad m \otimes x \mapsto \sum_{g \in G} m\widehat{g} \otimes \widehat{g}^{-1}x, \tag{6.2}$$

where $\widehat{g}$ is any element in Γ, which is mapped under the projection $\Gamma \to G$ to g. The definition (6.2) is independent of the choice of $\widehat{g}$, since for $\omega \in \pi$, $m \in M$ and $x \in C_*$ we get in $i^*M \otimes_{\mathbb{Z}\pi} i^*C_*$

$$m\widehat{g}\omega \otimes (\widehat{g}\omega)^{-1}x = m\widehat{g}\omega \otimes \omega^{-1}\widehat{g}^{-1} = m\widehat{g} \otimes \widehat{g}^{-1}x.$$

One easily checks that both definitions are compatible with the tensor relations and define indeed $\mathbb{Z}$-chain maps. Moreover, i_* and trf_* are natural in both C_* and M and satisfy

$$i_* \circ \operatorname{trf}_* = |G| \cdot \operatorname{id}_{M \otimes_{\mathbb{Z}\Gamma} C_*} \tag{6.3}$$

Applying this to the $\mathbb{Z}\Gamma$-chain complexes $C_*(X, \partial X)$ for a slice complement model $(X, \partial X)$, to $C_*(E\Gamma, \partial E\Gamma)$, to $C_*(\underline{E}\Gamma, \partial\underline{E}\Gamma)$ and to $C_*(\underline{E}\Gamma)$ and take the d-th homology, we obtain the following commutative diagram of $\mathbb{Z}$-modules, whose vertical arrows are the obvious maps

(6.4)

$$\begin{tikzcd}
H_d^{\Gamma}(X,\partial X;\mathbb{Z}^w) \arrow[r, "H_d(\mathrm{trf}_*)"] \arrow[d, "\cong"] & H_d^{\pi}(i^*X,i^*\partial X;\mathbb{Z}^v) \arrow[r, "H_d(i_*)"] \arrow[d, "\cong"] & H_d^{\Gamma}(X,\partial X;\mathbb{Z}^w) \arrow[d, "\cong"] \\
H_d^{\Gamma}(E\Gamma,\partial E\Gamma;\mathbb{Z}^w) \arrow[r, "H_d(\mathrm{trf}_*)"] \arrow[d, "\cong"] & H_d^{\pi}(i^*E\Gamma,i^*\partial E\Gamma;\mathbb{Z}^v) \arrow[r, "H_d(i_*)"] \arrow[d, "\cong"] & H_d^{\Gamma}(E\Gamma,\partial X;\mathbb{Z}^w) \arrow[d, "\cong"] \\
H_d^{\Gamma}(\underline{E}\Gamma,\partial\underline{E}\Gamma;\mathbb{Z}^w) \arrow[r, "H_d(\mathrm{trf}_*)"] & H_d^{\pi}(i^*\underline{E}\Gamma,i^*\partial\underline{E}\Gamma;\mathbb{Z}^v) \arrow[r, "H_d(i_*)"] & H_d^{\Gamma}(\underline{E}\Gamma,\partial\underline{E}\Gamma;\mathbb{Z}^w) \arrow[d, leftrightarrow, "\cong"] \\
H_d^{\Gamma}(\underline{E}\Gamma;\mathbb{Z}^w) \arrow[r, "H_d(\mathrm{trf}_*)"] \arrow[u, "\cong"] & H_d^{\pi}(i^*\underline{E}\Gamma;\mathbb{Z}^v) \arrow[r, "H_d(i_*)"] \arrow[u, "\cong"] & H_d^{\Gamma}(\underline{E}\Gamma;\mathbb{Z}^w)
\end{tikzcd}$$

such that in each row the composite of the two horizontal maps is $|G| \cdot \mathrm{id}$. The vertical arrows from the second row to the third row and the composite of the arrows from the first row to the second row with the arrows from the second row to the third row are all bijective by excision applied to the homotopy Γ-pushouts appearing in Proposition 2.1 and in (4.3). The horizontal arrows from the fourth row to the third row are bijective, since $\partial\underline{E}\Gamma$ is zero-dimensional. Hence all vertical arrows are bijective.

6.2. **Some necessary conditions.** In this subsection we assume that we have a slice complement model $(X,\partial X)$ for $\underline{E}\Gamma$ with respect to the free d-dimensional slice system $\mathcal{S}$ such that the quotient space $(X/\Gamma,\partial X\Gamma)$ carries the structure of a finite Poincaré pair. Recall from Remark 5.5 that there is only one choice possible for the orientation homomorphism $w\colon \Gamma \to \{\pm 1\}$. Namely, the abelian group $H_0(C^{d-*}_{\mathrm{untw}}(X,\partial X))$ must be infinite cyclic and as a $\mathbb{Z}\Gamma$-module it must be isomorphic to $\mathbb{Z}^w$. Since $(X,\partial X)$ is a slice complement model for $\underline{E}\Gamma$, the projection $\mathrm{pr}\colon (X,\partial X) \to (\underline{E}\Gamma,\partial\underline{E}\Gamma)$ is a $\mathbb{Z}\Gamma$-chain homotopy equivalence because of the homotopy Γ-pushout (4.3) and induces an $\mathbb{Z}\Gamma$-isomorphism $H_0(C^{d-*}_{\mathrm{untw}}(X,\partial X)) \xrightarrow{\cong_{\mathbb{Z}\Gamma}} H_0(C^{d-*}_{\mathrm{untw}}(\underline{E}\Gamma,\partial\underline{E}\Gamma))$. Since $\partial\underline{E}\Gamma$ is zero-dimensional, the obvious inclusion induces an isomorphism of $\mathbb{Z}\Gamma$-modules $H_0(C^{d-*}_{\mathrm{untw}}(\underline{E}\Gamma,\partial\underline{E}\Gamma)) \xrightarrow{\cong} H_0(C^{d-*}_{\mathrm{untw}}(\underline{E}\Gamma))$. Hence we obtain an isomorphism of $\mathbb{Z}\Gamma$-modules

$$H_0(C^{d-*}_{\mathrm{untw}}(X,\partial X)) \xrightarrow{\cong_{\mathbb{Z}\Gamma}} H_0(C^{d-*}_{\mathrm{untw}}(\underline{E}\Gamma)).$$

If we define $w\colon \Gamma \to \{\pm\}$ by requiring that the $\mathbb{Z}\Gamma$-module $H_0(C^{d-*}_{\mathrm{untw}}(\underline{E}\Gamma))$ is isomorphic to $\mathbb{Z}^w$, then w is defined in terms of Γ only and has to be the orientation homomorphism for any structure of a finite Poincar'e pair on $(X/\Gamma,\partial X/\Gamma)$ for any slice complement model $(X,\partial X)$.

Note that for a finite Poincaré pair $(X/\Gamma,\partial X/\Gamma)$ there is the induced structure of a finite Poincare complex on $\partial X/\Gamma$ and the orientation homomorphism for $\partial X/\Gamma$ is obtained by the one for $(X/\Gamma,\partial X/\Gamma)$ by restriction. Recall that $\partial X/\Gamma$ is $\coprod_{F\in\mathcal{M}} S_F/F$ and that we have figured out the first Stiefel-Whitney class for S_F/F in Lemma 3.3.

Since $(X/\Gamma,\partial X/\Gamma)$ is a finite d-dimensional Poincaré pair, the same is true for $(X/\pi,\partial X/\pi)$ by [17, Theorem H] and the homomorphisms of infinite cyclic groups $H_d(\mathrm{trf}_*)\colon H_d^{\Gamma}(X,\partial X;\mathbb{Z}^w) \to H_d^{\pi}(i^*X,i^*\partial X;\mathbb{Z}^w)$ sends the fundamental classes to one another and hence is bijective. We conclude from (6.4) that the map $H_d(\mathrm{trf}_*)\colon H_d^{\Gamma}(\underline{E}\Gamma;\mathbb{Z}^w) \xrightarrow{\cong} H_d^{\pi}(i^*\underline{E}\Gamma;\mathbb{Z}^w)$ is bijective. Since $i^*\underline{E}\Gamma$ is a model for $B\pi$, we conclude from Lemma 5.8 (1) applied to the restriction to π of the

Γ-pushout

$$\begin{array}{ccc}\coprod_{F\in\mathcal{M}}\Gamma\times_F S_F & \longrightarrow & X\\ \downarrow & & \downarrow\\ \coprod_{F\in\mathcal{M}}\Gamma\times_F D_F & \longrightarrow & \underline{E}\Gamma.\end{array}$$

and Remark 5.5 that the restriction of w to π must be the orientation homomorphism $v\colon \pi \to \{\pm 1\}$ of $B\pi$. To summarize, we have the following necessary conditions for the existence of a slice complement model $(X, \partial X)$ such that the quotient space $(X/\Gamma, \partial X/\Gamma)$ carries the structure of a finite Poincaré pair:

(i) The abelian group $H_0(C^{d-*}_{\mathrm{untw}}(\underline{E}\Gamma))$ is infinite cyclic. Let $w\colon \Gamma \to \{\pm 1\}$ be the homomorphisms uniquely determined by the property that the $\mathbb{Z}\Gamma$-module $H_0(C^{d-*}_{\mathrm{untw}}(\underline{E}\Gamma))$ is $\mathbb{Z}\Gamma$-isomorphic to $\mathbb{Z}^w$.
(ii) There is a finite d-dimensional Poincaré CW-complex model for $B\pi$ with respect to the orientation homomorphisms $v\colon \pi \to \{\pm 1\}$;
(iii) We have $v = w|_\pi$;
(iv) Consider any $F \in \mathcal{M}$. The restriction of w to F is trivial, if d is even. If d is odd, $F \cong \mathbb{Z}/2$ and restriction of w to F is non-trivial;
(v) The transfer map

$$H_d(\mathrm{trf}_*)\colon H^\Gamma_d(\underline{E}\Gamma;\mathbb{Z}^w) \xrightarrow{\cong} H^\pi_d(i^*\underline{E}\Gamma;\mathbb{Z}^v) = H^\pi_d(E\pi;\mathbb{Z}^v)$$

is bijective.

These necessary conditions motivate the material of the next subsections.

6.3. **The orientation homomorphism.** For the remainder of this section we will make the following assumptions

Assumption 6.5.

- *There exists a finite Γ-CW-model for $\underline{E}\Gamma$ of dimension d such that its singular Γ-subspace $\underline{E}\Gamma^{>1}$ is $\coprod_{F\in\mathcal{M}}\Gamma/F$;*
- *There is a finite d-dimensional Poincaré CW-complex model for $B\pi$ with respect to the orientation homomorphisms $v\colon \pi \to \{\pm 1\}$ and fundamental class $[B\pi] \in H^\pi_d(E\pi;\mathbb{Z}^v)$.*

Let $\mathrm{trunc}\colon \mathbb{Z}\Gamma \to \mathbb{Z}\pi$ be the homomorphisms of $\mathbb{Z}\pi$-$\mathbb{Z}\pi$-bimodules, which sends $\sum_{\gamma\in\Gamma}\lambda_\gamma\cdot\gamma$ to $\sum_{\gamma\in\pi}\lambda_\gamma\cdot\gamma$. Consider $\gamma_0 \in \Gamma$. Let $r_{\gamma_0^{-1}}\colon \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma$ be the $\mathbb{Z}\Gamma$ automorphism of left $\mathbb{Z}\Gamma$-modules sending $\sum_\gamma \lambda_\gamma\cdot\gamma$ to $\sum_\gamma\lambda_\gamma\cdot\gamma\gamma_0^{-1}$. Denote by $l_{\gamma_0^{-1}}\colon M \to M$ the automorphism of abelian groups sending u to $\gamma_0^{-1}u$. Let $c_{\gamma_0}\colon \pi \xrightarrow{\cong} \pi$ be the group automorphism sending u to $\gamma_0 u\gamma_0^{-1}$. Denote by $\mathbb{Z}[c_{\gamma_0}]\colon \mathbb{Z}\pi \xrightarrow{\cong} \mathbb{Z}\pi$ the induced ring automorphism. Let M be a $\mathbb{Z}\Gamma$-module. We get by composition and precomposition maps of abelian groups

$$\begin{array}{rcl}\mathrm{trunc}_*\colon \hom_{\mathbb{Z}\Gamma}(M,\mathbb{Z}\Gamma) & \to & \hom_{\mathbb{Z}\pi}(i^*M,\mathbb{Z}\pi);\\ (r_{\gamma_0^{-1}})_*\colon \hom_{\mathbb{Z}\Gamma}(M,\mathbb{Z}\Gamma) & \to & \hom_{\mathbb{Z}\Gamma}(M,\mathbb{Z}\Gamma);\\ \mathbb{Z}[c_{\gamma_0}]_*\circ(l_{\gamma_0^{-1}})^*\colon \hom_{\mathbb{Z}\pi}(i^*M,\mathbb{Z}\pi) & \to & \hom_{\mathbb{Z}\pi}(i^*M,\mathbb{Z}\pi).\end{array}$$

The map $\mathbb{Z}[c_{\gamma_0}]_* \circ (l_{\gamma_0^{-1}})^*$ is indeed well-defined as the following calculation shows for $f \in \hom_{\mathbb{Z}\pi}(i^*M, \mathbb{Z}\pi)$, $\omega \in \pi$, and $x \in M$

$$\begin{aligned}
\big(\mathbb{Z}[c_{\gamma_0}]_* \circ (l_{\gamma_0^{-1}})^*(f)\big)(\omega x) &= \mathbb{Z}[c_{\gamma_0}](f(\gamma_0^{-1}\omega x)) \\
&= \mathbb{Z}[c_{\gamma_0}](f(\gamma_0^{-1}\omega\gamma_0\gamma_0^{-1}x)) \\
&= \mathbb{Z}[c_{\gamma_0}](\gamma_0^{-1}\omega\gamma_0 \cdot f(\gamma_0^{-1}x)) \\
&= \mathbb{Z}[c_{\gamma_0}](\gamma_0^{-1}\omega\gamma_0) \cdot \mathbb{Z}[c_{\gamma_0}](f(\gamma_0^{-1}x)) \\
&= \omega \cdot \big(\mathbb{Z}[c_{\gamma_0}]_* \circ (l_{\gamma_0^{-1}})^*(f)\big)(x).
\end{aligned}$$

Lemma 6.6. *(1) The map* $\operatorname{trunc}_*\colon \hom_{\mathbb{Z}\Gamma}(M, \mathbb{Z}\Gamma) \xrightarrow{\cong} \hom_{\mathbb{Z}\pi}(i^*M, \mathbb{Z}\pi)$ *is bijective;*

(2) The following diagram commutes

$$\begin{tikzcd}
\hom_{\mathbb{Z}\Gamma}(M, \mathbb{Z}\Gamma) \arrow[r, "\operatorname{trunc}_*", "\cong"'] \arrow[d, "(r_{\gamma_0^{-1}})_*"'] & \hom_{\mathbb{Z}\pi}(i^*M, \mathbb{Z}\pi) \arrow[d, "\mathbb{Z}[c_{\gamma_0}]_* \circ (l_{\gamma_0^{-1}})^*"] \\
\hom_{\mathbb{Z}\Gamma}(M, \mathbb{Z}\Gamma) \arrow[r, "\cong", "\operatorname{trunc}_*"'] & \hom_{\mathbb{Z}\pi}(i^*M, \mathbb{Z}\pi).
\end{tikzcd}$$

Proof. (1) Since π has finite index in Γ, the coinduction $i_!\mathbb{Z}\pi$ of $\mathbb{Z}\pi$ is $\mathbb{Z}\Gamma$-isomorphic to $\mathbb{Z}\Gamma$, see [5, Proposition 5.8 in III.5 on page 70]. One has the adjunction isomorphism $(i^*, i_!)$, see [5, (3.6) in III.3 on page 64]

$$\hom_{\mathbb{Z}\Gamma}(M, \mathbb{Z}\Gamma) \xrightarrow{\cong} \hom_{\mathbb{Z}\pi}(i^*M, \mathbb{Z}\pi).$$

One easily checks by going through the definitions, that this isomorphism is the map trunc_*.

(2) This follows from the following computation for $f \in \hom_{\mathbb{Z}\Gamma}(M, \mathbb{Z}\Gamma)$ and $x \in M$. Write $f(x) = \sum_{\gamma\in\Gamma} \lambda_\gamma \cdot \gamma$. Then we get $f(x)\gamma_0^{-1} = \sum_{\gamma\in\Gamma} \lambda_\gamma \cdot \gamma\gamma_0^{-1}$ and $f(\gamma_0^{-1}x) = \sum_{\gamma\in\Gamma} \lambda_\gamma \cdot \gamma_0^{-1}\gamma$. This implies

$$\begin{aligned}
\big(\operatorname{trunc}_* \circ (r_{\gamma_0^{-1}})_*(f)\big)(x) &= \sum_{\substack{\gamma\in\Gamma \\ \gamma\gamma_0^{-1}\in\pi}} \lambda_\gamma \cdot \gamma\gamma_0^{-1}; \\
\big((l_{\gamma_0^{-1}})^* \circ \operatorname{trunc}_*(f)\big)(x) &= \sum_{\substack{\gamma\in\Gamma \\ \gamma_0^{-1}\gamma\in\pi}} \lambda_\gamma \cdot \gamma_0^{-1}\gamma.
\end{aligned}$$

Now the claim follows from the computation

$$\begin{aligned}
\big(\mathbb{Z}[c_{\gamma_0}] \circ l^*_{\gamma_0^{-1}} \circ \operatorname{trunc}_*(f)\big)(x) &= \mathbb{Z}[c_{\gamma_0}]\Bigg(\sum_{\substack{\gamma\in\Gamma \\ \gamma_0^{-1}\gamma\in\pi}} \lambda_\gamma \cdot \gamma_0^{-1}\gamma\Bigg) \\
&= \sum_{\substack{\gamma\in\Gamma \\ \gamma_0^{-1}\gamma\in\pi}} \lambda_\gamma \cdot \gamma\gamma_0^{-1} \\
&= \sum_{\substack{\gamma\in\Gamma \\ \gamma\gamma_0^{-1}\in\pi}} \lambda_\gamma \cdot \gamma\gamma_0^{-1} \\
&= \big(\operatorname{trunc}_* \circ (r_{\gamma_0^{-1}})_*(f)\big)(x).
\end{aligned}$$

□

Note that Lemma 6.6 (1) implies that the abelian group underlying the $\mathbb{Z}\pi$-module $H_0(i^*C^{d-*}_{\mathrm{untw}}(\underline{E}\Gamma))$ is infinite cyclic, since $H_0(C^{d-*}_{\mathrm{untw}}(E\pi))$ is infinite cyclic. Hence the following notation makes sense

Notation 6.7. The Γ-homomorphism $w\colon \Gamma \to \{\pm 1\}$ is uniquely determined by the property that the $\mathbb{Z}\Gamma$-module $H_0(C^{d-*}_{\mathrm{untw}}(\underline{E}\Gamma))$ is $\mathbb{Z}\Gamma$-isomorphic to $\mathbb{Z}^w$.

