Presentation of two topics for Bachelor thesis

Wolfgang Lück
Bonn
Germany
email wolfgang.lueck@him.uni-bonn.de

slides available on
http://www.him.uni-bonn.de/lueck/

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The simplicial volume

- Define the L^1-seminorm of an element y in the p-th singular homology $H_p^{\text{sing}}(X;\mathbb{R}) := H_p(C_*^{\text{sing}}(X;\mathbb{R}))$ by

$$||y||_1 \quad := \quad \inf\{||x||_1 \mid x \in C_p^{\text{sing}}(X;\mathbb{R}), \partial_p(x) = 0, y = [x]\}.$$

Definition (Simplicial volume)

Let M be a closed connected orientable manifold of dimension n. Define its simplicial volume to be the non-negative real number

$$||M|| := ||j([M])||_1 \qquad \in [0,\infty)$$

for any choice of fundamental class $[M] \in H_n^{\text{sing}}(M;\mathbb{Z})$ and $j\colon H_n^{\text{sing}}(M;\mathbb{Z}) \to H_n^{\text{sing}}(M;\mathbb{R})$ the change of coefficients map associated to the inclusion $\mathbb{Z} \to \mathbb{R}$.

Lemma

Let $f\colon M \to N$ be a map of closed connected oriented manifolds of the same dimension n. Let $\deg(f)$ be the degree of f. Then

$$||M|| \ge |\deg(f)| \cdot ||N||.$$

In particular the simplicial volume is a homotopy invariant.

Proof.

For any $x \in C_n^{\text{sing}}(X;\mathbb{R})$ we get $||C_n^{\text{sing}}(f)(x)||_1 = ||x||_1$. □

Corollary

If $f\colon M \to M$ is a selfmap of a closed connected orientable manifold of degree different from -1, 0 and 1, then $||M|| = 0$. In particular we get $||S^1 \times M|| = 0$.

Lemma

Let $p\colon M \to N$ be a d-sheeted covering of closed connected orientable manifolds. Then

$$||M|| = d \cdot ||N||.$$

Theorem

Let M be a connected closed orientable manifold. If M carries a non-trivial S^1-action, then

$$||M|| = 0.$$

Theorem

Let M be a closed connected orientable manifold of dimension ≥ 1 with amenable fundamental group. Then

$$||M|| = 0.$$

Theorem (Simplicial volume of hyperbolic manifolds)

Let M be a closed hyperbolic orientable manifold of dimension n. Then

$$||M|| = \frac{\mathrm{vol}(M)}{v_n}.$$

for a specific dimension constant $v_n > 0$.

Theorem (Betti numbers and simplicial volume)

Let M be a complete connected orientable Riemannian manifold of dimension n with finite volume. Let $k_1 \geq k_2 > 0$ be positive constants such that the sectional curvature satisfies $-k_1 \leq \sec(M) \leq -k_2$. Then there is a constant $C(n, k_1/k_2)$, which depends only on n and the ratio k_1/k_2 but not on M, such that

$$\sum_{i \geq 0} b_p(M) \;\leq\; C(n, k_1/k_2) \cdot ||M||.$$

- Literature: [2] and [4, Section 14.1].

The group cohomology of certain crystallographic groups

- Give a basic introduction to group (co)homology.
- This includes projective resolutions, classifying spaces of groups, Tate cohomology, transfer, Hochschild-Serre spectral sequence, Tate cohomology
- Literature: Brown [1].

- Let $G \cong \mathbb{Z}/m$ be a finite cyclic group of order m and let $L \cong \mathbb{Z}^n$ be a finitely generated free abelian group of rank n. Let $\rho\colon G \to \mathrm{aut}_{\mathbb{Z}}(L)$ be a group homomorphism. Let Γ be the associated semi-direct product $L \rtimes_\rho G$.

Conjecture (Adem-Ge-Pan-Petrosyan)

The Lyndon-Hochschild-Serre spectral sequence associated to the semi-direct product $L \rtimes_\rho G$ collapses in the strongest sense, i.e., all differentials in the E_r-term for $r \geq 2$ are trivial and all extension problems at the E_∞-level are trivial. In particular we get for all $k \geq 0$

$$H^k(\Gamma;\mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G;H^j(L)).$$

Theorem (Langer-Lück [3])

Conjecture of Adem-Ge-Pan-Petrosyan is true, provided that the G-action on L is free outside the origin.

Theorem (Langer-Lück [3])

Consider the special case $n = 6$ and $m = 4$, where ρ is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Then the second differential in the Lyndon-Hochschild-Serre spectral sequence associated to the semi-direct product $L \rtimes_\rho G$ is non-trivial.

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