Introduction to the Farrell-Jones Conjecture

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$K_0(R)$ and the Idempotent Conjecture

- Given a ring R and a group G, denote by RG or $R[G]$ the group ring.
- An RG-module is the same as G-representation with coefficients in R, i.e., an R-module with G-action by R-linear maps.
- If $\overline{X} \to X$ is a G-covering of a CW-complex X, then the cellular chain complex of $\overline{X}$ is a free $\mathbb{Z}G$-chain complex.

- If g has finite order $|g|$ and F is a field of characteristic zero, then we get an idempotent in FG by

$$x = \frac{1}{|g|} \cdot \sum_{i=0}^{|g|-1} g^i.$$

- Are there other idempotents?

Conjecture (Idempotent Conjecture)

The Kaplansky Conjecture says that for a torsionfree group G and a field F of characteristic zero the elements 0 and 1 are the only idempotents in FG.

Definition (Projective class group $K_0(R)$)

Define the projective class group of a ring R

$$K_0(R)$$

to be the following abelian group:

- Generators are isomorphism classes $[P]$ of finitely generated projective R-modules P;
- The relations are $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \to P_0 \to P_1 \to P_2 \to 0$ of finitely generated projective R-modules.

Definition (Reduced Projective class group $\widetilde{K}_0(R)$)

The reduced projective class group

$$\widetilde{K}_0(R) = \operatorname{cok}\bigl(K_0(\mathbb{Z}) \to K_0(R)\bigr)$$

is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R-modules.

- Let P be a finitely generated projective R-module. It is stably free, i.e., $P \oplus R^m \cong R^n$ for some $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\widetilde{K}_0(R)$.

Conjecture (Vanishing of reduced projective class group for torsionfree G)

If G is torsionfree, then $\widetilde{K}_0(\mathbb{Z}G)$ and $\widetilde{K}_0(FG)$ for a field F of characteristic zero vanish.

- The last conjecture implies the Idempotent Conjecture.

Definition (K_1-group $K_1(R)$)

Define the K_1-group of a ring R

$$K_1(R)$$

to be the abelian group whose generators are conjugacy classes $[f]$ of automorphisms $f \colon P \to P$ of finitely generated projective R-modules with the following relations:

- Given an exact sequence $0 \to (P_0, f_0) \to (P_1, f_1) \to (P_2, f_2) \to 0$ of automorphisms of finitely generated projective R-modules, we get $[f_0] + [f_2] = [f_1]$;
- $[g \circ f] = [f] + [g]$.

- Put $GL(R) := \bigcup_{n \geq 1} GL_n(R)$. The obvious maps $GL_n(R) \to K_1(R)$ induce an isomorphism

$$GL(R)/[GL(R), GL(R)] \xrightarrow{\cong} K_1(R).$$

- An invertible matrix $A \in GL(R)$ can be reduced by elementary row and column operations and (de-)stabilization to the trivial empty matrix if and only if $[A] = 0$ holds in the reduced K_1-group

$$\widetilde{K}_1(R) := K_1(R)/\{\pm 1\} = \operatorname{cok}\left(K_1(\mathbb{Z}) \to K_1(R)\right).$$

- The assignment $A \mapsto [A] \in K_1(R)$ can be thought of as the universal determinant for R.

Definition (Whitehead group)

The Whitehead group of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

Theorem (s-Cobordism Theorem, Barden, Mazur, Stallings, Kirby-Siebenmann)

Let M be a closed smooth or topological manifold of dimension ≥ 5. Then the so called Whitehead torsion yields a bijection

$$\tau \colon \mathcal{H}(M) \xrightarrow{\cong} \mathrm{Wh}(\pi_1(M))$$

where $\mathcal{H}(M)$ is the set of h-cobordisms over M modulo diffeomorphisms or homeomorphisms relative M.