Lemma 6.8. *The restriction i^*w of w to π is $v\colon \pi \to \{\pm 1\}$.*

Proof. If γ_0 belongs to π, then the following diagram commutes

$$\begin{tikzcd} \hom_{\mathbb{Z}\Gamma}(M,\mathbb{Z}\Gamma) \arrow[r, "\mathrm{trunc}_*", "\cong"'] \arrow[d, "(r_{\gamma_0^{-1}})_*"'] & \hom_{\mathbb{Z}\pi}(i^*M,\mathbb{Z}\pi) \arrow[d, "(r_{\gamma_0^{-1}})_*"] \\ \hom_{\mathbb{Z}\Gamma}(M,\mathbb{Z}\Gamma) \arrow[r, "\cong", "\mathrm{trunc}_*"'] & \hom_{\mathbb{Z}\pi}(i^*M,\mathbb{Z}\pi) \end{tikzcd}$$

by Lemma 6.6 (2), since in this special case $\mathbb{Z}[c_{\gamma_0}]_* \circ (l_{\gamma_0^{-1}})^*$ reduces to $(r_{\gamma_0^{-1}})_*$. Recall that the free π-CW-complex $i^*\underline{E}\Gamma$ is a model for $E\pi$. Hence we get an isomorphism of $\mathbb{Z}\pi$-chain complexes

$$i^*C^{d-*}_{\mathrm{untw}}(\underline{E}\Gamma) \xrightarrow{\cong_{\mathbb{Z}\pi}} C^{d-*}_{\mathrm{untw}}(E\pi),$$

which induces an isomorphism of $\mathbb{Z}\pi$-modules

$$H_0(i^*C^{d-*}_{\mathrm{untw}}(\underline{E}\Gamma)) \xrightarrow{\cong_{\mathbb{Z}\pi}} H_0(C^{d-*}_{\mathrm{untw}}(E\pi)).$$

This implies $w|_\pi = v$. □

Next we want to express w in terms of π, where we take only the Γ-action on π by conjugation into account and do not refer to $\underline{E}\Gamma$.

Fix $\gamma_0 \in \Gamma$. The group automorphism $c_{\gamma_0^{-1}}\colon \pi \xrightarrow{\cong} \pi$ induces a π-homotopy equivalence $Ec_{\gamma_0^{-1}}\colon E\pi \to c^*_{\gamma_0^{-1}}E\pi$, where $\omega \in \pi$ acts on $c^*_{\gamma_0^{-1}}E\pi$ by the action of $\gamma_0^{-1}\omega\gamma_0$ on $E\pi$. If γ_0 belongs to π, then $Ec_{\gamma_0^{-1}}$ is π-homotopic to the map $l_{\gamma_0^{-1}}\colon E\pi \to c^*_{\gamma_0^{-1}}E\pi$ given by left multiplication with γ_0^{-1}. By assumption there is a finite Poincaré structure on $B\pi$ with respect to the orientation homomorphism $v\colon \pi \to \{\pm 1\}$ and fundamental class $[B\pi] \in H^\pi_d(E\pi;\mathbb{Z}^v)$. The π-homotopy equivalence $Ec_{\gamma_0^{-1}}$ induces an isomorphism of abelian groups

$$H^\pi_d(Ec_{\gamma_0^{-1}};\mathbb{Z}^v)\colon H^\pi_d(E\pi;\mathbb{Z}^v) \xrightarrow{\cong_{\mathbb{Z}}} H^\pi_d(c^*_{\gamma_0^{-1}}E\pi;\mathbb{Z}^v). \tag{6.9}$$

There is an obvious isomorphism of abelian groups

$$a'\colon H^\pi_d(c^*_{\gamma_0^{-1}}E\pi;\mathbb{Z}^{c^*_{\gamma_0^{-1}}(v)}) \xrightarrow{\cong} H^\pi_d(E\pi;\mathbb{Z}^v) \tag{6.10}$$

coming from the identification

$$\mathbb{Z}^v \otimes_{\mathbb{Z}\pi} C_*(E\pi) \xrightarrow{\cong} \mathbb{Z}^{c^*_{\gamma_0^{-1}}(v)} \otimes_{\mathbb{Z}\pi} C_*(c^*_{\gamma_0^{-1}}E\pi), \quad n \otimes x \mapsto n \otimes x.$$

We get

$$(c^{-1}_{\gamma_0})^*(v) = v \tag{6.11}$$

from the following calculation for $\omega \in \pi$

$$(c^{-1}_{\gamma_0})^*(v)(\omega) = v(\gamma_0^{-1}\omega\gamma_0) = w(\gamma_0^{-1}\omega\gamma_0) = w(\gamma_0^{-1}) \cdot w(\omega) \cdot w(\gamma_0) = w(\omega) = v(\omega).$$

Putting (6.10) and (6.11) together yields an isomorphism of abelian groups

$$a\colon H^\pi_d(c^*_{\gamma_0^{-1}}E\pi;\mathbb{Z}^v) \xrightarrow{\cong} H^\pi_d(E\pi;\mathbb{Z}^v). \tag{6.12}$$

We obtain an automorphism

$$a \circ H^\pi_d(Ec_{\gamma_0^{-1}};\mathbb{Z}^v)\colon H^\pi_d(E\pi;\mathbb{Z}^v) \xrightarrow{\cong} H^\pi_d(E\pi;\mathbb{Z}^v) \tag{6.13}$$

of an infinite cyclic group $H_d^\pi(Ec_{\gamma_0^{-1}};\mathbb{Z}^v)$ with the generator $[B\pi]$ by composing the isomorphisms (6.9) and (6.12). Now define

$$u(\gamma_0) \in \{\pm 1\} \tag{6.14}$$

by the equation

$$a \circ H_d^\pi(c^*_{\gamma_0^{-1}}E\pi;\mathbb{Z}^v)([B\pi]) = u(\gamma_0)\cdot [B\pi].$$

Lemma 6.15. *We have $w(\gamma_0) = u(\gamma_0)$ for every $\gamma_0 \in \Gamma$.*

Proof. Let $[c^*_{\gamma_0}E\pi] \in H_d^\pi(c^*_{\gamma_0^{-1}}E\pi;\mathbb{Z}^v)$ be the image of $[B\pi]$ under $H_d^\pi(Ec_{\gamma_0^{-1}};\mathbb{Z}^v)$. Then we get a diagram of $\mathbb{Z}\pi$-chain complexes which commutes up to π-homotopy

$$\begin{tikzcd}
C^{d-*}(E\pi) \arrow[d, "-\cap[B\pi]"', "\simeq_\pi"] & C^{d-*}(c^*_{\gamma_0^{-1}}E\pi) \arrow[l, "C^{d-*}(Ec_{\gamma_0^{-1}})"'] \arrow[d, "\simeq_\pi"', "-\cap[B\pi]"] \\
C_*(E\pi) \arrow[r, "C_*(Ec_{\gamma_0^{-1}})"'] & C_*(c^*_{\gamma_0^{-1}}E\pi)
\end{tikzcd} \tag{6.16}$$

where the dual $\mathbb{Z}\pi$-chain complexes are taken with respect to the v-twisted involution and $B\pi$ on the left side corresponds to $E\pi/\pi$ and on the right side to $(c_{\gamma_0^{-1}}E\pi)/\pi$

The following diagram of $\mathbb{Z}\pi$-chain complexes commutes up to $\mathbb{Z}\pi$-chain homotopy

$$\begin{tikzcd}
C^{d-*}(c^*_{\gamma_0^{-1}}E\pi) \arrow[d, "-\cap[B\pi]"', "\simeq_{\mathbb{Z}\pi}"] & c^*_{\gamma_0^{-1}}C^{d-*}(E\pi) \arrow[l, "\mathbb{Z}[c_{\gamma_0}]_*"'] \arrow[d, "\simeq_{\mathbb{Z}\pi}"', "c^*_{\gamma_0^{-1}}(-\cap a([B\pi]))"] \\
C_*(c^*_{\gamma_0^{-1}}E\pi) = c^*_{\gamma_0^{-1}}C_*(E\pi) \arrow[r, "="'] & c^*_{\gamma_0^{-1}}C_*(E\pi)
\end{tikzcd} \tag{6.17}$$

where $\mathbb{Z}[c_{\gamma_0^{-1}}]_*$ is given by composing with the ring automorphism $\mathbb{Z}[c_{\gamma_0^{-1}}]\colon \mathbb{Z}\pi \xrightarrow{\cong} \mathbb{Z}\pi$ induced by the group automorphism $c_{\gamma_0^{-1}}\colon \pi \xrightarrow{\cong} \pi$.

By putting (6.14), (6.16), and (6.17) together and using the equality $\mathbb{Z}[c_{\gamma_0}]_* \circ C^{d-*}(Ec_{\gamma_0^{-1}}) = C^{d-*}(Ec_{\gamma_0^{-1}}) \circ \mathbb{Z}[c_{\gamma_0}]_*$, yields a commutative diagram of $\mathbb{Z}\pi$-chain complex which commutes up to $\mathbb{Z}\pi$-chain homotopy

$$\begin{tikzcd}
C^{d-*}(E\pi) \arrow[d, "-\cap[B\pi]"', "\simeq_{\mathbb{Z}\pi}"] & c^*_{\gamma_0^{-1}}C^{d-*}(E\pi) \arrow[l, "\mathbb{Z}[c_{\gamma_0}]_*\circ C^{d-*}(Ec_{\gamma_0^{-1}})"'] \arrow[d, "\simeq_{\mathbb{Z}\pi}"', "u(\gamma_0)\cdot c^*_{\gamma_0^{-1}}(-\cap[B\pi])"] \\
C_*(E\pi) \arrow[r, "C_*(Ec_{\gamma_0^{-1}})"'] & c^*_{\gamma_0^{-1}}C_*(E\pi) = C_*(c^*_{\gamma_0^{-1}}E\pi).
\end{tikzcd} \tag{6.18}$$

If we apply H_0 to the diagram above, we obtain a commutative diagram of $\mathbb{Z}\pi$-modules taking into account that the $\mathbb{Z}\pi$-module $H_0(C_*(E\pi))$ is isomorphic to the $\mathbb{Z}\pi$-module $\mathbb{Z}$ given by $\mathbb{Z}$ equipped with the trivial π-action and the map $H_0(C_*(Ec_{\gamma_0^{-1}}))\colon H_0(E\pi) \to H_0(E\pi)$ is the identity

$$\begin{tikzcd}
H_0(C^{d-*}(E\pi)) \arrow[d, "H_0(-\cap[B\pi])"', "\simeq_{\mathbb{Z}\pi}"] & c^*_{\gamma_0^{-1}}H_0(C^{d-*}(E\pi)) \arrow[l, "H_0(\mathbb{Z}[c_{\gamma_0}]_*\circ C^{d-*}(Ec_{\gamma_0^{-1}}))"'] \arrow[d, "\simeq_{\mathbb{Z}\pi}"', "u(\gamma_0)\cdot c^*_{\gamma_0^{-1}}H_0(-\cap[B\pi])"] \\
\mathbb{Z} \arrow[r, "\mathrm{id}_{\mathbb{Z}}"'] & \mathbb{Z}.
\end{tikzcd} \tag{6.19}$$

This implies that the map of infinite cyclic groups

$$H_0(\mathbb{Z}[c_{\gamma_0}]_* \circ C^{d-*}(Ec_{\gamma_0^{-1}}))\colon H_0(C^{d-*}(E\pi)) \xrightarrow{\cong} H_0(C^{d-*}(E\pi))$$

is multiplication with $u(\gamma_0)$.

The Γ-map $l_{\gamma_0^{-1}}\colon \underline{E}\Gamma \to c^*_{\gamma_0^{-1}}\underline{E}\Gamma$ is after restriction with i π-homotopic to the π-map $Ec_{\gamma_0^{-1}}\colon E\pi \to c^*_{\gamma_0^{-1}} E\pi$ taking into account that $i^*\underline{E}\Gamma$ is a model for $E\pi$. Hence the following diagram of $\mathbb{Z}$-chain complexes commutes up to $\mathbb{Z}$-chain homotopy

$$\begin{tikzcd}
C^{d-*}(\underline{E}\Gamma) \arrow[r, "{\mathbb{Z}[c_{\gamma_0}]_* \circ C^{d-*}(l_{\gamma_0^{-1}})}"] \arrow[d, "\cong"'] & C^{d-*}(\underline{E}\Gamma) \arrow[d, "\cong"] \\
C^{d-*}(E\pi) \arrow[r, "{\mathbb{Z}[c_{\gamma_0}]_* \circ C^{d-*}(Ec_{\gamma_0^{-1}})}"'] & C^{d-*}(E\pi)
\end{tikzcd}$$

where both vertical arrows are the $\mathbb{Z}$-chain isomorphism coming from Lemma 6.6 (1). The upper arrow is multiplication with γ_0 on $H_0(C^{d-*}_{\mathrm{untw}}(\underline{E}\Gamma))$ by Lemma 6.6 (2). Hence it induces on H_0 multiplication with $w(\gamma_0)$ by definition. We have already shown that the lower arrow induces on H_0 multiplication with $u(\gamma_0)$. Hence $w(\gamma_0) = u(\gamma_0)$. $\square$

The Hochschild-Serre spectral sequence applied to the group extension $1 \to \pi \xrightarrow{i} \Gamma \xrightarrow{p} G \to 1$ and the $\mathbb{Z}\Gamma$-module $\mathbb{Z}^w$ has a E^2-term $E^2_{p,q} = H^G_p(EG, H^\pi_q(E\pi;\mathbb{Z}^v))$ and converges to $H^\Gamma_{p+q}(E\Gamma;\mathbb{Z}^w)$. The G-action on $H^\pi_q(E\pi;\mathbb{Z}^v)$ can be computed as follows.

The composite $E\Gamma \to B\Gamma \to BG$ has a preimage of the base point in BG the space $\Gamma \times_\pi E\pi$. The Γ-equivariant fiber transport along loops in BG assigns to each $g \in G = \pi_1(BG)$ a unique Γ-homotopy class $[t_g]$ of Γ-maps $\Gamma \times_\pi E\pi \to \Gamma \times_\times E\pi$. A representative t_g is given for $g \in G$ by the Γ-map

$$t_g\colon \Gamma \times_\pi E\pi \to \Gamma \times_\pi E\pi, \quad (\gamma, x) \to (\gamma\widehat{g}^{-1}, Ec_{\widehat{g}}(x))$$

for any $\widehat{g} \in \Gamma$, which is mapped under the projection $\Gamma \to G$ to g, see for instance [20, Sections 1 and Subsection 7A]. The $\mathbb{Z}\Gamma$-map t_g induces a $\mathbb{Z}$-chain map

$$\mathrm{id}_{\mathbb{Z}^w} \otimes_{\mathbb{Z}\Gamma} C_*(t_g)\colon \mathbb{Z}^w \otimes_{\mathbb{Z}\Gamma} C_*(\Gamma \times_\pi E\pi) \to \mathbb{Z}^w \otimes_{\mathbb{Z}\Gamma} C_*(\Gamma \times_\pi E\pi).$$

Under the obvious identification

$$\mathbb{Z}^w \otimes_{\mathbb{Z}\Gamma} C_*(\Gamma \times_\pi E\pi) = \mathbb{Z}^w \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\pi} C_*(E\pi) = \mathbb{Z}^v \otimes_{\mathbb{Z}\pi} C_*(E\pi)$$

this becomes the $\mathbb{Z}$-chain map

$$w(\gamma_0^{-1}) \cdot \mathrm{id}_{\mathbb{Z}^v} \otimes_{\mathbb{Z}\pi} C_*(Ec_{\widehat{g}})\colon \mathbb{Z}^v \otimes_{\mathbb{Z}\pi} C_*(E\pi) \to \mathbb{Z}^v \otimes_{\mathbb{Z}\pi} C_*(E\pi).$$

Hence multiplication with $g \in G$ on $H^\pi_d(E\pi;\mathbb{Z}^v)$ is given by

$$w(\gamma_0^{-1}) \cdot \left(a \circ H^\pi_d(Ec_{\gamma_0^{-1}};\mathbb{Z}^v)\right)^{-1}$$

for the automorphism $a \circ H^\pi_d(Ec_{\gamma_0^{-1}};\mathbb{Z}^v)$ defined in (6.13). From the definition (6.14) of $u(\gamma_0)$ we get that $a \circ H^\pi_d(c^*_{\gamma_0^{-1}}E\pi;\mathbb{Z}^v)$ is multiplication with $u(\gamma_0)$. Lemma 6.15 implies that the G-action on $H^\pi_d(E\pi;\mathbb{Z}^v)$ is trivial. Hence $H_0(BG, H^\pi_d(E\pi;\mathbb{Z}^v)) = H^\pi_d(E\pi;\mathbb{Z}^v)$. Since $H_p(BG; H^\pi_q(E\pi;\mathbb{Z}^v))[1/|G|]$ vanishes for $p \geq 1$ and $i^*E\Gamma$ is a model for $E\pi$, we conclude from the Hochschild-Serre spectral sequence that the map $H_d(i_*)\colon H^\pi_d(i^*E\Gamma;\mathbb{Z}^v) \to H^\Gamma_d(E\Gamma;\mathbb{Z}^w)$ of (6.1) induces an isomorphism

$$H_d(i_*)[1/|G|]\colon H^\pi_d(E\pi;\mathbb{Z}^v)[1/|G|] \xrightarrow{\cong} H^\Gamma_d(E\Gamma;\mathbb{Z}^w)[1/|G|]. \tag{6.20}$$

6.4. **The fundamental classes.**

Lemma 6.21. *Let* $\mathrm{pr}\colon E\Gamma \to \underline{E}\Gamma$ *be the projection. Then the maps*

$$\begin{array}{rcl} H_d(i_*)\colon H_d^{\pi}(E\pi;\mathbb{Z}^v) & \to & H_d^{\Gamma}(E\Gamma;\mathbb{Z}^w); \\ H_d(i_*)\colon H_d^{\pi}(E\pi;\mathbb{Z}^v) & \to & H_d^{\Gamma}(\underline{E}\Gamma;\mathbb{Z}^w); \\ H_d(\mathrm{pr})\colon H_d^{\Gamma}(E\Gamma;\mathbb{Z}^w) & \to & H_d^{\Gamma}(\underline{E}\Gamma;\mathbb{Z}^w), \end{array}$$

are inclusion of infinite cyclic groups such that the index of the image in the target divides $|G|$.

