The Farrell-Jones Conjecture for the Hecke algebras of reductive p-adic groups

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Dictionary

td-groups	discrete groups
Smooth G-representations over the ring R	G-representations over the ring R
Hecke algebra $\mathcal{H}(G;R)$	group ring RG
$\exists$ approximate unit	$\exists$ unit
{smooth G-representations} = {n.d. $\mathcal{H}(G;R)$-modules}	$\{G$-representations$\}$ = $\{RG$-modules$\}$

reductive p-adic groups	CAT(0)-groups
Examples: $\mathrm{GL}_n(\mathbb{Q}_p)$, $\mathrm{SL}_n(\mathbb{Q}_p)$	Examples: Fundamental groups of closed manifolds with non-sectional curvature
Cocompact proper smooth action on the associated Bruhat-Tits building	Cocompact smooth proper action on a CAT(0)-space
Family $\mathcal{COM}$ of compact open subgroups	Family $\mathcal{FIN}$ of finite subgroups

A sequence of subgroups $K \supseteq K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ of a compact td-group does in general *not* stabilize after finitely many steps.	A sequence of subgroups $F \supseteq F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ of a finite group F stabilizes after finitely many steps.
The space $G/H \times G/K$ with the diagonal G-action is in general *not* G-homotopy equivalent to a G-CW-complex.	The space $G/H \times G/K$ with the diagonal G-action is a zero-dimensional G-CW-complex.
The classifying spaces $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$ are *not* G-homotopy equivalent in general. Fortunately $E_{\mathcal{COM}}(G)$ and $J_{\mathcal{COM}}(G)$ are G-homotopy equivalent.	The classifying spaces $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$ are G-homotopy equivalent.

The Hecke algebra is only functorial under open group homomorphisms. In particular it is not functorial under inclusions of subgroups $H \subseteq G$, unless H is open in G.	The group ring RG is functorial under any group homomorphism.
$\exists$ unit $\Leftrightarrow$ G discrete	$\exists$ unit
Reductive p-adic groups contain interesting closed but not open subgroups such as the Borel subgroup	not applicable

Prominent Conjectures about group rings

Conjecture (Idempotent Conjecture)

The Idempotent Conjecture says that for a torsionfree group G and a field F the elements 0 and 1 are the only idempotents in FG.

Conjecture (Unit Conjecture)

The Unit Conjecture says that for a torsionfree group G and an integral domain R every unit in RG is trivial, i.e., of the form $r \cdot g$ for $r \in R^{\times}$ and $g \in G$.

Conjecture (Stable Unit Conjecture)

The Stable Unit Conjecture says that for a torsionfree group G and an integral domain R we can find for every unit u in RG a trivial unit v in RG such that one can pass from the invertible $(1,1)$ matrix (u) to the invertible $(1,1)$ matrix (v) by elementary row and column operation and taking the block sum $A \mapsto A \oplus (1)$ or the inverse operation.

Definition (Projective class group $K_0(R)$)

Define the projective class group of a ring R

$$K_0(R)$$

to be the following abelian group:

- Generators are isomorphism classes $[P]$ of finitely generated projective R-modules P;
- The relations are $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

Definition (Reduced Projective class group $\widetilde{K}_0(R)$)

The reduced projective class group

$$\widetilde{K}_0(R) = \text{cok}\big(K_0(\mathbb{Z}) \to K_0(R)\big)$$

is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R-modules.

- Let P be a finitely generated projective R-module. It is stably free, i.e., $P \oplus R^m \cong R^n$ for some $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$.

Conjecture (Vanishing of reduced projective class group for torsionfree G)

If G is torsionfree, then $\widetilde{K}_0(\mathbb{Z}G)$ and $\widetilde{K}_0(FG)$ for a field F of characteristic zero vanish.

- The last conjecture implies the Idempotent Conjecture.

- There is the so called Whitehead group $\mathrm{Wh}(G)$ which is the quotient of $K_1(\mathbb{Z}G)$ by the subgroup given by the trivial units $\pm g$.
- It is the obstruction group for the obstruction of a homotopy equivalence of between finite CW-complexes to be a simple homotopy equivalence and for an h-cobordism over a closed manifold to be trivial.
- Therefore the following conjecture plays a prominent role for the classification of closed manifolds because of the so called s-Cobordism Theorem

Conjecture (Vanishing of $\mathrm{Wh}(G)$ for torsionfree G)

If G is torsionfree, then

$$\mathrm{Wh}(G) = \{0\}.$$

Motivation and Statement of the Farrell-Jones Conjecture for torsionfree groups

- For some time G is a discrete group.
- There are K-groups $K_n(R)$ for every $n \in \mathbb{Z}$.
- Can one identify $K_n(RG)$ with more accessible terms?
- If G_0 and G_1 are torsionfree and R is regular, one gets isomorphisms

$$\begin{aligned} K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R); \\ \widetilde{K}_n(R[G_0 * G_1]) &\cong \widetilde{K}_n(RG_0) \oplus \widetilde{K}_n(RG_1). \end{aligned}$$