Conjecture (Vanishing of $\mathrm{Wh}(G)$ for torsionfree G)

If G is torsionfree, then

$$\mathrm{Wh}(G) = \{0\}.$$

Lemma

Let G be finitely presented and $d \geq 5$ be any natural number. Then the following statements are equivalent:

- *The Whitehead group $\mathrm{Wh}(G)$ vanishes;*
- *For one closed manifold M of dimension d with $G \cong \pi_1(M)$ every h-cobordism over M is trivial;*
- *For every closed manifold M of dimension d with $G \cong \pi_1(M)$ every h-cobordism over M is trivial.*

Conjecture (Unit Conjecture)

The Unit Conjecture says that for a torsionfree group G and an integral domain R every unit in RG is trivial, i.e., of the form $r \cdot g$ for $r \in R^\times$ and $g \in G$.

Conjecture (Stable Unit Conjecture)

The Stable Unit Conjecture says that for a torsionfree group G and an integral domain R we can find for every unit u in RG a trivial unit v in RG such that one can pass from the invertible $(1, 1)$ matrix (u) to the invertible $(1, 1)$ matrix (v) by elementary row and column operation and taking the block sum $A \mapsto A \oplus (1)$ or the inverse operation.

- The Farrell-Jones Conjecture implies the Stable Unit Conjecture.
- Gardam has recently shown for a group G which contains $\mathbb{Z}^3$ as subgroup of finite index that the Unit Conjecture is false.
- For this group the Farrell-Jones Conjecture and hence the Stable Unit Conjecture are true.

Motivation and Statement of the Farrell-Jones Conjecture for torsionfree groups

- There are K-groups $K_n(R)$ for every $n \in \mathbb{Z}$.
- Can one identify $K_n(RG)$ with more accessible terms?
- If G_0 and G_1 are torsionfree and R is regular, one gets isomorphisms

$$\begin{aligned} K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R); \\ \widetilde{K}_n(R[G_0 * G_1]) &\cong \widetilde{K}_n(RG_0) \oplus \widetilde{K}_n(RG_1). \end{aligned}$$

- If $\mathcal{H}$ is any (generalized) homology theory, then

$$\begin{aligned} \mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\mathrm{pt}) \oplus \mathcal{H}_{n-1}(\mathrm{pt}); \\ \widetilde{\mathcal{H}}_n(B(G_0 * G_1)) &\cong \widetilde{\mathcal{H}}_n(BG_0) \oplus \widetilde{\mathcal{H}}_n(BG_1). \end{aligned}$$

- Question: Can we find $\mathcal{H}_*$ with $\mathcal{H}_n(BG) \cong K_n(RG)$, provided that G is torsionfree and R is regular.
- Of course such $\mathcal{H}_*$ has to satisfy $\mathcal{H}_n(\mathrm{pt}) = K_n(R)$.
- So the only reasonable candidate is $H_n(-;\mathbf{K}_R)$.

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG;\mathbf{K}_R) \to K_n(RG)$$

is bijective for every $n \in \mathbb{Z}$.

- There is also an *L*-theory version.

Applications of the Farrell-Jones Conjecture

- The conjectures above about the vanishing of $\widetilde{K}_0(\mathbb{Z}G)$ and $\mathrm{Wh}(G)$ for torsionfree G do follow from the Farrell-Jones Conjecture above.

- The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG;\mathbf{K}_R)$ whose E^2-term is given by

$$E^2_{p,q} = H_p(BG, K_q(R)),$$

using

$$K_n(\mathbb{Z}) = \begin{cases} \{0\} & n \leq -1; \\ \mathbb{Z} & n = 0; \\ \{\pm 1\} & n = 1. \end{cases}$$

Definition (Topologically rigid)

A closed topological manifold N is called topologically rigid if any homotopy equivalence $f\colon M \to N$ with a closed manifold M as source is homotopic to a homeomorphism.

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts that an aspherical closed manifold with fundamental group G is topologically rigid.