Proof. From the Mayer-Vietoris sequence associated to the Γ-pushout with an inclusion of free Γ-CW-complexes as upper horizontal arrow, see Proposition 2.1

$$\begin{tikzcd} \coprod_{F\in\mathcal{M}} \Gamma\times_F EF \arrow[r] \arrow[d] & E\Gamma \arrow[d] \\ \coprod_{F\in\mathcal{M}} \Gamma/F \arrow[r] & \underline{E}\Gamma \end{tikzcd}$$

we obtain the exact sequence sequence

$$\prod_{F\in F} H_d^F(EF;\mathbb{Z}^{w|_F}) \to H_d^{\Gamma}(E\Gamma;\mathbb{Z}^w) \to H_d^{\Gamma}(\underline{E}\Gamma;\mathbb{Z}^w) \to \prod_{F\in F} H_{d-1}^F(EF;\mathbb{Z}^{w|_F}).$$

If d is even, then $H_d^F(EF;\mathbb{Z}^{w|_F}) = H_d(BF)$ by Lemma 3.3 and $H_d(BF)=0$ holds, as d is even and F has periodic homology, see [5, Exercise 4 in Section VI.9 on page 159]. If d is odd, $F\cong\mathbb{Z}/2$ and $w|_F$ is non-trivial by Lemma 3.3 and a direct computation shows $H_d^F(EF;\mathbb{Z}^{w|_F})=0$. Hence we get the short exact sequence

$$1 \to H_d^{\Gamma}(E\Gamma;\mathbb{Z}^w) \to H_d^{\Gamma}(\underline{E}\Gamma;\mathbb{Z}^w) \to \prod_{F\in F} H_{d-1}^F(EF;\mathbb{Z}^{w|_F}).$$

Since there is a cocompact d-dimensional model for $\underline{E}\Gamma$ with zero-dimensional $\underline{E}\Gamma^{>1}$, the abelian group $H_d(C_*(\underline{E}\Gamma)\otimes_{\mathbb{Z}\Gamma}\mathbb{Z}^w)$ is finitely generated free. Hence also the group $H_d^{\Gamma}(E\Gamma;\mathbb{Z}^w)$ is finitely generated free and has the same rank as $H_d(C_*(\underline{E}\Gamma)\otimes_{\mathbb{Z}\Gamma}\mathbb{Z}^w)$. The group $H_d^{\pi}(E\pi;\mathbb{Z}^v)$ is infinite cyclic. Since $i^*E\Gamma$ and $i^*\underline{E}\Gamma$ are models for $E\pi$, we conclude from (6.20) that all three groups $H_d^{\Gamma}(E\Gamma;\mathbb{Z}^w)$, $H_d(C_*(\underline{E}\Gamma)\otimes_{\mathbb{Z}\Gamma}\mathbb{Z}^w)$ and $H_d^{\pi}(E\pi;\mathbb{Z}^v)$ is infinite cyclic. The index of the inclusion $H_d(i_*)\colon H_d^{\pi}(E\pi;\mathbb{Z}^v) \to H_d^{\Gamma}(\underline{E}\Gamma;\mathbb{Z}^w)$ divides $|G|$ by (6.4). Since it factorizes as the composite of $H_d(i_*)\colon H_d^{\pi}(E\pi;\mathbb{Z}^v)\to H_d^{\Gamma}(E\Gamma;\mathbb{Z}^w)$ and $H_d(\mathrm{pr})\colon H_d^{\Gamma}(E\Gamma;\mathbb{Z}^w)\to H_d^{\Gamma}(\underline{E}\Gamma;\mathbb{Z}^w)$, also these two maps are inclusions of infinite cyclic groups, whose index divides $|G|$. This finishes the proof of e Lemma 6.21. □

We will improve Lemma 6.21 in Theorem 7.19 (2).

Notation 6.22. The choice of the fundamental class $[B\pi]\in H_d^{\pi}(E\pi;\mathbb{Z}^v)$ determines preferred choices of generators of infinite cyclic groups

$$\begin{array}{rcl} [\underline{E}\Gamma/\Gamma] & \in & H_d^{\Gamma}(\underline{E}\Gamma;\mathbb{Z}^w); \\ [\underline{E}\Gamma/\Gamma,\partial\underline{E}\Gamma/\Gamma] & \in & H_d^{\Gamma}(\underline{E}\Gamma,\partial\underline{E}\Gamma;\mathbb{Z}^w); \\ [E\Gamma/\Gamma] & \in & H_d^{\Gamma}(E\Gamma;\mathbb{Z}^w); \\ [E\Gamma/\Gamma,\partial E\Gamma/\Gamma] & \in & H_d^{\Gamma}(E\Gamma,\partial E\Gamma;\mathbb{Z}^w), \end{array}$$

by the injections or bijections of infinite cyclic groups appearing in (6.4) and in Lemma 6.21 by requiring that the injection $H_d(i_*)\colon H_d^{\pi}(E\pi;\mathbb{Z}^v)\to H_d^{\Gamma}(\underline{E}\Gamma;\mathbb{Z}^w)$ sends $[B\pi]$ to $n\cdot[\underline{E}\Gamma/\Gamma]$ for some integer n satisfying $n\geq 1$.

If $(X,\partial X)$ is a slice complement model for $\underline{E}\Gamma$, it also inherits a generator of an infinite cyclic group

$$[X/\Gamma,\partial X/\Gamma] \quad\in\quad H_d^{\Gamma}(X,\partial X;\mathbb{Z}^w)$$

from (6.4) and $[\underline{E}\Gamma, \partial\underline{E}\Gamma]$. We also obtain a preferred generator

$$[X/\pi, \partial X/\pi] \quad \in \quad H_d^{\pi}(X, \partial X; \mathbb{Z}^v),$$

namely, the one, which is mapped under the isomorphism

$$H_d^{\pi}(X, \partial X; \mathbb{Z}^v) \xrightarrow{\cong} H_d^{\pi}(\underline{E}\Gamma, \partial\underline{E}\Gamma; \mathbb{Z}^v) \xleftarrow{\cong} H_d^{\pi}(\underline{E}\Gamma; \mathbb{Z}^v) = H_d(E\pi; \mathbb{Z}^v).$$

to $[B\pi]$. Equivalently, one can define $[X/\pi, \partial X/\pi]$ by requiring that the injection of infinite cyclic groups $H_d(i_*)\colon H_d^{\pi}(X, \partial X; \mathbb{Z}^v) \to H_d^{\Gamma}(X, \partial X; \mathbb{Z}^w)$ sends $[X/\pi, \partial X/\pi]$ to $n \cdot [X/\Gamma, \partial X/\Gamma]$ for some integer n satisfying $n \geq 1$.

We call these generators fundamental classes as well.

Example 6.23 (Special case of trivial v)**.** As an illustration we explain, what happens in the special case that v is trivial. Then $[B\pi]$ is a generator of the infinite cyclic group $H_d^{\pi}(E\pi; \mathbb{Z}^v) = H_d(B\pi)$. The Γ-action on π by conjugation induces a G-action on $H_d(B\pi)$. Since $H_d(B\pi)$ is infinite cyclic, there is precisely one group homomorphism $\overline{w}\colon G \to \{\pm 1\}$, for which the $\mathbb{Z}G$-module $H_d(B\pi)$ is $\mathbb{Z}G$-isomorphic to $\mathbb{Z}^{\overline{w}}$. Then w is the composite $\Gamma \xrightarrow{\mathrm{pr}} G \xrightarrow{\overline{w}} \{\pm 1\}$.

The homomorphisms appearing in Lemma 6.21 boil down to homomorphisms

$$\begin{array}{rcl} H_d(i_*)\colon H_d(B\pi) & \to & H_d^G(B\pi; \mathbb{Z}^{\overline{w}}); \\ H_d(i_*)\colon H_d(B\pi) & \to & H_d^G(\underline{E}\Gamma/\pi; \mathbb{Z}^{\overline{w}}); \\ H_d(\mathrm{pr})\colon H_d^G(B\pi; \mathbb{Z}^{\overline{w}}) & \to & H_d^G(\underline{E}\Gamma/\pi; \mathbb{Z}^{\overline{w}}). \end{array}$$

The string of isomorphisms given by the left column in the diagram (6.4) reduces to the string of isomorphisms

$$\begin{aligned} H_d^G(X/\pi, \partial X/\pi; \mathbb{Z}^{\overline{w}}) &\xrightarrow{\cong} H_d^G(E\Gamma/\pi, \partial E\Gamma/\pi; \mathbb{Z}^{\overline{w}}) \\ &\xrightarrow{\cong} H_d^G(\underline{E}\Gamma/\pi, \partial\underline{E}\Gamma/\pi; \mathbb{Z}^{\overline{w}}) \xleftarrow{\cong} H_d^G(\underline{E}\Gamma/\pi; \mathbb{Z}^{\overline{w}}). \end{aligned}$$

If we furthermore assume that $\overline{w}$ is trivial, this reduces further to

$$\begin{array}{rcl} H_d(Bi)\colon H_d(B\pi) & \to & H_d(B\Gamma); \\ H_d(\mathrm{pr} \circ Bi)\colon H_d(B\pi) & \to & H_d(\underline{B}\Gamma); \\ H_d(\mathrm{pr})\colon H_d(B\Gamma) & \to & H_d(\underline{E}\Gamma), \end{array}$$

and

$$H_d(X/\Gamma, \coprod_{F \in \mathcal{M}} S_F/F) \xrightarrow{\cong} H_d^G(B\Gamma, \coprod_{F \in \mathcal{M}} BF) \xrightarrow{\cong} H_d(\underline{B}\Gamma, \coprod_{F \in \mathcal{M}} \{\bullet\}) \xleftarrow{\cong} H_d(\underline{B}\Gamma).$$

7. Constructing slice complement models

In this section we construct appropriate slice complement models, for which we will later show that they carry the desired structure of a finite d-dimensional Poincaré pair. Not every slice complement model has such a structure. Moreover, we will classify slice complement models up to (simple) Γ-homotopy equivalence in terms of the underlying free d-dimensional slice system in Sections 8 and 9.

Throughout this section we will make the following assumptions:

Assumption 7.1.

- *The natural number d is even and satisfies $d \geq 4$;*
- *The group Γ satisfies conditions (M) and (NM), see Notation 1.8;*
- *The homomorphism $w\colon \Gamma \to \{\pm 1\}$ of Notation 6.7 has the property that $w|_F$ is trivial for every $F \in \mathcal{M}$;*

- *The composite*

$$H_d^\Gamma(E\Gamma, \partial E\Gamma; \mathbb{Z}^w) \xrightarrow{\partial} H_{d-1}^\Gamma(\partial E\Gamma; \mathbb{Z}^w) \xrightarrow{\cong} \bigoplus_{F \in \mathcal{M}} H_{d-1}(BF) \to H_{d-1}(BF)$$

  of the boundary map, the inverse of the obvious isomorphism and the projection to the summand of $F \in \mathcal{M}$ is surjective for all $F \in \mathcal{M}$;
- *There exists a finite Γ-CW-model for $\underline{E}\Gamma$ of dimension d such that its singular Γ-subspace $\underline{E}\Gamma^{>1}$ is $\coprod_{F \in \mathcal{M}} \Gamma/F$. (This condition is discussed and simplified in Theorem 1.12 and implies conditions (M) and (NM), see Remark 1.13.)*
- *There is a finite d-dimensional Poincaré CW-complex model for $B\pi$ with respect to the orientation homomorphisms $v = w|_\pi \colon \pi \to \{\pm 1\}$. We fix a choice of a fundamental class $[B\pi] \in H_d^\pi(E\pi; \mathbb{Z}^v)$.*

Note that the assumption that d is even implies, that F acts orientation preserving on S_F by Lemma 3.3.

Remark 7.2 (Reformulation of (H))**.** One can easily check using (6.4) that the map appearing in condition (H) can be identified with the map

$$H_d^\Gamma(\underline{E}\Gamma; \mathbb{Z}^w) \xrightarrow{\partial} H_{d-1}^\Gamma(\partial E\Gamma; \mathbb{Z}^w) \xrightarrow{\cong} \bigoplus_{F \in \mathcal{M}} H_{d-1}(BF) \to H_{d-1}(BF),$$

where the first map is the boundary map of the Mayer-Vietoris sequence associated to the Γ-pushout appearing in Proposition 2.1 and the second map is the projection onto the summand belonging to F. Recall that $H_d^\Gamma(\underline{E}\Gamma; \mathbb{Z}^w)$ is infinite cyclic. From this Mayer Vietoris sequence, we also conclude that condition (H) is satisfied, if and only if the kernel of the map $\bigoplus_{F \in \mathcal{M}} H_{d-1}(BF) \to H_{d-1}^\Gamma(E\Gamma; \mathbb{Z}^w)$ induced by the various inclusions $F \to \Gamma$ contains an element $\{\kappa_F \mid F \in \mathcal{M}\}$ such that each $\kappa_F \in H_{d-1}(BF)$ is a generator of the finite cyclic group $H_{d-1}(BF)$ of order $|F|$.

7.1. **Invariants associated to slice complement models.**

Notation 7.3. Let $\kappa_F \in H_d(BF)$ be the image of $[E\Gamma/\Gamma, \partial E\Gamma/\Gamma]$ defined in Notation 6.22 under the composite

$$H_d^\Gamma(E\Gamma, \partial E\Gamma; \mathbb{Z}^w) \xrightarrow{\partial} H_{d-1}^\Gamma(\partial E\Gamma; \mathbb{Z}^w) \xrightarrow{\cong} \bigoplus_{F \in \mathcal{M}} H_{d-1}(BF) \to H_{d-1}(BF)$$

Note that κ_F is a generator of the finite cyclic group $H_{d-1}(BF)$ of order $|F|$, since the composite above is surjective by assumption.

Let $(X, \partial X)$ be a slice complement model with respect to the slice system $\mathcal{S}$. Fix an orientation on $\mathcal{S}$ in the sense of Definition 3.1, i.e., a choice of fundamental class $[S_F] \in H_{d-1}(S_F)$ for every $F \in \mathcal{M}$. Recall that this is the same as a choice of fundamental class $[S_F/F] \in H_{d-1}(S_F/F)$ for every $F \in \mathcal{M}$, as explained after Definition 3.1.

Define

$$\mu(X, \partial X) = (\mu(X, \partial X)_F)_{F \in \mathcal{M}} \in H_{d-1}(\partial X/\Gamma) = \bigoplus_{F \in \mathcal{M}} H_d(S_F/F) \tag{7.4}$$

to be the image of $[X/\Gamma, \partial X/\Gamma]$ defined in Notation 6.22 under the boundary map $H_d^\Gamma(X, \partial X; \mathbb{Z}^w) \to H_{d-1}^\Gamma(\partial X; \mathbb{Z}^w) = H_{d-1}(\partial X/\Gamma)$. Define the element

$$s \in H_{d-1}^\Gamma(\partial X; \mathbb{Z}^w) = \bigoplus_{F \in \mathcal{M}} H_d(S_F/F) \tag{7.5}$$

by $([S_F/F])_{F \in \mathcal{M}}$. For $F \in \mathcal{M}$ denote by

$$m_F(X, \partial X) \in \mathbb{Z} \tag{7.6}$$

the integer, for which $\mu(X,\partial X)_F = m_F(X,\partial X)\cdot[S_F/F]$ holds.

Lemma 7.7. *Let $(X,\partial X)$ be a slice complement model with respect to the oriented free d-dimensional slice system $\mathcal{S}=\{S_F \mid F\in\mathcal{M}\}$. Then we get for every $F\in\mathcal{M}$*

$$\kappa_F = m_F(X,\partial X)\cdot d(S_F)$$

in $H_d(BF)$, where the invariant $d(S_F)$ has been defined in (3.2).

Proof. We have the commutative diagram

$$\begin{tikzcd} H_d^\Gamma(X,\partial X;\mathbb{Z}^w) \arrow[r, "\partial"] \arrow[d, "H_d^\Gamma(c(X),\partial c(\partial X);\mathbb{Z}^w)"', "\cong"] & H_{d-1}^\Gamma(\partial X;\mathbb{Z}^w) = \bigoplus_{F\in\mathcal{M}} H_d(S_F/F) \arrow[d, "\bigoplus_{F\in\mathcal{M}} H_d(c_{S_F})"] \\ H_d^\Gamma(E\Gamma,\partial E\Gamma;\mathbb{Z}^w) \arrow[r, "\partial"'] & H_{d-1}^\Gamma(\partial E\Gamma;\mathbb{Z}^w) = \bigoplus_{F\in\mathcal{M}} H_d(BF), \end{tikzcd}$$

where the maps $c(X)$, $\partial c(\partial X)$, and $c(S_F)$ are given by classifying maps. The left vertical arrow is an isomorphism, see (6.4) and sends $[X/\Gamma,\partial X/\Gamma]$ to $[E\Gamma/\Gamma,\partial E\Gamma/\Gamma]$ by definition. Now $\kappa_F = m_F(X,\partial X)\cdot d(S_F)$ follows from the definitions of $d(S_F)$, κ_F, and $m_F(X,\partial X)$. □

Definition 7.8 (Poincaré slice complement model)**.** We call a slice complement model $(X,\partial X)$ a *Poincaré slice complement model* if $(X/\Gamma,\partial X/\Gamma)$ carries the structure of a finite Poincaré pair.