- If $\mathcal{H}$ is any (generalized) homology theory, then

$$\begin{aligned} \mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\mathrm{pt}) \oplus \mathcal{H}_{n-1}(\mathrm{pt}); \\ \widetilde{\mathcal{H}}_n(B(G_0 * G_1)) &\cong \widetilde{\mathcal{H}}_n(BG_0) \oplus \widetilde{\mathcal{H}}_n(BG_1). \end{aligned}$$

- Question: Can we find $\mathcal{H}_*$ with $\mathcal{H}_n(BG) \cong K_n(RG)$, provided that G is torsionfree and R is regular.
- Of course such $\mathcal{H}_*$ has to satisfy $\mathcal{H}_n(\mathrm{pt}) = K_n(R)$.
- So the only reasonable candidate is $H_n(-;\mathbf{K}_R)$.

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG;\mathbf{K}_R) \to K_n(RG)$$

is bijective for every $n \in \mathbb{Z}$.

- There is also an *L*-theory version.

Applications of the Farrell-Jones Conjecture

- The conjectures above about the vanishing of $\widetilde{K}_0(\mathbb{Z}G)$, $\widetilde{K}_0(FG)$ and $\mathrm{Wh}(G)$ for torsionfree G are consequences of the the Farrell-Jones Conjecture above.

- This follows from the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG; \mathbf{K}_{\mathbb{Z}})$. It is a first quadrant spectral sequence with E^2-term

$$\begin{array}{ccc}
K_2(\mathbb{Z}) \cong \mathbb{Z}/2 & H_1(BG; K_2(\mathbb{Z})) & H_2(BG, K_1(\mathbb{Z}/2) \\
 & & \\
K_1(\mathbb{Z}) \cong \{\pm 1\} & H_1(BG; K_1(\mathbb{Z})) & H_2(BG, K_1(\mathbb{Z}/2) \\
 & & \\
K_0(\mathbb{Z}) \cong \mathbb{Z} & H_1(BG; \mathbb{Z}) & H_2(BG, K_1(\mathbb{Z})
\end{array}$$

- The Farrell-Jones Conjecture implies the Stable Unit Conjecture.
- Gardam has recently shown for a group G which contains $\mathbb{Z}^3$ as subgroup of finite index that the Unit Conjecture is false.
- For this group the Farrell-Jones Conjecture and hence the Stable Unit Conjecture are true.

Definition (Topologically rigid)

A closed topological manifold N is called topologically rigid if any homotopy equivalence $f\colon M \to N$ with a closed manifold M as source is homotopic to a homeomorphism.

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts that an aspherical closed manifold with fundamental group G is topologically rigid.

- In particular the Borel Conjecture predicts that two aspherical closed manifolds are homeomorphic if and only if their fundamental groups are isomorphic.
- The Farrell-Jones Conjecture implies the Borel Conjecture in dimensions ≥ 5.

There are many other applications of the Farrell-Jones Conjecture, for instance:

- Novikov Conjecture.
- Bass Conjecture.
- Moody's Induction Conjecture.
- Serre's Conjecture.
- Classification of manifolds.
- Poincaré duality groups.
- κ-classes for aspherical manifolds.
- Hyperbolic groups with a sphere as boundary.
- Rational calculation of the homotopy groups of the space of automorphism of aspherical closed manifolds in a certain range.
- Stable Cannon Conjecture.

- One can formulate a version of the Farrell-Jones Conjecture which makes sense for all groups G and all rings R.

Conjecture (K-theoretic Farrell-Jones-Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG).$$

is bijective for every $n \in \mathbb{Z}$.

- $H_*^G(-;\mathbf{K}_R)$ is a G-homology theory satisfying

$$H_n^G(G/H;\mathbf{K}_R) \cong K_n(RH).$$

- The Farrell-Jones Conjecture is equivalent to the homotopy theoretic version that we have a weak homotopy equivalence

$$\operatorname{hocolim}_{G/V \in \mathrm{Or}_{\mathcal{VC}\mathrm{yc}}(G)} K_R(G/V) \xrightarrow{\simeq} \mathbf{K}_R(G/G).$$

- There is also an L-theory version.

- One can also allow twisted group rings and orientation characters.

- In the sequel the Full Farrell-Jones Conjecture refers to the most general version for both K-theory and L-theory, namely, with coefficients in additive G-categories (with involution) and finite wreath products.

Status of the Full Farrell-Jones Conjecture

Theorem (Bartels, Bestvina, Farrell, Kammeyer, Lück, Reich, Rüping, Wegner)

Let $\mathcal{FJ}$ be the class of groups for which the Full Farrell-Jones Conjecture holds. Then $\mathcal{FJ}$ contains the following groups:

- *Hyperbolic groups;*
- $\operatorname{CAT}(0)$*-groups;*
- *Solvable groups;*
- *(Not necessarily uniform) lattices in almost connected Lie groups;*
- *Fundamental groups of (not necessarily compact) d-dimensional manifolds (possibly with boundary) for $d \leq 3$;*
- *Subgroups of $GL_n(\mathbb{Q})$ and of $GL_n(F[t])$ for a finite field F;*
- *All S-arithmetic groups;*
- *mapping class groups.*

Theorem (continued)

Moreover, $\mathcal{FJ}$ has the following inheritance properties:

- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *If $H \subseteq G$ is a subgroup of G with $[G : H] < \infty$ and $H \in \mathcal{FJ}$, then $G \in \mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*

- Many more mathematicians have made important contributions to the Farrell-Jones Conjecture, e.g., Bökstedt, Carlsson, Farrell, Hsiang, Jones, Kasprowski, Linnell, Madsen, Pedersen, Quinn, Ranicki, Reich, Rognes, Tessera, Varisco, Weinberger, Yu, Wu.