- In particular the Borel Conjecture predicts that two aspherical closed manifolds are homeomorphic if and only if their fundamental groups are isomorphic.
- The Poincaré Conjecture is equivalent to the statement that S^n is topologically rigid.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity.

  A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds of dimension ≥ 3 is homotopic to an isometric diffeomorphism.

- The Borel Conjecture is not true in the smooth category by results of Farrell-Jones.

- The Borel Conjecture follows in dimension ≥ 5 from the Farrell-Jones Conjecture.

- The Borel Conjecture follows in dimension 3 from the Perelman's proof of the Thurston Geometrization Conjecture

Theorem (Bartels-Lück-Weinberger)

Let G be a torsionfree hyperbolic group and let n be an integer ≥ 6.

Then the following statements are equivalent:

- *The boundary ∂G is homeomorphic to S^{n-1};*
- *There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.*

The manifold above is unique up to homeomorphism.

Theorem (Homotopy groups of automorphism groups of aspherical manifolds)

Let M be an orientable closed aspherical (smooth) manifold of dimension > 10 with fundamental group G. Suppose that G satisfies the K-and the L-theoretic Farrell Jones Conjecture.

Then for $1 \le i \le (\dim M - 7)/3$ one has

$$\pi_i(\mathrm{Top}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathrm{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ 0 & \text{if } i > 1, \end{cases}$$

and

$$\pi_i(\mathrm{Diff}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathrm{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ \bigoplus_{j=1}^{\infty} H_{(i+1)-4j}(M;\mathbb{Q}) & \text{if } i > 1,\ \dim M \text{ odd}; \\ 0 & \text{if } i > 1,\ \dim M \text{ even}. \end{cases}$$

Conjecture (Cannon's Conjecture in the torsionfree case)

A torsionfree hyperbolic group G has S^2 as boundary if and only if it is the fundamental group of a closed hyperbolic 3-manifold.

Theorem (Ferry-Lück-Weinberger, Stable Cannon Conjecture)

Let G be a hyperbolic 3-dimensional Poincaré duality group. Let N be any smooth, PL or topological manifold respectively which is closed and whose dimension is ≥ 2.

Then there is a closed smooth, PL or topological manifold M and a normal map of degree one

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}^a} & \xrightarrow{\underline{f}} & \xi \times TN \\ \downarrow & & \downarrow \\ M & \xrightarrow[\simeq_s]{f} & BG \times N \end{array}$$

such that the map f is a simple homotopy equivalence.

Theorem (Stable Cannon Conjecture, continued)

Moreover:

Let $\widehat{M} \to M$ *be the G-covering associated to the composite of the isomorphism* $\pi_1(f) \colon \pi_1(M) \xrightarrow{\cong} G \times \pi_1(N)$ *with the projection* $G \times \pi_1(N) \to G$*. Suppose additionally that N is aspherical and* $\dim(N) \geq 3$*.*

Then $\widehat{M}$ *is homeomorphic to* $\mathbb{R}^3 \times N$*.*

There are many other applications of the Farrell-Jones Conjecture, for instance:

- Novikov Conjecture.
- Bass Conjecture.
- Moody's Induction Conjecture.
- Serre's Conjecture.
- Classification of certain classes of manifolds with infinite fundamental group.
- Classification of Poincaré duality groups.
- κ-classes for aspherical manifolds.

The general version the Farrell-Jones Conjecture

- One can formulate a version of the Farrell-Jones Conjecture which makes sense for all groups G and all rings R.

Conjecture (K-theoretic Farrell-Jones-Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG).$$

is bijective for every $n \in \mathbb{Z}$.

- There is also an L-theory version.
- One can also allow twisted group rings and orientation characters.
- In the sequel the Full Farrell-Jones Conjecture refers to the most general version for both K-theory and L-theory, namely, with coefficients in additive G-categories (with involution) and finite wreath products.
- All conjectures or results mentioned in this talk follow from the Full Farrell-Jones Conjecture.