Recall that the orientation homomorphism underlying the Poincaré structure on $(X/\Gamma,\partial X/\Gamma)$ must be the map $w\colon \Gamma\to\{\pm1\}$ defined in (6.7) by Remark 5.5. Moreover, we have a preferred fundamental class $[X/\Gamma,\partial X/\Gamma]\in H_d^\Gamma(X,\partial X;\mathbb{Z}^w)$, see Notation 6.22.

Definition 7.9 (Condition (S))**.** An oriented free d-dimensional slice system satisfies condition (S) if $d(S_F)=\kappa_F$ holds for all $F\in\mathcal{M}$.

Note that condition (S) determines each S_F up to oriented F-homotopy equivalence. It determines also the orientation $[S_F]$ for those $F\in\mathcal{M}$, for which $|F|\geq 3$ holds. If $|F|$ has order 2, then replacing $[S_F]$ by $-[S_F]$ does not affect condition (S).

Lemma 7.10. *Let $(X,\partial X)$ be a Poincaré slice complement model with respect to the free d-dimensional slice system $\mathcal{S}=\{S_F\mid F\in\mathcal{M}\}$. Then there is an orientation on $\mathcal{S}$ such that $m_F(X,\partial X)=1$ holds for every $F\in\mathcal{M}$ and $\mathcal{S}$ satisfies condition (S).*

Proof. Since $(X/\Gamma,\partial X/\Gamma)$ admits the structure of a finite Poincaré pair, it is part of the definition that the boundary map

$$H_d^\Gamma(X,\partial X;\mathbb{Z}^w)\to H_{d-1}^\Gamma(\partial X;\mathbb{Z}^w)=\bigoplus_{F\in\mathcal{M}} H_d(S_F/F)$$

sends the fundamental class $[X,\partial X]$ to an element whose component for $F\in\mathcal{M}$ is a generator of $H_{d-1}(S_F/F)$ for every $F\in\mathcal{M}$. Choose some orientation on $\mathcal{S}$. With respect to it we get $m_F(X,\partial X)\in\{\pm1\}$. Then we get the desired orientation by replacing $[S_F]$ by $m_F(X,\partial X)\cdot[S_F]$. Namely, with this new orientation we have $m_F(X,\partial X)=1$ for every $F\in\mathcal{M}$ and (S) holds by Lemma 7.7. □

Remark 7.11 (On the condition (S))**.** Our goal is to construct a Poincaré slice complement model $(X,\partial X)$ with respect to some free d-dimensional slice system $\mathcal{S}=\{S_F\mid F\in\mathcal{M}\}$ By Lemma 7.10 there exists an orientation on $\mathcal{S}$ such that condition (S) holds. This motivates that in the sequel we will consider only oriented systems $\mathcal{S}$ satisfying condition (S). Note that condition (S) implies by Lemma 7.7

$$m_F(X, \partial X) \equiv 1 \mod |F|$$

for every $F \in \mathcal{M}$, since κ_F is a generator of the finite cyclic group $H_d(BF)$ of order $|F|$ for every $F \in \mathcal{M}$.

The main result of this section is

Theorem 7.12 (Existence of Poincaré slice complement models). *Suppose that Assumption 7.1 is satisfied and let $\mathcal{S}$ be an oriented free d-dimensional slice system $\mathcal{S}$ satisfying condition (S).*

Then there exists a Poincaré slice complement model $(X, \partial X)$ with respect to $\mathcal{S}$ such that $m_F(X, \partial X) = 1$ holds for every $F \in \mathcal{M}$.

Remark 7.13 (Basic strategy). Consider any oriented slice system $\mathcal{S}$ satisfying condition (S). Let $(X, \partial X)$ be a slice complement model with respect to $\mathcal{S}$. Then $m_F(X, \partial X) \equiv 1 \mod |F|$ holds for every $F \in \mathcal{M}$ as explained in Remark 7.11. So our basic strategy will be to construct some slice complement model $(X, \partial X)$ with respect to $\mathcal{S}$ and then to modify it using $m_F(X, \partial X) \equiv 1 \mod |F|$ such that we have get $m_F(X, \partial X) = 1$ for every $F \in \mathcal{M}$. This will be done in Theorem 7.14 (1). Then we will show in Theorem 7.19 that $(X, \partial X)$ carries the structure of a finite d-dimensional Poincaré pair.

Hence Theorem 7.12 will be a direct consequence of Theorem 7.14 and Theorem 7.19 (1).

7.2. **Constructing slice complement models.**

Theorem 7.14 (Constructing slice complement models). *Suppose that Assumption 7.1 is satisfied. Let $\mathcal{S}$ be an oriented free d-dimensional slice system $\mathcal{S}$ satisfying condition (S).*

Then there exists a slice complement model $(X, \partial X)$ for $\underline{E}\Gamma$ with respect to $\mathcal{S}$ such that $m_F(X, \partial X) = 1$ holds for every $F \in \mathcal{M}$.

Proof. We begin with constructing a Γ-CW-pair $(X, \partial X)$ together with a cellular Γ-map of Γ-CW-pairs

$$(u, \partial u) \colon (X, \partial X) \to (\underline{E}\Gamma, \partial \underline{E}\Gamma).$$

Note for the sequel that $\underline{E}\Gamma_n \cup \partial \underline{E}\Gamma$ is $\partial \underline{E}\Gamma$ for $n = -1$ and is $\underline{E}\Gamma_n$ for $n = 0, 1, 2 \ldots, d$.

We will construct by induction over $n = -1, 0, 1, \ldots, d$ Γ-CW-pairs $(X_n, \partial X)$ and Γ-maps $u_n \colon X_n \to \underline{E}\Gamma_n \cup \partial \underline{E}\Gamma$ satisfying

- $X_{n-1} \subseteq X_n$ holds for $n = 0, 1, 2 \ldots, d$;
- $u_n|_{X_{n-1}} = u_{n-1}$ holds for $n = 0, 1, 2 \ldots, d$;
- u_n is $(d-1)$-connected for $n = -1, 0, 1, 2 \ldots, d$;
- $H_n(u_n, u_{n-1}) \colon H_n(X_n, X_{n-1}) \to H_n(\underline{E}\Gamma_n, \underline{E}\Gamma_{n-1} \cup \partial \underline{E}\Gamma)$ is bijective for $n = 0, 1, 2 \ldots, d$. (Actually, the source and target will come with explicit $\mathbb{Z}\pi$-bases, which are respected by this map.)

Since $\underline{E}\Gamma = \underline{E}\Gamma_d$, we then can and will define X to be X_d, and u to be u_d.

The induction beginning $n = -1$ is given by $X_{-1} = \partial X$ and $u_{-1} = \partial u \colon \partial X \to \partial \underline{E}\Gamma$ is the coproduct over $F \in \mathcal{M}$ of the projections $\Gamma \times_F S_F \to \Gamma/F$. The induction step from $(n-1)$ to n for $n = 0, 1, 2, \ldots, d$ is done as follows.

Choose a finite index set I_n and for $i \in I_n$ a map of pairs

$$(Q_i^n, q_i^n) \colon (D^n, S^{n-1}) \to (\underline{E}\Gamma_n, \underline{E}\Gamma_{n-1} \cup \partial \underline{E}\Gamma)$$

such that for the induced Γ-maps

$$(\widehat{Q_i^n}, \widehat{q_i^n}) \colon (\Gamma \times D^n, \Gamma \times S^{n-1}) \to (\underline{E}\Gamma_n, \underline{E}\Gamma_{n-1} \cup \partial \underline{E}\Gamma)$$

sending (γ, x) to $\gamma \cdot Q_i^n(x)$ for $\gamma \in \Gamma$ and $x \in D^n$ we get a Γ-pushout

$$\begin{tikzcd}
\coprod_{i \in I_n} \Gamma \times S^{n-1} \arrow[r, "\coprod_{i \in I_n} \widehat{q_i^n}"] \arrow[d] & \underline{E}\Gamma_{n-1} \cup \partial \underline{E}\Gamma \arrow[d] \\
\coprod_{i \in I_n} \Gamma \times D^n \arrow[r, "\coprod_{i \in I_n} \widehat{Q_i^n}"] & \underline{E}\Gamma_n.
\end{tikzcd}$$

Since u_{n-1} is $(d-1)$-connected by induction hypothesis, we can find for $i \in I_n$ a map $p_i^n \colon S^{n-1} \to X_{n-1}$ and a homotopy

$$h_i^n \colon S^n \times [0,1] \to \underline{E}\Gamma_{n-1} \cup \partial \underline{E}\Gamma$$

from q_i^n to $u_{n-1} \circ p_i^n$. By the Cellular Approximation Theorem, we can assume without loss of generality that the image of p_i^n is contained in $X_{n-1} \cap (\partial X)_{n-1}$. Now define X_n by the Γ-pushout

$$\begin{tikzcd}
\coprod_{i \in I_n} \Gamma \times S^{n-1} \arrow[r, "\coprod_{i \in I_n} \widehat{p_i^n}"] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_{i \in I_n} \Gamma \times D^n \arrow[r, "\coprod_{i \in I_n} \widehat{P_i^n}"] & X_n
\end{tikzcd}$$

for an appropriate extension $P_i^n \colon D^n \to X_n$ of p_i^n. In order to define the extension $u_n \colon X_n \to \underline{E}\Gamma_n$ of $u_{n-1} \colon X_{n-1} \to \underline{E}\Gamma_{n-1} \cup \partial \underline{E}\Gamma$, we have to specify for each $i \in I_n$ a map $v_i^n \colon D^n \to \underline{E}\Gamma_n$ whose restriction to S^{n-1} is $u_{n-1} \circ p_i^n$. It is given by sending $x \in D^n$ to $h_i^n(x, 2 \cdot |x| - 1)$ if $1/2 \leq |x| \leq 1$ and to $Q_i^n(2 \cdot x)$ if $0 \leq |x| \leq 1/2$.

We get for each $i \in I_n$ an element $e_i^n \in H_n(\underline{E}\Gamma_n, \underline{E}\Gamma_{n-1} \cup \partial \underline{E}\Gamma)$, namely the image of the class of (Q_i^n, q_i^n) under the Hurewicz homomorphism $\pi_n(\underline{E}\Gamma_n, \underline{E}\Gamma_{n-1} \cup \partial \underline{E}\Gamma) \to H_n(\underline{E}\Gamma_n, \underline{E}\Gamma_{n-1} \cup \partial \underline{E}\Gamma)$. Analogously get for each $i \in I_n$ an element $x_i^n \in H_n(X_n, X_{n-1})$, namely the image of the class of (P_i^n, p_i^n) under the Hurewicz homomorphism $\pi_n(X_n, X_{n-1}) \to H_n(X_n, X_{n-1})$. One easily checks that $\{e_i^n \mid i \in I_n\}$ and $\{x_i^n \mid i \in I_n\}$ are $\mathbb{Z}\Gamma$-basis for the finitely generated free $\mathbb{Z}\Gamma$-modules $H_n(\underline{E}\Gamma_n, \underline{E}\Gamma_{n-1} \cup \partial \underline{E}\Gamma)$ and $H_n(X_n, X_{n-1})$ and that $H_n(u_n, u_{n-1})$ sends x_i^n to e_i^n. In particular $H_n(u_n, u_{n-1})$ is bijective.

It is not hard to show that the commutative diagram

$$\begin{tikzcd}
X_{n-1} \arrow[r, "u_{n-1}"] \arrow[d] & \underline{E}\Gamma_{n-1} \cup \partial \underline{E}\Gamma \arrow[d] \\
X_n \arrow[r, "u_n"] & \underline{E}\Gamma_n
\end{tikzcd} \tag{7.15}$$

is a homotopy pushout, actually a Γ-homotopy pushout. The reason is essentially that changing the attaching maps of the Γ-cells for a Γ-CW-complex by a Γ-homotopy does not change the Γ-homotopy type. Now one easily checks that u_n is $(d-1)$-connected using the induction hypothesis that u_{n-1} is $(d-1)$-connected. This finishes the construction of the map $(u, \partial u)$.

Next we analyze this construction in the top dimension closer. Since $\underline{E}\Gamma$ is a finite d-dimensional Γ-CW-complex, the inclusion $(\underline{E}\Gamma, \partial \underline{E}\Gamma) \to (\underline{E}\Gamma, \underline{E}\Gamma_{d-1})$ induces an exact sequence of finitely generated abelian groups

$$0 \to H_d^\Gamma(\underline{E}\Gamma, \partial\underline{E}\Gamma; \mathbb{Z}^w) \to H_d^\Gamma(\underline{E}\Gamma, \underline{E}\Gamma_{d-1}; \mathbb{Z}^w) = \mathbb{Z}^w \otimes_{\mathbb{Z}\Gamma} C_d(\underline{E}\Gamma, \partial\underline{E}\Gamma) \xrightarrow{c_d \otimes_{\mathbb{Z}\pi} \mathrm{id}_{\mathbb{Z}^w}} \mathbb{Z}^w \otimes_{\mathbb{Z}\Gamma} C_{d-1}(\underline{E}\Gamma, \partial\underline{E}\Gamma),$$

where $c_d \colon C_d(\underline{E}\Gamma, \partial\underline{E}\Gamma) \to C_{d-1}(\underline{E}\Gamma, \partial\underline{E}\Gamma)$ is the d-th differential of the cellular $\mathbb{Z}\Gamma$-chain complex of the Γ-CW-pair $(\underline{E}\Gamma, \partial\underline{E}\Gamma)$. Since the image of $c_d \otimes_{\mathbb{Z}\pi} \mathrm{id}_{\mathbb{Z}^w}$ is a free $\mathbb{Z}$-module, the $\mathbb{Z}$-map $H_d^\Gamma(\underline{E}\Gamma, \partial\underline{E}\Gamma; \mathbb{Z}^w) \to H_d^\Gamma(\underline{E}\Gamma, \underline{E}\Gamma_{d-1}; \mathbb{Z}^w)$ is split injective. Recall that $H_d^\Gamma(\underline{E}\Gamma, \partial\underline{E}\Gamma; \mathbb{Z}^w)$ is an infinite cyclic group and comes with a preferred generator $[\underline{E}\Gamma/\Gamma, \partial\underline{E}\Gamma/\Gamma]$ and that $\{1 \otimes e_i^d \mid i \in I_d\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}^w \otimes_{\mathbb{Z}\Gamma} C_d(\underline{E}\Gamma, \partial\underline{E}\Gamma)$. Hence we can find integers λ_i and μ_i for $i \in I_d$ such that the image of $[\underline{E}\Gamma/\Gamma, \partial\underline{E}\Gamma/\Gamma]$ under

$$H_d^\Gamma(\underline{E}\Gamma, \partial\underline{E}\Gamma; \mathbb{Z}^w) \to H_d^\Gamma(\underline{E}\Gamma, \underline{E}\Gamma_{d-1}; \mathbb{Z}^w) = \mathbb{Z}^w \otimes_{\mathbb{Z}\Gamma} C_d(\underline{E}\Gamma, \partial\underline{E}\Gamma)$$

is $\sum_{i\in I_d} \lambda_i \cdot (1 \otimes e_i^d)$ and $\sum_{i \in I_d} \lambda_i \cdot \mu_i = 1$ holds.

The map u induces a commutative diagram

$$\begin{tikzcd}
H_d^\Gamma(X, \partial X; \mathbb{Z}^w) \arrow[r] \arrow[d, "\cong"', "H_d^\Gamma(u,\partial u;\mathbb{Z}^w)"] & H_d^\Gamma(X, X_{d-1}; \mathbb{Z}^w) = \mathbb{Z}^w \otimes_{\mathbb{Z}\Gamma} C_d(X, \partial X) \arrow[d, "\cong"', "H_d^\Gamma(u,u_{d-1};\mathbb{Z}^w) = \mathrm{id}_{\mathbb{Z}^w} \otimes_{\mathbb{Z}\Gamma} C_d(u,\partial u)"] \\
H_d^\Gamma(\underline{E}\Gamma, \partial\underline{E}\Gamma; \mathbb{Z}^w) \arrow[r] & H_d^\Gamma(\underline{E}\Gamma, \underline{E}\Gamma_{d-1}; \mathbb{Z}^w) = \mathbb{Z}^w \otimes_{\mathbb{Z}\Gamma} C_d(\underline{E}\Gamma, \partial\underline{E}\Gamma_{d-1})
\end{tikzcd}$$

where the left vertical arrow sends $[X/\Gamma, \partial X/\Gamma]$ to $[\underline{E}\Gamma/\Gamma, \partial\underline{E}\Gamma/\Gamma]$ and the right vertical arrow $1 \otimes x_i^d$ to $1 \otimes e_i^d$. Hence the map

$$H_d^\Gamma(X, \partial X; \mathbb{Z}^w) \to H_d^\Gamma(X, X_{d-1}; \mathbb{Z}^w) = \mathbb{Z}^w \otimes_{\mathbb{Z}\Gamma} C_d(X, \partial X)$$

sends $[X/\Gamma, \partial X/\Gamma]$ to $\sum_{i\in I_d} \lambda_i \cdot (1 \otimes x_i^d)$. The following diagram commutes

$$\begin{tikzcd}
H_d^\Gamma(X, \partial X; \mathbb{Z}^w) \arrow[r] \arrow[d] & H_{d-1}^\Gamma(\partial X; \mathbb{Z}^w) = H_{d-1}(\partial X/\Gamma) \arrow[d, "H_{d-1}^\Gamma(j;\mathbb{Z}^w)"] \\
H_d^\Gamma(X, X_{d-1}; \mathbb{Z}^w) \arrow[r] & H_{d-1}^\Gamma(X_{d-1}; \mathbb{Z}^w),
\end{tikzcd}$$

where $j \colon \partial X \to X_{d-1}$ is the inclusion and the horizontal arrows are boundary homomorphism of pairs. Let $\mathrm{pr}_{\partial X} \colon \partial X \to \partial X_{d-1}/\Gamma$ be the projection. One easily checks that the lower horizontal arrow sends $1 \otimes x_i^d$ to the image of the class $[p_i^d]$ of p_i^d under the composite

$$\pi_{d-1}(X_{d-1}) \xrightarrow{h_{d-1}[X_{d-1}]} H_{d-1}(X_{d-1}) \xrightarrow{H_d(f_*)} H_d^\Gamma(X_{d-1}; \mathbb{Z}^w),$$

where $h_{d-1}[X_{d-1}]$ is the Hurewicz homomorphism and f_* is the obvious chain map $f_* \colon C_*(X_{d-1}) \to C_*(X_{d-1}) \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}^w$. Recall that $\mu(X, \partial X)$ is the image of $[X/\Gamma, \partial X/\Gamma]$ under the upper horizontal arrow in the diagram above. Hence we get for the image of $\mu(X, \partial X)$ under the map

$$H_{d-1}^\Gamma(j; \mathbb{Z}^w) \colon H_{d-1}^\Gamma(\partial X; \mathbb{Z}^w) = H_{d-1}(\partial X/\Gamma) \to H_{d-1}^\Gamma(X_{d-1}; \mathbb{Z}^w)$$

the equality

$$(7.16) \qquad H_{d-1}^\Gamma(j; \mathbb{Z}^w)(\mu(X, \partial X)) = \sum_{i \in I_d} \lambda_i \cdot H_{d-1}(f_*) \circ h_{d-1}[X_{d-1}]([p_i^d]).$$

Note that $H_{d-1}(j; \mathbb{Z}^w)$ is injective, as $j \colon \partial X \to X_{d-1}$ is an inclusion of $(d-1)$-dimensional Γ-CW-complexes. Hence we have expressed $\mu(X, \partial X)$ in terms of the attaching maps $p_i^d \colon S^{d-1} \to X_{d-1}$.