- The Farrell-Jones Conjecture is open for:
  - Out(F_n);
  - amenable groups;
  - Thompson's groups;
- It would be great to find a counterexample.

Towards the Farrell-Jones Conjecture for reductive *p*-adic groups

Conjecture (Farrell-Jones Conjecture for fields of characteristic zero)

Let F be a field of characteristics zero. Then the assembly map

$$H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_F) \xrightarrow{\cong} K_n(FG)$$

is an isomorphism for $n \in \mathbb{Z}$.

It implies:

Conjecture (Moody's Induction Conjecture)

For every field F of characteristic zero we get an isomorphism

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{FIN}}(G)} K_0(FH) \xrightarrow{\cong} K_0(FG).$$

Theorem (Bernstein, Dat)

Let G be a reductive p-adic group. Then the canoncial map

$$\operatorname*{colim}_{G/K \in \mathrm{Or}_{\mathcal{COM}}(G)} K_0(\mathcal{H}(K;\mathbb{C})) \xrightarrow{\cong} K_0(\mathcal{H}(G;\mathbb{C}))$$

is rationally an isomorphism

- Often a smooth G-representation regarded as a module over $\mathcal{H}(G;\mathbb{C})$ has a finite projective resolution and hence defines an element in $K_0(\mathcal{H}(G;\mathbb{C}))$.
- There is the conjecture, often attributed to Bernstein that every irreducible super-cuspidal representation of a reductive p-adic group is (compactly) induced from some compact open subgroup.

Theorem (Bartels-Lück)

Let G be a closed subgroup of a reductive p-adic group. Let R be a regular ring satisfying $\mathbb{Q} \subseteq R$. Then the canoncial map

$$\operatorname*{colim}_{G/K \in \mathrm{Or}_{\mathcal{COM}}(G)} K_0(\mathcal{H}(K;R)) \xrightarrow{\cong} K_0(\mathcal{H}(G;R))$$

is an isomorphism.

Conjecture (Farrell-Jones Conjecture for td-groups)

Let G be a td-group.
Then for any regular ring R satisfying $\mathbb{Q} \subseteq R$ and $n \in \mathbb{Z}$ the assembly map

$$H_n^G(E_{\mathcal{COM}}(G); \mathbf{K}_R) \xrightarrow{\cong} H_n^G(G/G; \mathbf{K}_R) = K_n(\mathcal{H}(G; R))$$

is an isomorphism.

- $H_*^G(-; \mathbf{K}_R)$ is a smooth G-homology theory satisfying $H_n^G(G/H); \mathbf{K}_R)) \cong K_n(\mathcal{H}(H; R))$ for every open subgroup $H \subseteq G$.

Theorem (Bartels-Lück)

The Farrell-Jones Conjecture holds for G if G is a closed subgroup of a reductive p-adic group.

Strategy of the proof

- The assembly map can be thought of an approximation of the algebraic K- or L-theory by a homology theory.
- The basic feature between the left and right side of the assembly map is that on the left side one has excision which is not present on the right side.
- In general excision is available if one can make representing cycles small.
- A best illustration for this is the proof of excision for simplicial or singular homology based on subdivision whose effect is to make the support of cycles arbitrary small.

- Then the basic goal of the proof is obvious: Find a procedure to make the support of a representing cocycle as small as possible without changing its class.

- Suppose that $G = \pi_1(M)$ for a closed Riemannian manifold with negative sectional curvature.

- The idea is to use the geodesic flow on the universal covering to gain the necessary control.

- We will briefly explain this in the case, where the universal covering is the two-dimensional hyperbolic space $\mathbb{H}^2$.

- Consider two points with coordinates (x_1, y_1) and (x_2, y_2) in the upper half plane model of two-dimensional hyperbolic space. We want to use the geodesic flow to make their distance smaller in a functorial fashion. This is achieved by letting these points flow towards the boundary at infinity along the geodesic given by the vertical line through these points, i.e., towards infinity in the y-direction.

- There is a fundamental problem: if $x_1 = x_2$, then the distance between these points is unchanged. Therefore we make the following prearrangement. Suppose that $y_1 < y_2$. Then we first let the point (x_1, y_1) flow so that it reaches a position where $y_1 = y_2$. Inspecting the hyperbolic metric, one sees that the distance between the two points (x_1, τ) and (x_2, τ) goes to zero if τ goes to infinity. This is the basic idea the negatively curved case to make the cycles small, or in other words, to gain control.

Thank you for your attention!