Status of the Full Farrell-Jones Conjecture

Theorem (Bartels, Bestvina, Farrell, Kammeyer, Lück, Reich, Rüping, Wegner)

Let $\mathcal{FJ}$ be the class of groups for which the Full Farrell-Jones Conjecture holds. Then $\mathcal{FJ}$ contains the following groups:

- *Hyperbolic groups;*
- $\operatorname{CAT}(0)$*-groups;*
- *Solvable groups;*
- *(Not necessarily uniform) lattices in almost connected Lie groups;*
- *Fundamental groups of (not necessarily compact) d-dimensional manifolds (possibly with boundary) for $d \leq 3$;*
- *Subgroups of $GL_n(\mathbb{Q})$ and of $GL_n(F[t])$ for a finite field F;*
- *All S-arithmetic groups;*
- *mapping class groups.*

Theorem (continued)

Moreover, $\mathcal{FJ}$ has the following inheritance properties:

- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *If $H \subseteq G$ is a subgroup of G with $[G : H] < \infty$ and $H \in \mathcal{FJ}$, then $G \in \mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*
- *Let $1 \to K \to G \to Q \to 1$ be an extension of groups such that Q belongs to $\mathcal{FJ}$ and H is virtually torsionfree hyperbolic. Then G belongs to $\mathcal{FJ}$.*

- Many more mathematicians have made important contributions to the Farrell-Jones Conjecture, e.g., Bökstedt, Carlsson, Davis, Jones, Ferry, Hambleton, Gandini, Hsiang, Jones, Kasprowski, Linnell, Madsen, Nicas, Pedersen, Quinn, Ranicki, Rognes, Roushon, Rosenthal, Stark, Tessera, Varisco, Weinberger, Yu, Wu.

The Farrell-Jones Conjecture is open for:

- $\mathrm{Out}(F_n)$;
- amenable groups;
- Thompson's groups;

- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.
- One example is the construction of groups with expanders due to Gromov, see Arzhantseva-Delzant. These yield counterexamples to the Baum-Connes Conjecture with coefficients due to Higson-Lafforgue-Skandalis.
- However, our results show that these groups do satisfy the Full Farrell-Jones Conjecture and hence also the other conjectures mentioned above.
- Many groups of the region 'Hic abundant leones' in the universe of groups in the sense of Bridson do satisfy the Full Farrell-Jones Conjecture.
- We have no good candidate for a group (or for a property of groups) for which the Farrell-Jones Conjecture may fail.

- Davis-Januszkiewicz have constructed exotic aspherical closed manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.
- However, in all cases the universal coverings are $\mathrm{CAT}(0)$-spaces and the fundamental groups are $\mathrm{CAT}(0)$-groups. Hence they satisfy the Full Farrell-Jones Conjecture and in particular the Borel Conjecture in dimension ≥ 5.

The K-theoretic Farrell-Jones Conjecture for Hecke algebras of reductive p-adic groups

Conjecture (Farrell-Jones Conjecture for td-groups)

For a td-groups G and any regular ring R satisfying $\mathbb{Q} \subseteq R$ and $n \in \mathbb{Z}$ the assembly map

$$H_n^G(E_{\mathcal{COM}}(G); \mathbf{K}_R) \xrightarrow{\cong} H_n^G(G/G; \mathbf{K}_R) = K_n(\mathcal{H}(G; R))$$

is an isomorphism.

- $H_*^G(-; \mathbf{K}_R)$ is a smooth G-homology theory satisfying $H_n^G(G/H); \mathbf{K}_R)) \cong K_n(\mathcal{H}(H; R))$ for every open subgroup $H \subseteq G$.

Theorem (Bartels-Lück (2022))

The Farrell-Jones Conjecture holds if G is a closed subgroup of a reductive p-adic group.