Next we investigate how we can change the maps p_i^d and how this change affects $\mu(X, \partial X)$. Since we have for $n = 0, 1, 2, \ldots, (d-1)$ the homotopy pushout (7.15), the following diagram is a homotopy pushout

$$\begin{tikzcd} \partial X \arrow[r, "u_{-1}"] \arrow[d, "j"'] & \partial \underline{E}\Gamma \arrow[d] \\ X_{d-1} \arrow[r, "u_{d-1}"] & \underline{E}\Gamma_{d-1}. \end{tikzcd}$$

Since $H_d(\underline{E}\Gamma_{d-1})$, $H_{d-2}(\partial X)$, and $H_{d-1}(\partial \underline{E}\Gamma)$ vanish, we get from the associated Mayer-Vietoris sequence a short exact sequence of $\mathbb{Z}\Gamma$-modules

$$0 \to H_{d-1}(\partial X) \xrightarrow{H_{d-1}(j)} H_{d-1}(X_{d-1}) \xrightarrow{H_{d-1}(u_{d-1})} H_{d-1}(\underline{E}\Gamma_{d-1}) \to 0.$$

Since $\underline{E}\Gamma$ is contractible and $u_{d-1}\colon X_{d-1} \to \underline{E}\Gamma_{d-1}$ is $(d-1)$-connected, we conclude that X_{d-1} and $\underline{E}\Gamma_{d-1}$ are $(d-2)$-connected. Hence we can extend the short exact sequence above to a commutative diagram whose vertical maps are Hurewicz isomorphisms

$$\begin{tikzcd} & \pi_{d-1}(X_{d-1}) \arrow[r, "\pi_{d-1}(u_{d-1})"] \arrow[d, "h_{d-1}{[X_{d-1}]}"', "\cong"] & \pi_{d-1}(\underline{E}\Gamma_{d-1}) \arrow[d, "h_{d-1}{[\underline{E}\Gamma_{d-1}]}"', "\cong"] \\ H_{d-1}(\partial X) \arrow[r, "H_{d-1}(j)"] & H_{d-1}(X_{d-1}) \arrow[r, "H_{d-1}(u_{d-1})"] & H_{d-1}(\underline{E}\Gamma_{d-1}) \end{tikzcd}$$

Note that the only requirement about p_i^d is that the image of the class $[p_i^d]$ under $\pi_{d-1}(u_{d-1})\colon \pi_{d-1}(X_{d-1}) \to \pi_{d-1}(\underline{E}\Gamma_{d-1})$ is $[q_i^d]$. Hence we can choose for every $i \in I_d$ an element $a_i \in H_{d-1}(\partial X)$ and replace p_i^d by any map $\widehat{p_i^d}\colon S^{d-1} \to X_{d-1}$ satisfying

$$h_{d-1}[X_{d-1}]([\widehat{p_i^d}] - [p_i^d]) = H_{d-1}(j)(a_i).$$

If we use the maps $\widehat{p_i^d}$, then we get a Γ-CW-pair $(\widehat{X}, \partial X)$ and equation (7.16) becomes

$$H_{d-1}^{\Gamma}(j; \mathbb{Z}^w)(\mu(\widehat{X}, \partial X)) = \sum_{i \in I_d} \lambda_i \cdot H_{d-1}(f_*) \circ h_{d-1}[X_{d-1}]([\widehat{p_i^d}]). \tag{7.17}$$

One easily checks

$$\begin{aligned} &H_{d-1}(f_*) \circ h_{d-1}[X_{d-1}]([\widehat{p_i^d}]) - H_{d-1}(f_*) \circ h_{d-1}[X_{d-1}]([p_i^d]) \\ &\qquad = H_{d-1}(f_*) \circ H_{d-1}(j)(a_i) = H_{d-1}^{\Gamma}(j; \mathbb{Z}^w) \circ H_{d-1}(\mathrm{pr}_{\partial X})(a_i). \end{aligned}$$

Since $H_{d-1}^{\Gamma}(j; \mathbb{Z}^w)$ is injective, we conclude from(7.16) and (7.17)

$$\mu(\widehat{X}, \partial X) - \mu(X, \partial X) = \sum_{i \in I_d} \lambda_i \cdot H_{d-1}(\mathrm{pr}_{\partial X})(a_i).$$

Now consider any element $a \in H_{d-1}(\partial X)$. If we choose $a_i = \mu_i \cdot a$, we get

$$\begin{aligned} \mu(\widehat{X}, \partial X) - \mu(X, \partial X) &= \sum_{i \in i_d} \lambda_i \cdot H_{d-1}(\mathrm{pr}_{\partial X})(\mu_i \cdot a) \\ &= \sum_{i \in I_d} \lambda_i \cdot \mu_i \cdot H_{d-1}(\mathrm{pr}_{\partial X})(a) \\ &= \left(\sum_{i \in I_d} \lambda_i \cdot \mu_i\right) \cdot H_{d-1}(\mathrm{pr}_{\partial X})(a) \\ &= H_{d-1}(\mathrm{pr}_{\partial X})(a). \end{aligned} \tag{7.18}$$

The map $H_{d-1}(\mathrm{pr}_{\partial X})\colon H_{d-1}(\partial X) \to H_{d-1}(\partial X/\Gamma)$ can be identified with the map

$$\bigoplus_{F\in\mathcal{M}} \pi_X[F]\colon \bigoplus_{F\in\mathcal{M}} \mathbb{Z}\Gamma \otimes_{\mathbb{Z}F} H_d(S_F) \to \bigoplus_{F\in\mathcal{M}} H_d(S_F/F)$$

where $\pi_X[F]\colon \mathbb{Z}\Gamma \otimes_{\mathbb{Z}F} H_{d-1}(S_F) \to H_{d-1}(S_F/F)$ sends (γ, x) to the image of x under $H_{d-1}(\mathrm{pr}_{S_F})\colon H_{d-1}(S_F) \to H_{d-1}(S_F/F)$ for the projection $\mathrm{pr}_{S_F}\colon S_F \to S_F/F$. Since $H_{d-1}(\mathrm{pr}_{S_F})\colon H_{d-1}(S_F) \to H_{d-1}(S_F/F)$ is the inclusion of infinite cyclic groups of index $[F]$, we conclude that an element $b = (b_F)_{F\in\mathcal{M}} \in \bigoplus_{F\in\mathcal{M}} H_{d-1}(S_F/F) = H_{d-1}(\partial X/\Gamma)$ lies in the image of $H_{d-1}(\mathrm{pr}_{\partial X})\colon H_{d-1}(\partial X) \to H_{d-1}(\partial X/\Gamma)$, if and only if $b_F = m_F\cdot[S_F/F]$ for some integer m_F satisfying $m_F \equiv 0 \mod |F|$.

Recall from Remark 7.11 that the integer $m_F(X,\partial X)$ defined by $\mu(X,\partial X)_F = m_F(X,\partial X)\cdot[S_F/F]$ satisfies $m_F(X,\partial X) \equiv 1 \mod |F|$. Hence the difference $s - \mu(X,\partial X)$ lies in the image of the map $H_{d-1}(\mathrm{pr}_{\partial X})$, where s has been defined in (7.5). If a is such a preimage, we have associated to it the new pair $(\widehat{X},\partial X)$. We conclude $m_F(\widehat{X},\partial X) = 1$ for every $F \in \mathcal{M}$ from (7.18). This finishes the proof of Theorem 7.14. □

7.3. **Checking Poincar'e duality.**

Theorem 7.19 (Checking Poincaré duality)**.** *Suppose that Assumption 7.1 is satisfied. Let $\mathcal{S}$ be an oriented free d-dimensional slice system satisfying condition (S). Let $(X,\partial X)$ be a slice complement model for $\underline{E}\Gamma$ with respect to $\mathcal{S}$ such that $m_F(X,\partial X) = 1$ holds for every $F \in \mathcal{M}$. Then:*

(1) *The Γ-CW-pair $(X/\Gamma,\partial X/\Gamma)$ carries the structure of a finite Poincaré pair with respect to the orientation homomorphism $w\colon \Gamma \to \{\pm 1\}$ of Notation 6.7 and the fundamental class $[X/\Gamma,\partial X/\Gamma] \in H_d^{\Gamma}(X,\partial X;\mathbb{Z}^w)$ defined in Notation 6.22;*

(2) *The map*

$$H_d(i_*(X,\partial X))\colon H_d^{\pi}(X,\partial X;\mathbb{Z}^v) \to H_d^{\Gamma}(X,\partial X;\mathbb{Z}^w)$$

induced by the chain map $i_*(X,\partial X)$ of (6.1) *is an inclusion of infinite cyclic groups of index $|G|$ and the map induced by the transfer chain map of* (6.2)

$$H_d(\mathrm{trf}_*(X,\partial X))\colon H_d^{\Gamma}(X,\partial X;\mathbb{Z}^w) \xrightarrow{\cong} H_d^{\pi}(X,\partial X;\mathbb{Z}^v)$$

is an isomorphism. Moreover, the map

$$H_d(i_*(\underline{E}\Gamma))\colon H_d^{\pi}(E\pi;\mathbb{Z}^v) \to H_d^{\Gamma}(\underline{E}\Gamma;\mathbb{Z}^w)$$

is an inclusion of infinite cyclic groups with index $|G|$.

Proof. Let $i_*(X,\partial X)\colon \mathbb{Z}^v \otimes_{\mathbb{Z}\pi} i^*C_*(X,\partial X) \to \mathbb{Z}^w \otimes_{\mathbb{Z}\Gamma} C_*(X,\partial X)$ be the $\mathbb{Z}$-chain map defined in (6.1). Define $i_*(\partial X)\colon \mathbb{Z}^v \otimes_{\mathbb{Z}\pi} i^*C_*(\partial X) \to \mathbb{Z}^w \otimes_{\mathbb{Z}\Gamma} C_*(\partial X)$ and $i_*(\underline{E}\Gamma,\partial\underline{E}\Gamma)\colon \mathbb{Z}^v\otimes_{\mathbb{Z}\pi} i^*C_*(\underline{E}\Gamma,\partial\underline{E}\Gamma) \to \mathbb{Z}^w\otimes_{\mathbb{Z}\Gamma} C_*(\underline{E}\Gamma,\partial\underline{E}\Gamma)$ analogously. Consider the following commutative diagram

$$\begin{tikzcd}
H_d^{\pi}(X,\partial X;\mathbb{Z}^v) \arrow[r, "\partial"] \arrow[d, "H_d(i_*(X,\partial X))"'] & H_{d-1}^{\pi}(\partial X;\mathbb{Z}^v) = H_{d-1}(\partial X/\pi) \arrow[d, "H_d(i_*(\partial X))=H_{d-1}(q)"] \\
H_d^{\Gamma}(X,\partial X;\mathbb{Z}^w) \arrow[r, "\partial"'] & H_{d-1}^{\Gamma}(\partial X;\mathbb{Z}^w) = H_{d-1}(\partial X/\Gamma),
\end{tikzcd}$$

where the horizontal arrows are boundary maps and $q\colon \partial X/\pi \to \partial X/\Gamma$ is the projection. We can determine $H_{d-1}(q)$ by

$$\begin{tikzcd} \bigoplus_{F\in\mathcal{M}} \bigoplus_{G/\operatorname{pr}(F)} H_{d-1}(S_F) \arrow[r, "\cong"] \arrow[d, "\bigoplus_{F\in\mathcal{M}} \bigoplus_{G/\operatorname{pr}(F)} H_{d-1}(\operatorname{pr}_{S_F})"'] & H_{d-1}(\partial X/\pi) \arrow[d, "H_{d-1}(q)"] \\ \bigoplus_{F\in\mathcal{M}} H_{d-1}(S_F/F) \arrow[r, "\cong"] & H_{d-1}(\partial X/\Gamma). \end{tikzcd}$$

We know already that $H_d^{\Gamma}(\underline{E}\Gamma,\partial\underline{E}\Gamma;\mathbb{Z}^w)$ is an infinite cyclic group. We conclude from (6.3) that the map

$$H_d(i_*(X,\partial X)_*)\colon H_d^{\Gamma}(X,\partial X;\mathbb{Z}^w) \to H_d^{\Gamma}(\underline{E}\Gamma,\partial\underline{E}\Gamma;\mathbb{Z}^w)$$

is injective and its image has finite index which divides $|G|$. Let the element $k\in\mathbb{Z}$ with $k\geq 0$ be uniquely determined by the equation

$$H_d(i_*(\underline{E}\Gamma,\partial\underline{E}\Gamma))([\underline{E}\Gamma/\pi,\partial\underline{E}\Gamma/\pi]) = k\cdot[\underline{E}\Gamma/\Gamma,\partial\underline{E}\Gamma/\Gamma].$$

Note that k divides $|G|$. We can identify $H_d(i_*(X,\partial X))$ and $H_d(i_*(\underline{E}\Gamma,\partial\underline{E}\Gamma))$ by

$$\begin{tikzcd} H_d^{\pi}(X,\partial X;\mathbb{Z}^v) \arrow[r, "H_d^{\pi}(p,\partial p;\mathbb{Z}^v)", "\cong"'] \arrow[d, "H_d(i_*(X,\partial X))"'] & H_d^{\pi}(\underline{E}\Gamma,\partial\underline{E}\Gamma;\mathbb{Z}^v) \arrow[d, "H_d(i_*(\underline{E}\Gamma,\partial\underline{E}\Gamma))"] \\ H_d^{\Gamma}(X,\partial X;\mathbb{Z}^w) \arrow[r, "\cong", "H_d^{\Gamma}(p,\partial p;\mathbb{Z}^w)"'] & H_d^{\Gamma}(\underline{E}\Gamma,\partial\underline{E}\Gamma;\mathbb{Z}^w) \end{tikzcd}$$

where $(p,\partial p)\colon (X,\partial X)\to(\underline{E}\Gamma,\partial\underline{E}\Gamma)$ is the projection. We conclude

$$H_d(i_*(X,\partial X))([X/\pi,\partial X/\pi]) = k\cdot[X/\Gamma,\partial X/\Gamma]. \tag{7.20}$$

Consider the composite

$$H_d^{\pi}(X,\partial X;\mathbb{Z}^v) \xrightarrow{\partial} H_d^{\pi}(\partial X;\mathbb{Z}^v) \xrightarrow{\cong} \bigoplus_{F\in\mathcal{M}} \bigoplus_{G/\operatorname{pr}(F)} H_{d-1}(S_F).$$

There is the G-action on its source given for $g\in G$ and any element $\widehat{g}\in\Gamma$, which is sent by the projection $\Gamma\to G$ to g, by the $\mathbb{Z}$-chain map

$$\mathbb{Z}^v\otimes_{\mathbb{Z}\pi} C_*(X,\partial X)\to \mathbb{Z}^v\otimes_{\mathbb{Z}\pi} C_*(X,\partial X), \quad (m\otimes x)\mapsto (m\cdot w(\widehat{g})\otimes \widehat{g}^{-1}x).$$

There is the G-action on the target given by permuting the summand according to the canonical G-action on $G/\operatorname{pr}(F)$. One easily checks that the composite above is compatible with these G-actions.

We have defined a specific G-action on $H_d^{\pi}(E\pi;\mathbb{Z}^v)$ at the end of Subsection 6.3 and shown that it is trivial. The isomorphism $H_d^{\pi}(E\pi;\mathbb{Z}^v) \xrightarrow{\cong} H_d^{\pi}(X,\partial X;\mathbb{Z}^v)$ obtained by the composite of the isomorphisms (or their inverses) appearing in the middle column of (6.4) is compatible with these G-actions. Therefore the G-action on $H_d^{\pi}(X,\partial X;\mathbb{Z}^v)$ is trivial as well. Hence we can find a collection of integers $\{n_F \mid F\in\mathcal{M}\}$ such that the image of $[X/\pi,\partial X/\pi]$ under the composite above has as entry in the summand $H_n(S_F)$ for $F\in\mathcal{M}$ and $g\operatorname{pr}(F)\in G/\operatorname{pr}(F)$ the element $n_F\cdot[S_F]$. This element is sent under

$$\bigoplus_{F\in\mathcal{M}} \bigoplus_{G/\operatorname{pr}(F)} H_{d-1}(\operatorname{pr}_{S_F})\colon \bigoplus_{F\in\mathcal{M}} \bigoplus_{G/\operatorname{pr}(F)} H_{d-1}(S_F) \to \bigoplus_{F\in\mathcal{M}} H_{d-1}(S_F/F)$$

to $\{n_F\cdot|G|\cdot[S_F/F] \mid F\in\mathcal{M}\}$, since $|G| = |F|\cdot|G/\operatorname{pr}(F)|$. Since the composite

$$H_d^{\Gamma}(X,\partial X;\mathbb{Z}^w) \xrightarrow{\partial} H_d^{\Gamma}(\partial X;\mathbb{Z}^w) \xrightarrow{\cong} \bigoplus_{F\in\mathcal{M}} H_{d-1}(S_F/F)$$

sends $[X/\Gamma, \partial X/\Gamma]$ to $\{[S_F] \mid F \in \mathcal{M}\}$, we get $k = n_F \cdot [G]$ for every $F \in \mathcal{M}$. Since k divides $[G]$ and k and n_F are positive, we conclude $n_F = 1$ for every $F \in \mathcal{M}$ and $k = |G|$.

This implies that the maps

$$\begin{array}{rcl} H_d(i_*(X, \partial X))\colon H_d^{\pi}(X, \partial X; \mathbb{Z}^v) & \to & H_d^{\Gamma}(X, \partial X; \mathbb{Z}^w); \\ H_d(i_*(\underline{E}\Gamma))\colon H_d^{\pi}(E\pi; \mathbb{Z}^v) & \to & H_d^{\Gamma}(\underline{E}\Gamma; \mathbb{Z}^w), \end{array}$$

are inclusions of infinite cyclic groups with index $|G|$. We conclude from (6.3) that the map $H_d(\mathrm{trf}_*)\colon H_d^{\Gamma}(X, \partial X; \mathbb{Z}^w) \xrightarrow{\cong} H_d^{\pi}(X, \partial X; \mathbb{Z}^v)$ is an isomorphism.

We also conclude that the composite

$$H_d^{\pi}(X, \partial X; \mathbb{Z}^v) \xrightarrow{\partial} H_d^{\pi}(\partial X; \mathbb{Z}^v) \xrightarrow{\cong} \bigoplus_{F \in \mathcal{M}} \bigoplus_{G/\operatorname{pr}(F)} H_{d-1}(S_F)$$

sends $[X/\pi, \partial X/\pi]$ to the element, which is given in any of the summands by the fundamental class $[S_F]$.

It remains to show that $(X/\Gamma, \partial X/\Gamma)$ carries the structure of a finite Poincaré pair with respect to the orientation homomorphism $w\colon \Gamma \to \{\pm 1\}$ and the fundamental class $[X/\Gamma, \partial X/\Gamma] \in H_d^{\Gamma}(X, \partial X; \mathbb{Z}^w)$. Because of [17, Theorem H] it suffices to show that $(X/\pi, \partial X\pi)$ carries the structure of a finite Poincaré pair with respect to the orientation homomorphism $v\colon \pi \to \{\pm 1\}$ and the fundamental class $[X/\pi, \partial X/\pi] \in H_d^{\pi}(X, \partial X; \mathbb{Z}^v)$. This follows from Lemma 5.8 (3) applied in the case

$$\begin{array}{rcl} Y & = & B\pi; \\ Y_1 & = & \coprod_{F \in \mathcal{M}} G/\operatorname{pr}(F) \times D^d; \\ Y_2 & = & X/\pi; \\ Y_0 & = & \partial X/\pi = \coprod_{F \in \mathcal{M}} G/\operatorname{pr}(F) \times S^{d-1}, \end{array}$$

using the assumption that there is a finite d-dimensional Poincaré CW-complex model for $B\pi$ with respect to the orientation homomorphism $v\colon \pi \to \{\pm 1\}$ and fundamental classes $[B\pi] \in H_d^{\pi}(E\pi; \mathbb{Z}^v)$, the fundamental class $[X/\pi, \partial X/\pi] \in H_d^{\pi}(X, \partial X; \mathbb{Z}^v)$, and the preimage under the composite of isomorphisms

$$\begin{aligned} \partial\colon H_d\Big(\coprod_{F \in \mathcal{M}} G/\operatorname{pr}(F) \times (D^d, S^{d-1})\Big) & \xrightarrow{\cong} H_{d-1}\Big(\coprod_{F \in \mathcal{M}} G/\operatorname{pr}(F) \times S^{d-1}\Big) \\ & \xrightarrow{\cong} \bigoplus_{F \in \mathcal{M}} \bigoplus_{G/\operatorname{pr}(F)} H_{d-1}(S_F) \end{aligned}$$

of the obvious element in $H_{d-1}(\coprod_{F \in \mathcal{M}} G/\operatorname{pr}(F) \times S^{d-1})$, which is given in any of the summands by the fundamental class $[S_F]$. Now the conditions about the fundamental classes appearing in Lemma 5.8 follow from the following commutative diagram with exact right row

$$\begin{tikzcd}
& 0 \arrow[d] \\
& H_d^\pi(\widetilde{B\pi};\mathbb{Z}^v) \arrow[d] \\
H_d^\pi(X,\partial X;\mathbb{Z}^v)\oplus H_d(\coprod_{F\in\mathcal{M}} G/\operatorname{pr}(F)\times(D^d,S^{d-1})) \arrow[r, "\cong"] \arrow[d, "\partial\oplus\partial"] & H_d^\pi(\widetilde{B\pi},\partial X;\mathbb{Z}^v) \arrow[d, "\partial"] \\
H_{d-1}^\pi(\partial X;\mathbb{Z}^v)\oplus H_{d-1}(\coprod_{F\in\mathcal{M}} G/\operatorname{pr}(F)\times S^{d-1}) \arrow[r, "\mathrm{id}\oplus -\mathrm{id}"] & H_{d-1}^\pi(\partial X;\mathbb{Z}^v),
\end{tikzcd}$$

where we identify $H_{d-1}^\pi(\partial X;\mathbb{Z}^v)$ and $H_{d-1}(\coprod_{F\in\mathcal{M}} G/\operatorname{pr}(F)\times S^{d-1})$ by the obvious isomorphism. $\square$

8. Homotopy classification of slice complement models

Throughout this section we make Assumption 7.1.

For the remainder of this section we fix two oriented free d-dimensional slice systems $\mathcal{S}$ and $\mathcal{S}'$ satisfying condition (S), see Notation 7.9. Recall that we have defined an orientation homomorphism $w\colon \Gamma\to\{\pm 1\}$ in Notation 6.7. Let $(X,\partial X)$ and $(X',\partial X')$ respectively be slice complement models for $\underline{E}\Gamma$ with respect to $\mathcal{S}$ and $\mathcal{S}'$ respectively. Then we have defined fundamental classes $[X/\Gamma,\partial X/\Gamma]$ and $[X/\Gamma,\partial X/\Gamma]$ in Notation 6.22, and integers $m(X,\partial X)_F$ and $m(X',\partial X')_F$ satisfying $m(X,\partial X)_F\equiv m(X',\partial X')_F\equiv 1 \mod |G|$ for $F\in\mathcal{M}$ in (7.6). The main result of this section will be

Theorem 8.1 (Homotopy classification of slice models)**.** *Suppose that Assumption 7.1 is satisfied. Let $(X,\partial X)$ and $(X',\partial X')$ respectively be slice complement models for $\underline{E}\Gamma$ with respect to $\mathcal{S}$ and $\mathcal{S}'$ respectively.*

Then the following two assertions are equivalent:

1. *There exists a Γ-homotopy equivalence $(f,\partial f)\colon (X,\partial X)\to(X',\partial X')$ of free Γ-CW-pairs with the properties that the Γ-map ∂f extends to a Γ-homotopy equivalence of Γ-CW-pairs $C(\partial X)\to C(\partial X')$ and the isomorphism $H_d^\Gamma(X,\partial X;\mathbb{Z}^w)\xrightarrow{\cong} H_d^\Gamma(X',\partial X';\mathbb{Z}^w)$ induced by $(f,\partial f)$ sends $[X,\partial X]$ to $[X',\partial X']$;*
2. *We have $m(X,\partial X)_F = m(X',\partial X')_F$ for every $F\in\mathcal{M}$ with $|F|\ge 3$ and $m(X,\partial X)_F=\epsilon_F\cdot m(X',\partial X')_F$ for every $F\in\mathcal{M}$ with $|F|=2$ for some $\epsilon_F\in\{\pm1\}$.*

Its proof needs some preparations. Recall that up F-homotopy there is precisely one orientation preserving F-homotopy equivalence, see Lemma 3.3,

$$s_F\colon S_F\to S'_F. \tag{8.2}$$

Lemma 8.3. *Let $\partial u\colon \partial X\to\partial X'$ be the Γ-homotopy equivalence given by the disjoint union of the Γ-homotopy equivalences $\mathrm{id}_\Gamma\times_F s_F\colon \Gamma\times_F S_F\to\Gamma\times_F S'_F$ for $F\in\mathcal{M}$. Suppose that there is a Γ-map $u\colon X\to X'$ extending $\partial u\colon\partial X\to\partial X'$.*

Then $(u,\partial u)\colon (X,\partial X)\to(X',\partial X')$ is a Γ-homotopy equivalence of Γ-CW-pairs and the isomorphism $H_d(X,\partial X;\mathbb{Z}^w)\xrightarrow{\cong} H_d(X',\partial X';\mathbb{Z}^w)$ induced by $(u,\partial u)$ sends $[X/\Gamma,\partial X/\Gamma]$ to $[X'/\Gamma,\partial X'/\Gamma]$.

Proof. Each map $s_F\colon S_F \to S'_F$ extends to F-map $D_F\colon D'_F$ by taking the cone. Hence there exists a Γ-homotopy equivalence $C(\partial u)\colon C(\partial X) \to C(\partial X')$ extending ∂u. We obtain a Γ-map

$$u \cup_{\partial u} C(\partial u)\colon X \cup_{\partial X} C(\partial X) \to X' \cup_{\partial X'} C(\partial X').$$

Since the source and target of this map are Γ-homotopy equivalent to $\underline{E}\Gamma$, it is a Γ-homotopy equivalence. Since $u \cup_{\partial u} C(\partial u)$, ∂u and $C(\partial u)$ induce homology equivalences, the map $H_n(u)\colon H_n(X) \to H_n(X')$ is bijective for all $n \geq 0$. Since X and X' are simply connected by Lemma 4.4, the map $u\colon X \to X'$ is a non-equivariant homotopy equivalence. Since X and X' are free Γ-CW-complexes, $u\colon X \to X'$ is a Γ-homotopy equivalence. Since $\partial u\colon \partial X \to \partial X'$ is a Γ-homotopy equivalence, $(u, \partial u)\colon (X, \partial X) \to (X', \partial X')$ is a Γ-homotopy equivalence of Γ-CW-pairs.

One easily checks by inspecting the definitions and the commutative diagram

$$\begin{tikzcd}
H_d^\Gamma(X, \partial X; \mathbb{Z}^w) \arrow[r, "\partial"] \arrow[d, "H_d^\Gamma(u,\partial u;\mathbb{Z}^w)"', "\cong"] & H_{d-1}^\Gamma(\partial X; \mathbb{Z}^w) \arrow[d, "\cong"', "H_d^\Gamma(\partial u;\mathbb{Z}^w)"] \\
H_d^\Gamma(X', \partial X'; \mathbb{Z}^w) \arrow[r, "\partial"'] & H_{d-1}^\Gamma(\partial X'; \mathbb{Z}^w)
\end{tikzcd}$$

that the isomorphism $H_d(X, \partial X; \mathbb{Z}^w) \xrightarrow{\cong} H_d(X', \partial X'; \mathbb{Z}^w)$ induced by $(u, \partial u)$ sends $[X/\Gamma, \partial X/\Gamma]$ to $[X'/\Gamma, \partial X'/\Gamma]$, since the horizontal arrows and the right vertical arrow respects the fundamental classes. $\square$

Lemma 8.4. *Suppose additionally that $(X, \partial X)$ is a Poincaré slice complement model and $m_F(X, \partial X) = 1$ hold for all $F \in \mathcal{M}$. (Such $(X, \partial X)$ exists by Lemma 7.12.) Let $v_F\colon S_F \to S'_F$ be the F-map uniquely determined up to F-homotopy by the property that it sends $[S_F/F]$ to $m_F(X', \partial X') \cdot [S'_F/F]$. (It exists, since we have $\mu_F(X', \partial X') \equiv 1 \mod |G|$.) Let $\partial u\colon \partial X \to \partial X'$ be the Γ-map given by the disjoint union of the Γ-maps $\Gamma \times_F v_F$.*

Then the following assertions are equivalent

1. *There exists a Γ-map $u\colon X \to X'$ extending the Γ-map $\partial u\colon \partial X \to \partial X'$;*
2. *There exists a π-map $u'\colon i^*X \to i^*X'$ extending the π-map $i^*u\colon i^*\partial X \to i^*\partial X'$.*

Proof. Obviously (1) implies (2). The implication (2) $\implies$ (1) is proved by equivariant obstruction theory as follows.

Note that $(X, \partial X)$ is a free Γ-CW-pair and $\dim(X) = d$. The Γ-CW-complex X' is $(d-2)$-connected by Lemma 4.4 (3). Hence we get from equivariant obstruction theory an exact sequence

$$[X, X']^\Gamma \to [\partial X, X']^\Gamma \xrightarrow{o^\Gamma} H^d_\Gamma(X, \partial X; \pi_{d-1}(X')).$$

This is explained for finite Γ for instance in [32, pages 119 - 120], the condition that Γ is finite is not needed at all. The construction is compatible with restriction. So we get a commutative diagram with exact rows

$$\begin{tikzcd}
{[X, X']^\Gamma} \arrow[r] \arrow[d, "i^*"] & {[\partial X, X']^\Gamma} \arrow[r, "o^\Gamma"] \arrow[d, "i^*"] & H^d_\Gamma(X, \partial X; \pi_{d-1}(X')) \arrow[d, "i^*"] \\
{[i^*X, i^*X']^\pi} \arrow[r] & {[i^*\partial X, i^*X']^\pi} \arrow[r, "o^\pi"] & H^d_\pi(i^*X, i^*\partial X; \pi_{d-1}(i^*X')).
\end{tikzcd}$$

Hence it suffices to show that $i^*\colon H^d_\Gamma(X, \partial X; \pi_{d-1}(X')) \to H^d_\pi(i^*X, i^*\partial X; \pi_{d-1}(i^*X'))$ is injective. This will be done by a cohomological version of the transfer argument appearing in Subsection 6.1, which we explain next. Recall from the definitions

$$\begin{aligned} H^d_\Gamma(X,\partial X;\pi_{d-1}(X')) &= H^d(\hom_{\mathbb{Z}\Gamma}(C_*(X,\partial X),\pi_{d-1}(X'))); \\ H^d_\Gamma(X,\partial X;\pi_{d-1}(X')) &= H^d(\hom_{\mathbb{Z}\pi}(i^*C_*(X,\partial X),i^*\pi_{d-1}(X'))). \end{aligned}$$

The group G-acts on $\hom_{\mathbb{Z}\pi}(i^*C_*(X,\partial X),i^*\pi_{d-1}(X'))$ in the obvious way. We have

$$\hom_{\mathbb{Z}\pi}(i^*C_*(X,\partial X),i^*\pi_{d-1}(X'))^G = \hom_{\mathbb{Z}\Gamma}(C_*(X,\partial X),\pi_{d-1}(X')).$$

If we put $D^* = \hom_{\mathbb{Z}\pi}(i^*C_*(X,\partial X),i^*\pi_{d-1}(X'))$, then $i^*\colon H^d_\Gamma(X,\partial X;\pi_{d-1}(X')) \to H^d_\pi(i^*X,i^*\partial X;\pi_{d-1}(i^*X'))$ can be identified with the map $H^d(j^*)\colon H^d((D^*)^G) \to H^d(D^*)$ for the inclusion $j^*\colon (D^*)^G \to D^*$. Multiplication with the norm element $N := \sum_{g\in G} g \in \mathbb{Z}G$ defines a $\mathbb{Z}$-chain map $t^*\colon D^* \to (D^*)^G$ such that $t^*\circ j^* = |G|\cdot \mathrm{id}_{(D^*)^G}$. Hence j^* is injective, if $|G|\cdot\mathrm{id}\colon H^d((D^*)^G) \to H^d((D^*)^G)$ is injective. Therefore it suffices to show that $H^d((D^*)^G) = H^d(\hom_{\mathbb{Z}\Gamma}(C_*(X,\partial X),\pi_{d-1}(X')))$ is torsionfree.

This follows from the following string of isomorphisms

$$\begin{aligned} H^d(\hom_{\mathbb{Z}\Gamma}(C_*(X,\partial X),\pi_{d-1}(X'))) &\cong H^d(\hom_{\mathbb{Z}\Gamma}(C_*(X,\partial X),H_{d-1}(X'))) \\ &\cong H^d(\hom_{\mathbb{Z}\Gamma}(C_*(X,\partial X),H_{d-1}(\partial X'))) \\ &\cong H_0(C_*(X)\otimes_{\mathbb{Z}\Gamma} H_{d-1}(\partial X')) \\ &\cong \mathbb{Z}\otimes_{\mathbb{Z}\Gamma} H_{d-1}(\partial X') \\ &\cong H_{d-1}(\partial X'/\Gamma) \\ &\cong \bigoplus_{F\in\mathcal{M}} H_{d-1}(S'_F/F). \end{aligned}$$

The first isomorphism comes from the Hurewicz homomorphism $\pi_{d-1}(X') \xrightarrow{\cong} H_{d-1}(X')$, which is bijective, as X' is $(d-2)$-connected by Lemma 4.4. The second isomorphism comes from the $\mathbb{Z}\Gamma$-isomorphism $H_{d-1}(\partial X') \xrightarrow{\cong} H_{d-1}(X')$, which is bijective, since for $n\in\{(d-1),d\}$ we get

$$H_n(X',\partial X') \cong H_n(\underline{E}\Gamma,\partial\underline{E}\Gamma) \cong H_n(\underline{E}\Gamma) = 0$$

using the homotopy Γ-pushout (4.3). The third isomorphism is a consequence of the assumption that $(X/\Gamma,\partial X/\Gamma)$ is a Poincaré pair. The fourth and fifth isomorphism come from the fact that X is a connected free Γ-CW-complex and the functor $-\otimes_{\mathbb{Z}\Gamma} H_{d-1}(\partial X')$ is right exact. The last isomorphism is obvious. Note that $H_{d-1}(S'_F/F)$ is infinite cyclic and hence torsionfree. □

Remark 8.5. Suppose that we are in the situation of Lemma 8.4. Consider the following composite

$$\alpha\colon [\partial X,\partial X']^\Gamma \xrightarrow{j_*} [\partial X,X']^\Gamma \xrightarrow{o^\Gamma} H^d_\Gamma(X,\partial X;\pi_{d-1}(X')) \xrightarrow{\cong} \bigoplus_{F\in\mathcal{M}} H_{d-1}(S'_F/F),$$

where the first map is given by composition with the inclusion $j\colon \partial X' \to X'$, the second is given by the equivariant obstruction, and the third map is the isomorphism appearing in the proof of Lemma 8.4. One may guess what this composition is for ∂u, namely, we expect

$$\alpha([\partial u]) = \mu(X',\partial X') - H_{d-1}(\partial u)(\mu(X,\partial X)). \tag{8.6}$$

This makes sense, since the existence of an extension of $j\circ\partial u$ to a Γ-map $X\to X'$ implies that $\mu(X',\partial X') - H_{d-1}(\partial u)(\mu(X,\partial X))$ vanishes. Since $(X,\partial X)$ is by assumption a Poincaré slice complement model, we conclude from Lemma 7.10 that $\mu(X,\partial X) = s$. This implies $\mu(X',\partial X') - H_{d-1}(\partial u)(\mu(X,\partial X)) = 0$ and hence the existence of the Γ-extension u of the Γ-map ∂u follows from obstruction theory.

However, it is not so easy to check equation (8.6) and we will not do this here. Instead we will construct the desired π-extension of the π-map $i^*\partial u$ directly and use the equivariant obstruction theory only to reduce the problem from Γ to π by Lemma 8.4.

In the next step we construct a specific model for $(i^*X, i^*\partial X)$.

Because of Lemma 5.10 we can assume that we have a special model for $B\pi = H \cup_z \widehat{B\pi}$ for $B\pi$. Put

$$J := \coprod_{F \in \mathcal{M}} G/\operatorname{pr}(F),$$

where $\operatorname{pr}\colon \Gamma \to G$ is the projection. Define

$$\tau\colon J \to \mathcal{F}$$

to be the obvious projection, which sends the summand $G/\operatorname{pr}(F)$ belonging to $F \in \mathcal{M}$ to F.

Choose for every $j \in J$ an embedded disk $D_j^H \subseteq H \setminus \partial H$ such that the disks for different j are disjoint. Put $S_j^H = \partial D_j^H$. Note that the superscript H shall remind the reader that these disks and spheres lie in the interior of H. Let

$$\overline{Y} = B\pi \setminus \coprod_{j \in J} \operatorname{int}(D_j^H)$$

be obtained from $B\pi = H \cup_z \widehat{B\pi}$ by deleting the interiors of these embedded disks D_j^H. Define

$$\begin{aligned} \overline{\partial Y} &= \coprod_{j \in J} S_j^H; \\ C(\overline{\partial Y}) &= \coprod_{j \in J} D_j^H. \end{aligned}$$

Then $\overline{Y} \cap H$ is a compact smooth manifold, whose boundary is the disjoint union of ∂H and $\overline{\partial Y}$ and we have

$$B\pi = (\overline{Y} \cap H) \cup_{\partial H \amalg \overline{\partial Y}} (\widehat{B\pi} \amalg C(\overline{\partial Y})) = \overline{Y} \cup_{\overline{\partial Y}} C(\overline{\partial Y}).$$

Let Y and ∂Y be the free π-CW-complexes obtained by taking the preimage of $\overline{Y}$ and $\overline{\partial Y}$ under the universal covering $E\pi \to B\pi$. Note that then we can identify $\overline{Y} = Y/\pi$ and $\overline{\partial Y} = \partial Y/\pi$ and $\partial \overline{Y} = \coprod_{j \in J} \pi \times S_j^H$.

Since $\partial X'/\Gamma = \coprod_{F \in \mathcal{M}} S'_F/F$ holds, the π-space $i^*\partial X'$ can be written as

$$i^*\partial X' = \coprod_{F \in \mathcal{M}} \Big(\coprod_{G/\operatorname{pr}(F)} \pi \times S'_F \Big) = \coprod_{j \in J} \pi \times S'_j,$$

where S'_j is a copy of $S'_{\tau(j)}$. For $j \in J$ choose a map $\partial v_j\colon S_j^H \to S_j$, whose degree is $m_{\tau(j)}(X', \partial X')$. Let $\partial v\colon \partial Y \to i^*\partial X'$ be the disjoint union of the π-maps $\mathrm{id}_\pi \times \partial v_j\colon \pi \times S_j^H \to \pi \times S'_j$. Since each map $\partial v_j\colon S_j^H \to S'_j$ can be extended to a map $D_j^H \to D'_j$ by coning, we get an extension of $\partial v\colon \partial Y \to \partial X'$ to a π-map $C(\partial v)\colon C(\partial Y) \to C(i^*\partial X')$.

Lemma 8.7. *The π-map $\partial v\colon \partial Y \to i^*\partial X'$ extends to a π-map $v\colon Y \to i^*X'$.*

Proof. We begin with explaining that we can assume without loss of generality that each S'_F is the standard sphere with its standard orientation and hence each D'_F is the standard disk. Namely, we can replace $(i^*X', i^*\partial X')$ by a π-homotopy equivalent pair $(X'', \partial X'')$ such that $\partial X''$ is a disjoint union of standard spheres, by the following construction. Choose for every j an orientation preserving homotopy equivalence $\partial g_j\colon S'_j \to S''_j$ with a copy of the standard sphere of dimension $(d -$

1) with its standard orientation as target. Define $\partial X'' = \coprod_{j\in J} \pi \times S''_j$ and let $\partial g\colon i^*\partial X' \to \partial X''$ be the π-homotopy equivalence given by $\coprod_{j\in J} \mathrm{id}_\pi \times \partial g_j$. Define X'' and the π-map $g\colon i^*X' \to X''$ by the π-pushout

$$\begin{tikzcd} i^*\partial X' \arrow[r, "\partial g"] \arrow[d] & \partial X'' \arrow[d] \\ i^*X' \arrow[r, "g"'] & X''. \end{tikzcd}$$

Since ∂g is a π-homotopy equivalence, $(g, \partial g)\colon (X', \partial X') \to (X'', \partial X'')$ is a π-homotopy equivalence of free π-CW-pairs. Obviously it suffices to show that the map $\partial g \circ \partial v\colon \partial Y \to \partial X''$ extends to a π-map $Y \to X''$. Because we may replace $(i^*X', i^*\partial X')$ with $(X'', \partial X'')$, we can assume without loss of generality that each S'_j is the $(d-1)$-dimensional standard sphere with its standard orientation and each D'_j is the d-dimensional standard disk.

Put $\overline{X} := X'/\pi$ and $\overline{\partial X} := \partial X'/\pi$. Let $\overline{\partial v}\colon \overline{\partial Y} \to \overline{X'}$ be $\partial v/\pi$. Since $\overline{Y} \cup_{\partial \overline{Y}} C(\partial\overline{Y})$ and $\overline{X'} \cup_{\partial\overline{X'}} C(\partial\overline{X'})$ are models for $B\pi$, the map

$$C(\overline{\partial v})\colon C(\partial\overline{Y}) = \coprod_{j\in J} D^H_j \to C(\partial\overline{X'}) = \coprod_{j\in J} D'_j$$

extends to a homotopy equivalence

$$f\colon \overline{Y} \cup_{\partial\overline{Y}} C(\partial\overline{Y}) \xrightarrow{\simeq} \overline{X'} \cup_{\partial\overline{X'}} C(\partial\overline{X'})$$

inducing the identity on π. Since the inclusion $\overline{X'} \to \overline{X'} \cup_{\coprod_{j\in J} S'_j} \coprod_F D'_j$ is $(d-1)$-connected, $\widehat{B\pi}$ is a $(d-2)$-dimensional CW-complex and $\partial H \to H$ is a cofibration, we can arrange that $f(\widehat{B\pi}) \subseteq \overline{X'}$ holds without altering f on H and the homotopy class of f. In particular $f(\partial H) \subseteq X'$.

Now we can change

$$f|_{\overline{Y}\cap H}\colon \overline{Y} \cap H = H \setminus \coprod_{j\in J} \mathrm{int}(D^H_j) \to \overline{X'} \cup_{\coprod_{j\in J} S'_j} \coprod_{j\in J} D'_j$$

up to homotopy relative ∂H such that it is transversal to each origine $0'_j \in D'_j$, since D_F is a compact smooth manifold containing 0_F in interior and $\overline{Y} \cap H$ is a compact smooth manifold with boundary $\partial(\overline{Y} \cap H)$ such that $f(\partial(\overline{Y} \cap H))$ does not contain any of the points o'_F. Furthermore we can arrange that for every $j \in J$ the preimage $(f|_{\overline{Y}\cap H})^{-1}(D'_j) = f^{-1}(D_j) \setminus D^H_F$ is a disjoint union of disks $\coprod_{i\in I_j} D^H_{j,i}$ for a finite set I_j and $f|_{D[H]_{j,i}}\colon (D^H_{j,i}, S^H_{j,i}) \to (D'_j, S'_j)$ is a homeomorphism of pairs for $S^H_{j,i} = \partial D^H_{j,i}$. Let $\delta_{j,i} \in \{\pm 1\}$ be the local degree of the homeomorphism $f|_{S^H_{j,i}}\colon S^H_{j,i} \to S'_j$. Let $\mathrm{int}(D^H_j)$, $\mathrm{int}(D^H_{j,i})$, and $\mathrm{int}(D'_j)$ denote the interior of D^H_j, $D^H_{j,i}$ and D'_j. We abbreviate

$$\begin{aligned} Z' &:= \overline{X'} \cup_{\partial\overline{X'}} C(\partial\overline{X'}) = \overline{X'} \cup_{\coprod_{j\in J} S'_j} \coprod_{j\in J} D'_j; \\ A &:= B\pi \setminus \big(\coprod_{j\in J} \mathrm{int}(D^H_j) \amalg \coprod_{i\in I_j} \mathrm{int}(D^H_{j,i})\big). \end{aligned}$$

Next we construct the following commutative diagram, where $p\colon E\pi \to B\pi$ and $p'\colon \widetilde{Z'} \to Z$ are the universal coverings

$$\begin{tikzcd}
H_d^{\pi}(E\pi;\mathbb{Z}^v) \arrow[r] \arrow[d] & H_d^{\pi}(\widetilde{Z'};\mathbb{Z}^v) \arrow[d] \\
H_d^{\pi}\big(E\pi, p^{-1}(A);\mathbb{Z}^v\big) \arrow[r] & H_d^{\pi}\big(\widetilde{Z'}, p'^{-1}(Z' \setminus (\coprod_{j\in J}\operatorname{int}(D'_j)));\mathbb{Z}^v\big) \\
H_d\big(\coprod_{j\in J}(D_j^H,S_j^H) \amalg (\coprod_{i\in I_j}(D_{j,i}^H,S_{j,i}^H))\big) \arrow[r] \arrow[u, "\cong"'] & H_d\big(\coprod_{j\in J}(D'_j,S'_j)\big) \arrow[u, "\cong"] \\
\bigoplus_{j\in J}\Big(H_d(D_j^H,S_j^H) \oplus \bigoplus_{i\in I_j} H_d(D_{j.i}^H,S_{j.i}^H)\Big) \arrow[r] \arrow[u] \arrow[d] & \bigoplus_{j\in J} H_d(D'_j,S'_j) \arrow[u] \arrow[d] \\
\bigoplus_{j\in J}\Big(H_{d-1}(S_j^H) \oplus \bigoplus_{i\in I_j} H_{d-1}(S_{j.i}^H)\Big) \arrow[r] & \bigoplus_{j\in J} H_{d-1}(S'_j)
\end{tikzcd}$$

The uppermost two vertical arrows are given by the obvious inclusions. The vertical arrows pointing upwards are the isomorphisms are given by excision or by the disjoint union axiom. The lower most vertical arrows are given by boundary homomorphisms. All vertical arrows are induced by the homotopy equivalence $f\colon B\pi = \overline{Y} \cup_{\overline{\partial Y}} C(\overline{\partial Y}) \to Z' := \overline{X'} \cup_{\partial\overline{X'}} C(\partial\overline{X'})$.

The fundamental class $[B\pi]$ is sent under the composite of the four vertical arrows (or their inverses) of the left column to the element in the left lower corner $\bigoplus_{j\in J}\Big(H_{d-1}(S_j^H) \oplus \bigoplus_{i\in I_j} H_{d-1}(S_{j.i}^H)\Big)$, which is given for each summand by the fundamental class of the corresponding sphere. The fundamental class $[B\pi]$ is sent under the uppermost vertical arrow to the fundamental class $[Z']$. The fundamental class of $[Z']$ is sent under the under the composite of the four vertical arrows (or their inverses) of the right column to $(m_{\tau(j)}(X',\partial X')\cdot[S_j])_{j\in J}$. Given $j\in J$, the lowermost vertical arrow sends by construction the fundamental class $[S_j^H]\in H_{d-1}(S_j^H)$ to $m_{\tau(j)}(X',\partial X')\cdot[S'_j]\in H_{d-1}(S'_j)$ and by the definition of $\delta_{j,i}$ the fundamental class $[S_{j,i}^H]\in H_{d-1}(S_{j,i}^H)$ to $\delta_{j,i}\cdot[S'_j]$ in $H_{d-1}(S'_j)$ for every $i\in F$. Since the diagram commutes, we conclude for every $j\in J$

$$m_{\tau(j)}(X',\partial X') + \sum_{i\in I_j}\delta_{j,i} = m_{\tau(j)}(X',\partial X').$$

Hence we get $\sum_{i\in I_j}\delta_{j,i} = 0$ for every $j\in J$.

Next we show that we can change f up to homotopy relative $\coprod_{j\in J} D_j^H \amalg \widehat{B\pi}$ such that each I_j is empty. We use induction over the cardinality of $\coprod_{j\in J} I_j$. The induction beginning $|\coprod_{j\in J} I_j| = 0$ is trivial, the induction step done as follows. Choose $j\in J$ with $I_j\neq\emptyset$. Since $\sum_{i\in I_j}\delta_{j,i}=0$ and each element $\delta_{j,i}$ belongs to $\{\pm 1\}$, we can find i_+ and $i_-\in I_j$ with $\delta_{j,i_+}=1$ and $\delta_{j,i_-}=-1$. Choose an embedded arc in H joining a point $x_+\in S_{j,i_+}^H$ to a point $x_-\in S_{j,i_-}^H$ such that the intersection of the arc with $\coprod_{j\in J} D_F^H \amalg (\coprod_{i\in I_j} D_{j,i}^H)$ is $\{x_+,x_-\}$ and the arc meets S_{j,i_+}^H and S_{j,i_-}^H transversely. Then we can thicken this arc to a small tube T in the obvious way such that the intersection of T with $\coprod_{j\in J} D_j^H \amalg (\coprod_{i\in I_j} D_{j,i}^H)$ is contained in small neighbourhoods of x_+ in S_{j,i_+}^H and x_- in S_{j,i_+}^H, which are diffeomorphic to

$(d-1)$-dimensional discs. The union $D^H_{j,i_+} \cup T \cup D^H_{i_-,j}$ is diffeomorphic to a disk D^d. One can change f up to homotopy on a small neighborhood of the tube such that f is constant on the tube. Then f induces a map $f_{D^d}\colon (D^d, S^{d-1}) \to (D^H_j, S^H_j)$ such that the degree of $f|_{S^{d-1}}\colon S^{d-1} \to S^H_j$ is $\delta_{j,i_+} + \delta_{j,i_-} = 0$. We conclude the map $f|_{S^{d-1}}\colon S^{d-1} \to S^H_F$ is nullhomotopic. Hence we can change f up to homotopy relative $B\pi \setminus D^d$ such that $f(D^d)$ does not meet $0'_j$. Thus we get rid of the points x_+ and x_- and have made the cardinality of $\coprod_{j\in J} I_j$ smaller. This finishes the proof that we can change f up to homotopy such that each I_j is empty, or, equivalently, such that $f^{-1}(0'_F) = \{0^H_F\}$ holds, where $0'_j \in D'_j$ and $0^H_j \in D^H_j$ are the origines.

Since the inclusion $\overline{X'} \to \overline{X'} \cup_{\coprod_{j\in J} S'_F} \coprod_{j\in J} D'_j \setminus \{0_F\}$ admits a retraction relative $\overline{X'}$ and $f(\coprod_{j\in J} S^H_j) = \coprod_{j\in J} S_j$, we can change $f|_{\overline{Y}}\colon \overline{Y} \to \overline{X'} \cup_{\coprod_{j\in J} S^H_j} \coprod_{j\in J} D^H_j \setminus \{0^H_j\}$ relative $\coprod_{j\in J} S^H_j$ to a map $v\colon \overline{Y} \to \overline{X'}$. By construction v extends $\partial v/\pi$. Hence by passing to the universal coverings, we obtain a π-map $v\colon Y \to i^*X'$ extending ∂v. $\square$

Lemma 8.8. *Suppose additionally that $(X, \partial X)$ is a Poincaré slice complement model and $m_F(X, \partial X) = 1$ holds for every $F \in \mathcal{M}$. Let $v_F\colon S_F \to S'_F$ be the F-map uniquely determined up to F-homotopy by the property that it sends $[S_F/F]$ to $m_F(X', \partial X') \cdot [S'_F/F]$. Let $\partial u\colon \partial X \to \partial X'$ be the Γ-map given by the disjoint union of the Γ-maps $\Gamma \times_F v_F$.*

Then there exists a Γ-map $u\colon X \to X'$ extending the Γ-map $\partial u\colon \partial X \to \partial X'$.

Proof. Firstly we apply Lemma 8.7 to $(i^*X, i^*\partial X)$ instead of $(i^*X, i^*\partial X)$. Since $m_F(X', \partial X') = 1$ holds for every $F \in \mathcal{M}$, the map $\partial v_X\colon \partial Y \to i^*\partial X$ appearing in Lemma 8.7 is a π-homotopy equivalence. Moreover, by Lemma 8.7 we get an extension of ∂v_X to a Γ-map $v_X\colon Y \to X$. The same argument as it appears in Lemma 8.3, but for π instead of Γ, shows that $(v_X, \partial v_X)\colon (Y, \partial Y) \to (i^*X, i^*\partial X)$ is a π-homotopy equivalence.

From Lemma 8.7 we obtain an extension of $\partial v\colon \partial Y \to \partial X'$ to a π-map $v\colon Y \to i^*X'$. Since $\partial u \circ \partial v_X$ and ∂v are π-homotopic, the π-map $i^*\partial u\colon \partial X \to \partial X'$ extends to a π-map $u'\colon i^*X \to i^*X'$. Finally we conclude from Lemma 8.4 that the Γ-map $\partial u\colon \partial X \to \partial X'$ extends to Γ-map $u\colon X \to X'$. $\square$

Now we are ready to give the proof of Theorem 8.1.

Proof of Theorem 8.1. The implication(1) $\implies$ (2) follows from the definitions, Lemma 3.4 (1) and the commutative diagram

$$\begin{tikzcd} H^\Gamma_d(X, \partial X; \mathbb{Z}^w) \arrow[r, "\partial"] \arrow[d, "H^\Gamma_d(f,\partial f;\mathbb{Z}^w)"', "\cong"] & H^\Gamma_{d-1}(\partial X; \mathbb{Z}^w) \arrow[d, "\cong"', "H^\Gamma_d(\partial f;\mathbb{Z}^w)"] \\ H^\Gamma_d(X', \partial X'; \mathbb{Z}^w) \arrow[r, "\partial"'] & H^\Gamma_{d-1}(\partial X'; \mathbb{Z}^w) \end{tikzcd}$$

The implication (2) $\implies$ (1) is proved as follows. After possibly changing the orientations of S_F for $F \in \mathcal{M}$ satisfying $|F| = 2$, we can find a Poincaré slice complement model $(Y, \partial Y)$ satisfying $m_F(Y, \partial Y) = 1$ for all $F \in \mathcal{M}$ by Theorem 7.14 and Theorem 7.19. Since we may change the orientations of S'_F for $F \in \mathcal{M}$ satisfying $|F| = 2$, we can asume without loss of generality that $m_F(X, \partial X) = m(X', \partial X')$ hold for every $F \in \mathcal{M}$. From Lemma 8.8 we obtain Γ-maps of Γ-CW-pairs

$$\begin{aligned} (U, \partial u)\colon (Y, \partial Y) &\to (X, \partial X); \\ (U', \partial u')\colon (Y, \partial Y) &\to (X', \partial X'). \end{aligned}$$

Since $m_F(X, \partial X) = m(X', \partial X')$ hold for every $F \in \mathcal{M}$ and there is for every $F \in \mathcal{M}$ an orientation preserving F-homotopy equivalence $S_F \to S'_F$, there is a Γ-homotopy equivalence $\partial f \colon \partial X \to \partial X'$ such that $\partial f \circ \partial u$ is Γ-homotopic to $\partial u'$ and ∂f extends to a Γ-homotopy equivalence $C(\partial X) \to C(\partial X')$. Now define Z and Z' by the Γ-pushouts

$$\begin{tikzcd} \partial Y \arrow[r, "\partial u"] \arrow[d] & \partial X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \partial Y \arrow[r, "\partial u'"] \arrow[d] & \partial X' \arrow[d] \\ Y \arrow[r] & Z' \end{tikzcd}$$

The Γ-maps $(U, \partial u)$ and $(U', \partial u')$ yield Γ-maps $(V, \mathrm{id}_{\partial X}) \colon (Z, \partial X) \to (X, \partial X)$ and $(V', \mathrm{id}_{\partial X'}) \colon (Z', \partial X') \to (X', \partial X')$. Note that $(Z, \partial X)$ and $(Z', \partial X')$ are slice complement models by Lemma 4.5 and the canoncial isomorphisms $H^\pi_d(Y, \partial Y; \mathbb{Z}^w) \xrightarrow{\cong} H^\pi_d(Z, \partial X; \mathbb{Z}^w)$ and $H^\pi_d(Y, \partial Y; \mathbb{Z}^w) \xrightarrow{\cong} H^\pi_d(Z', \partial X'; \mathbb{Z}^w)$ respect the fundamental classes. Lemma 8.3 implies that $(V, \mathrm{id}_{\partial X})$ and $(V', \mathrm{id}_{\partial X'})$ are Γ-homotopy equivalences of Γ-CW-pairs and respect the fundamental classes. Obviously the Γ-homotopy equivalence $\partial f \colon \partial X \to \partial X'$ satisfying $\partial f \circ \partial u \simeq_\Gamma \partial u'$ extends to a Γ-homotopy equivalence of Γ-CW-pairs $(f, \partial f) \colon (X, \partial X) \to (X', \partial X')$ such that $(f, \partial f) \circ (V, \mathrm{id}_{\partial X})$ and $(V', \mathrm{id}_{\partial X})$ are Γ-homotopic. This finishes the proof of Theorem 8.1. □

Theorem 8.9 (Uniqueness of Poincare slice complement models)**.** *Suppose that Assumption 7.1 is satisfied. Let $(X, \partial X)$and $(X', \partial X')$ be two Poincaré slice models with respect to the slice systems $\mathcal{S}$ and $\mathcal{S}'$.Then:*

1. *(1) The slice systems $\mathcal{S}$ and $\mathcal{S}'$ can be oriented in such a way that $m_F(X, \partial X) = m_F(X', \partial X') = 1$ holds for $F \in \mathcal{M}$ and both satisfy condition (S). Moreover, $\mathcal{S}$ and $\mathcal{S}'$ are oriented homotopy equivalent in the sense that for every $F \in \mathcal{M}$ there exists an orientation preserving F-homotopy equivalence $S_F \to S'_F$;*
2. *(2) There exists a Γ-homotopy equivalence $(f, \partial f) \colon (X, \partial X) \to (X', \partial X')$ of free Γ-CW-pairs with the properties that the Γ-map ∂f extends to a Γ-homotopy equivalence of Γ-CW-pairs $C(\partial X) \to C(\partial X')$ and the isomorphism $H^\Gamma_d(X, \partial X; \mathbb{Z}^w) \xrightarrow{\cong} H^\Gamma_d(X', \partial X'; \mathbb{Z}^w)$ induced by $(f, \partial f)$ sends $[X, \partial X]$ to $[X', \partial X']$.*

Proof. (1) This follows from Lemma 3.3 and Lemma 7.10.

(2) This follows from assertion (1) and Theorem 8.1. □

9. Simple homotopy classification of slice complement models

Theorem 9.1 (Simple homotopy classification)**.** *Suppose that Assumption 7.1 is satisfied. Let $(X, \partial X)$ and $(X', \partial X')$ be Poincaré slice complement models with respect to the slice systems $\mathcal{S}$ and $\mathcal{S}'$. Suppose that $\mathcal{S}$ and $\mathcal{S}'$ satisfy condition (S). Assume that the following conditions are satisfied:*

- *The Farrell-Jones Conjecture for K-theory holds for $\mathbb{Z}\Gamma$;*
- *For all $F \in \mathcal{M}$ the 2-Sylow subgroup of F is cyclic;*
- *The Poincaré structures on $(X, \partial X)$ and $(X', \partial X')$ are simple;*
- *For every $F \in \mathcal{F}$ the F-homotopy equivalence $v_F \colon S_F \to S'_F$, which is uniquely determined by the property that it sends $[S_F]$ to $[S'_F]$, is a simple F-homotopy equivalence.*

Then the Γ-homotopy equivalence $(f, \partial f) \colon (X, \partial X) \to (X', \partial X')$ of Theorem 8.9 (2) *is a simple homotopy equivalence of free Γ-CW-pairs, i.e., both ∂f and f are simple Γ-homotopy equivalences.*

Proof. For $M \in \mathcal{M}$, let $i(F)\colon F \to \Gamma$ be the inclusion and $i(F)_*\colon \operatorname{Wh}(F) \to \operatorname{Wh}(\Gamma)$ be the induced homomorphism on the Whitehead groups. The map

$$\bigoplus_{F \in \mathcal{M}} i(F)_*\colon \bigoplus_{F \in \mathcal{M}} \operatorname{Wh}(F) \to \operatorname{Wh}(\Gamma) \tag{9.2}$$

is bijective. This follows by inspecting the proof of [8, Theorem 5.1 (d)], which works also for $\Lambda = \mathbb{Z}$ in the notation used there, or from [9, Theorem 5.1].

We get from the assumptions $\tau(v_F) = 0$ in $\operatorname{Wh}(F)$. Since we have equipped $\mathcal{S}$ and $\mathcal{S}'$ with their canoncial orientations, Lemma 3.4 (1) implies $\partial f = \coprod_{F \in \mathcal{M}} \operatorname{id}_\Gamma \times_F v_F$. Hence we get

$$\tau(\partial f) = \sum_{F \in \mathcal{M}} i(F)_*(\tau(\partial v_F)) = 0. \tag{9.3}$$

Equip $\operatorname{Wh}(F)$ and $\operatorname{Wh}(\Gamma)$ with the involutions coming from the w-twisted involution on $\mathbb{Z}\Gamma$ and the untwisted involution on $\mathbb{Z}F$. Recall that $w|_F = 0$ holds by assumption. Hence the isomorphism (9.2) is compatible with the involutions.

Since $(X, \partial X)$ and $(X', \partial X')$ are simple Poincaré pairs by assumption, we conclude

$$\tau(f) + (-1)^d * (\tau(f)) = \tau(\partial f) = 0 \tag{9.4}$$

from equation (9.3) and the diagram of $\mathbb{Z}\Gamma$-chain homotopy equivalences, which commutes up to $\mathbb{Z}\Gamma$-chain homotopy,

$$\begin{tikzcd}
C^{d-*}(X) \arrow[d, "-\cap[X/\Gamma, \partial X/\Gamma]"'] & C^{d-*}(X') \arrow[l, "C^{d-*}(f)"'] \arrow[d, "-\cap[X'/\Gamma, \partial X'/\Gamma]"] \\
C_d(X, \partial X) \arrow[r, "C_*(f, \partial f)"'] & C_*(X', \partial X').
\end{tikzcd}$$

Since F acts freely on S_F and S_F is homotopy equivalent to the $(d-1)$-dimensional standard sphere, the cohomology of F is periodic. This implies that the p-Sylow subgroup of F is finite cyclic if p is odd, see [5, Proposition 9.5 in Chapter VI on page 157]. Since the 2-Sylow subgroup is cyclic by assumption, $SK_1(\mathbb{Z}F)$ vanishes, see Oliver [29, Theorem 14.2 (i) on page 330]. Hence $\operatorname{Wh}(F)$ is a finitely generated free abelian group and agrees with $\operatorname{Wh}'(F) := \operatorname{Wh}(F)/\operatorname{tors}(\operatorname{Wh}(F))$. The involution on $\operatorname{Wh}'(F)$ and hence also on $\operatorname{Wh}(F)$ is trivial, see [29, Corollary 7.5 on page 182]. Hence the involution on $\operatorname{Wh}(\Gamma)$ is trivial and $\operatorname{Wh}(\Gamma)$ is torsionfree because of the isomorphism (9.2), which is compatible with the involutions. Since d is even, the equation (9.4) boils down to $2 \cdot \tau(f) = 0$. Since $\operatorname{Wh}(\Gamma)$ is torsionfree, we conclude $\tau(f) = 0$. □

10. The case of odd d

Recall that from Section 7 on we have assumed that $d \geq 4$ and d is even. We want to explain the case that d is odd and $d \geq 3$ in this section. Recall from Lemma 3.3 that then each element $F \in \mathcal{M}$ is cyclic of order two and F acts orientation reversing on S_F. Let $H^F_{d-1}(EF; \mathbb{Z}^-)$ be the homology of EF with coefficients in the $\mathbb{Z}F$-module $\mathbb{Z}^-$, whose underlying abelian group is $\mathbb{Z}$ and on which the generator of $F = \mathbb{Z}/2$ acts by $-\operatorname{id}$. Hence instead of Assumption 7.1 we will make in this section the following assumption:

Assumption 10.1.

- *The natural number d is odd and satisfies $d \geq 3$;*
- *The group Γ satisfies conditions (M) and (NM), see Notation 1.8;*

- *There exists a finite Γ-CW-model for $\underline{E}\Gamma$ of dimension d such that its singular Γ-subspace $\underline{E}\Gamma^{>1}$ is $\coprod_{F\in\mathcal{M}} \Gamma/F$. (This condition is discussed and simplified in Theorem 1.12 and Remark 1.14 and implies conditions (M) and (NM), see Remark 1.13.)*
- *There is a finite d-dimensional Poincaré CW-complex model for $B\pi$ with respect to the orientation homomorphisms $v\colon \pi \to \{\pm 1\}$. We have made a choice of a fundamental class $[B\pi] \in H_d^{\pi}(E\pi;\mathbb{Z}^v)$;*
- *The homomorphism $w\colon \Gamma \to \{\pm 1\}$ of Notation 6.7 has the property that $w|_F$ is non-trivial for every $F \in \mathcal{M}$;*
- *The composite*

$$H_d^{\Gamma}(E\Gamma,\partial E\Gamma;\mathbb{Z}^w) \xrightarrow{\partial} H_{d-1}^{\Gamma}(\partial E\Gamma;\mathbb{Z}^w) \xrightarrow{\cong} \bigoplus_{F\in\mathcal{M}} H_{d-1}^F(EF;\mathbb{Z}^-) \to H_{d-1}^F(EF;\mathbb{Z}^-)$$

of the boundary map, the inverse of the obvious isomorphism, and the projection to the summand of $F \in \mathcal{M}$ is surjective for all $F \in \mathcal{M}$.

Besides the simplification that each element $F \in \mathcal{M}$ is cyclic of order two, it is also convenient that we have to consider only one slice system as explained next. Recall that S_F is homotopy equivalent to S^{d-1}. The antipodal action of $F = \mathbb{Z}/2$ on S^{d-1} is free and reverses the orientation. Equivariant obstruction theory implies that there is an F-homotopy equivalence $S_F \xrightarrow{\simeq_F} S^{d-1}$, see [32, Theorem 4.11 on page 126] or [21, Theorem 3.5 on page 139]. Moreover, any F-selfhomotopy equivalence $S^{d-1} \xrightarrow{\simeq_F} S^{d-1}$ is homotopic to an F-homeomorphisms, namely to the identity or the antipodal selfmap. This implies that we only have to consider only one slice system $\mathcal{S}^{\mathrm{st}} = \{S_F^{\mathrm{st}} \mid F \in \mathcal{M}\}$, namely the one, where each S_F^{st} is the standard $(d-1)$-dimensional sphere S^{d-1} with the antipodal $F = \mathbb{Z}/2$-action. An orientation for it is a choice of fundamental class $[S_F^{\mathrm{st}}]$ in the infinite cyclic group $H_{d-1}^F(S_F^{\mathrm{st}};\mathbb{Z}^-)$, which corresponds to a choice of fundamental class $[S_F^{\mathrm{st}}]$ in the infinite cyclic group $H_{d-1}(S_F^{\mathrm{st}})$, since the canonical map $H_{d-1}(S_F^{\mathrm{st}}) \to H_{d-1}^F(S_F^{\mathrm{st}};\mathbb{Z}^-)$ is an inclusion of infinite cyclic groups with index 2. The third simplification is that $\mathrm{Wh}_n(F)$ vanishes for $n \le 1$. Hence $\mathrm{Wh}_n(\Gamma)$ vanish for $n \le 1$, see [8, Theorem 5.1 (d)] or [9, Theorem 5.1]. This is interesting in view of Remark 1.14.

The proofs of Theorem 7.12 and Theorem 8.9 carry directly over (and actually simplify) to the following theorems; one has to replace $H_{d-1}(BF)$ and $H_{d-1}(S_F/F)$ by $H_{d-1}^F(EF;\mathbb{Z}^-)$ and $H_{d-1}^F(S_F,\mathbb{Z}^-)$ everywhere.

Theorem 10.2 (Existence of Poincaré slice complement models in the odd dimensional case)**.** *Suppose that Assumption 10.1 is satisfied. Then there exists a Poincaré slice complement model $(X,\partial X)$ with respect to $\mathcal{S}^{\mathrm{st}}$.*

Theorem 10.3 (Homotopy classification of Poincaré slice complement models in the odd dimensional case)**.** *Suppose that Assumption 10.1 is satisfied. Let $(X,\partial X)$ and $(X',\partial X')$ respectively be Poincaré slice complement models for $\underline{E}\Gamma$ with respect to $\mathcal{S}^{\mathrm{st}}$.*

Then there exists a simple Γ-homotopy equivalence $(f,\partial f)\colon (X,\partial X) \to (X',\partial X')$ of free Γ-CW-pairs such that ∂f is the identity on $\coprod_{F\in\mathcal{M}} \Gamma \times_F S_F$ and the isomorphism $H_d^{\Gamma}(X,\partial X;\mathbb{Z}^w) \xrightarrow{\cong} H_d^{\Gamma}(X',\partial X';\mathbb{Z}^w)$ induced by $(f,\partial f)$ sends $[X,\partial X]$ to $[X',\partial X']$.

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